3102 Homework set II (due Feb. 6)

You need to work out one of the three problems to get full credits. You should choose the suitably challenging one for your own sake. You are of course encouraged to work out as much as you can.

Level I Problem:
(1) Look up the values of the CKM matrix elements from the review by the *Particle Data Group*. Notice the hierarchical structure of the inter-generation entries, such as V_{ud}, V_{us}, V_{ub} (which are contrasted with the mixing elements of MNS in the neutrino sector). Suggest some physical processes to determine each one of the three mixing quantities. [You could search from the literature and find the processes that have been adopted to best measure those CKM elements.]

(2) The "*Exercise*" on page 57 of *Collider Physics*.

Level II Problem:
(1) Let Φ be an SU(2) doublet that transforms as

$$\Phi \to \Phi' = U\Phi, \quad U = \exp(-\frac{i}{2}g\vec{\alpha} \cdot \vec{\sigma}),$$

show that so does $\tilde{\Phi} = \epsilon\Phi^*$, where $\epsilon = i\sigma_2$. [You may use the property of infinitesimal transformations.]

(2) Based on the SM field contents, construct a lowest dimensional operator that is Lorentz-invariant, Hermitian, and SM gauge-invariant and that leads to a neutrino mass after the spontaneous gauge symmetry breaking.
(i) If such an operator exists, why wasn't it included in the SM Lagrangian?
(ii) What new interactions would this operator introduce beyond the known SM interactions?

Level III Problem:
(1) Let χ be a Weyl spinor that transforms under SU(2)⊗SU(2) as (1/2,0), $\chi \to U(\theta)\chi$, where $U(\theta) = \exp(-i\vec{\theta} \cdot \vec{\sigma}/2)$ is a Lorentz group transformation, with θ being real for rotation and imaginary for boost.
(i) Show that $\chi^T \epsilon \chi$ is Lorentz-invariant.
(ii) Show that this term is forbidden if χ carries a conserved charge q.

(iii) Let ξ be another Weyl spinor with the same Lorentz property, but with an opposite charge $-q$. Show that $\xi^T \epsilon \chi$ is Lorentz-invariant, and it is charge-conserving.

(2) Based on the observation in (1), a Majorana mass term is constructed with a single Weyl spinor; while a Dirac mass term needs two Weyl spinors,

$$\mathcal{L}_M = \frac{1}{2}m(\chi^T \epsilon \chi + h.c.), \quad \mathcal{L}_D = m(\xi^T \epsilon \chi + h.c.) \tag{1}$$

In terms of the 4-component notation, we thus construct the Majorana and Dirac spinors as, respectively,

$$\psi_M = \begin{pmatrix} \chi \\ \epsilon \chi^* \end{pmatrix}, \quad \psi = \begin{pmatrix} \chi \\ \epsilon \xi^* \end{pmatrix}.$$

(i) Define the charge conjugate of a fermion field in the Weyl (or chiral) representation as

$$\psi^c \equiv C\gamma_0 \psi^* = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \chi^* \\ \epsilon \xi \end{pmatrix} = \begin{pmatrix} \xi \\ \epsilon \chi^* \end{pmatrix}, \quad \overline{\psi^c} = \psi^T C.$$

Show that

$$(\psi^c)_{L,R} = (\psi_{R,L})^c; \quad \psi_M^c = \psi_M \text{ (the Majorana condition)}.$$

(ii) Show that the 4-component expressions for the Majorana and Dirac mass terms equivalent to Eq. (1) can be written as

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\begin{aligned}
\mathcal{L}_M &= -\frac{1}{2}m\bar{\psi}_M\psi_M = -\frac{1}{2}m(\overline{\psi_R^c}\,\psi_L + h.c.) = -\frac{1}{2}m(\psi_L^T C\psi_L + h.c.), \\
\mathcal{L}_D &= -m\bar{\psi}\psi = -\frac{1}{2}m(\overline{\psi_R}\,\psi_L + h.c.) = -\frac{1}{2}m((\psi^c)_L^T C\psi_L + h.c.) \\
&= -\frac{1}{2}m(\bar{\psi}\psi + \overline{\psi^c}\psi^c) = -\frac{1}{2}m(\overline{\psi_M^1}\psi_M^1 + \overline{\psi_M^2}\psi_M^2),
\end{aligned}
$$

where $\psi_M^{1,2} \equiv (\psi \pm \psi^c)/\sqrt{2}$.

Two points to take from these expressions: A Majorana mass can be written in terms of a Dirac spinor, but one chiral state (say ψ_L) is enough; A Dirac mass can be written in terms of two degenerate Majorana spinors.

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