Solutions for homework 11

1. Section 8.3 An introduction to systems. Qualitative analysis.

(i) Plot the nullclines for each equation in the given system of differential equations. Use different colors or linestyles for the x-nullcline and the y-nullcline.
(ii) Calculate the coordinates of the equilibrium points. Plot each equilibrium point in your sketch from part (i) and label it with its coordinates.

1.

$$x' = 0.2x - 0.04xy$$
$$y' = -0.1y + 0.005xy$$

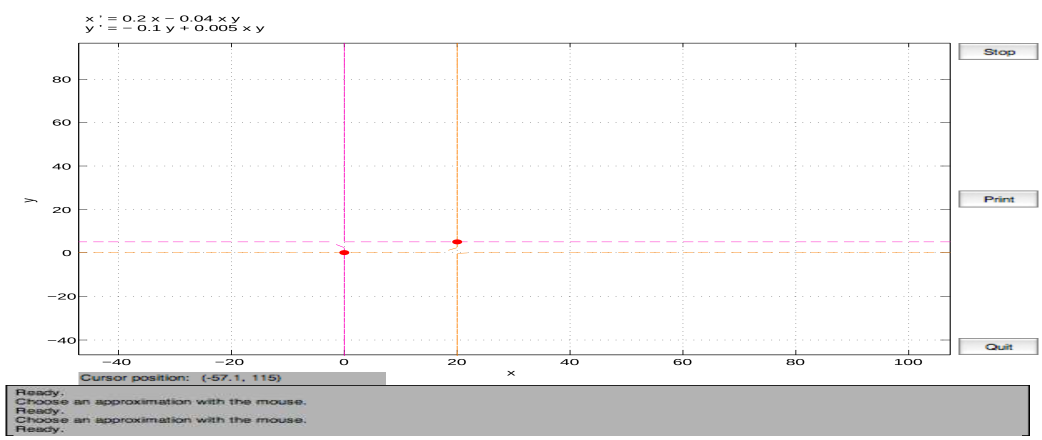
Solving the system of algebraic equations

$$\begin{aligned} 0.2x - \quad 0.04xy &= 0 \\ -0.1y + 0.005xy &= 0 \end{aligned}$$

we obtain the equilibrium points

$$(0,0) \qquad \text{and} \qquad (20,50).$$

Now we use `pplane8.m` to plot the nullclines and the equilibrium points:



3.

$$x' = x - y - x^3$$
$$y' = x$$

Similarly we solve the system of algebraic equations

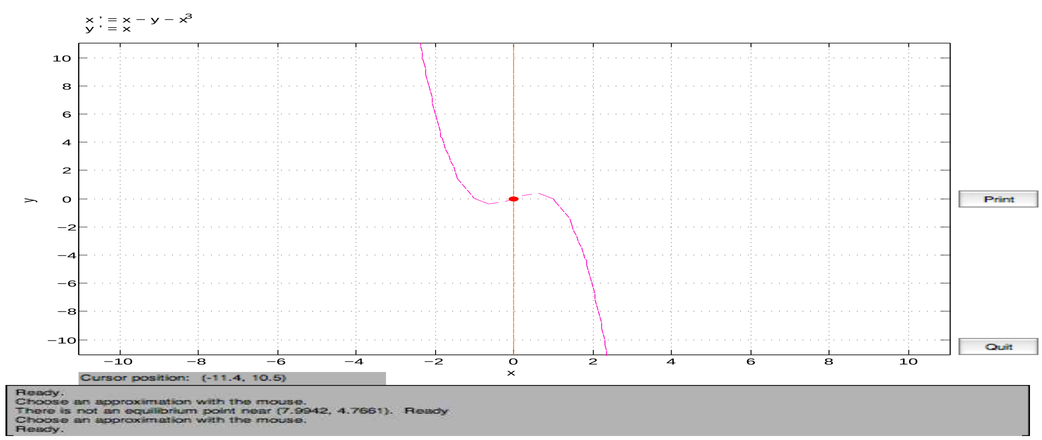
$$x - y - x^3 = 0$$

$$x = 0$$

we obtain the equilibrium point

$$(0, 0).$$

Now we use `pplane8.m` to plot the nullclines and the equilibrium point:



