MATH 1080 Midterm Exam

sample problems

10:00 to 10:50

Instructions

1. You may use a one page formula sheet. Formula sheets may not be shared.
2. Before you begin, enter your name in the space below.
3. Show all your work on the exam itself. If you need additional space, use the backs of the pages.
4. You may not use books or notes on the exam.

Name	

1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. Carry out Gaussian elimination with scaled partial pivoting on the matrix:

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 4 \\ 5 & 2 & 0 \end{bmatrix}$$

Show intermediate matrices.

2. Carry out Gaussian elimination with scaled partial pivoting on the matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix}$$

Show intermediate matrices.

3. The lower triangular matrix L in the LU decomposition of matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.40000 & 1 & 0 \\ 0.32000 & 1.7333 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 25 & 5 & 4 \\ 0 & 6 & 14.400 \\ 0 & 0 & -4.2400 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 10 & 1 & 0 \\ 8 & 12 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 8 & 12 & 1 \end{bmatrix}$$

4. The lower triangular matrix L in the LU decomposition of matrix given below

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & -3 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

5. (a) The lower triangular matrix L in the LU decomposition of matrix given below

$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

is

i. $$\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

ii. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

iii. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & -3 & 1 \end{bmatrix}$$

iv. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

(b) Compute the determinant

$$\det(A) = \dots$$

of the matrix A, which has the following factorization $A = LDL^T$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 7 & 1 & 0 & 0 \\ -4 & 3 & 1 & 0 \\ 13 & -6 & 5 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

6. (a) The conditioning number of matrix A is $\kappa(A) := \|A\| \cdot \|A^{-1}\|$.
The matrix A is said to be *ill-conditioned* (yielding a linear system $Ax = b$ sensitive to perturbations in the elements of A or components of b) if the conditioning number (choose one option):

$\kappa(A) = -1$; $\kappa(A) = 0$; $\kappa(A) = 1$; $\kappa(A) =$ large number.

(b) In order to carry out the 1st step of Gaussian elimination with scaled partial pivoting on the matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix},$$

determine:

- the scale vector $s = [\quad,\quad,\quad,\quad]$;
- the pivot row $\{\quad,\quad,\quad,\quad\}$;
- the new index vector $\ell = [\quad,\quad,\quad,\quad]$.

(c) **TRUE** or **FALSE**:

For diagonally dominant tridiagonal coefficient matrices, partial pivoting is NOT necessary because zero divisors will NOT be encountered.

Recall: $A = (a_{ij})_{n\times n}$ is diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq i} |a_{ij}|, \quad \forall 1 \leq i \leq n.$$

7. To ensure that the following system of equations

$$\begin{aligned} 3x_1 + 7x_2 - 11x_3 &= 6 \\ x_1 + 4x_2 + x_3 &= -5 \\ 9x_1 + 5x_2 + 2x_3 &= 17 \end{aligned}$$

converges using Gauss-Seidel method, one can write the above equations as follows:

(a)

$$\begin{bmatrix} 3 & 7 & -11 \\ 1 & 4 & 1 \\ 9 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 9 & 5 & 2 \\ 1 & 4 & 1 \\ 3 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \\ 6 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 9 & 5 & 2 \\ 1 & 4 & 1 \\ 3 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$$

(d) The equations cannot be written in a form to ensure convergence.

8. Necessary and sufficient conditions for SOR method, with $0 < \omega < 2$, to work on the linear system $\mathbf{Ax} = \mathbf{b}$.

   a. $\mathbf{A}$ is diagonally dominant.

   b. $\rho(\mathbf{A}) < \mathbf{1}$.

   c. $\mathbf{A}$ is symmetric positive definite.

   d. $\mathbf{x}^{(0)} = 0$.

   e. None of these.

9. Compute the first two steps of the Jacobi and the Gauss-Seidel methods with starting vector $[0, \ldots, 0]^T$.

   (a) $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$
   (*Hint* : Solution Jacobi = [7/3 ; 17/6], GS = [47/18 ; 119/36] .)

