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ON THE ROBUST OPTIMAL STOPPING PROBLEM*

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Abstract. We study a robust optimal stopping problem with respect to a set $\mathcal{P}$ of mutually singular probabilities. This can be interpreted as a zero-sum controller-stopper game in which the stopper is trying to maximize its payoff while an adverse player wants to minimize this payoff by choosing an evaluation criteria from $\mathcal{P}$. We show that the *upper Snell envelope* $\overline{Z}$ of the reward process Y is a supermartingale with respect to an appropriately defined nonlinear expectation $\underline{\mathscr{E}}$, and $\overline{Z}$ is further an $\underline{\mathscr{E}}$-martingale up to the first time τ^* when $\overline{Z}$ meets Y. Consequently, τ^* is the optimal stopping time for the robust optimal stopping problem and the corresponding zero-sum game has a value. Although the result seems similar to the one obtained in the classical optimal stopping theory, the mutual singularity of probabilities and the game aspect of the problem give rise to major technical hurdles, which we circumvent using some new methods.

Key words. robust optimal stopping, zero-sum game of control and stopping, volatility uncertainty, dynamic programming principle, Snell envelope, nonlinear expectation, weak stability under pasting, path-dependent stochastic differential equations with controls

AMS subject classifications. Primary, 60G40, 93E20; Secondary, 49L20, 91A15, 60G44, 91G80

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1. Introduction.

We solve a continuous-time *robust* optimal stopping problem with respect to a nondominated set $\mathcal{P}$ of mutually singular probabilities on the canonical space Ω of continuous paths. This optimal stopping problem can also be interpreted as a zero-sum controller-stopper game in which the stopper is trying to maximize its payoff while an adverse player wants to minimize this payoff by choosing an evaluation criteria from $\mathcal{P}$. In our main result, Theorem 5.1, we construct an optimal stopping time and show that the corresponding game has a value. More precisely, we obtain that

$$\sup_{\tau\in\mathcal{T}}\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\big[Y_\tau\big]=\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\big[Y_{\tau^*}\big]=\inf_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathbb{P}}\big[Y_\tau\big]. \tag{1.1}$$

Here $\mathcal{T}$ denotes the set of all stopping times with respect to the natural filtration $\mathbf{F}$ of the canonical process B, Y is an $\mathbf{F}$-adapted right-continuous-with-left-limits (RCLL) (càdlàg) process satisfying an one-sided uniform continuity condition (see (3.1)), and τ^* is the first time Y meets its *upper Snell envelope* $\overline{Z}_t(\omega)\stackrel{\triangle}{=}\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_\tau]$, $(t,\omega)\in[0,T]\times\Omega$. (Please refer to sections 2–5 for the notation.)

The proof of this result turns out to be quite technical for three reasons. First, since the probability set $\mathcal{P}$ does not admit a dominating probability, there is no dominated convergence theorem for the nonlinear expectation $\underline{\mathscr{E}}_t[\cdot](\omega)\stackrel{\triangle}{=}\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}[\cdot]$,

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$(t,\omega) \in [0,T] \times \Omega$. So we cannot follow techniques similar to the ones used in the classical theory of optimal stopping due to El Karoui [15] to obtain the martingale property of the upper Snell envelope $\overline{Z}$. Second, we do not have a measurable selection theorem for stopping strategies, which complicates the proof of the dynamic programming principle. Moreover, the local approach that used comparison principle of viscosity solutions to show the existence of game value (see, e.g., [16] and [1]) does not work for our path-dependent set-up.

In Theorem 5.1, we demonstrate that $\overline{Z}$ is an $\underline{\mathscr{E}}$-supermartingale, and an $\underline{\mathscr{E}}$-martingale up to τ^*, the first time $\overline{Z}$ meets Y, from which (1.1) immediately follows. To prove this theorem, we use a more global approach rather than the local approach. We start with a dynamic programming principle, see Proposition 4.1, whose "super-solution" part is technically difficult due to the lack of measurable selection for stopping times. We overcome this issue by using a countable dense subset of $\mathcal{T}^t$ to construct a suitable approximation. This dynamic programming result is used to show the continuity of the upper Snell envelope, which plays an important role in the main theorem as our results heavily rely on the construction of approximating stopping times for τ^*. However, the dynamic programming principle directly enters the proof of Theorem 5.1 to show the supermartingale property of $\overline{Z}$ only after we upgrade the dynamic programming principle for random horizons in Proposition 4.3. We would like to emphasize that the submartingale property of the upper Snell envelope $\overline{Z}$ until τ^* does not directly follow from the dynamic programming principle. Instead, we build a delicate approximation scheme that involves carefully pasting probabilities and leveraging the martingale property of the single-probability Snell envelopes until they meet Y.

Let us say a few words about our assumptions. It should not come us a surprise that as a function of (t,ω), the probability set $\mathcal{P}(t,\omega)$ needs to be adapted. The most important assumption on the probability class

$$\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$$

is the weak stability under pasting; see (P2) in section 3. It is hard to envision that a dynamic programming result could hold without a stability under *pasting* assumption. This assumption along with the aforementioned continuity assumption (3.1) on Y (the regularity assumptions on the reward are common and can be verified for example of payoffs of all financial derivatives) allows us to construct approximate strategies for the controller by appropriately choosing its conditional distributions. Our stability assumption is weaker than its counterpart in Ekren, Touzi, and Zhang [12]; see, for example, our Remark 3.4 for a further discussion. We show in section 6 that this assumption (along with other assumptions we make on the probability class) are satisfied for some path-dependent SDEs with controls, which represents a large class of models on simultaneous drift and volatility uncertainty. (A stronger stability assumption as in [12] leads to results which is applicable only for volatility uncertainty.) We see section 6, which we dedicate one third of our paper to, as one of the main contributions of our paper. Another assumption we make on the probability class is that the augmentation of the filtration generated by the canonical process with respect to each probability in the class is right-continuous. This is because, as mentioned above, we exploit the results from the classic optimal stopping theory on the martingale property of the Snell envelopes for a given probability. Again, the example in section 6 is shown to satisfy this assumption.

Relevant literature. Since the seminal work [34], the martingale approach was extensively used in optimal stopping theory (see, e.g., [27, 15], and Appendix D of

[21]) and has been applied to various problems stemming from mathematical finance, the most important example of which is the computation of the super hedging price of the American contingent claims [7, 18, 19, 23]. Optimal stopping under Knightian uncertainty/nonlinear expectations/risk measures or the closely related controller-stopper-games have attracted a lot of attention in the recent years [24, 25, 17, 9, 10, 32, 2, 3, 4, 5, 8, 26]. In this literature, the set of probabilities is assumed to be dominated by a single probability or the controller is only allowed to influence the drift.

When the set of probabilities contain mutually singular probabilities or the controller can influence not only the drift but also the volatility, results are available only in some particular cases. Karazas and Sudderth [22] considered the controller-stopper-game in which the controller is allowed to control the volatility as well as the drift and resolved the saddle point problem for case of one-dimensional state variable using the characterization of the value function in terms of the scale function of the state variable. In the multidimensional case, [1] showed the existence of the value of a game using a comparison principle for viscosity solutions.

Our technical set-up follows closely that of [12], which analyzed a control problem with discretionary stopping (i.e., $\sup_{\tau\in\mathcal{T}}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}[Y_\tau]$) in a non-Markovian framework with mutually singular probability priors. (The solution of this problem was an important technical step in extending the notion of viscosity solutions to the fully nonlinear path-dependent PDEs in [13] and [14].) Nutz and Zhang [30] independently and around the same time addressed the problem we are considering by using a different (and an elegant) approach: They exploited the "tower property" of the nonlinear expectation $\underline{\mathscr{E}}$ developed in [29] to derive the $\underline{\mathscr{E}}$-martingale property of the *discrete time version* of the *lower Snell envelope* $\underline{Z}_t(\omega)\stackrel{\triangle}{=}\sup_{\tau\in\mathcal{T}^t}\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_\tau]$, $(t,\omega)\in[0,T]\times\Omega$. In contrast, we take an approach we consider to be very natural: We work with the upper Snell envelope and build our approximations *directly* in continuous time leveraging the known results from the classical optimal stopping theory. In the introduction, [30] states that they cannot work on upper Snell envelope due to the measurability selection issue; see paragraph 3 on page 3 of that paper. Our paper overcomes this issue. A major benefit of our approach is that we do not have to assume that the reward process is bounded since we do not have to rely on the approximation from discrete to continuous time. Another benefit is the weaker continuity assumption we impose on the value function in the path; compare Assumption 4.1 in our paper and Assumption 3.2 in [30]. The latter requires the value of any stopping strategy to be continuous with the same modulus of continuity, which is an assumption that is not easily verifiable. One strong suit of [30] is the saddle point analysis, which works under the weak formulation of the problem.

