MATH 1080: Numerical Linear Algebra

Take-Home Final, April 2024

1. Consider the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, & \forall x \in (0,1), t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0 & \forall t > 0 \\ u(x,0) = \sin(\pi x) & \forall x \in [0,1], \end{cases}$$

   (a) Show that

$$u_{\text{exact}}(x,t) = e^{-\pi^2 t} \sin(\pi x).$$

   is the exact solution (i.e., it satisfies the equation, the boundary conditions, and the initial condition).

   (b) Solve using a forward-Euler in time and central-difference in space discretization on $x \in (0,1)$ and $t \in [0, 0.2]$ with $m = 900$ intervals in time, and different $n = 4, 8, 16, 32$ space intervals.

   - Plot the solutions and the errors in time and space.
   - Show that the ratios of the errors corresponding to consecutive finer space meshes $\mathcal{E}(8)/\mathcal{E}(4)$, $\mathcal{E}(16)/\mathcal{E}(8)$, and $\mathcal{E}(32)/\mathcal{E}(16)$ are about 4.

$$\mathcal{E}(m) = \left(\frac{0.2}{m} \sum_{j=1}^{m} \frac{1}{n} \sum_{i=1}^{n} \left| u_{ij} - u_{\text{exact}}(x_i, t_j) \right|^2 \right)^{1/2}.$$

   (Hint: see the code `PDE_heat_Forward_Difference`.)

2. Consider the (elliptic) Poisson problem:

$$\begin{cases} \Delta u(x,y) = f(x,y), & \forall (x,y) \in \Omega = [0,1] \times [1,2] \\ u(0,y) = \sin(7\pi y), \quad u(1,y) = -\sin(7\pi y), \quad u(x,1) = 0, \quad u(x,2) = 0, \end{cases}$$

   where the right-hand side function is $f(x,y) = -74\pi^2 \cos(5\pi x) \cdot \sin(7\pi y)$.

   (a) Show that

$$u(x,y) := \cos(5\pi x) \cdot \sin(7\pi y), \qquad 0 \le x \le 1, \quad 1 \le y \le 2,$$

   is the exact solution to this Poisson problem, i.e., it satisfies the equation and all the boundary conditions (on the left, right, top and bottom sides).

   (b) Discretize the square spatial domain $\Omega = [0,1] \times [1,2]$ into M subintervals on the x-axis and N subintervals on the y axis.
   Take $M = N = 10, 20, 40, 80$, use the Finite Difference Method and plot the numerical solution for each mesh.

(c) Compute the errors for each mesh and show that the errors decay by a factor of $1/4$ each time the mesh size is halved.
(Hint: see the code `PDE_Poisson`. Compute and record $\log\left(\mathcal{E}(10)/\mathcal{E}(20)\right)/\log(2)$, $\log\left(\mathcal{E}(20)/\mathcal{E}(40)\right)/\log(2)$, $\log\left(\mathcal{E}(40)/\mathcal{E}(80)\right)/\log(2)$, and comment on how these values behave with respect to the theoretical value of the convergence rate, namely $p = 2$.)

3. Solve the elliptic (boundary value) problem:

$$\begin{cases} -u''(x) + u(x) = f(x), \quad 0 < x < 1, \\ u(0) = 1, u(1) = -1, \end{cases}$$

with the function $f(x) = \pi^2 \cos(\pi x) + \cos(\pi x)$.
Discretize the interval $(0, 1)$ into N subintervals with $N = 4, 8, 16, 32$.

(a) Plot the numerical solution for each mesh.

(b) Compute the 'root-mean-square error'

$$E = \left(\frac{1}{N} \sum_{i=1}^{N} \left(u(x_i) - u_h(x_i)\right)^2 \right)^{1/2}$$

for each mesh, knowing that the exact solution is $u(x) = \cos(\pi x)$.

(c) Show that the numerical method converges with the correct rate.

You will do the computations using the Finite Difference Method.

4. (*extra credit*) The Euler-Bernoulli beam is a simple model for bending under stress. The vertical displacement of the beam is represented by a function $y(x)$ where $0 \leq x \leq L$ along the beam of length L. The displacement $y(x)$ satisfies the Euler-Bernoulli equation

$$EIy''''(x) = f(x)$$

where E, the Young's modulus of the material and I, the area moment of inertia, are constant along the beam. The right-hand side $f(x)$ is the applied load, including the weight of the beam, in force per unit length. Consider the boundary conditions

$$y(0) = y'(0) = y(L) = y'(L) = 0,$$

which model a beam fixed at both end-points.

Using a finite diference discretization, set $h = L/N$ with N positive integer. Let $x_i = ih$ for $i = 0, \ldots, N$. For the nodes $x_2, \ldots, x_{N-2}$, consider the following approximation of y'''':

$$y'''' \approx \frac{y(x-2h) - 4y(x-h) + 6y(x) - 4y(x+h) + y(x+2h)}{h^4}.$$

For the first interior node x_1 consider the following approximation of y'''':

$$y'''' \approx \frac{12y(x) - 6y(x+h) + \frac{4}{3}y(x+2h)}{h^4}.$$

For the last interior node x_{N-1} consider the following approximation of y'''':

$$y'''' \approx \frac{-12y(x) + 6y(x-h) + \frac{4}{3}y(x-2h)}{h^4}.$$

(a) Write down the resulting linear system.
(b) Consider a solid steel beam of length $L = 10$ meters, with depth $d = .05$ meters and width $b = 0.1$ meters. Young's modulus of steel is approximately 2×10^{11} Newton$/m^2$. The area moment of inertia I is $bd^3/12$. Assume that the force $f(x)$ represents only the weight of the beam itself. The weight of the beam per meter of length is the constant $7850bd$ multiplied by the acceleration of gravity 9.81. Solve the system for displacements $y_i \approx y(x_i)$ with $N = 10$ and with both the Jacobi and Gauss-Seidel methods. Which one converges? How many steps are required to converge to six decimal places?