5.

$$\begin{aligned} x' &= y \\ y' &= -\sin x - y \end{aligned}$$

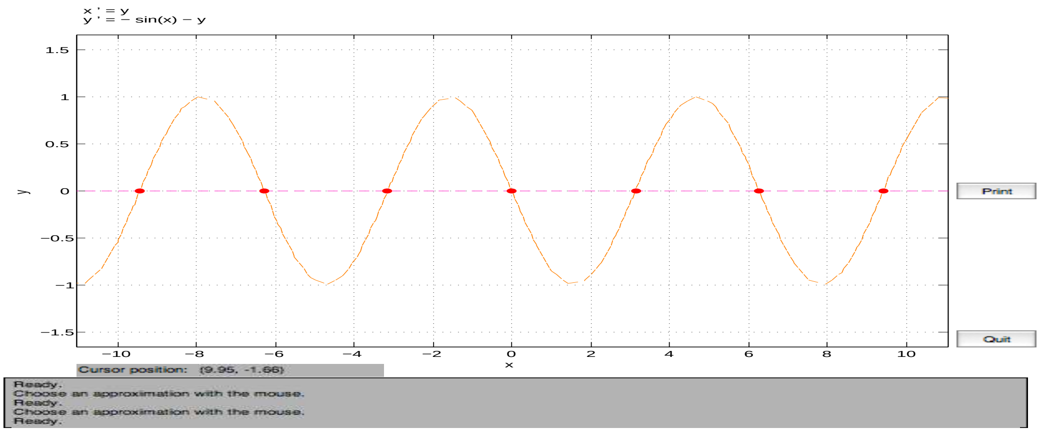
The system of algebraic equations

$$\begin{aligned} y &= 0 \\ -\sin x - y &= 0 \end{aligned}$$

gives the equilibrium points

$$(n\pi, 0), \text{ for all integers } n.$$

`pplane8.m` plots the nullclines and the equilibrium points:


x ' = y
y ' = − sin(x) − y
1.5
1
0.5
0
−0.5
−1
−1.5
y
−10
−8
−6
−4
−2
0
2
4
6
8
10
x
Print
Quit
Cursor position: (9.95, -1.66)
Ready.
Choose an approximation with the mouse.
Ready.
Choose an approximation with the mouse.
Ready.

2. Section 9.1 LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS. OVERVIEW OF THE TECHNIQUE.

3. *Use hand calculations to find the characteristic polynomial and eigenvalues for the matrix*

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -5 \end{pmatrix}.$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 7\lambda + 10,$$

and the eigenvalues are

$$\lambda_1 = -2, \qquad \lambda_2 = -5$$

real, distinct.

5. *Use hand calculations to find the characteristic polynomial and eigenvalues for the matrix*

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 - \lambda - 2,$$

and the eigenvalues are

$$\lambda_1 = -1, \qquad \lambda_2 = 2$$

real, distinct.

17. *Use hand calculations to find a fundamental set of solutions for the system* $\mathbf{y}' = A\mathbf{y}$*, where* A *is the matrix*

$$A = \begin{pmatrix} 6 & -8 \\ 0 & -2 \end{pmatrix}.$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 - 4\lambda - 12,$$

and the eigenvalues are

$$\lambda_1 = -2, \qquad \lambda_2 = 6$$

real, distinct.