The rest of the paper is organized as follows: In section 2 we will introduce notation and some preliminary results such as the regular conditional probability distribution. In section 3, we set up the stage for our main result by imposing some assumptions on the reward process and the classes of mutually singular probabilities. Then section 4 studies properties of the upper Snell envelope of the reward process such as path regularity and dynamic programming principles. They are the essence to resolve our main result on the robust optimal stopping problem stated in section 5. In section 6, we give an example of path-dependent SDEs with controls that satisfies all our assumptions. The proofs of our results are deferred to section 7, and the appendix contains some technical lemmata needed for the proofs of the main results.

2. Notation and preliminaries. Let $(\mathbb{M},\varrho_{\mathbb{M}})$ be a generic metric space and let $\mathscr{B}(\mathbb{M})$ be the Borel σ-field of $\mathbb{M}$. For any $x\in\mathbb{M}$ and $\delta>0$, $O_\delta(x)\stackrel{\triangle}{=}\{x'\in$

$\mathbb{M} : \varrho_{\mathbb{M}}(x,x') < \delta\}$ and $\overline{O}_\delta(x) \stackrel{\triangle}{=} \{x' \in \mathbb{M} : \varrho_{\mathbb{M}}(x,x') \le \delta\}$, respectively, denote the open and closed ball centered at x with radius δ. Fix $d \in \mathbb{N}$. Let $\mathcal{S}_d^{>0}$ stand for all $\mathbb{R}^{d\times d}$-valued positively definite matrices. We denote by $\mathscr{B}(\mathcal{S}_d^{>0})$ the Borel σ-field of $\mathcal{S}_d^{>0}$ under the relative Euclidean topology.

Given $0 \le t \le T < \infty$, let $\Omega^{t,T} \stackrel{\triangle}{=} \{\omega \in \mathbb{C}([t,T];\mathbb{R}^d) : \omega(t) = 0\}$ be the canonical space over the period $[t,T]$, whose null path $\omega(\cdot) \equiv 0$ will be denoted by $\mathbf{0}^{t,T}$. For any $t \le s \le S \le T$, we introduce a seminorm $\|\cdot\|_{s,S}$ on $\Omega^{t,T}$: $\|\omega\|_{s,S} \stackrel{\triangle}{=} \sup_{r\in[s,S]}|\omega(r)|$ for all $\omega \in \Omega^{t,T}$. In particular, $\|\cdot\|_{t,T}$ is a norm on $\Omega^{t,T}$, called uniform norm, under which $\Omega^{t,T}$ is a separable complete metric space. Also, the *truncation* mapping $\Pi^{t,T}_{s,S}$ from $\Omega^{t,T}$ to $\Omega^{s,S}$ is defined by

$$\big(\Pi^{t,T}_{s,S}(\omega)\big)(r) \stackrel{\triangle}{=} \omega(r) - \omega(s) \quad \forall\, \omega \in \Omega^{t,T}, \ \forall\, r \in [s,S].$$

The canonical process $B^{t,T}$ on $\Omega^{t,T}$ is a d-dimensional Brownian motion under the Wiener measure $\mathbb{P}^{t,T}_0$ on $(\Omega^{t,T}, \mathscr{B}(\Omega^{t,T}))$. Let $\mathbf{F}^{t,T} = \{\mathcal{F}^{t,T}_s \stackrel{\triangle}{=} \sigma(B^{t,T}_r; r \in [t,s])\}_{s\in[t,T]}$ be the natural filtration of $B^{t,T}$ and let $\mathcal{C}^{t,T}$ collect all *cylinder* sets in $\mathcal{F}^{t,T}_T$: $\mathcal{C}^{t,T} \stackrel{\triangle}{=} \{\bigcap_{i=1}^{m} (B^{t,T}_{t_i})^{-1}(\mathcal{E}_i) : m \in \mathbb{N},\ t < t_1 < \cdots < t_m \le T,\ \{\mathcal{E}_i\}_{i=1}^m \subset \mathscr{B}(\mathbb{R}^d)\}$. It is well known that

$$\mathscr{B}(\Omega^{t,T}) = \sigma(\mathcal{C}^{t,T}) = \sigma\Big\{\big(B^{t,T}_r\big)^{-1}(\mathcal{E}) : r \in [t,T], \mathcal{E} \in \mathscr{B}(\mathbb{R}^d)\Big\} = \mathcal{F}^{t,T}_T.$$

Let $\mathscr{P}^{t,T}$ denote the $\mathbf{F}^{t,T}$-progressively measurable σ-field of $[t,T] \times \Omega^{t,T}$ and let $\mathcal{T}^{t,T}$ collect all $\mathbf{F}^{t,T}$-stopping times. We set $\mathcal{T}^{t,T}_s \stackrel{\triangle}{=} \{\tau \in \mathcal{T}^{t,T} : \tau \ge s\}$ for each $s \in [t,T]$ and will use the convention $\inf \emptyset \stackrel{\triangle}{=} \infty$.

From now on, we shall fix a time horizon $T \in (0,\infty)$ and drop it from the above notation, i.e., $(\Omega^{t,T}, \mathbf{0}^{t,T}, \|\ \|_{t,T}, B^{t,T}, \mathbb{P}^{t,T}_0, \mathbf{F}^{t,T}, \mathscr{P}^{t,T}, \mathcal{T}^{t,T}_s) \longrightarrow (\Omega^t, \mathbf{0}^t, \|\ \|_t, B^t, \mathbb{P}^t_0, \mathbf{F}^t, \mathscr{P}^t, \mathcal{T}^t_s)$. When $S = T$, $\Pi^{t,T}_{s,T}$ will be simply denoted by Π^t_s. For any $0 \le t \le s \le T$, $\omega \in \Omega^t$, and $\delta > 0$, define $O^s_\delta(\omega) \stackrel{\triangle}{=} \{\omega' \in \Omega^t : \|\omega' - \omega\|_{t,s} < \delta\}$ (In particular, $O^T_\delta(\omega) = O_\delta(\omega) = \{\omega' \in \Omega^t : \|\omega' - \omega\|_{t,T} < \delta\}$). Since Ω^t is the set of $\mathbb{R}^d$-valued continuous functions on $[t,T]$ starting from 0,

$$\begin{aligned}
O^s_\delta(\omega) &= \bigcup_{n\in\mathbb{N}} \big\{\omega' \in \Omega^t : \|\omega' - \omega\|_{t,s} \le \delta - \delta/n\big\} \\
&= \bigcup_{n\in\mathbb{N}} \bigcap_{r\in(t,s)\cap\mathbb{Q}} \big\{\omega' \in \Omega^t : |\omega'(r) - \omega(r)| \le \delta - \delta/n\big\} \\
&= \bigcup_{n\in\mathbb{N}} \bigcap_{r\in(t,s)\cap\mathbb{Q}} \big\{\omega' \in \Omega^t : B^t_r(\omega') \in \overline{O}_{\delta-\delta/n}\big(\omega(r)\big)\big\} \in \mathcal{F}^t_s. \qquad (2.1)
\end{aligned}$$

We fix a countable dense subset $\{\widehat{\omega}^t_j\}_{j\in\mathbb{N}}$ of Ω^t under $\|\cdot\|_t$, and set $\Theta^t_s \stackrel{\triangle}{=} \{O^s_\delta(\widehat{\omega}^t_j) : \delta \in \mathbb{Q}_+,\ j \in \mathbb{N}\} \subset \mathcal{F}^t_s$.

Given $t \in [0,T]$ and a probability $\mathbb{P}$ on $(\Omega^t, \mathscr{B}(\Omega^t)) = (\Omega^t, \mathcal{F}^t_T)$, let us set $\mathscr{N}^{\mathbb{P}} \stackrel{\triangle}{=} \{\mathcal{N} \subset \Omega^t : \mathcal{N} \subset A \text{ for some } A \in \mathcal{F}^t_T \text{ with } \mathbb{P}(A) = 0\}$. The $\mathbb{P}$-augmentation $\mathbf{F}^{\mathbb{P}}$ of $\mathbf{F}^t$ consists of $\mathcal{F}^{\mathbb{P}}_s \stackrel{\triangle}{=} \sigma(\mathcal{F}^t_s \cup \mathscr{N}^{\mathbb{P}})$, $s \in [t,T]$. (In particular, we will write $\overline{\mathscr{N}}^t$ for $\mathscr{N}^{\mathbb{P}^t_0}$ and $\overline{\mathbf{F}}^t = \{\overline{\mathcal{F}}^t_s\}_{s\in[t,T]}$ for $\mathbf{F}^{\mathbb{P}^t_0} = \{\mathcal{F}^{\mathbb{P}^t_0}_s\}_{s\in[t,T]}$.) We denote by $\mathcal{T}^{\mathbb{P}}$ the collection of all $\mathbf{F}^{\mathbb{P}}$-stopping times and set $\mathcal{T}^{\mathbb{P}}_s \stackrel{\triangle}{=} \{\tau \in \mathcal{T}^{\mathbb{P}} : \tau \ge s\}$ for each $s \in [t,T]$.

