10. Semi-Markov Processes

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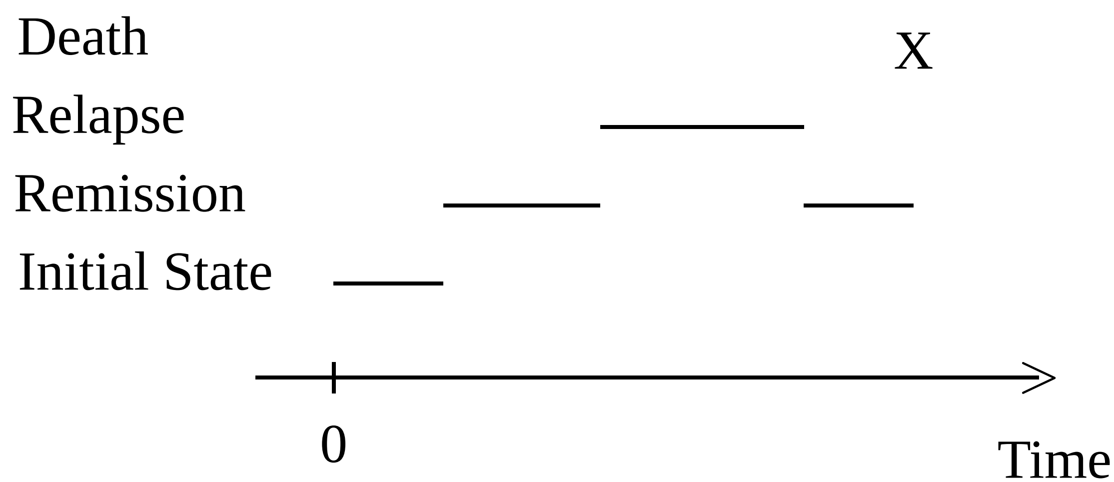
10.5 Alternate Model: Semi-Markov II

10.1 Introduction

Example: Clinical Trial. Consider a clinical trial in which a patient can be in one of several states when being followed in time; i.e.

- Initial State
- Partial Remission
- Complete Remission
- Progressive Disease
- Relapse
- Death



Example: Idealized Natural History(Chronic Disease)

S_0: Disease Free

S_p: Pre-clinical disease

S_c: Clinical disease

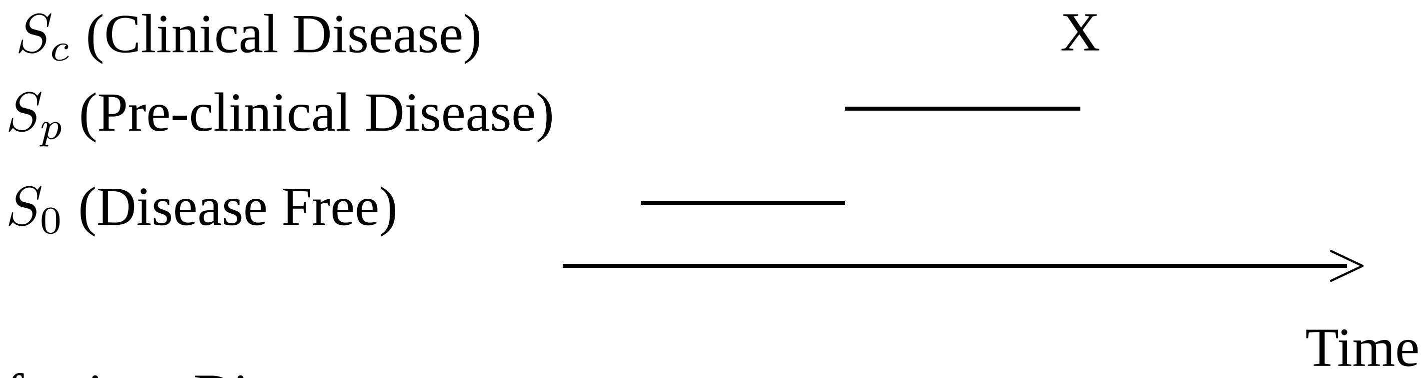
S_{d1}: Death with disease

S_{d2}: Death without disease

$$\begin{array}{ccccc} S_0 & \longrightarrow & S_p & \longrightarrow & S_c \\ & & \downarrow & \swarrow & \downarrow \\ & & S_{d2} & & S_{d1} \end{array}$$

Example: Early Detection (Progressive Disease Model)

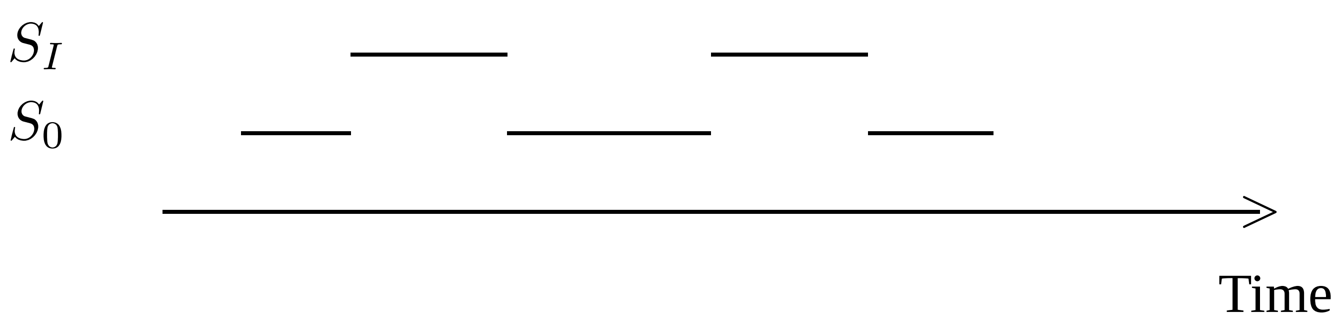
$$S_0 \longrightarrow S_p \longrightarrow S_c$$