   (b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

10. Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of the Jacobi and the Gauss-Seidel methods from the starting vector $[0, \ldots, 0]^T$.

(a) $\left\{ \begin{array}{rcrcr} x_1 & + & 3x_2 & = & -1 \\ 5x_1 & + & 4x_2 & = & 6 \end{array} \right.$

(b) $\left\{ \begin{array}{rcrcrcr} x_1 & - & 8x_2 & - & 2x_3 & = & 1 \\ x_1 & + & x_2 & + & 5x_3 & = & 4 \\ 3x_1 & - & x_2 & - & x_3 & = & -2 \end{array} \right.$

(c) $\left\{ \begin{array}{rcrcrcr} x_1 & + & 4x_2 & & & = & 5 \\ & & x_2 & + & 2x_3 & = & 2 \\ 4x_1 & & & + & 3x_3 & = & 0 \end{array} \right.$

11. (a) (Jacobi iteration) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$. Starting with initial vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, carry out **one** Jacobi iteration.

(b) To ensure that the following system of equations

$$\begin{cases} 3x_1 + 7x_2 - 11x_3 = 6 \\ x_1 + 4x_2 + x_3 = -5 \\ 9x_1 + 5x_2 + 2x_3 = 17 \end{cases}$$

converges using Gauss-Seidel method, one can write the above equations as follows:

i.

$$\begin{bmatrix} 3 & 7 & -11 \\ 1 & 4 & 1 \\ 9 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$$

ii.

$$\begin{bmatrix} 9 & 5 & 2 \\ 1 & 4 & 1 \\ 3 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \\ 6 \end{bmatrix}$$

iii.

$$\begin{bmatrix} 9 & 5 & 2 \\ 1 & 4 & 1 \\ 3 & 7 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 17 \end{bmatrix}$$

iv. The equations cannot be written in a form to ensure convergence.

(c) Necessary and sufficient conditions for SOR method, with $0 < \omega < 2$, to work on the linear system $\mathbf{Ax} = \mathbf{b}$.

i. $\mathbf{A}$ is diagonally dominant.
ii. $\rho(\mathbf{A}) < \mathbf{1}$.
iii. $\mathbf{A}$ is symmetric positive definite.
iv. $\mathbf{x}^{(0)} = 0$.
v. None of these.

12. Solve the problems by carrying out the Conjugate Gradient method by hand (with starting vector $[0, \ldots, 0]^T$).

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

($Hint$: Solution (a) = [3 ; -1], Solution (b) = [-1 ; 1] .)

(c) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

($Hint$: Solution (c) = [1 ; 1], Solution (d) = [-1 ; 1] .)

13. (a) Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 6 & -8 \\ 0 & -2 \end{bmatrix}.$$

(b) **TRUE** or **FALSE**::
Gershgorin's Theorem asserts that every eigenvalue λ *of an* $n \times n$ *matrix* A *must satisfy one of these inequalities:*

$$|\lambda - a_{ii}| = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad \text{for } 1 \leq i \leq n.$$

(c) The sum of the eigenvalues of the matrix $\begin{bmatrix} 1+i & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1-i \end{bmatrix}$ is (choose one of the following options):

$$1, \qquad 1+i, \qquad 1-i, \qquad 2, \qquad 3+i, \qquad 4-i.$$

(d) To compute the eigenvalue closest to a given number $\mu \in \mathbb{C}$, one should produce a sequence by $\{x^k\}_{k\geq 0}$ starting from a given x^0 by (choose) one of the following iterations:

i. $x^{k+1} = x^k$
ii. $x^{k+1} = Ax^k$
iii. $x^{k+1} = (A - \mu I)x^k$
iv. $(A - \mu I)x^{k+1} = x^k$

14. (a) Prove that if $d > 4$, the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & d \end{bmatrix}$ is positive-definite.