The completion of $(\Omega^t, \mathcal{F}^t_T, \mathbb{P})$ is the probability space $(\Omega^t, \mathcal{F}^{\mathbb{P}}_T, \overline{\mathbb{P}})$ with $\overline{\mathbb{P}}|_{\mathcal{F}^t_T} = \mathbb{P}$; we still write $\mathbb{P}$ for $\overline{\mathbb{P}}$ for convenience. In particular, the expectation on $(\Omega^t, \overline{\mathcal{F}}^t_T, \mathbb{P}^t_0)$

will be simply denoted by $\mathbb{E}_t$. A probability space $(\Omega^t, \mathcal{F}', \mathbb{P}')$ is called an extension of $(\Omega^t, \mathcal{F}^t_T, \mathbb{P})$ if $\mathcal{F}^t_T \subset \mathcal{F}'$ and $\mathbb{P}'|_{\mathcal{F}^t_T} = \mathbb{P}$.

For any metric space $\mathbb{M}$ and any $\mathbb{M}$-valued process $X = \{X_s\}_{s\in[t,T]}$, we set $\mathbf{F}^X = \{\mathcal{F}^X_s \stackrel{\triangle}{=} \sigma(X_r; r \in [t,s])\}_{s\in[t,T]}$ as the natural filtration of X and let $\mathbf{F}^{X,\mathbb{P}} = \{\mathcal{F}^{X,\mathbb{P}}_s \stackrel{\triangle}{=} \sigma(\mathcal{F}^X_s \cup \mathscr{N}^{\mathbb{P}})\}_{s\in[t,T]}$. If X is $\mathbf{F}^{\mathbb{P}}$-adapted, it holds for any $s \in [t,T]$ that $\mathcal{F}^X_s \subset \mathcal{F}^{\mathbb{P}}_s$ and thus $\mathcal{F}^{X,\mathbb{P}}_s \subset \mathcal{F}^{\mathbb{P}}_s$.

The following spaces about $\mathbb{P}$ will be frequently used in what follows:

(1) For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}^{\mathbb{P}}_T$, let $L^1(\mathcal{G},\mathbb{P})$ be the space of all real-valued, $\mathcal{G}$-measurable random variables ξ with $\|\xi\|_{L^1(\mathcal{G},\mathbb{P})} \stackrel{\triangle}{=} \mathbb{E}_{\mathbb{P}}[|\xi|] < \infty$.

(2) Let $\mathbb{D}^1(\mathbf{F}^t,\mathbb{P})$ (resp., $\mathbb{S}^1(\mathbf{F}^t,\mathbb{P})$) be the space of all real–valued, $\mathbf{F}^t$-adapted processes $\{X_s\}_{s\in[t,T]}$ whose paths are all right-continuous (resp., continuous) and satisfy $\mathbb{E}_{\mathbb{P}}[X_*] < \infty$, where $X_* \stackrel{\triangle}{=} \sup_{s\in[t,T]}|X_s|$. Also, by setting $\phi(x) = x\ln^+(x)$, $x \in [0,\infty)$, we define $\widehat{\mathbb{D}}(\mathbf{F}^t,\mathbb{P}) \stackrel{\triangle}{=} \{X \in \mathbb{D}^1(\mathbf{F}^t,\mathbb{P}) : \mathbb{E}_{\mathbb{P}}[\phi(X_*)] < \infty\}$. For any $x,y \in [0,\infty)$, if $z \stackrel{\triangle}{=} x \vee y < 2$, $\phi(x+y) \le \phi(2z) < \phi(4)$; otherwise, if $z \ge 2$, $\phi(x+y) \le \phi(2z) = 2z\ln(2z) < 2z\ln z^2 = 4z\ln z = 4\phi(z) \le 4(\phi(x)+\phi(y))$. So

$$\phi(x+y) \le 4\phi(x) + 4\phi(y) + \phi(4). \tag{2.2}$$

If the superscript $t=0$, we will drop them from the above notation. For example, $\mathbf{0} = \mathbf{0}^{0,T}$ and $\mathcal{T} = \mathcal{T}^{0,T}$.

2.1. Concatenation of sample paths. In the rest of this section, let us fix $0 \le t \le s \le T$. We concatenate an $\omega \in \Omega^t$ and an $\widetilde{\omega} \in \Omega^s$ at time s by

$$\big(\omega \otimes_s \widetilde{\omega}\big)(r) \stackrel{\triangle}{=} \omega(r)\,\mathbf{1}_{\{r\in[t,s)\}} + \big(\omega(s) + \widetilde{\omega}(r)\big)\,\mathbf{1}_{\{r\in[s,T]\}} \quad \forall\, r \in [t,T],$$

which is still of Ω^t. For any nonempty $\widetilde{A} \subset \Omega^s$, we set $\omega \otimes_s \emptyset = \emptyset$ and $\omega \otimes_s \widetilde{A} \stackrel{\triangle}{=} \{\omega \otimes_s \widetilde{\omega} : \widetilde{\omega} \in \widetilde{A}\}$.

The next result shows that $A \in \mathcal{F}^t_s$ consists of elements $\omega \otimes_s \Omega^s$ with $\omega \in A$.

Lemma 2.1. *Let $A \in \mathcal{F}^t_s$. If $\omega \in A$, then $\omega \otimes_s \Omega^s \subset A$. Otherwise, if $\omega \notin A$, then $\omega \otimes_s \Omega^s \subset A^c$.*

For any $\mathcal{F}^t_s$-measurable random variable η, since $\{\omega' \in \Omega^t : \eta(\omega') = \eta(\omega)\} \in \mathcal{F}^t_s$, Lemma 2.1 shows that

$$\omega \otimes_s \Omega^s \subset \{\omega' \in \Omega^t : \eta(\omega') = \eta(\omega)\}, \quad \text{i.e.,} \quad \eta(\omega \otimes_s \widetilde{\omega}) = \eta(\omega) \quad \forall\, \widetilde{\omega} \in \Omega^s. \tag{2.3}$$

On the other hand, for any $A \subset \Omega^t$ we set $A^{s,\omega} \stackrel{\triangle}{=} \{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in A\}$ as the projection of A on Ω^s along ω. In particular, $\emptyset^{s,\omega} = \emptyset$.

For any $r \in [s,T]$, the operation $(\ )^{s,\omega}$ projects an $\mathcal{F}^t_r$-measurable set to an $\mathcal{F}^s_r$-measurable set while the operation $\omega \otimes_s \cdot$ takes an $\mathcal{F}^s_r$-measurable set as input and returns an $\mathcal{F}^t_r$-measurable set.

Lemma 2.2. *Given $\omega \in \Omega^t$ and $r \in [s,T]$, we have $A^{s,\omega} \in \mathcal{F}^s_r$ for any $A \in \mathcal{F}^t_r$, and $\omega \otimes_s \widetilde{A} \in \mathcal{F}^t_r$ for any $\widetilde{A} \in \mathcal{F}^s_r$.*

Corollary 2.1. *Given $\tau \in \mathcal{T}^t$ and $\omega \in \Omega^t$, if $\tau(\omega \otimes_s \Omega^s) \subset [r,T]$ for some $r \in [s,T]$, then $\tau^{s,\omega} \in \mathcal{T}^s_r$.*

For any $\mathcal{D} \subset [t,T] \times \Omega^t$, we accordingly set $\mathcal{D}^{s,\omega} \stackrel{\triangle}{=} \{(r,\widetilde{\omega}) \in [s,T] \times \Omega^s : (r, \omega \otimes_s \widetilde{\omega}) \in \mathcal{D}\}$.

2.2. Regular conditional probability distributions. Let $\mathbb{P}$ be a probability on $(\Omega^t, \mathscr{B}(\Omega^t))$. In virtue of Theorem 1.3.4 and (1.3.15) of [37], there exists a family $\{\mathbb{P}^\omega_s\}_{\omega\in\Omega^t}$ of probabilities on $(\Omega^t, \mathscr{B}(\Omega^t))$, called the *regular conditional probability distribution* of $\mathbb{P}$ with respect to $\mathcal{F}^t_s$, such that

(i) for any $A \in \mathcal{F}^t_T$, the mapping $\omega \to \mathbb{P}^\omega_s(A)$ is $\mathcal{F}^t_s$-measurable;

(ii) for any $\xi \in L^1(\mathcal{F}^t_T, \mathbb{P})$, $\mathbb{E}_{\mathbb{P}^\omega_s}[\xi] = \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}^t_s](\omega)$ for $\mathbb{P}$-a.s. $\omega \in \Omega^t$;

(iii) for any

$$\omega \in \Omega^t,\ \mathbb{P}^\omega_s(\omega \otimes_s \Omega^s) = 1. \tag{2.4}$$

Given $\omega \in \Omega^t$, by Lemma 2.2, $\omega \otimes_s \widetilde{A} \in \mathcal{F}^t_T$ for any $\widetilde{A} \in \mathcal{F}^s_T$. So we can deduce from (2.4) that

$$\mathbb{P}^{s,\omega}\big(\widetilde{A}\big) \stackrel{\triangle}{=} \mathbb{P}^\omega_s\big(\omega \otimes_s \widetilde{A}\big) \quad \forall\, \widetilde{A} \in \mathcal{F}^s_T \tag{2.5}$$

defines a probability on $(\Omega^s, \mathcal{F}^s_T)$. The Wiener measures, however, are invariant under path shift.