Ex. Infectious Disease

S_0 : Disease free S_I : Infectious Disease

$$S_0 \longleftrightarrow S_I$$

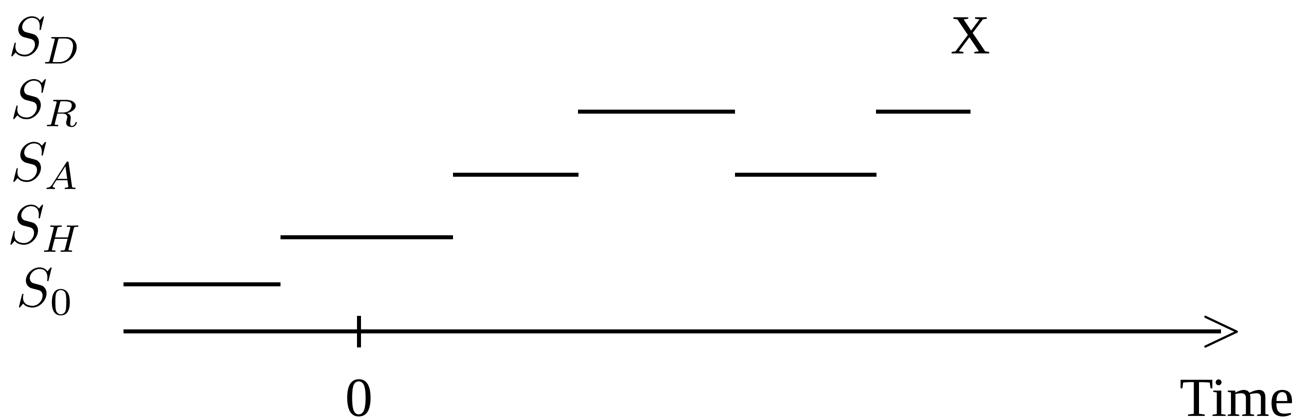


Example: AIDS

S_0: Disease Free; S_H: HIV Positive; S_R: Remission

S_A: AIDS; S_D: Death

$$
\begin{tikzcd}
S_0 \arrow[r] & S_H \arrow[r] \arrow[d] \arrow[dr] & S_A \arrow[d] \\
& S_R \arrow[r] & S_D
\end{tikzcd}
$$



$t = 0$ corresponds to beginning of observation period.

Note person was in S_H before beginning of observation period.

At time t, $X(t)$ is defined as the state of the system. $X(t)$ is right continuous, i.e.

$$X(t) = \lim_{\epsilon \to} X(t + \epsilon) \text{ for } \epsilon > 0$$

Also note that the system has an embedded Markov Chain with possible transition probabilities $P = (p_{ij})$. We will take $p_{ii} = 0$ for transient states.

The system starts in a state $X(0)$, stays there for a length of time, moves to another state, stays there for a length of time, etc. This system or process is called a Semi-Markov Process.

10.2 Theoretical Developments

Consider process as having n transient states and m absorbing states

	X_n :	state at n^{th} transition
or	$X(t)$:	state at time t
	T_n :	Sojourn time for n^{th} transition

History of process

$$H_n = \{X_0, t_0; X_1, t_1; \ldots ; X_n, t_n\}$$

Problems

Probability distribution of reaching a state

Probability distribution of returning to state

Probability distribution of time in state

First passage time probabilities, etc.

<u>Assume:</u>

$$P\{X_n = j,\, t < T_n \le t{+}\Delta t | H_{n-1}\} = P\{X_n = j,\, t < T_n \le t{+}\Delta t | X_{n-1}\}$$

Joint probability depends on *History* only through previous state

$$\pi_{ij}(t) = \lim_{\Delta t \to 0} \frac{P(X_n = j, t < T_n \le t + \Delta t | X_{n-1} = i)}{\Delta t}$$

Since $\pi_{ij}(t)$ does not depend on n (n^{th} transition) the process is time homogeneous.

$$p_{ij} = \int_0^\infty \pi_{ij}(t)dt = P\{X_n = j | X_{n-1} = i\} \;\; i \ne j, \;\; \text{where} \;\;\; p_{ii} = 0$$

$\therefore \qquad \{X_n, n \ge 0\}$ form a Markov chain

$$P\{t < T_n \le t + \Delta t | X_n = j,\; X_{n-1} = i\} = \frac{\pi_{ij}(t)dt}{p_{ij}} = q_{ij}(t)dt$$

$q_{ij}(t)$: *pdf* of sojourn time in state j conditional on coming from state i.

$$\boxed{\pi_{ij}(t)dt = p_{ij}q_{ij}(t)dt}$$

$$\boxed{\pi_{ij}(t) = p_{ij}q_{ij}(t)}$$

$q_{ij}(t)$ is a pdf of time spent in j conditional on previous transition being in i. $q_{ij}(t)$ may be defined by:

$$q_{ij}(t) = q_j(t)$$

$$q_{ij}(t) = q_i(t)$$

$$q_{ij}(t) = q(t)$$

If $q_{ij}(t) = \lambda_{ij}e^{-\lambda_{ij}t}$ the stochastic process is called a Markov Process. Many authors define a Markov Process when $\lambda_{ij} = \lambda_j$.

Remark: Sojourn time in state depends on current and previous state.

Note: Two Time Scales — Internal Time and External Time.

External Time: Time leaving or entering state

Internal Time: Time in a state or time to return to state

Def.

$w_{ij}(t)dt = P\{\text{system leave state } j \text{ during } (t, t+dt) | X_0 = i\}$

$w_{ij}(t)$ is defined by external time

$$N_{ij}(T) = \int_0^T w_{ij}(t)dt = \text{ Expected no. of times system leaves state } j \text{ conditional on } X_0 = i \text{ over } (0, T)$$

$$N_{ij} = \lim_{T\to\infty} N_{ij}(t)$$

$= $ Expected no. of times system leaves state j over $[0, \infty)$ conditional on $X_0 = i$.

Definition:

$q_{oi}(t)$: pdf of time spent in the initial state $X_0 = i$

$q_{ij}(t)$: pdf of time spent in state j conditional on coming from state i.

$$Q_{oi}(t) = P\{T_0 > t | X_0 = i\} = \int_t^\infty q_{oi}(x)dx$$

$$Q_{ij}(t) = P\{T_n > t, x_n = j | x_0 = i\} = \int_t^\infty q_{ij}(x)dx$$

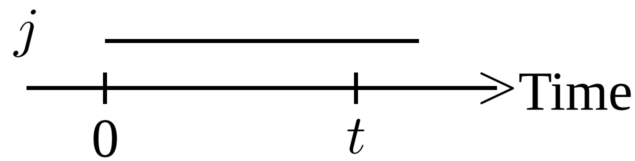
Def.: $U_{ij}(t) = P\{$system in state j at time $t | X_0 = i\}$

$U_{ij}(t)$ is a function of external time; $q_{oi}(t)$, $q_{ij}(t)$ are functions of internal time.

A. <u>Finding $U_{ij}(t)$</u>

Suppose $x_0 = j$. Then to be in j at time t either:

(a) system never left $x_0 = j$



$U_{ii}(t) = \delta_{ij}U_{ij}(t) = Q_{0i}(t)\delta_{ij}$

or

(b) system left state k at time τ $(w_i(\tau))$, entered state j (p_{kj}) and stayed at least $(t-\tau)$ units of time $Q_{kj}(t-\tau)$; i.e. $w_{ik}(\tau)d\tau p_{kj}Q_{kj}(t-\tau)$. Then integrate over possible values of τ $(0, t)$ and sum over all transient states

$$\boxed{\sum_{k=1}^{n} p_{kj} \int_0^t w_{ik}(\tau)Q_{kj}(t-\tau)d\tau}$$

Since (a) and (b) describe mutually exclusive events

$$\boxed{\begin{aligned} U_{ij}(t) &= Q_{0i}(t)\delta_{ij} + \sum_{k=1}^{n} p_{kj} \int_0^t w_{ik}(\tau) Q_{kj}(t-\tau)\,d\tau \\ & \qquad i, j = 1, 2, \ldots, n \end{aligned}}$$

