Solutions for homework 10

1. Section 5.7 CONVOLUTIONS.

DEFINITION 1.1. *The convolution of two piecewise continuous functions f and g is the function $f * g$ defined by*

$$f * g(t) = \int_0^t f(u)g(t-u)du.$$

6. *Use the Definition 1.1 to calculate the convolution of the given function*

$$f(t) = t - 1, \qquad g(t) = t - 2.$$

From the definition we have

$$\begin{aligned} f * g\,(t) &= \int_0^t \underbrace{f(u)}_{u-1}\underbrace{g(t-u)}_{(t-u)-2}\,du = \int_0^t (u-1)(t-u-2)du = \int_0^t (u-1)(t-u-2)du \\ &= \int_0^t (ut - u^2 - 2u - t + u + 2)\,du = \int_0^t (-u^2 - u + ut - t + 2)\,du \\ &= -\frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{2}t^2 t - t^2 + 2t = \frac{1}{6}t^3 - \frac{3}{2}t^2 + 2t. \end{aligned}$$

8. *Use the Definition to calculate the convolution of the given function*

$$f(t) = t, \qquad g(t) = e^t.$$

Using the definition and integration by parts we obtain

$$\begin{aligned} f * g\,(t) &= \int_0^t \underbrace{f(u)}_{u}\underbrace{g(t-u)}_{e^{t-u}}\,du = \int_0^t ue^{t-u}du = e^t\int_0^t ue^{-u}du \\ &= -e^t\int_0^t u(e^{-u})'du = -e^t\Big(ue^{-u}\Big|_{u=0}^{u=t} - \int_0^t e^{-u}du\Big) \\ &= -e^t\Big(te^{-t} - \int_0^t (-e^{-u})'du\Big) = -e^t\Big(te^{-t} + e^{-u}\Big|_0^t\Big) \\ &= -e^t\Big(te^{-t} + e^{-t} - 1\Big) = -t - 1 + e^t. \end{aligned}$$

10. *Let*

$$f(t) = e^t \qquad \text{and} \qquad g(t) = e^{2t}.$$

Let $F(s)$ and $G(s)$ be the Laplace transforms of $f(t)$ and $g(t)$, respectively.
(a) *Compute $F(s)G(s)$.*
(b) *Compute $\mathcal{L}\{f(t)g(t)\}$ and compare with the result fond in part (a). What point is made by this example?*
Solution:

(a) Since

$$F(s) = \mathcal{L}\{e^t\}(s) = \frac{1}{s-1},$$

and

$$G(s) = \mathcal{L}\{e^{2t}\}(s) = \frac{1}{s-2}$$

we have

$$F(s)G(s) = \frac{1}{(s-1)(s-2)}.$$

(b) Using the properties of the Laplace transform we have

$$\mathcal{L}\{f(t)g(t)\}(s) = \mathcal{L}\{e^t e^{2t}\}(s) = \mathcal{L}\{e^{3t}\}(s) = \frac{1}{s-3}$$

and conclude that

$$\mathcal{L}\{f\}\mathcal{L}\{g\} \neq \mathcal{L}\{fg\},$$

namely: **the Laplace transform of the product of two functions is NOT the product of the Laplace transforms of each respective function**.

2. Section 8.1 An introduction to systems: Definitions and examples.

5. *Indicate the dimension and whether or not the system*

$$\begin{aligned}
u_1' &= u_2\\
u_2' &= -\frac{1}{2}u_1 + \frac{1}{2}u_3\\
u_3' &= u_4\\
u_4' &= \frac{3}{2}u_1 + \frac{1}{2}u_3
\end{aligned}$$

is autonomous. Assume the independent variable is t.
This is a linear autonomous system of dimension 4.

7. *Show that the functions $x(t) = 2e^{2t} - 2e^{-t}$ and $y(t) = -e^{-t} + 2e^{2t}$ are solution of the system*

$$\begin{aligned}
x' &= -4x + 6y\\
y' &= -3x + 5y,
\end{aligned}$$

satisfying the initial conditions $x(0) = 0$ and $y(0) = 1$.
The given functions

$$x(t) = 2e^{2t} - 2e^{-t} \text{ and } y(t) = -e^{-t} + 2e^{2t}$$

satisfy the initial conditions:

$$\begin{aligned}
x(0) &= 2e^0 - 2e^{-0} = 2 - 2 = 0,\\
y(0) &= -e^{-0} + 2e^0 = -1 + 2 = 1
\end{aligned}$$

By calculating the derivatives

$$\begin{aligned}
x'(t) &= (2e^{2t} - 2e^{-t})' = 4e^{2t} + 2e^{-t}\\
y'(t) &= (-e^{-t} + 2e^{2t})' = e^{-t} + 4e^{2t}
\end{aligned}$$

and substituting in the system

$$\begin{aligned}
-4x(t) + 6y(t) &= -4(2e^{2t} - 2e^{-t}) + 6(-e^{-t} + 2e^{2t}) = 2e^{-t} + 4e^{2t}\\
-3x(t) + 5y(t) &= -3(2e^{2t} - 2e^{-t}) + 5(-e^{-t} + 2e^{2t}) = 4e^{2t} + e^{-t}
\end{aligned}$$

we obtain the result.