Lemma 2.3. *Let $0 \le t \le s \le T$. It holds for $\mathbb{P}^t_0$-a.s. $\omega \in \Omega^t$ that $(\mathbb{P}^t_0)^{s,\omega} = \mathbb{P}^s_0$.*

Thanks to the existence of regular conditional probability distribution we can define conditional distributions using (2.5). Then by introducing path regularity for the reward process Y, one can treat path-dependent problems in ways similar to state-dependent problems. This can be seen as the general idea behind a dynamic programming in the path-dependent setting and the path-dependent PDEs introduced in [11].

2.3. Shifted random variables and shifted processes. Given a random variable ξ and a process $X = \{X_r\}_{r\in[t,T]}$ on Ω^t, for any $\omega \in \Omega^t$ we define the shifted random variable $\xi^{s,\omega}$ by $\xi^{s,\omega}(\widetilde{\omega}) \stackrel{\triangle}{=} \xi(\omega \otimes_s \widetilde{\omega})$, $\forall\, \widetilde{\omega} \in \Omega^s$ and the shifted process $X^{s,\omega}$ by $X^{s,\omega}_r(\widetilde{\omega}) = X(r, \omega \otimes_s \widetilde{\omega})$, $(r, \widetilde{\omega}) \in [s, T] \times \Omega^s$.

In light of Lemma 2.2 and the regular conditional probability distribution, shifted random variables/processes "inherit" measurability and integrability as follows.

Proposition 2.1. *Let $\mathbb{M}$ be a generic metric space and let $\omega \in \Omega^t$.*

(1) *If an $\mathbb{M}$-valued random variable ξ on Ω^t is $\mathcal{F}^t_r$-measurable for some $r \in [s, T]$, then $\xi^{s,\omega}$ is $\mathcal{F}^s_r$-measurable.*

(2) *If an $\mathbb{M}$-valued process $\{X_r\}_{r\in[t,T]}$ is $\mathbf{F}^t$-adapted (resp., $\mathbf{F}^t$-progressively measurable), then the shifted process $\{X^{s,\omega}_r\}_{r\in[s,T]}$ is $\mathbf{F}^s$-adapted (resp., $\mathbf{F}^s$-progressively measurable).*

Proposition 2.2. *If $\xi \in L^1(\mathcal{F}^t_T, \mathbb{P})$ for some probability $\mathbb{P}$ on $(\Omega^t, \mathscr{B}(\Omega^t))$, then it holds for $\mathbb{P}$-a.s. $\omega \in \Omega^t$ that the shifted random variable $\xi^{s,\omega} \in L^1(\mathcal{F}^s_T, \mathbb{P}^{s,\omega})$ and*

$$\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\xi^{s,\omega}\big] = \mathbb{E}_{\mathbb{P}}\big[\xi\big|\mathcal{F}^t_s\big](\omega) \in \mathbb{R}. \tag{2.6}$$

As a consequence of (2.6), a shifted $\mathbb{P}^t_0-$null set (or $dr \times d\mathbb{P}^t_0$-null set) also has zero measure.

Lemma 2.4. *For any $\mathcal{N} \in \overline{\mathscr{N}}^t$, it holds for $\mathbb{P}^t_0$-a.s. $\omega \in \Omega^t$ that $\mathcal{N}^{s,\omega} \in \overline{\mathscr{N}}^s$; for any $\mathcal{D} \in \mathscr{B}([t, T]) \otimes \mathcal{F}^t_T$ with $(dr \times d\mathbb{P}^t_0)(\mathcal{D} \cap ([s, T] \times \Omega^t)) = 0$, it holds for $\mathbb{P}^t_0$-a.s. $\omega \in \Omega^t$ that $(dr \times d\mathbb{P}^s_0)(\mathcal{D}^{s,\omega}) = 0$.*

The proofs of results in this section can be found in [36, 35], see also [6].

In the next three sections, we will gradually provide the technical set-up and preparation for our main result (Theorem 5.1) on the robust optimal stopping problem.

3. Weak stability under pasting. In the proof of Theorem 5.1, we will use an approximation scheme which exploits results from the classic optimal stopping theory for a given probability. For this purpose, we consider the following probability set.

DEFINITION 3.1. *For any $t \in [0,T]$, let $\mathfrak{P}_t$ collect all probabilities $\mathbb{P}$ on $(\Omega^t, \mathscr{B}(\Omega^t))$ such that $\mathbf{F}^{\mathbb{P}}$ is right-continuous.*

We will also need some regularity assumption on the reward process.

Standing assumptions on reward process $\boldsymbol{Y}$.

(**Y**) Y is an **F**-adapted process that satisfies an one-sided continuity condition in (t,ω) with respect to some modulus of continuity function ρ_0 in the following sense:

$$Y_{t_1}(\omega_1) - Y_{t_2}(\omega_2) \le \rho_0\Big(\mathbf{d}_\infty\big((t_1,\omega_1),(t_2,\omega_2)\big)\Big) \quad \forall\, 0 \le t_1 \le t_2 \le T, \ \ \forall\, \omega_1, \omega_2 \in \Omega, \tag{3.1}$$

where $\mathbf{d}_\infty((t_1,\omega_1),(t_2,\omega_2)) \stackrel{\triangle}{=} (t_2 - t_1) + \|\omega_1(\cdot \wedge t_1) - \omega_2(\cdot \wedge t_2)\|_{0,T}$.

Remark 3.1.

(1) As pointed out in Remark 3.2 of [12], (3.1) implies that each path of Y is RCLL with positive jumps.

(2) Also, one can deduce from (3.1) that the process Y is left upper semicontinuous, i.e., for any $(t,\omega) \in (0,T] \times \Omega$, $Y_t(\omega) \ge \overline{\lim}_{s \nearrow t} Y_s(\omega)$. It follows that the shifted process $Y^{t,\omega}$ is also left upper semicontinuous. Then we can apply the classical optimal stopping theory to $Y^{t,\omega}$ under each $\mathbb{P} \in \mathfrak{P}_t$. Actually, the proof of Theorem 5.1 relies on the comparison of $\overline{Z}^{t,\omega}$ with the Snell envelope of $Y^{t,\omega}$ under each $\mathbb{P} \in \mathfrak{P}_t$.

The next result shows that $\mathbb{L}\ln\mathbb{L}$-integrability of shifted reward process is independent of the given path history.

LEMMA 3.1. *Assume (Y). For any $t \in [0,T]$ and any probability $\mathbb{P}$ on $(\Omega^t, \mathscr{B}(\Omega^t))$, if $Y^{t,\omega} \in \widehat{\mathbb{D}}(\mathbf{F}^t, \mathbb{P})$ for some $\omega \in \Omega$, then $Y^{t,\omega'} \in \widehat{\mathbb{D}}(\mathbf{F}^t, \mathbb{P})$ for all $\omega' \in \Omega$.*

We shall focus on the following subset of $\mathfrak{P}_t$ that makes the shifted reward process $\mathbb{L}\ln\mathbb{L}$-integrable.

Assumption 3.1. For any $t \in [0,T]$, the set $\mathfrak{P}^Y_t \stackrel{\triangle}{=} \{\mathbb{P} \in \mathfrak{P}_t : Y^{t,\mathbf{0}} \in \widehat{\mathbb{D}}(\mathbf{F}^t, \mathbb{P})\}$ is not empty.

Remark 3.2.

(1) If $Y \in \widehat{\mathbb{D}}(\mathbf{F}, \mathbb{P}_0)$, then Lemma 2.3, (2.6), and Lemma 3.1 imply that $\mathbb{P}^t_0 \in \mathfrak{P}^Y_t$ for any $t \in [0,T]$.

(2) As we will see in Lemma 6.1, when the modulus of continuity ρ_0 has polynomial growth, the laws of solutions to the controlled SDEs (6.1) over period $[t,T]$ belong to $\mathfrak{P}^Y_t$.

Under (Y) and Assumption 3.1, we see from Lemma 3.1 that for any $t \in [0,T]$ and $\mathbb{P} \in \mathfrak{P}^Y_t$,

$$Y^{t,\omega} \in \widehat{\mathbb{D}}\big(\mathbf{F}^t, \mathbb{P}\big) \quad \forall\, \omega \in \Omega. \tag{3.2}$$

Next, we need the probability classes to be adapted and weakly stable under pasting in the following sense.

Standing assumptions on probability class.

(**P0**) For any $t \in [0,T]$, let us consider a family $\{\mathcal{P}(t,\omega) = \mathcal{P}_Y(t,\omega)\}_{\omega \in \Omega}$ of subsets of $\mathfrak{P}^Y_t$ which is adapted in the sense that $\mathcal{P}(t,\omega_1) = \mathcal{P}(t,\omega_2)$ if $\omega_1|_{[0,t]} = \omega_2|_{[0,t]}$. So $\mathcal{P} \stackrel{\triangle}{=} \mathcal{P}(0,\mathbf{0}) = \mathcal{P}(0,\omega)$ for all $\omega \in \Omega$.