Taking LaPlace Transforms

$$\boxed{U^*_{ij}(s) = Q^*_{oi}(s)\delta_{ij} + \sum_{k=1}^{n} p_{kj} w^*_{ik}(s) Q^*_{kj}(s)}$$

for $i, j = 1, 2, \ldots, n$ (only over transient states)

B. Finding $w_{ij}(t)dt$

If $x_0 = j$ then to leave state j in interval $(t, t+dt)$ either:

(a) system never left state i before time t and emerged for first time at $(t, t+dt)$; i.e. $P\{t < T_0 \leq t + dt | x_0 = j\}$

$$w_{ii}(t) = \delta_{ij} w_{ij}(t) = q_{oi}(t) dt \delta_{ij}$$

or

(b) system leaves state k at time τ $(w_{ik}(\tau)d\tau)$; enters state j (p_{kj}) and stays there for $(t-\tau, t-\tau+dt)$ time units $(q_{kj}(t-\tau)dt)$. Hence

$$\boxed{w_{ij}(t)dt \quad = q_{oi}(t) dt \delta_{ij} + \sum_{k=1,\ k \neq j}^{n} p_{kj} \int_0^t w_{ik}(\tau) q_{kj}(t-\tau) d\tau dt}$$

$$
\boxed{\begin{aligned} w_{ij}(t) \quad &= q_{oi}(t)\delta_{ij} + \sum_{k=1}^{n} p_{kj} \int_0^t w_{ik}(\tau) q_{kj}(t-\tau) d\tau \\ &\qquad\qquad i, j = 1, 2, \ldots, n \end{aligned}}
$$

Taking Laplace Transforms

$$
\boxed{\begin{aligned} w^*_{ij}(s) = q^*_{oi}(s)\delta_{ij} + \sum_{k=1}^{n} p_{kj} w^*_{ik}(s) q^*_{kj}(s) \\ i, j = 1, 2, \ldots, n \end{aligned}}
$$

Remark: The equations in the time domain are two sets of coupled integral equations. However in the S domain, we have two sets of linear equations.

C. Marginal Probabilities

(1) $$U_{ij}(t) = \delta_{ij} Q_{oi}(t) + \sum_{k=1}^{t} p_{kj} \int_0^t w_{ik}(\tau) Q_{kj}(t-\tau)\, d\tau$$

(2) $$w_{ij}(t) = \delta_{ij} q_{oi}(t) + \sum_{k=1}^{n} p_{kj} \int_0^t w_{ik}(\tau) q_{kj}(t-\tau)\, d\tau$$

$$i, j = 1, 2, \ldots, n \quad \text{(Transient states only)}$$

Suppose $a_i = P\{X_0 = i\}$. Then we can find

$$\left.\begin{aligned} U_j(t) &= \sum_{i=1}^{n} a_i U_{ij}(t) \\ w_j(t) &= \sum_{i=1}^{n} a_i w_{ij}(t) \end{aligned}\right\} \quad \text{Marginal Probabilities}$$

Multiplying (1) by a_i and summing over i

$$U_j(t) = \sum_{i=1}^{n} \delta_{ij} a_i Q_{oi}(t) + \sum_{k=1}^{n} p_{kj} \int_0^t w_k(\tau) Q_{kj}(t-\tau) d\tau$$

Since $\sum_{i=1}^{n} \delta_{ij} a_i Q_{oi}(t) = a_j Q_{oj}(t)$

$$\boxed{U_j(t) = a_j Q_{oj}(t) + \sum_{k=1}^{n} p_{kj} \int_0^t w_k(\tau) Q_{kj}(t-\tau) d\tau,\ j = 1, 2, \ldots, n}$$

Similarly

$$\boxed{w_j(t) = a_j q_{oj}(t) + \sum_{k=1}^{n} p_{kj} \int_0^t w_k(\tau) q_{kj}(t-\tau) d\tau,\ j = 1, 2, \ldots, n}$$

$$\boxed{w_i^* = D(q_0^*) e_i + \pi^{*'} w_i^*}$$

Consider $n = 3$ and $i = 2$

$$
\begin{bmatrix} w_{i1}^*(s) \\ w_{i2}^*(s) \\ w_{i3}^*(s) \end{bmatrix} = \begin{bmatrix} q_{01}^*(s) & 0 & 0 \\ 0 & q_{02}^*(s) & 0 \\ 0 & 0 & q_{03}^*(s) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & p_{21}q_{21}^*(s) & p_{31}q_{31}^* \\ p_{12}q_{12}^*(s) & 0 & p_{32}q_{32}^* \\ p_{13}q_{13}^*(s) & p_{23}q_{23}^*(s) & 0 \end{bmatrix} \begin{bmatrix} w_{i1}^* \\ w_{i2}^* \\ w_{i3}^* \end{bmatrix}
$$

$$
w_i^* = \begin{bmatrix} \sum_{j=1}^3 p_{j1}q_{j1}^*(s)w_{2j}^*(s) \\ q_{02}^*(s) + \sum_{j=1}^3 p_{j2}q_{j2}^*(s)w_{2j}^*(s) \\ \sum_{j=1}^3 p_{j3}q_{j3}^*(s)w_{2j}^*(s) \end{bmatrix}
$$

Similarly $\boxed{U_i = D(Q_0^*)e_i + \Phi^{*'} w_i^*}$

D. Matrix Relations and Solutions

Consider the S - domain equations:

$$\boxed{\begin{aligned} U^*_{ij}(s) &= Q^*_{oi}(s)\delta_{ij} + \sum_{k=1}^{n} p_{kj} w^*_{ik}(s) Q^*_{kj}(s) \\ w^*_{ij}(s) &= q^*_{oi}(s)\delta_{ij} + \sum_{k=1}^{n} p_{kj} w^*_{ik}(s)\, q^*_{kj}(s) \end{aligned}}$$

Define:

$$
\begin{aligned}
U_i^* &= (U_{i1}^*(s), U_{i2}^*(s), \ldots, U_{in}^*(s))' \\
w_i^* &= (w_{i1}^*(s), \ldots, w_{in}^*(s))' \\
Q_o^* &= (Q_{o1}^*(s), \ldots, Q_{on}^*(s))' \\
P &= (p_{ij}),\ \ \Phi_{ij}^*(s) = p_{ij} Q_{ij}^*(s),\ \Phi^* = (\Phi_{ij}^*) \\
\pi_{ij}^*(s) &= p_{ij} q_{ij}^*(s),\ \pi^* = (\pi_{ij}(s)) \\
e_i &= (0, 0, \ldots, 1, 0 \ldots 0)' \quad 1 \text{ in } i^{th} \text{ position}
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{\begin{aligned} U_i^* = D(Q_0^*) e_i + \Phi^{*'} w_i^* \\ w_i^* = D(q_0^*) e_i + \pi^{*'} w_i^* \end{aligned}}
$$

$$
\boxed{w_i^* = D(q_0^*) e_i + \pi^{*'} w^*}
$$

$$
(I - \pi^{*'}) w_i^* = D(q_o^*) e_i
$$

<u>Def.</u> $N^* = (I - \pi^{*'})^{-1}$

$$\boxed{w_i^* = (I - \pi^{*'})^{-1} D(q_0^*) e_i = N^* D(q_0^*) e_i}$$

$$w_{ij}^*(s) = \int_0^\infty e^{-st} w_{ij}(t) dt$$

$$w_{ij}^*(0) = \int_0^\infty w_{ij}(t) dt = \text{Expected no. of visits to } j \text{ conditional on } x_0 = i$$