For the eigenvalue $\lambda = -2$, the eigenspace is the nullspace of

$$A + 2I = \begin{pmatrix} 6+2 & -8 \\ 0 & -2+2 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ 0 & 0 \end{pmatrix},$$

which is generated by the vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For the eigenvalue $\lambda = 6$, the eigenspace is the nullspace of

$$A - 6I = \begin{pmatrix} 6-6 & -8 \\ 0 & -2-6 \end{pmatrix} = \begin{pmatrix} 0 & -8 \\ 0 & -8 \end{pmatrix},$$

which is generated by the vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Therefore a fundamental set of solutions is

$$\mathbf{y}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{y}_2(t) = e^{6t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

19. *Use hand calculations to find a fundamental set of solutions for the system* $\mathbf{y}' = A\mathbf{y}$, *where* A *is the matrix*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 2\lambda + 1,$$

and the eigenvalues are

$$\lambda_1 = -1, \qquad \lambda_2 = -1$$

real, repeated.
Therefore a fundamental set of solutions is

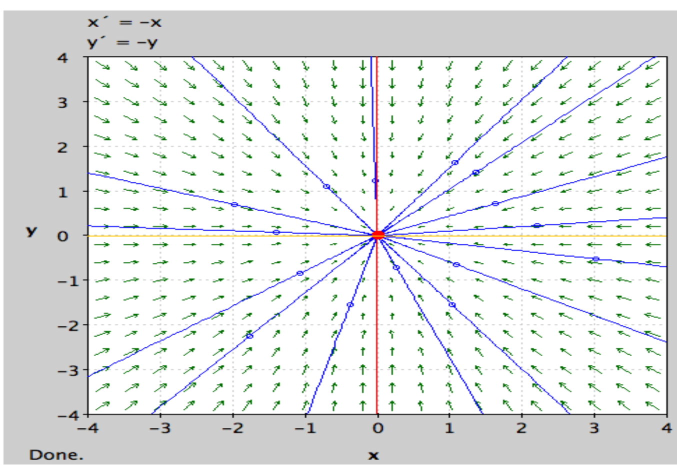
$$\mathbf{y}_1(t) = e^{-t}\mathbf{y}_0 \qquad \text{and} \qquad \mathbf{y}_2(t) = te^{-t}\begin{pmatrix} -1+1 & 0 \\ 0 & -1+1 \end{pmatrix}\mathbf{y}(0) = \mathbf{0},$$

hence

$$\mathbf{y}_1(t) = e^{-t}\mathbf{y}_0$$

for any nonzero vector $\mathbf{y}_0$.

`Note that the given system is a decoupled system of DE` $\begin{cases} y_1' = -y_1 \\ y_2' = -y_2 \end{cases}$



3. Section 9.2 LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS: PLANAR SYSTEMS.

3. *Find the general solution of the system* $\mathbf{y}' = A\mathbf{y}$.

$$A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix}.$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 + 7\lambda + 12,$$

and the eigenvalues are

$$\lambda_1 = -3, \qquad \lambda_2 = -4$$

real, distinct.
For the eigenvalue $\lambda = -3$, the eigenspace is the nullspace of

$$A + 3I = \begin{pmatrix} -5+3 & 1 \\ -2 & -2+3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix},$$

which is generated by the vector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
For the eigenvalue $\lambda = -4$, the eigenspace is the nullspace of

$$A + 4I = \begin{pmatrix} -5+4 & 1 \\ -2 & -2+4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix},$$

which is generated by the vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Therefore the general solution is

$$\mathbf{y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

13. *The complex values vector* $\mathbf{z}(t)$ *is given. Find the real and imaginary parts of* $\mathbf{z}(t)$.

$$\mathbf{z}(t) = e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

`Solution.` Using Euler's formula, we have

$$\begin{aligned} \mathbf{z}(t) &= e^{(0+2i)t}\begin{pmatrix} 1+0i \\ 1+1i \end{pmatrix} \\ &= e^{0t}(\cos 2t + i\sin 2t)\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\ &= \left\{\cos 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} + i\left\{\cos 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \end{aligned}$$

therefore the real part is

$$Re\big(\mathbf{z}(t)\big) = \cos 2t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix}$$

and the imaginary part is

$$Im\big(\mathbf{z}(t)\big) = \cos 2t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin 2t \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}.$$