We further assume that the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfy the following two conditions for some modulus of continuity function $\widehat{\rho}_0$: for any $0\le t<s\le T$, $\omega\in\Omega$ and $\mathbb{P}\in\mathcal{P}(t,\omega)$.

(**P1**) There exist an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of $(\Omega^t,\mathcal{F}^t_T,\mathbb{P})$ and $\Omega'\in\mathcal{F}'$ with $\mathbb{P}'(\Omega')=1$ such that for any $\widetilde{\omega}\in\Omega'$, $\mathbb{P}^{s,\widetilde{\omega}}\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$.

(**P2**) For any $\delta\in\mathbb{Q}_+$ and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^{\lambda}$ be a $\mathcal{F}^t_s$-partition of Ω^t such that for $j=1,\dots,\lambda$, $\mathcal{A}_j\subset O^s_\delta(\widetilde{\omega}_j)$ for some $\widetilde{\omega}_j\in\Omega^t$. Then for any $\mathbb{P}_j\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega}_j)$, $j=1,\dots,\lambda$, there exists a $\widehat{\mathbb{P}}\in\mathcal{P}(t,\omega)$ such that

(i) $\widehat{\mathbb{P}}(A\cap\mathcal{A}_0)=\mathbb{P}(A\cap\mathcal{A}_0)$, $\forall\,A\in\mathcal{F}^t_T$;

(ii) for any $j=1,\dots,\lambda$ and $A\in\mathcal{F}^t_s$, $\widehat{\mathbb{P}}(A\cap\mathcal{A}_j)=\mathbb{P}(A\cap\mathcal{A}_j)$ and

$$\begin{aligned}(3.3)\qquad &\sup_{\tau\in\mathcal{T}^t_s}\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y^{t,\omega}_\tau\big]\\ &\qquad\le\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\left(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta\big]+\widehat{\rho}_0(\delta)\right)\right].\end{aligned}$$

From now on, when writing $Y^{t,\omega}_\tau$, we mean $(Y^{t,\omega})_\tau$ not $(Y_\tau)^{t,\omega}$.

Remark 3.3.

(1) As we will show in section 7, both sides of (3.3) are finite. In particular, the expectation on right-hand side is well-defined since the mapping $\widetilde{\omega}\to\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta]$ is continuous.

(2) The condition (P2) can be viewed as a weak stability under pasting since it is implied by the stability under finite pasting (see, e.g., (4.18) of [35]): for any $0\le t<s\le T$, $\omega\in\Omega$, $\mathbb{P}\in\mathcal{P}(t,\omega)$, $\delta\in\mathbb{Q}_+$, and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^{\lambda}$ be a $\mathcal{F}^t_s$-partition of Ω^t such that for $j=1,\dots,\lambda$, $\mathcal{A}_j\subset O^s_\delta(\widetilde{\omega}_j)$ for some $\widetilde{\omega}_j\in\Omega^t$. Then for any $\mathbb{P}_j\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega}_j)$, $j=1,\dots,\lambda$, there exists a $\widehat{\mathbb{P}}\in\mathcal{P}(t,\omega)$ such that

$$(3.4)\qquad \widehat{\mathbb{P}}(A)=\mathbb{P}(A\cap\mathcal{A}_0)+\sum_{j=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_j\}}\mathbb{P}_j\big(A^{s,\widetilde{\omega}}\big)\Big]\quad\forall\,A\in\mathcal{F}^t_T.$$

Remark 3.4. The reason we assume (P2) rather than the stability of finite pasting (3.4) lies in the fact that the latter does not hold for our example of path-dependent SDEs with controls (section 6) as pointed out in Remark 3.6 of [28], while the former is sufficient for our approximation methods in proving the main results.

4. The dynamic programming principle. The key to solving problem (1.1) is the following upper Snell envelope of the reward processes:

$$(4.1)\qquad \overline{Z}_t(\omega)\stackrel{\triangle}{=}\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_\tau\big]\quad\forall\,(t,\omega)\in[0,T]\times\Omega.$$

In this section, we derive some basic properties of $\overline{Z}$ and the dynamic programming principles it satisfies. These results will provide an important technical step for the proof of Theorem 5.1. Let (Y), (P0), (P1), and (P2) hold throughout the section.

Given $(t,\omega)\in[0,T]\times\Omega$, since Y_t is $\mathcal{F}_t$-measurable, (2.3) implies that $Y^{t,\omega}_t=Y_t(\omega)$. It then follows from (4.1) that

$$(4.2)\qquad \overline{Z}_t(\omega)\ge\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_t\big]=Y_t(\omega)\quad\forall\,(t,\omega)\in[0,T]\times\Omega.$$

We need two additional assumptions on $\overline{Z}$ before discussing its path regularity properties and dynamic programming principle.

Assumption 4.1. There exists a modulus of continuity function $\rho_1 \ge \rho_0$ such that for any $t \in [0,T]$

$$\big|\overline{Z}_t(\omega_1) - \overline{Z}_t(\omega_2)\big| \le \rho_1\big(\|\omega_1 - \omega_2\|_{0,t}\big) \quad \forall\, \omega_1, \omega_2 \in \Omega. \tag{4.3}$$

Remark 4.1. If $\mathcal{P}(t,\omega)$ does not depend on ω for all $t\in[0,T]$, then (3.1) implies Assumption 4.1.

Remark 4.2. Assumption 4.1 on $\overline{Z}$ implies that $\overline{Z}$ is $\mathbf{F}$-adapted.

Assumption 4.2. For any $\alpha > 0$, there exists a modulus of continuity function ρ_α such that for any $t \in [0,T)$

$$\sup_{\omega\in O^t_\alpha(\mathbf{0})} \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}_\mathbb{P}\left[\rho_1\left(\delta + 2 \sup_{r\in[t,(t+\delta)\wedge T]} |B^t_r|\right)\right] \le \rho_\alpha(\delta) \quad \forall\, \delta \in (0,T]. \tag{4.4}$$

Similar to (3.2), one has the following integrability result of shifted processes of $\overline{Z}$.

LEMMA 4.1. *Given $(t,\omega) \in [0,T]\times\Omega$, it holds for any $\mathbb{P} \in \mathcal{P}(t,\omega)$ and $s \in [t,T]$ that $\mathbb{E}_\mathbb{P}[|\overline{Z}^{t,\omega}_s|] < \infty$.*

As to the dynamic programming principle, we present first a basic version in which the transit horizon is deterministic.

PROPOSITION 4.1. *For any $0 \le t \le s \le T$ and $\omega \in \Omega$,*

$$\overline{Z}_t(\omega) = \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<s\}} Y^{t,\omega}_\tau + \mathbf{1}_{\{\tau\ge s\}} \overline{Z}^{t,\omega}_s\Big]. \tag{4.5}$$

Consequently, all paths of $\overline{Z}$ are continuous.

PROPOSITION 4.2. *For any $(t,\omega) \in [0,T]\times\Omega$ and $\mathbb{P} \in \mathcal{P}(t,\omega)$, $\overline{Z}^{t,\omega} \in \mathbb{S}^1(\mathbf{F}^t,\mathbb{P})$.*

The continuity of $\overline{Z}$ allows us to derive a general version of dynamic programming principle with random horizons.

PROPOSITION 4.3. *For any $(t,\omega) \in [0,T]\times\Omega$ and $\nu \in \mathcal{T}^t$,*

$$\overline{Z}_t(\omega) \ge \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<\nu\}} Y^{t,\omega}_\tau + \mathbf{1}_{\{\tau\ge \nu\}} \overline{Z}^{t,\omega}_\nu\Big]. \tag{4.6}$$

The reverse inequality holds under an additional condition; see [6] *for details.* (*But this is not needed for our main result.*)

5. Robust optimal stopping. In this section, we state our main result on robust optimal stopping problem. Let (Y), (P0), (P1), (P2), and Assumptions 3.1–4.2 hold throughout the section.

For any $t \in [0,T]$, we set $\mathscr{L}_t \stackrel{\triangle}{=} \{$random variable ξ on $\Omega : \xi^{t,\omega} \in L^1(\mathcal{F}^t_T,\mathbb{P})$ for all $\omega \in \Omega,\ \mathbb{P} \in \mathcal{P}(t,\omega)\}$ and define on $\mathscr{L}_t$ a nonlinear expectation: $\underline{\mathscr{E}}_t[\xi](\omega) \stackrel{\triangle}{=} \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_\mathbb{P}[\xi^{t,\omega}]$ for all $\omega\in\Omega,\ \xi\in\mathscr{L}_t$.

Remark 5.1. Given $\tau\in\mathcal{T}$, $Y_\tau, \overline{Z}_\tau \in \mathscr{L}_t$ for any $t \in [0,T]$, thanks to (3.2) and Proposition 4.2.