$\therefore$ putting $s = 0$ in: $w_i^*, D(q_0^*),\ N^*$

$$w_i^*(0) = N^*(0) e_i \quad \text{as} \quad q_{0i}^*(0) = 1,\ \ N^*(0) = (I - P')^{-1}$$

<u>i.e.</u>

$$N^*(0) = [I - \pi^{*'}(0)]^{-1} \quad , \quad \pi^*(s) = (p_{ij} q_{ij}^*(s))$$

$$\pi^*(0) = P,\ \pi^*(0)' = P'$$

$\therefore$ $N(0) = (I - P')^{-1}$

Define

$$
\begin{aligned}
N_{ij} &= \text{mean number of times } j \text{ is visited where } x_0 = i \\
e_i &= \text{column vector of zeros except the } i^{th} \text{ element is unity} \\
N = (N_{ij}) &= (I - P')^{-1}
\end{aligned}
$$

and

$$e_i' N = (N_{i1} \;\; N_{i2} \ldots N_{in}) = N_i'$$

Therefore

$$w_i^*(0) = (I - P')^{-1} e_i = \begin{bmatrix} N_{i1} \\ N_{i2} \\ \vdots \\ N_{in} \end{bmatrix} = N_i$$

$$\boxed{w_i^* = N^* D(q_0^*) e_i}$$

$$\begin{aligned} U_i^* &= D(Q_0^*)e_i + \Phi'^* w_i^*, \quad \Phi^*(p_{ij} Q_{ij}^*(s)) \\ &= D(Q_0^*)e_i + \Phi'^* N^* D(q_0^*) e_i \end{aligned}$$

$$\boxed{U_i^* = [D(Q_0^*) + \Phi'^* N^* D(q_0^*)] e_i}$$

<u>Recall</u>: $U_{ij}^*(s) = \int_0^\infty U_{ij}(t) e^{-st} dt$

$U_{ij}^*(0) = \int_0^\infty U_{ij}(t) dt =$ mean time spent in j conditional on $X_0 = i$

$Q_{oi}^*(0) = \int_0^\infty Q_{oi}(t) dt = m_{oi}$

$\Phi^*(s) = (p_{ij} Q_{ij}^*(s)), \Phi^*(0) = (p_{ij} m_{ij})$, where m_{ij} is mean of $q_{ij}(t)$.

<u>Ex.</u> $n = 3,\ X_0 = 2,\ e_2 = (0\ 1\ 0)'$

$U_2^*(0) = [D(m_0) + \Phi'^*(0) N^*(0)] e_2$

$$\begin{bmatrix} U_{21}^*(0) \\ U_{22}^*(0) \\ U_{23}^*(0) \end{bmatrix} = \begin{pmatrix} m_{01} & 0 & 0 \\ 0 & m_{02} & 0 \\ 0 & 0 & m_{03} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & p_{21}m_{21} & p_{31}m_{31} \\ p_{12}m_{12} & 0 & p_{32}m_{32} \\ p_{13}m_{13} & p_{23}m_{23} & 0 \end{bmatrix} \begin{bmatrix} N_{21} \\ N_{22} \\ N_{23} \end{bmatrix}$$

as $\quad N^*(0)e_i = \begin{pmatrix} N_{11} & N_{21} & N_{31} \\ N_{12} & N_{22} & N_{32} \\ N_{13} & N_{23} & N_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} N_{21} \\ N_{22} \\ N_{23} \end{bmatrix}$

$$\begin{bmatrix} U_{21}^*(0) \\ U_{22}^*(0) \\ U_{23}^*(0) \end{bmatrix} = \begin{bmatrix} 0 \\ m_{02} \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{3} p_{j1} m_{j1} N_{2j} \\ \sum_{j=1}^{3} p_{j2} m_{j2} N_{2j} \\ \sum_{j=1}^{3} p_{j3} m_{j3} N_{2j} \end{bmatrix} = \begin{array}{c} \text{mean time in state j = 1,2,3} \\ \text{starting from } X_0 = 2 \end{array}$$

Interpretation

$$U^*_{21}(0) = \sum_{j=1}^{3} p_{j1} m_{j1} N_{2j} = \text{mean time spent in state 1 if } X_0 = 2.$$

$$m_{j1} N_{2j} = (\text{meantime in 1 coming from } j) \times (\text{Expected no. of visits from 2 to } j)$$

$$p_{j1} m_{j1} N_{2j} = E[\text{Total no. of visits to } j \text{ starting from state } 2] \times \text{Probability } \{j \to 1\} \times \text{mean time in state 1 coming from } j$$

$$U^*_{22}(0) = m_{02} + \sum_{j=1}^{3} p_{j2} m_{j2} N_{2j}$$

Note additional term because $X_0 = 2$.

Homework

Consider the 3 state model with transition probabilities

$$P = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ \beta & 0 & 1-\beta \\ 0 & 0 & 1 \end{bmatrix}$$

States 1 and 2 are transient states and state 3 is an absorbing state.

Define $q_{ij}(t) = q_j(t) = \lambda_j e^{-\lambda_j t}$

Find $U_{ij}(t)$ and $w_{ij}(t)$ for $i, j = 1, 2$.

E. Matrix Note

All results depended on the inversion of $[I - \pi_{ij}^{*'}(s)]$. Does inverse exist?

Consider matrix $(n \times n)$ A.

$$\begin{aligned} L_n &= (I-A)(I + A + A^2 + \ldots + A^{n-1}) \\ &= (I + A + \ldots + A^{n-1}) - (A + A^2 + \ldots + A^n) = I - A^n \end{aligned}$$

If $A^n \to 0$ as $n \to \infty \Rightarrow L_n \to I$ and $(I-A)^{-1} = I + A + A^2 + \ldots$ exists.

Now consider $(I - \pi^{*'})$, $\pi^* = (\pi_{ij}^*) = (p_{ij} q_{ij}^*(s))$

Since $q_{ij}^*(s) = \int_0^\infty e^{-st} q_{ij}(t) dt$, $|q_{ij}^*(s)| \leq 1 \quad \therefore |\pi_{ij}^*| \leq p_{ij}$

But $P^n = (p_{ij}^{(n)}) \to 0$ as $n \to 0$ as there exists at least one absorbing state and with prob $= 1$, system will eventually be in absorbing state.