13. *Write the initial value problem*

$$x'' + \delta x' - x + x^3 = \gamma \cos \omega t, \qquad x(0) = x_0, x'(0) = v_0$$

as a system of first-order equations using vector notation.

We define the new variables

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

and obtain the linear system

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} u_2 \\ -\delta u_2 + u_1 - u_1^3 + \gamma \cos \omega t \end{pmatrix}$$

and initial conditions

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}.$$

15. *Write the initial value problem*

$$\omega''' = \omega, \qquad \omega(0) = \omega_0, \omega'(0) = \alpha_0, \omega''(0) = \gamma_0$$

as a system of first-order equations using vector notation.

With the new variables

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \omega \\ \omega' \\ \omega'' \end{pmatrix}$$

we obtain the linear system

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}' = \begin{pmatrix} u_2 \\ u_3 \\ u_1 \end{pmatrix}$$

with the initial conditions

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}.$$

3. Section 8.2 An introduction to systems. Geometric interpretation of solutions.

11.

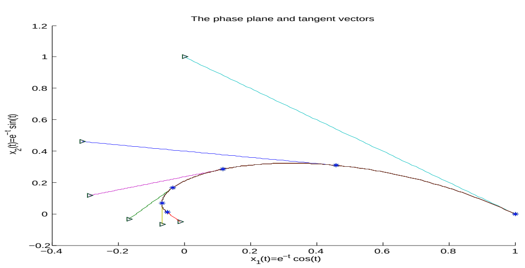
$$\mathbf{x}(t) = \begin{pmatrix} e^{-t}\cos t \\ e^{-t}\sin t \end{pmatrix}.$$

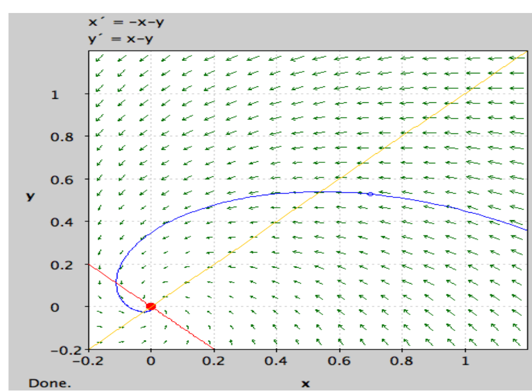
(i) Find the derivative of $\mathbf{x}(t)$*.*
(ii) Sketch the curve $t \to (x_1(t), x_2(t))^T$ *in the* x_1x_2 *phase plane. At incremental points along the curve, use the derivative of* $\mathbf{x}(t)$ *to calculate and plot the tangent vector to the curve*

(i) The derivative is

$$\mathbf{x}'(t) = \begin{pmatrix} (e^{-t}\cos t)' \\ (e^{-t}\sin t)' \end{pmatrix} = \begin{pmatrix} -e^{-t}\cos t + e^{-t}(-\sin t) \\ -e^{-t}\sin t + e^{-t}\cos t \end{pmatrix} = \begin{pmatrix} -e^{-t}(\cos t + \sin t) \\ e^{-t}(-\sin t + \cos t) \end{pmatrix}$$

(ii) The tangent vectors are plotted in figure.

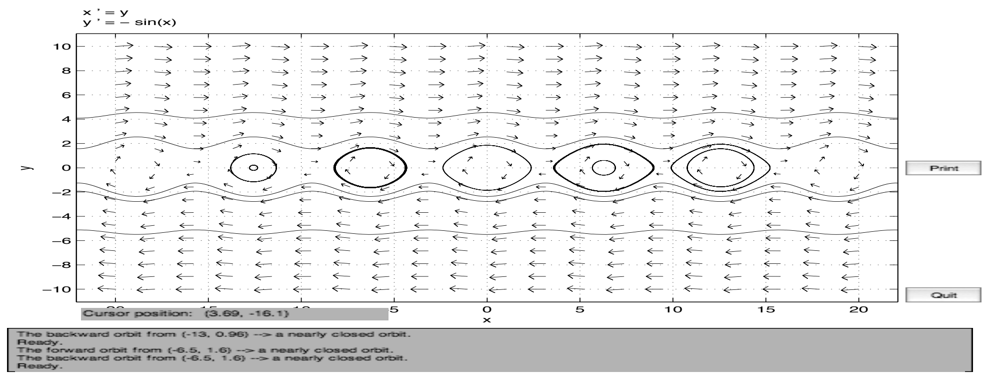




13. *Use your numerical solver to draw the direction field for the given planar, autonomous system*

$$\theta' = \omega \qquad \text{and} \qquad \omega' = -\sin\theta.$$

Superimpose solution trajectories for several initial conditions of your choice.
Using `pplane8.m` for the system above one gets the following direction field



15. *Use your numerical solver to draw the direction field for the given planar, autonomous system*

$$x' = (0.4 - 0.01y)x \qquad \textit{and} \qquad y' = (0.005x - 0.3)y.$$

Superimpose solution trajectories for several initial conditions of your choice.
Using `pplane8.m` for the system above one gets the following direction field