Similar to the classic optimal stopping theory, we will show that the first time $\overline{Z}$ meets Y

$$\tau^* \stackrel{\triangle}{=} \inf\{t \in [0,T] : \overline{Z}_t = Y_t\}$$

is an optimal stopping time for (1.1), and the upper Snell envelope $\overline{Z}$ has a martingale characterization with respect to the nonlinear expectation $\underline{\mathscr{E}} \stackrel{\triangle}{=} \{\underline{\mathscr{E}}_t\}_{t\in[0,T]}$.

THEOREM 5.1. *Let* (Y), (P0), (P1), (P2) *and Assumptions* 3.1–4.2 *hold. If* $\sup_{(t,\omega)\in[0,T]\times\Omega} Y_t(\omega) = \infty$, *we further assume that for some* $L > 0$

$$Y_{t_2}(\omega) - Y_{t_1}(\omega) \le L + \phi\left(\sup_{r\in[0,t_1]} |Y_r(\omega)|\right) + \rho_1\left(\sup_{r\in[t_1,t_2]} \big|\omega(r) - \omega(t_1)\big|\right) \qquad \forall\, 0 \le t_1 \le t_2 \le T,\ \ \forall\, \omega \in \Omega. \tag{5.1}$$

Then $\overline{Z}$ *is an* $\underline{\mathscr{E}}$*-supermartingale and* $\{\overline{Z}^*_t \stackrel{\triangle}{=} \overline{Z}_{t\wedge\tau^*}\}_{t\in[0,T]}$ *is an* $\underline{\mathscr{E}}$*-martingale in the sense that*

$$\overline{Z}_t(\omega) \ge \underline{\mathscr{E}}_t\big[\,\overline{Z}_\tau\big](\omega) \quad \text{and} \quad \overline{Z}^*_t(\omega) = \underline{\mathscr{E}}_t\big[\,\overline{Z}^*_\tau\big](\omega) \qquad \forall\, (t,\omega) \in [0,T]\times\Omega,\ \ \forall\, \tau \in \mathcal{T}_t. \tag{5.2}$$

In particular, the $\mathbf{F}$*-stopping time* τ^* *satisfies* (1.1).

Remark 5.2.

(1) Similar to [30], we can apply (1.1) to subhedging of American options in a financial market with volatility uncertainty.

(2) As to a worst-case risk measure $\mathfrak{R}(\xi) \stackrel{\triangle}{=} \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[-\xi]$ defined for any bounded financial position ξ, applying (1.1) to a given bounded reward process Y yields that $\inf_{\tau\in\mathcal{T}} \mathfrak{R}(Y_\tau) = -\sup_{\tau\in\mathcal{T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[Y_\tau] = -\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[Y_{\tau^*}] = \mathfrak{R}(Y_{\tau^*})$. So τ^* is also an optimal stopping time for the optimal stopping problem of $\mathfrak{R}$.

(3) From the perspective of a zero-sum controller-stopper game in which the stopper chooses the termination time while the controller selects the distribution law from $\mathcal{P}$, (1.1) shows that such a game has a value $\underline{\mathscr{E}}_0[Y_{\tau^*}] = \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[Y_{\tau^*}]$ as its lower value $\sup_{\tau\in\mathcal{S}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_\mathbb{P}[Y_\tau]$ coincides with the upper one $\inf_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} \mathbb{E}_\mathbb{P}[Y_\tau]$.

6. Example: Path-dependent controlled SDEs. In this section we will present an example of the probability class $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ in the case of path-dependent SDEs with controls.

Let $\kappa > 0$ and let $b\colon [0,T]\times\Omega\times\mathbb{R}^{d\times d} \to \mathbb{R}^d$ be a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^{d\times d})/\mathscr{B}(\mathbb{R}^d)$-measurable function such that

$$|b(t,\omega,u) - b(t,\omega',u)| \le \kappa\|\omega-\omega'\|_{0,t} \quad \text{and} \quad |b(t,\mathbf{0},u)| \le \kappa(1+|u|) \qquad \forall\, \omega,\omega' \in \Omega,\ (t,u) \in [0,T]\times\mathbb{R}^{d\times d}.$$

Let $(t,\omega) \in [0,T]\times\Omega$. $b^{t,\omega}(r,\widetilde{\omega},u) \stackrel{\triangle}{=} b(r,\omega\otimes_t\widetilde{\omega},u)$, $(r,\widetilde{\omega},u) \in [t,T]\times\Omega^t\times\mathbb{R}^{d\times d}$ is clearly a $\mathscr{P}^t\otimes\mathscr{B}(\mathbb{R}^{d\times d})/\mathscr{B}(\mathbb{R}^d)$- measurable function that satisfies

$$|b^{t,\omega}(r,\widetilde{\omega},u) - b^{t,\omega}(r,\widetilde{\omega}',u)| \le \kappa\|\widetilde{\omega}-\widetilde{\omega}'\|_{t,r} \text{ and } |b^{t,\omega}(r,\mathbf{0}^t,u)| \le \kappa\big(1+\|\omega\|_{0,t}+|u|\big),\ \forall\, \widetilde{\omega},\widetilde{\omega}' \in \Omega^t,\ (r,u) \in [t,T]\times\mathbb{R}^{d\times d}.$$

For any $t \in [0,T]$, let $\mathcal{U}_t$ collect all $\mathcal{S}_d^{>0}$-valued, $\mathbf{F}^t$-progressively measurable processes $\{\mu_s\}_{s\in[t,T]}$ such that $|\mu_s| \le \kappa$, $ds\times d\mathbb{P}^t_0$-a.s. Given $\mu \in \mathcal{U}_t$, similar to the classical SDE theory, an application of fixed-point iteration shows that the following SDE on the probability space $(\Omega^t, \mathcal{F}^t_T, \mathbb{P}^t_0)$:

$$X_s = \int_t^s b^{t,\omega}(r, X, \mu_r)dr + \int_t^s \mu_r \, dB^t_r, \quad s \in [t, T], \tag{6.1}$$

admits a unique solution $X^{t,\omega,\mu}$, which is an $\overline{\mathbf{F}}^t$-adapted continuous process. Note that the SDE (6.1) depends on $\omega|_{[0,t]}$ via the generator $b^{t,\omega}$.

Without loss of generality, we assume that all paths of $X^{t,\omega,\mu}$ are continuous and starting from 0. (Otherwise, by setting $\mathcal{N} \stackrel{\triangle}{=} \{\omega \in \Omega^t : X^{t,\omega,\mu}_t(\omega) \neq \mathbf{0}$ or the path $X^{t,\omega,\mu}_\cdot(\omega)$ is not continuous$\} \in \overline{\mathscr{N}}^t$, one can take $\widetilde{X}^{t,\omega,\mu}_s \stackrel{\triangle}{=} \mathbf{1}_{\mathcal{N}^c} X^{t,\omega,\mu}_s$, $s \in [t, T]$. It is an $\overline{\mathbf{F}}^t$-adapted process that satisfies (6.1) and whose paths are all continuous and starting from 0.)

Applying the Burkholder–Davis–Gundy inequality and Gronwall's inequality and using the Lipschitz continuity of b in ω, one can easily derive the following estimates for $X^{t,\omega,\mu}$: for any $p \geq 1$

$$\begin{aligned} \mathbb{E}_t\left[\sup_{r\in[t,s]} |X^{t,\omega,\mu}_r|^p\right] \leq \varphi_p\big(\|\omega\|_{0,t}\big)(s-t)^{p/2} \ \text{ and } \ \mathbb{E}_t\left[\sup_{r\in[t,s]} \big|X^{t,\omega,\mu}_r - X^{t,\omega',\mu}_r\big|^p\right] \\ \leq C_p \|\omega-\omega'\|^p_{0,t}\,(s-t)^p \quad \forall\, \omega' \in \Omega, \end{aligned} \tag{6.2}$$

where φ_p is a modulus of continuity function depending on p, κ, T and C_p denotes a constant depending on p, κ, T.

Similar to Lemma 3.3 of [30], the following result shows that the shift of $X^{t,\omega,\mu}$ is exactly the solution of SDE (6.1) with shifted drift coefficient and shifted control. (See [6] for its proof.)