15. *The system*

$$\mathbf{y}' = \begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \mathbf{y} \tag{3.1}$$

has complex solution

$$\mathbf{z}(t) = e^{3it} \begin{pmatrix} -1-i \\ 2 \end{pmatrix}.$$

Verify, by direct substitution, that the real and the imaginary parts of this solution are solutions of system (3.1). Then use Proposition 5.12 in Section 8.5 to verify that they are linearly independent solutions.
`Solution.` Using Euler's formula, we have

$$\begin{aligned} \mathbf{z}(t) &= e^{(0+3i)t} \begin{pmatrix} -1-i \\ 2+0i \end{pmatrix} \\ &= (\cos 3t + i \sin 3t) \left[\begin{pmatrix} -1 \\ 2 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] \\ &= \left\{ \cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} + i \left\{ \cos 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}, \end{aligned}$$

therefore the real part is

$$\mathbf{y}_1(t) = \cos 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos 3t + \sin 3t \\ 2\cos 3t \end{pmatrix}$$

and the imaginary part is

$$\mathbf{y}_2(t) = \cos 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\cos 3t - \sin 3t \\ 2\sin 3t \end{pmatrix}.$$

We have that

$$\mathbf{y}_1'(t) = \begin{pmatrix} -\cos 3t + \sin 3t \\ 2\cos 3t \end{pmatrix}' = \begin{pmatrix} 3\sin 3t + 3\cos 3t \\ -6\sin 3t \end{pmatrix}$$

and

$$\mathbf{y}_2'(t) = \begin{pmatrix} -\cos 3t - \sin 3t \\ 2\sin 3t \end{pmatrix}' = \begin{pmatrix} 3\sin 3t - 3\cos 3t \\ 6\cos 3t \end{pmatrix}.$$

On the other hand

$$\begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \mathbf{y}_1(t) = \begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} -\cos 3t + \sin 3t \\ 2\cos 3t \end{pmatrix} = \begin{pmatrix} 3\sin 3t + 3\cos 3t \\ -6\sin 3t \end{pmatrix},$$

and respectively

$$\begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \mathbf{y}_2(t) = \begin{pmatrix} 3 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} -\cos 3t - \sin 3t \\ 2\sin 3t \end{pmatrix} = \begin{pmatrix} 3\sin 3t - 3\cos 3t \\ 6\cos 3t \end{pmatrix},$$

which shows that the real and imaginary parts are solutions of the given system.
To check the linear independence of $\mathbf{y}_1(t), \mathbf{y}_2(t)$ it is sufficient to check it at one point. At $t = 0$,

$$\mathbf{y}_1(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{y}_2(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

which are linearly independent:

$$det \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix} = 2 \neq 0.$$

59. *Figure 8 shows two tanks, each containing 360 liters of salt solution. Pure water pours into tank A at a rate of 5L/min. There are two pipes connecting tank A to tank B. The first pumps salt from tank B into tank A at a rate of 4L/min. The second pumps salt solution from tank A into tank B at a rate of 9L/min. Finally, there is a drain on tank B from which salt solution drains at a rate of 5L/min. Thus, each tank maintains a constant volume of 360 liters of salt solution. Initially there are 60 kg of salt present in tank A, but tank B contains pure water.*

(a) Set up, in matrix-vector form, an initial value problem that models the salt content in each tank over time.

(b) Find the eigenvalues and eigenvectors of the coefficient matrix in part (a)*, the find the general solution in vector form. Find the solution that satisfies the initial conditions posed in part* (a)*.*

(c) Plot each component of your solution in part (b) *over a period of four time constants (see Section 4.7 or Section 2.2, Exercise 29)* $[0, 4T_c]$*. what is the eventual salt content in each tank? Why? Give both a physical and mathematical reason for your answer.*

`Solution.`

(a) Let $x(t)$= salt in 1st tank, $y(t)$= salt in 2nd tank.