(b) Find all numbers d such that $A = \begin{bmatrix} 1 & -2 \\ -2 & d \end{bmatrix}$ is positive-definite.

15. Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 7 \\ 2 & -5 \end{bmatrix}$$

B. (*True or false*) Gershgorin's Theorem asserts that every eigenvalue λ of an $n \times n$ matrix A must satisfy one of these inequalities:

$$|\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad \text{for } 1 \leq i \leq n.$$

16. To compute the eigenvalue closest to a given number μ one should produce a sequence by

    (a) $x^{k+1} = x^k$

    (b) $x^{k+1} = Ax^k$

    (c) $x^{k+1} = (A - \mu I)x^k$

    (d) $(A - \mu I)x^{k+1} = x^k$

17. If A is a 6×6 matrix with eigenvalues $-6, -3, 1, 2, 5, 7$, which eigenvalue of A will the following algorithms find?

    (a) Power Iteration

    (b) Inverse Power Iteration with shift $s = 4$

    (c) Find the linear convergence rates of the two computations. Which converges faster?

18. Euler's method can be derived from using first two terms of Taylor series of writing the value of y_{i+1}, that is the value of y at x_{i+1}, in terms of y_i and all the derivatives of y at x_i. If $h = x_{i+1} - x_i$, the explicit expression for y_{i+1} if the first three terms of the Taylor series are chosen for the ordinary differential equation

$$2\frac{dy}{dx} + 3y = e^{-5x}, y(0) = 7$$

would be

(a)

$$y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h$$

(b)

$$y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h - \left(\frac{5}{2}e^{-5x_i}\right)\frac{h^2}{2}$$

(c)

$$y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h + \left(-\frac{13}{4}e^{-5x_i} + \frac{9}{4}y_i\right)\frac{h^2}{2}$$

(d)

$$y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h - 3y_i\frac{h^2}{2}.$$

19. A. In solving a differential equation by Taylor series method of order 2 one needs to determine x'' when $x' = xt^2 + x^3 + e^x t$.

$x'' =$

B. Put the differential equation

$$x + 2xx' - x' = 0$$

in a form suitable for numerical solution by the Runge-Kutta method.

20. (a) In solving a differential equation by Taylor series method of order 2, one needs to determine x'' when $x' = xt^2 + x^2 + t\sin(x)$.

$x'' =$

(b) Compute $x(2)$ using the (forward-explicit) Euler method for

$$x'(t) = x(t),$$
$$x(0) = 1$$

with $\Delta t = 1$.

21. (a) The conditioning number of matrix A is $\kappa(A) := \|A\| \cdot \|A^{-1}\|$.
The matrix A is said to be *well-conditioned* (yielding a linear system $Ax = b$ insensitive to perturbations in the elements of A or components of b) if the conditioning number (choose one option):

$\kappa(A) =$ negative number smaller than -1;

$\kappa(A) =$ positive number smaller than 1;

$\kappa(A) = 1000$;

$\kappa(A) =$ small positive number larger than 1.

(b) Prove that

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

does not have an LU factorization.

22. (a) Determine whether the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix}$ is strictly diagonally dominant.

(b) Apply one step of the Gauss-Seidel method to the system

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

with the initial guess $\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) Will the Gauss-Seidel iteration converge to the solution $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$? Justify your answer.

23. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) Apply three steps of Power Iteration Method with initial vector $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) At each step, approximate the eigenvalue by the current Rayleigh quotient $\lambda_i = \frac{x_i^T A x_i}{x_i^T x_i}$.

(d) Predict the result of applying the Inverse Power Iteration with shift $\mu = 0$.

24. (a) Find a polynomial p with the property $p - p' = t^3 + t^2 - 2t$.

$p(t) =$

(b) Compute $x(2)$ using the (forward-explicit) Euler method for

$$x'(t) = -2x(t) + 1,$$
$$x(0) = 0$$

with $\Delta t = 1$.