Proposition 6.1. *Given $0 \leq t \leq s \leq T$, $\omega \in \Omega$, and $\mu \in \mathcal{U}_t$, let $\mathcal{X} \stackrel{\triangle}{=} X^{t,\omega,\mu}$. It holds for $\mathbb{P}^t_0$-a.s. $\widetilde{\omega} \in \Omega^t$ that $\mu^{s,\widetilde{\omega}} \in \mathcal{U}_s$ and that $\mathcal{X}^{s,\widetilde{\omega}} = X^{s,\omega\otimes_t \mathcal{X}(\widetilde{\omega}),\mu^{s,\widetilde{\omega}}} + \mathcal{X}_s(\widetilde{\omega})$.*

As a mapping from Ω^t to Ω^t, $X^{t,\omega,\mu}$ is $\overline{\mathcal{F}}^t_s/\mathcal{F}^t_s$-measurable for any $s \in [t, T]$: To see this, let us pick up an arbitrary $\mathcal{E} \in \mathscr{B}(\mathbb{R}^d)$. The $\overline{\mathbf{F}}^t$-adaptness of $X^{t,\omega,\mu}$ shows that for any $r \in [t, s]$

$$\begin{aligned} \big(X^{t,\omega,\mu}\big)^{-1}\Big(\big(B^t_r\big)^{-1}(\mathcal{E})\Big) &= \big\{\widetilde{\omega} \in \Omega^t : X^{t,\omega,\mu}(\widetilde{\omega}) \in \big(B^t_r\big)^{-1}(\mathcal{E})\big\} \\ &= \big\{\widetilde{\omega} \in \Omega^t : X^{t,\omega,\mu}_r(\widetilde{\omega}) \in \mathcal{E}\big\} \in \overline{\mathcal{F}}^t_s. \end{aligned} \tag{6.3}$$

Thus $(B^t_r)^{-1}(\mathcal{E}) \in \mathcal{G}^{X^{t,\omega,\mu}}_s \stackrel{\triangle}{=} \{A \subset \Omega^t : (X^{t,\omega,\mu})^{-1}(A) \in \overline{\mathcal{F}}^t_s\}$, a σ-field of Ω^t. It follows that $\mathcal{F}^t_s \subset \mathcal{G}^{X^{t,\omega,\mu}}_s$, i.e.,

$$\big(X^{t,\omega,\mu}\big)^{-1}(A) \in \overline{\mathcal{F}}^t_s \quad \forall\, A \in \mathcal{F}^t_s, \tag{6.4}$$

proving the measurability of the mapping $X^{t,\omega,\mu}$. We define the law of $X^{t,\omega,\mu}$ under $\mathbb{P}^t_0$ by

$$\mathfrak{p}^{t,\omega,\mu}(A) \stackrel{\triangle}{=} \mathbb{P}^t_0 \circ \big(X^{t,\omega,\mu}\big)^{-1}(A) \quad \forall\, A \in \mathcal{G}^{X^{t,\omega,\mu}}_T,$$

and denote by $\mathbb{P}^{t,\omega,\mu}$ the restriction of $\mathfrak{p}^{t,\omega,\mu}$ on $(\Omega^t, \mathcal{F}^t_T)$. The filtrations $\mathbf{F}^{\mathbb{P}^{t,\omega,\mu}}$ are all right-continuous.

Proposition 6.2. *For any $(t, \omega) \in [0, T] \times \Omega$ and $\mu \in \mathcal{U}_t$, $\mathbb{P}^{t,\omega,\mu}$ belongs to $\mathfrak{P}_t$.*

Remark 6.1. The reason we consider the law of $X^{t,\omega,\mu}$ under $\mathbb{P}^t_0$ over $\mathcal{G}^{X^{t,\omega,\mu}}_T$ (the largest σ-field to induce $\mathbb{P}^t_0$ under the mapping $X^{t,\omega,\mu}$) rather than $\mathcal{F}^t_T$ is as follows. Our proofs for Propositions 6.2 and 6.3 rely heavily on the inverse mapping $W^{t,\omega,\mu}$ of $X^{t,\omega,\mu}$, which is an $\mathbf{F}^t$-progessively measurable processes having only $\mathfrak{p}^{t,\omega,\mu}$-a.s. continuous paths. Consequently, as we will show in the proof of the following Proposition 6.3, it holds for $\mathfrak{p}^{t,\omega,\mu}$-a.s. $\widetilde{\omega}\in\Omega^t$ that the shifted probability $(\mathbb{P}^{t,\omega,\mu})^{s,\widetilde{\omega}}$ is the law of the solution to the shifted SDE and thus belongs to $\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$. This explains why our assumption (P1) needs an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of the probability space $(\Omega^t,\mathcal{F}^t_T,\mathbb{P})$.

Given $\varpi\ge 1$, let ρ_0 be a modulus of continuity function such that

$$\rho_0(\delta)\le\kappa(1+\delta^{\varpi})\quad\forall\,\delta>0,\tag{6.5}$$

and let Y satisfy (Y) with ρ_0. We set $\mathcal{P}(t,\omega)\stackrel{\triangle}{=}\{\mathbb{P}^{t,\omega,\mu}\colon\mu\in\mathcal{U}_t\}$.

Lemma 6.1. *Assume* (Y) *and* (6.5)*. For any* $(t,\omega)\in[0,T]\times\Omega$*, we have* $\mathcal{P}(t,\omega)\subset\mathfrak{P}^Y_t$.

For any $\omega_1,\omega_2\in\Omega$ with $\omega_1|_{[0,t]}=\omega_2|_{[0,t]}$, since (6.1) depends only on $\omega|_{[0,t]}$, we see that $X^{t,\omega_1,\mu}=X^{t,\omega_2,\mu}$ and thus $\mathbb{P}^{t,\omega_1,\mu}=\mathbb{P}^{t,\omega_2,\mu}$ for any $\mu\in\mathcal{U}_t$. It follows that $\mathcal{P}(t,\omega_1)=\mathcal{P}(t,\omega_2)$. So assumption (P0) is satisfied.

Proposition 6.3. *Assume* (Y) *and* (6.5)*. Then the probability class* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies* (P1), (P2), *and Assumptions* 4.1 *and* 4.2.

7. Proofs.

7.1. Proofs of the results in section 3.

Proof of Lemma 3.1. Let $t\in[0,T]$ and $\mathbb{P}$ be a probability on $(\Omega^t,\mathscr{B}(\Omega^t))$. Suppose that $Y^{t,\omega}\in\widehat{\mathbb{D}}(\mathbf{F}^t,\mathbb{P})$ for some $\omega\in\Omega$ and fix $\omega'\in\Omega$. The $\mathbf{F}$-adaptness of Y and Proposition 2.1 (2) show that $Y^{t,\omega'}$ is $\mathbf{F}^t$-adapted. Given $\widetilde{\omega}\in\Omega^t$, (3.1) implies that for any $s\in[t,T]$

$$\begin{aligned}\big|Y^{t,\omega'}_s(\widetilde{\omega})-Y^{t,\omega}_s(\widetilde{\omega})\big|&=\big|Y_s(\omega'\otimes_t\widetilde{\omega})-Y_s(\omega\otimes_t\widetilde{\omega})\big|\\&\le\rho_0\big(\|\omega'\otimes_t\widetilde{\omega}-\omega\otimes_t\widetilde{\omega}\|_{0,s}\big)=\rho_0\big(\|\omega'-\omega\|_{0,t}\big).\end{aligned}\tag{7.1}$$

It follows that $Y^{t,\omega'}_*(\widetilde{\omega})=\sup_{s\in[t,T]}|Y^{t,\omega'}_s(\widetilde{\omega})|\le\sup_{s\in[t,T]}|Y^{t,\omega}_s(\widetilde{\omega})|+\rho_0(\|\omega'-\omega\|_{0,t})=Y^{t,\omega}_*(\widetilde{\omega})+\rho_0(\|\omega'-\omega\|_{0,t})$. Then (2.2) implies that $\mathbb{E}_{\mathbb{P}}[\phi(Y^{t,\omega'}_*)]\le 4\mathbb{E}_{\mathbb{P}}[\phi(Y^{t,\omega}_*)]+4\phi(\rho_0(\|\omega'-\omega\|_{0,t}))+\phi(4)<\infty$. So $Y^{t,\omega'}\in\widehat{\mathbb{D}}(\mathbf{F}^t,\mathbb{P})$. □

Proof of Remark 3.3.

(1) Let $\widetilde{\omega}_1,\widetilde{\omega}_2\in\Omega^t$. For any $\zeta\in\mathcal{T}^s$, similar to (7.1), we can deduce that

$$\begin{aligned}\big|Y^{s,\omega\otimes_t\widetilde{\omega}_1}_\zeta(\widehat{\omega})-Y^{s,\omega\otimes_t\widetilde{\omega}_2}_\zeta(\widehat{\omega})\big|&\le\rho_0\big(\|(\omega\otimes_t\widetilde{\omega}_1)\otimes_s\widehat{\omega}-(\omega\otimes_t\widetilde{\omega}_2)\otimes_s\widehat{\omega}\|_{0,\zeta(\widehat{\omega})}\big)\\&=\rho_0\big(\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{t,s}\big)\quad\forall\,\widehat{\omega}\in\Omega^s.\end{aligned}$$

It follows that

$$\mathbb{E}_{\mathbb{P}_j}\big[Y^{s,\omega\otimes_t\widetilde{\omega}_1}_\zeta\big]\le\mathbb{E}_{\mathbb{P}_j}\big[Y^{s,\omega\otimes_t\widetilde{\omega}_2}_\zeta\big]+\rho_0\big(\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{t,s}\big).\tag{7.2}$$

Taking supremum over $\zeta\in\mathcal{T}^s$ yields that $\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}[Y^{s,\omega\otimes_t\widetilde{\omega}_1}_\zeta]\le\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}[Y^{s,\omega\otimes_t\widetilde{\omega}_2}_\zeta]+\rho_0(\|\widetilde{\omega}_1-\widetilde{\omega}_2\|_{t,T})$. Exchanging the roles of $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ shows that the mapping $\widetilde{\omega}\to\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta]$ is continuous and thus $\mathcal{F}^t_T$-measurable. Then the expectation on the right-hand side of (3.3) is well-defined.