10.3 Simplified Model

Suppose $q_{ij}(t) = q_j(t)$ and $q_{0j}(t) = q(j(t)$; i.e. sojourn time only depends on the state process is in. Then

$$\begin{aligned}
\pi_{ij}(t) &= p_{ij}q_{ij}(t) = p_{ij}q_j(t) \\
\Phi_{ij}(t) &= p_{ij}Q_{ij}(t) = p_{ij}Q_j(t) \\
\pi(t) = \pi_{ij}(t)) &= \begin{bmatrix} 0 & p_{12}q_2(t) & p_{13}q_3(t) & \dots & p_{1n}q_n(t) \\ p_{21}q_1(t) & 0 & p_{23}q_3(t) & \dots & p_{2n}q_n(t) \\ \vdots & & & & \\ p_{n1}q_1(t) & p_{n2}q_2(t) & & \dots & 0 \end{bmatrix}
\end{aligned}$$

$$\pi(t) = PD(q), \;\; D(q) = \begin{bmatrix} q_1(t) & 0 \;\ldots & 0 \\ 0 & q_2(t) \;\ldots & 0 \\ \vdots & & \\ 0 & \ldots & q_n(t) \end{bmatrix}$$

and $\pi^* = PD(q^*)$

Similarly $\quad \Phi(t) = PD(Q), \;\; \Phi^*(s) = PD(Q^*)$

$\pi^* = PD(q^*), \qquad \Phi^* = PD(Q^*)$

$D(q_0^*) = D(q^*), \quad D(Q_0^*) = D(Q^*)$

Since $w_i^* = (I - {\pi'}^*)^{-1} D(q_0^*) e_i$

we can write

$$w_i^* = [I - D(q^*)P']^{-1} D(q^*) e_i$$

Consider

$$
\begin{aligned}
[I - D(q^*)P']^{-1} D(q^*) \quad &= [I + DP' + (DP')^2 + \ldots]D \\
&= D[I + P'D + (P'D)^2 + \ldots] \\
&= D^*(q^*)[I - P'D(q^*)]^{-1} = D(q^*)M^* \\
\text{where} \qquad M^* \quad &= [I - P'D(q^*)]^{-1} \\
&\therefore \boxed{w_i^* = D(q^*)M^* e_i}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
U_i^* &= D(Q_0^*)e_i^* + \Phi'^* w_i^* \\
&= D(Q^*)e_i + D(Q^*)P'D(q^*)M^* e_i \\
&= D(Q^*)[I + P'D(q^*)M^*]e_i
\end{aligned}
$$

$$
\begin{aligned}
U_i^* \quad &= D(Q^*)[I + P'D(q^*)M^*]e_i \\
\text{where} \quad M^* \quad &= [I - P'D(q^*)]^{-1} \\
\text{But} \quad P'DM^* \quad &= P'D[I - P'D]^{-1} = P'D[I + P'D + (P'D)^2 + \ldots] \\
P'DM^* \quad &= P'D + (P'D]^2 + (P'D]^3 + \ldots \\
&= [I - P'D]^{-1} - I = M^* - I \\
&\therefore \ \boxed{U_i^* = D(Q)^* M^* e_i}
\end{aligned}
$$

$\therefore$ When the sojourn time only depends on the state the process is in we also have

$$\boxed{w_i^* = D(q^*)M^* e_i}.$$

Then $w_i^*(0) = (I - P')^{-1} e_i = \begin{pmatrix} N_{i1} \\ N_{i2} \\ \vdots \\ N_{in} \end{pmatrix} = N_i$

$$U_i^*(0) = D(m) N_i = \begin{pmatrix} m_1 & & 0 & \\ & m_2 & & \\ 0 & & \ddots & \\ & & & m_n \end{pmatrix} \begin{pmatrix} N_{i1} \\ N_{i2} \\ \vdots \\ N_{in} \end{pmatrix} = \begin{bmatrix} m_1 N_{i1} \\ m_2 N_{i2} \\ \vdots \\ m_n N_{in} \end{bmatrix}$$

First Passage Time Problems

A. Time from $X_0 = i$ to being absorbed

To simplify the problem, suppose that there is only one absorbing state which is $n+1$.

Define $T_i =$ random variable to Absorption conditional on $X_0 = i$.

$$G_i(t) = P\{T_i > t | X_0 = i\} = P\{T_{X_0} > t | X_0 = i\}$$

$U_{i,n+1}(t) =$ Prob. system is in state $n+1$ at time t.

$1 - U_{i,n+1}(t) =$ Prob. system is not in $(n+1)$ at t

$$\Rightarrow \quad G_i(t) = 1 - U_{i,n+1}(t) = \sum_{k=1}^{n} U_{ik}(t)$$

as $\quad \sum_{k=1}^{n} U_{ik}(t) + U_{i,n+1}(t) = 1$

$$\boxed{G_i(t) = \sum_{k=1}^{n} U_{ik}(t)}$$

$$G_i^*(s) = \sum_{k=1}^{n} U_{ik}^*(s)$$

Since $G_i^*(0) = \int_0^\infty G_i(t)dt = \sum_{j=1}^{n} U_{ij}^*(0) =$ mean time to Absorption

But we had found

$$\begin{aligned} U_{ij}^*(0) \quad &= \text{mean time in } j \text{ starting from } X_0 = i \\ &= \delta_{ij} m_{oi} + \sum_{k=1}^{n} p_{kj} m_{kj} N_{ik} \end{aligned}$$

$$\therefore \quad \boxed{G_i^*(0) = \sum_{j=1}^{n}\sum_{k=1}^{n} p_{kj} m_{kj} N_{ik} + m_{oi}}$$

If $m_{kj} = m_j$,

$$G_i^*(0) = \sum_{j=1}^{n} m_j \sum_{k=1}^{n} p_{kj} N_{ik} + m_{oi}$$

Recall

$$\begin{aligned} w_{ij}^*(s) &= q_{oi}^*(s)\delta_{ij} + \sum_{k=1}^{n} p_{kj} w_{ik}^*(s) q_{kj}^*(s) \\ w_{ij}^*(0) &= \delta_{ij} + \sum_{k=1}^{n} p_{kj} w_{ik}^*(0) \end{aligned}$$

or equivalently

$$\begin{aligned} N_{ij} &= \sum_{k=1}^{n} p_{kj} N_{ik} \quad i \neq j \\ N_{ii} &= 1 + \sum_{k=1}^{n} p_{ki} N_{ik} \end{aligned}$$

$$G_i^*(0) = \sum_{j=1}^{n} m_j \sum_{k=1}^{n} p_{kj} N_{ik} + m_{oi}$$

$$
\begin{aligned}
\text{If} \quad m_{kj} &= m_j \\
N_{ij} &= \sum_{k=1}^{n} p_{kj} N_{ik} \quad i \neq j \\
N_{ii} &= 1 + \sum_{k=1}^{n} p_{kj} N_{ik} \\
\therefore \qquad G_i^*(0) &= \sum_{j=1, j\neq i}^{n} m_j \left[\sum_{k=1}^{n} p_{kj} N_{ik} \right] + m_i \sum_{k=1}^{n} p_{ki} N_{ii} + m_{oi} \\
&= \sum_{j \neq i} m_j N_{ij} + m_i [N_{ii} - 1] + m_{oi}
\end{aligned}
$$