The initial conditions are then $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$, and the equations write

$$\frac{dx}{dt} = \frac{y}{360}4 - \frac{x}{360}9$$

$$\frac{dy}{dt} = \frac{x}{360}9 - \frac{9}{360}(5+4)$$

hence

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -\frac{1}{40} & \frac{1}{90} \\ \frac{1}{40} & -\frac{1}{40} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) The characteristic equation is

$$0 = \lambda^2 - \lambda\, \mathtt{Tr}(A) + \mathtt{det}(A) \equiv \lambda^2 - \lambda(\frac{-2}{40}) + (\frac{1}{1600} - \frac{1}{3600}) = \lambda^2 + \lambda(\frac{1}{20}) + \frac{5}{14400},$$

hence the eigenvalues/eigenvectors are

$$\lambda_1 = -\frac{1}{24}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\lambda_2 = -\frac{1}{120}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

The general solution is

$$\mathbf{y}(t) = c_1 e^{-\frac{t}{24}} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 e^{-\frac{t}{120}} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

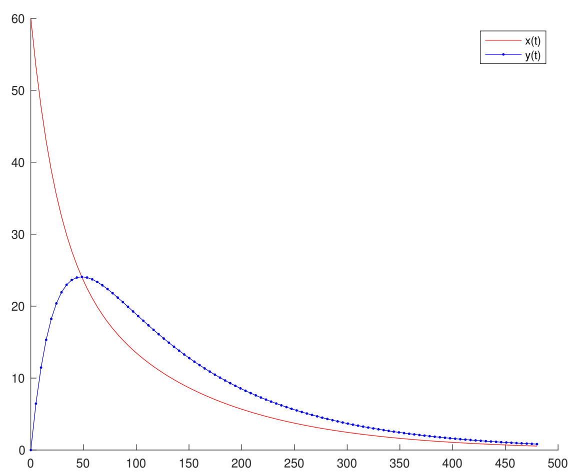
The solution corresponding to the initial condition is obtained from

$$\begin{pmatrix} 60 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

i.e., $c_1 = c_2 = 15$, hence

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 15e^{-\frac{t}{24}} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + 15e^{-\frac{t}{120}} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 30e^{-\frac{t}{24}} + 30e^{-\frac{t}{120}} \\ -45e^{-\frac{t}{24}} + 45e^{-\frac{t}{120}} \end{pmatrix}$$

(c) $T_{c_1} = 24, T_{c_2} = 120$, therefore we plot the solution on $[0, 4T_{c_2}] = [0, 480]$.



41. *Find the general solution of the system* $\mathbf{y}' = A\mathbf{y}$ *for the matrix*

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}.$$

`Solution.` The characteristic polynomial is

$$\lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 - 8\lambda + 16,$$

and the eigenvalues are

$$\lambda_1 = \lambda_2 = 4$$

real, repeated.
The general solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{4t}\mathbf{y}_0 + te^{4t}(A - 4I)\mathbf{y}_0 \\ &= e^{4t}\mathbf{y}_0 + te^{4t}\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\mathbf{y}_0, \end{aligned}$$

for any vector $\mathbf{y}_0$ such that $\mathbf{y}(t) = \mathbf{y}_0$.
49. *Find the solution of the initial value problem for system* $\mathbf{y}' = A\mathbf{y}$ *with matrix*

$$A = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix}$$

and the initial value $\mathbf{y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
`Solution.` The solution is

$$\begin{aligned} \mathbf{y}(t) &= e^{4t}\mathbf{y}_0 + te^{4t}\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\mathbf{y}_0 \\ &= e^{4t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + te^{4t}\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= e^{4t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + te^{4t}\begin{pmatrix} -2 \\ -1 \end{pmatrix} \\ &= e^{4t}\begin{pmatrix} 3 - 2t \\ 1 - t \end{pmatrix}. \end{aligned}$$