$$
\boxed{G_i^*(0) = \sum_{j=1}^{n} m_j N_{ij} + (m_{oi} - m_i)}
$$

If $m_{oi} = m_i$, $\boxed{G_i^*(0) = \sum_{j=1}^{n} m_j N_{ij}}$

B. Variance of Time to Absorption

$$G_i^*(s) = \sum_{j=1}^{n} U_{ij}^*(s), \quad G_i(t) = P\{T_i > t | X_0 = i\}$$

If $g_i(t)$ is pdf of time to Absorption with $X_0 = i$, we know that

$$G_i^*(s) = \frac{1 - g_i^*(s)}{s} = \frac{1 - [1 - sm_1 + \frac{s^2}{2} m_2 + \cdots]}{s} = m_1 - \frac{s}{2} m_2 + O(s^2)$$

$$\frac{dG_i^*(s)}{ds} = -\frac{m_2}{2} + O(s)$$

and taking $s = 0 \qquad \Rightarrow \quad m_2 = -2\left(\frac{dG_i^*(s)}{ds}\right)_{s=0}$

$$\therefore \ Var\, T_i = m_2 - m_1^2 = -2G_i^*(0) - [G_i^*(0)]^2$$

We have found $G_i^*(0)$, it is only necessary to find $-2\frac{dG_t^*(s)}{ds}$ evaluated at $s = 0$

To simplify problem we will consider the special case

$$m_{kj} = m_j,\ m_{oj} = m_j$$

Then

$U_i^* = D(Q^*)M^* e_i, \quad M^* = [I - P' D(q^*)]^{-1}$

$$-2\frac{dU_i^*}{ds} = -2\frac{d}{ds}D(Q^*)M^* e_i - 2D(Q^*)\frac{dM^*}{ds}e_i$$

Recall $-2\dfrac{dQ_j^*(s)}{ds}$ =second moment of $q_j(t) = (\sigma_j^2 + m_j^2), s = 0$

$$\therefore\ -2\frac{d}{ds}D(Q^*)_{s=0} = D(\sigma^2 + m^2) = \begin{bmatrix} \sigma_1^2 + m_1^2 & & \\ & \sigma_2^2 + m_2^2 & 0 \\ 0 & \ddots & \\ & & \sigma_k^2 + m_k^2 \end{bmatrix}$$

To evaluate $\dfrac{dM^*}{ds}$, consider $M^* {M^*}^{-1} = I$

$$\frac{dM^*}{ds}{M^*}^{-1} + M^* \frac{d}{ds}{M^*}^{-1} = 0$$

$$\boxed{\frac{dM^*}{ds} = -M^* \left(\frac{d}{ds}\ {M^*}^{-1} \right) M^*}$$

Since ${M^*}^{-1} = I - P'D(q^*)$, $\quad \left(\dfrac{d{M^*}^{-1}}{ds} \right)_{s=0} = +P'D(m)$

$$\frac{dM^*}{ds} = -M^* \left(\frac{d}{ds}{M^*}^{-1} \right) M^*$$

and setting $s = 0$

$$\left(\frac{dM^*}{ds} \right)_0 = -(I - P')^{-1} P'D(m)(I - P')^{-1} = -NP'D(m)N$$

$$\therefore \quad -2\frac{dU_i^*}{ds} = \left[-2\frac{d}{ds}D(Q^*)M^*, \quad -2D(Q^*)\frac{dM^*}{ds}\right]e_i$$

Setting $\quad s = 0$

$$= [D(\sigma^2 + m^2)N + 2D(m)NP'D(m)N]e_i$$

Now $\; Ne_i = \begin{pmatrix} N_{i1} \\ N_{i2} \\ \vdots \\ N_{ik} \end{pmatrix}$

$$-2\frac{dU_i^*}{ds} = [D(\sigma^2 + m^2) + 2D(m)NP'D(m)]N_i$$

Also $\quad G_i^*(0) = \sum U_i^*(0) = \sum_1^N N_i m_i = N_i' m$

$$\therefore\ VarT_i = -2\sum\left(\frac{dU_i^*(s)}{ds}\right)_{s=0} - m'N_iN_i'm$$

$$= \sum_{j=1}^{n}\sigma_j^2 N_{ij} + \sum_{j}^{N} m_j^2 N_{ij} + 2m'[N-I]D(m)N_i - m'N_iN_i'm$$

as $NP' = (I-P')^{-1}P' = P' + {P'}^2 + \ldots = N - I$

$$\therefore\ VarT_0 = \sum\sigma_j^2 N_{ij} + \sum_j m_j^2 N_{ij} + 2m'[N-I]D(m)N_i - m'N_iN_i'm$$

Simplifying $m'[N-I]D(m)N_i$

$$
\begin{aligned}
D(m)N_i &= D(N_i)m \\
m'[N-I]D(m)N_i &= m'ND(N_i)m - m'D(N_i)m \\
\text{Note:}\quad m'D(N_i)m &= \sum_{j=1}^{n} m_j^2 N_{ij} \\
VarT_i &= \sum \sigma_j^2 N_{ij} + 2m'ND(N_i)m - \textstyle\sum_j m_j^2 N_{ij} \\
&\qquad - m'N_iN_i'm \\
&= \textstyle\sum_j \sigma_j^2 N_{ij} + m'[2N-I]D(N_i)m - m'N_iN_i'm
\end{aligned}
$$

$$
\boxed{VarT_i = \sum_j \sigma_j^2 N_{ij} + m'\{[2N-I]D(N_i) - N_iN_i'\}m}
$$

C. Time to go from any state j to being absorbed conditional on $X_0 = i$

Earlier we had found the prob. distribution of being absorbed starting out from $X_0 = i$ at time 0.

Now we wish to generalize the result of finding the time to Absorption for an arbitrary state j with $X_0 = i$.

Define

$g_{j,n+1}(t) = g_j(t) =$ pdf of time to being absorbed beginning with the time the system enters state j.

There are two ways of going from j to the absorbing state $n+1$; i.e.

(a) The state after j is the absorbing state. Hence the time to being absorbed is T_j (time spent in j), or
(b) The state after j is another transient state r and the time to being absorbed is $T_j + T_{r,n+1}$ where $T_{r,n+1}$ is the time to Absorption from state r.
Assume that the state prior to entering state j is state k.

$$
\begin{aligned}
g_{j,n+1}(t) &= g_j(t) \\
&= \sum_{k=1}^{n} p_{kj} q_{kj}(t) p_{j,n+1} + \sum_{k=1}^{n} \sum_{r=1}^{n} p_{kj} \int_0^t q_{kj}(\tau) p_{jr} g_r(t-\tau) d\tau
\end{aligned}
$$

$$
g_j(t) = p_{j,n+1} q_j(t) + \sum_{r=1}^{n} p_{jr} \int_0^t q_j(\tau) g_r(t-\tau) d\tau
$$

where $q_j(t) = \sum_{k=1}^{n} p_{kj} q_{kj}(t)$

$$\boxed{g_j(t) = p_{j,n+1} q_j(t) + \sum_{r=1}^{n} p_{jr} \int_0^t q_j(\tau) g_r(t-\tau) d\tau \qquad j = 1, \ldots, n}$$

where $\quad q_j(t) = \sum_{k=1}^{n} p_{kj} q_{kj}(t).$

Note that before entering j the process was in state k. Thus the pdf of the stay in j is $q_{kj}(t)$. However it came from state k with prob. p_{kj}. It is also necessary to sum over all possible values of k. This leads to the marginal distribution $q_j(t)$.

Taking LaPlace Transforms results in

$$\boxed{g_j^*(s) = p_{j,n+1} q_j^*(s) + \sum_{r=1}^{n} p_{jr} q_j^*(s) g_r^*(s)}$$

for $\quad j = 1, \ldots, n.$

Substituting $p_{j,n+1} = 1 - \sum_{r=1}^{n} p_{jr}$, subtracting 1 from both sides and multiplying by $1/s$ results in

$$\frac{1 - g_j^*(s)}{s} = \frac{1 - q_j^*(s)}{s} + q_j^*(s) \sum_{r=1}^{n} \left(\frac{1 - g_r^*(s)}{s} \right) p_{jr}$$

or $\boxed{G_j^*(s) = Q_j^*(s) + q_j^*(s) \sum_{r=1}^{n} p_{jr} G_r^*(s)}$

where we have written $G_j^*(s)$ for $G_{j,n+1}^*(s)$

Writing $G_{n+1}^* = (G_1^*(s), G_2^*, \ldots, G_n^*(s))'$ we have

$\boxed{G_{n+1}^* = Q^* + D(q^*) P G_{n+1}^*}$ or $[I - D(q^*)P] G_{n+1}^* = Q^*$

Earlier we had defined $M^* = [I - P' D(q^*)]^{-1}$

$\therefore \quad G_{n+1}^* = [I - D(q^*)P]^{-1} Q^* = M'^{*} Q^*$

$$\boxed{G^*_{n+1} = M'^* Q^*} \quad , \quad M'^* = [I - D(q^*)P]^{-1}$$

Since $m_{j,n+1} =$ mean time to go from j to absorbing state we can write

$$\boxed{G^*_{n+1}(0) = m_{n+1} = (I - P)^{-1} m}$$

where $m_{n+1} = (m_{1,n+1}, m_{2,n+1}, \ldots, m_{n,n+1})'$

and $m = (m_1, m_2, \ldots, m_n)$.

Furthermore $(I - P)^{-1} = (N_{ij}) =$ (Expected no. of visits from i to j).
Therefore

$$\boxed{m_{j,n+1} = \sum_{r=1}^{n} N_{jr} m_r}$$

The variance of the time to be absorbed from j can be found from
$-2\left(\dfrac{dG^*}{ds}\right)_{s=0} - G^*(0)^2$

D. First passage time to go from $X_0 = i$ to an arbitrary Absorption state

Consider n transient states and m absorbing states. The transient states will be written as the first n states. Assume $X_0 = i$ and it is desired to find the first passage time starting from $X_0 = i$ to being absorbed by state r, conditional on reaching state r.

Define

$$
\begin{aligned}
f_{ir} = &\quad \text{Prob. of being absorbed by state } r \\
&\quad \text{conditional on } X_0 = i \\
f_{ir} = &\quad p_{ir} + \sum_{j=1}^{n} p_{ij} p_{jr} + \sum_{j,k} p_{ik} p_{kj} p_{jr} + \sum_{j,k,l} p_{il} p_{lk} p_{kj} p_{jr} \\
= &\quad p_{ir} + \sum_{j} p_{ij} p_{jr} + \sum_{j} p_{ij}^{(2)} p_{jr} + \sum_{j} p_{ij}^{(3)} p_{jr} + \ldots
\end{aligned}
$$

$$f_{ir} = p_{ir} + \sum_{j=1}^{n} p_{ij} p_{jr} + \sum_{j} p_{ij}^{(2)} p_{jr} + \sum_{j} p_{ij}^{(3)} p_{jr} + \ldots$$

where $p_{ij}^{(m)}$ = Prob. of $i \to j$ in m steps

$p_{ij}^{(m)}$ is i, j^{th} element of P^m.

Hence if

$$\begin{aligned} f_r = & \quad (f_{1r}, f_{2r}, \ldots, f_{nr})' \\ R = & \quad (p_{1r}, p_{2r}, \ldots, p_{nr})' \\ P = & \quad (p_{ij}) \quad i, j = 1, 2, \ldots, n \\ f_r = & \quad R + PR + P^2R + \ldots = (I + P + P^2 + \ldots)R \end{aligned}$$

$\boxed{f_r = (I - P)^{-1} R}$ i.e. $f_{ir} = \sum_{j=1}^{n} N_{ij} p_{jr}$

$$f_{ir} = e_i'(I - P)^{-1} R = R'(I - P')^{-1} e_i$$

$$e_i'(I - P)^{-1} = (N_{i1}, N_{i2}, \ldots, N_{im})$$

Define

$g_{ir}(t)$: pdf of time to being absorbed in state r with $X_0 = i$, and conditional on being absorbed in state r.

$$\boxed{g_{ir}(t) = \sum_{j=1}^{n} w_{ij}(t) p_{jr} / f_{ir} = w_i'(t) R / f_{ir}}$$

where $w_i(t) = (w_{i1}(t), w_{i2}(t), \ldots, w_{in}(t))'$

$R = (p_{1r}, p_{2R}, \ldots, p_{nr})'$

$$g_{ir}^*(s) = w_i^*(s)' R / f_{ir} = \boxed{R' w_i^*(s) / f_{ir}}$$

where $w_i^*(s) = N^* D(q_0^*) e_i$

Note: $g_{ir}^*(0) = \dfrac{R' w_i^*(0)}{f_{ir}} = \displaystyle\sum_{j=1}^{n} \frac{N_{ij} p_{jr}}{f_{ir}} = 1$

E. Other First Passage Time Problems

1. Suppose $i, j \in T$ (T= Transient States)

Define $T' = (n-1)$ transient states omitting j. A is set of absorbing states.

$$
\begin{aligned}
f_{ij} = &\quad \text{Prob. of } i \to j \text{ (Probability of eventually reaching } j \text{ from } i\text{)} \\
f_{ij} = &\quad p_{ij} + \sum_{k \in T'} p_{ik} p_{kj} + \sum_{k,l \in T'} p_{ik} p_{kl} p_{lj} + \ldots .
\end{aligned}
$$

Partition the transition matrix P

$$
P = \begin{array}{cccc}
 & T' & j & A \\
T' & \bar{P} & \alpha & \gamma \\
j & \beta' & 0 & \delta \\
A & 0 & 0 & I
\end{array} , \qquad
\begin{array}{c}
\alpha' = (p_{1j}, p_{2j}, \ldots, p_{nj}) \\
\text{omit } p_{jj} \\
\beta' = (p_{j1}, p_{j2}, \ldots, p_{jn})
\end{array}
$$

Define $e_i' = (n-1) \times 1$ vector with i in the i^{th} place and $0's$ everywhere else.

$$
\begin{aligned}
p_{ij} &= e_i'\alpha \\
f_{ij} &= e_i'\alpha + e_i'\bar{P}\alpha + e_i'\bar{P}^2\alpha + \ldots \\
&= e_i'(I - \bar{P})^{-1}\alpha \\
g_{ij}(t) &= \sum_{k \in T'} w_{ik}(t) p_{kj} / f_{ij}
\end{aligned}
$$

2. <u>First passage time to return to j</u>

$$g_{jj}(t) = \sum_{k \in T'} w_{jk}(t) p_{kj} / f_{jj}$$

f_{jj} = Prob. of visiting j after initial visit

$$f_{jj} = \sum_{k \in T'} p_{jk} p_{kj} + \sum_{k,l, \in T'} p_{jk} p_{kl} p_{lj} + \ldots$$

$$\boxed{f_{jj} = \beta' \alpha + \beta' \bar{P} \alpha + \beta' \bar{P}^2 \alpha + \ldots = \beta' (I - \bar{P})^{-1} \alpha}$$

10.5 Alternate Model: Semi-Markov II

Suppose time in state depends on current state and the next state. Our models up to now have assumed time in state depends on current state and state before current state; i.e.

$$P\{t < T_{X_n} \leq t + dt, X_n = j | X_{n-1} = i\} = \pi_{ij}(t)dt.$$

Now assume

$$P\{t < T_{X_n} \leq t + dt, X_{n+1} = j | X_n = i\} = \pi_{ij}(t)dt$$

$$P\{t < T_{X_n} \leq t + dt | X_n = i, X_{n+1} = j\} = q_{ij}(t)dt$$

$$\boxed{\pi_{ij}(t) = p_{ij}q_{ij}(t)}$$

$q_{ij}(t)$: refers to the *pdf* of time in i where j is the next transition.

Suppose $X_0 = i$. If next state is k, then time spent in i is $q_{ik}^{(0)}(t)$ where the (0) indicates the initial state with probabiblity p_{ik}.

Define $U_{ij}(t),\ w_{ij}(t)$ as before where $i, j \in$ Transient states.

$$U_{ij}(t) = \left[\sum_{k=1}^{n} p_{ik} Q_{ik}^{(0)}(t) \right] \delta_{ij}$$
$$+ \sum_{k=1}^{n} \sum_{l=1}^{n} \int_0^t w_{ik}(\tau) p_{kj} Q_{jl}(t-\tau) p_{jl} d\tau$$

<u>Initial state $X_0 = i$</u>: If $j = i, \delta_{ij} = 1$ and $Q_{ik}^{0}(t)$ where k is next state with Prob. = p_{ik}.

<u>Other</u>: At time t, system is in state j. Hence at time $< t$, it entered j from a state k and left state k at time $\tau(\tau < t)$. Also the state after j is l with Prob. p_{jl}.

$$U_{ij}(t) = \left[\sum_{k=1}^{n} p_{ik} Q_{ik}^{(0)}(t)\right] \delta_{ij} + \sum_{k=1}^{n} p_{kj} \int_0^t w_{ik}(\tau) \left[\sum_{l=1}^{n} p_{jl} Q_{jl}(t-\tau)\right] d\tau$$

Define

$$Q_{i0}(t) = \sum_{k=1}^{n} p_{ik} Q_{ik}^{(0)}(t) \qquad\qquad Q_j(t) = \sum_{l=1} p_{jl} Q_{jl}(t)$$

Then
$$\boxed{U_{ij}(t) = \delta_{ij} Q_{i0}(t) + \sum_{k=1}^{n} p_{kj} \int_0^t w_{ik}(\tau) Q_j(t-\tau) d\tau}$$

These are same equations as our other model except the sojourn time need only be marginal distributions.

Taking LaPlace Transforms

$$\boxed{U_{ij}^*(s) = \delta_{ij} Q_{i0}^*(s) + Q_j^*(s) \sum_{k=1}^{n} p_{kj} w_{ik}^*(s)}$$

Similarly we have

$$w_{ij}(t) = \delta_{ij} q_{i0}(t) + \sum_k p_{kj} \int_0^t w_{ik}(\tau) q_j(t-\tau) d\tau$$

resulting in

$$w_{ij}(s) = \delta_{ij} q_{i0}^*(s) + q_j^*(s) \sum_{k=1}^n p_{kj} w_{ik}^*(s)$$

Hence in matrix notation

$$\boxed{w_i^* = D(q_0^*) e_i + D(q^*) P' w_i^*}$$

$$[I - D(q^*)P'] w_i^* = D(q_0^*) e_i$$

$$\boxed{w_i^* = N^* D(q_0^*) e_i), \quad N^* = [I - D(q^*)P']^{-1}}$$

$$\boxed{w_i^* = N^* D(q_0^*) e_i), \quad N^* = [I - D(q^*)P']^{-1}}$$

$$U_{ij}^*(s) = \delta_{ij} Q_{io}^*(s) + Q_j^*(s) \sum_{k=1}^{n} p_{kj} w_{ik}^*(s)$$

or in matrix notation $\boxed{U_i^* = D(Q_0^*)e_i + D(Q^*)P' w_i^*}$

Substituting for w^*

$$U_i^* = D(Q_0^*)e_i + D(Q^*)P' N^* D(q_0^*) e_i$$

Now if $Q_{0i}^*(s) = Q_i^*(s)$

$$U_i^* = \quad [D(Q^*) + D(Q^*)P' N^* D(q^*)] e_i$$

$$U_i^* = \quad D(Q^*)[I + P' N^* D(q^*)] e_i$$

Since

$$\begin{aligned} P'N^*D(q^*) = &\quad P'[I + DP' + (DP')^2 + \ldots]D \\ = &\quad P'D + (P'D)^2 + \ldots = (I - P'D)^{-1} - I = M^* - I \end{aligned}$$

$$U_i^* = \quad D(Q^*)M^*e_i, \quad M^* = [I - P'D(q^*)]^{-1}$$

$$\boxed{U_i^* = D(Q^*)M^*e_i}, \quad M^* = [I - P'D(q^*)]^{-1}$$

Now

$$\boxed{w_i^* = N^*D(q_0^*)e_i}, \quad N^* = [I - D(q^*)P']^{-1}$$

Consider

$$\begin{aligned} N^*D &= [I + DP' + (DP')^2 + \ldots]D \\ &= D[I + P'D + \ldots] = D(I - P'D)^{-1} \\ N^*D(q^*) &= D(q^*)M^* \end{aligned}$$

Therefore $\boxed{w_i^* = D(q^*)M^*e_i}$ when $q_{0i}^* = q_i^*$