Next, let us show that both sides of (3.3) are finite: For any $\tau \in \mathcal{T}_s^t$, (3.2) shows that $|\mathbb{E}_{\widehat{\mathbb{P}}}[\mathbf{1}_{A\cap\mathcal{A}_j} Y_\tau^{t,\omega}]| \le \mathbb{E}_{\widehat{\mathbb{P}}}[|Y_\tau^{t,\omega}|] \le \mathbb{E}_{\widehat{\mathbb{P}}}[Y_*^{t,\omega}] < \infty$, which leads to the fact that $-\infty < -\mathbb{E}_{\widehat{\mathbb{P}}}[Y_*^{t,\omega}] \le \sup_{\tau\in\mathcal{T}_s^t}\mathbb{E}_{\widehat{\mathbb{P}}}[\mathbf{1}_{A\cap\mathcal{A}_j} Y_\tau^{t,\omega}] \le \mathbb{E}_{\widehat{\mathbb{P}}}[Y_*^{t,\omega}] < \infty$.

On the other hand, given $\widetilde{\omega} \in A \cap \mathcal{A}_j$ and $\zeta \in \mathcal{T}^s$, applying (7.2) with $(\widetilde{\omega}_1, \widetilde{\omega}_2) = (\widetilde{\omega}, \widetilde{\omega}_j)$ and $(\widetilde{\omega}_1, \widetilde{\omega}_2) = (\widetilde{\omega}_j, \widetilde{\omega})$, respectively, yields that

$$\begin{aligned}\Big|\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]\Big| &\le \Big|\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}_j}\big]\Big| + \Big|\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}} - Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}_j}\big]\Big| \\ &\le \mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] + \rho_0\big(\|\widetilde{\omega}-\widetilde{\omega}_j\|_{t,s}\big) \le \mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] + \rho_0(\delta).\end{aligned}$$

It then follows from (3.2) that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\left(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big] + \widehat{\rho}_0(\delta)\right)\right] \\ &\quad\le \Big(\mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] + \rho_0(\delta) + \widehat{\rho}_0(\delta)\Big)\mathbb{P}(A\cap\mathcal{A}_j) < \infty, \text{ and} \\ &\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\left(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big] + \widehat{\rho}_0(\delta)\right)\right] \\ &\quad\ge \Big(-\mathbb{E}_{\mathbb{P}_j}\big[Y_*^{s,\omega\otimes_t\widetilde{\omega}_j}\big] - \rho_0(\delta) + \widehat{\rho}_0(\delta)\Big)\mathbb{P}(A\cap\mathcal{A}_j) > -\infty.\end{aligned}$$

(2) Given $A \in \mathcal{F}_T^t$, for any $j = 1,\dots,\lambda$ and $\widetilde{\omega} \in \mathcal{A}_j$, since $\mathcal{A}_j \in \mathcal{F}_s^t$, Lemma 2.1 shows that $(\mathcal{A}_j)^{s,\widetilde{\omega}} = \Omega^s$ (or $(\mathbf{1}_{\mathcal{A}_j})^{s,\widetilde{\omega}} \equiv 1$), which implies that $(A\cap\mathcal{A}_0)^{s,\widetilde{\omega}} = \emptyset$. So it is easy to calculate that $\widehat{\mathbb{P}}(A\cap\mathcal{A}_0) = \mathbb{P}(A\cap\mathcal{A}_0)$.

Next, let $j = 1,\dots,\lambda$ and $A \in \mathcal{F}_s^t$. We see from Lemma 2.1 again that

$$\text{if } \widetilde{\omega} \in A\cap\mathcal{A}_j \text{ (resp., } \notin A\cap\mathcal{A}_j\text{), and then } (A\cap\mathcal{A}_j)^{s,\widetilde{\omega}} = \Omega^s \text{ (resp., } = \emptyset). \tag{7.3}$$

Then $\widehat{\mathbb{P}}(A\cap\mathcal{A}_j) = \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{P}_{j'}((A\cap\mathcal{A}_j)^{s,\widetilde{\omega}})] = \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{P}_{j'}(\Omega^s)] = \mathbb{P}(A\cap\mathcal{A}_j)$.

Given $\tau \in \mathcal{T}_s^t$, since $\tau^{s,\widetilde{\omega}} \in \mathcal{T}^s$ by Corollary 2.1, we can deduce from (7.3) again that

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\Big] &= \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{E}_{\mathbb{P}_{j'}}\Big[\big(\mathbf{1}_{A\cap\mathcal{A}_j}Y_\tau^{t,\omega}\big)^{s,\widetilde{\omega}}\Big]\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbb{E}_{\mathbb{P}_j}\Big[(Y_\tau^{t,\omega})^{s,\widetilde{\omega}}\Big]\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbb{E}_{\mathbb{P}_j}\Big[Y_{\tau^{s,\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}\Big]\right] \\ &\le \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\Big]\right],\end{aligned}$$

where we used the fact that $(Y_\tau^{t,\omega})^{s,\widetilde{\omega}}(\widehat{\omega}) = Y_\tau^{t,\omega}(\widetilde{\omega}\otimes_s\widehat{\omega}) = Y(\tau(\widetilde{\omega}\otimes_s\widehat{\omega}), \omega\otimes_t(\widetilde{\omega}\otimes_s\widehat{\omega})) = Y(\tau^{s,\widetilde{\omega}}(\widehat{\omega}), (\omega\otimes_t\widetilde{\omega})\otimes_s\widehat{\omega}) = Y^{s,\omega\otimes_t\widetilde{\omega}}(\tau^{s,\widetilde{\omega}}(\widehat{\omega}), \widehat{\omega}) = Y_{\tau^{s,\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}(\widehat{\omega})\ \ \forall\,\widehat{\omega}\in\Omega^s$. ∎

7.2. Proofs of the results in section 4.

Proof of Remark 4.1. Let $t \in [0,T]$ and $\omega_1, \omega_2 \in \Omega$. For any $\mathbb{P} \in \mathcal{P}_t$, $\tau \in \mathcal{T}^t$, and $\widetilde{\omega} \in \Omega^t$, (7.1) shows that $|Y_s^{t,\omega_1}(\widetilde{\omega}) - Y_s^{t,\omega_2}(\widetilde{\omega})| \le \rho_0(\|\omega_1-\omega_2\|_{0,t})$, $\forall\, s \in [t,T]$. In particular, $|Y^{t,\omega_1}(\tau(\widetilde{\omega}),\widetilde{\omega}) - Y^{t,\omega_2}(\tau(\widetilde{\omega}),\widetilde{\omega})| \le \rho_0(\|\omega_1-\omega_2\|_{0,t})$. It then follows that

$$\mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega_1}\big] \le \mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega_2}\big] + \rho_0\big(\|\omega_1-\omega_2\|_{0,t}\big). \tag{7.4}$$

Taking supremum over $\tau \in \mathcal{T}^t$ and then taking infimum over $\mathbb{P} \in \mathcal{P}_t$ yields that $\overline{Z}_t(\omega_1) \le \overline{Z}_t(\omega_2) + \rho_0(\|\omega_1 - \omega_2\|_{0,t})$. Exchanging the role of ω_1 and ω_2, we obtain (4.3) with $\rho_1 = \rho_0$. $\square$

Proof of Lemma 4.1. Let $0 \le t \le s \le T$, $\omega \in \Omega$, and $\mathbb{P} \in \mathcal{P}(t,\omega)$. If $t = s$, as $\overline{Z}_t$ is $\mathcal{F}_t$-measurable by Remark 4.2, (2.3) shows that $\mathbb{E}_{\mathbb{P}}[|\overline{Z}^{t,\omega}_t|] = \mathbb{E}_{\mathbb{P}}[|\overline{Z}_t(\omega)|] = |\overline{Z}_t(\omega)| < \infty$. So let us assume $t < s$. For any $\widetilde{\omega} \in \Omega^t$, one can deduce that

$$
\begin{aligned}
Y^{s,\omega\otimes_t\widetilde{\omega}}_*(\widehat{\omega}) &= \sup_{r\in[s,T]} \big|Y\big(r, \omega \otimes_t (\widetilde{\omega} \otimes_s \widehat{\omega})\big)\big| \le \sup_{r\in[t,T]} \big|Y^{t,\omega}_r(\widetilde{\omega} \otimes_s \widehat{\omega})\big| \\
&= Y^{t,\omega}_*(\widetilde{\omega} \otimes_s \widehat{\omega}) = \big(Y^{t,\omega}_*\big)^{s,\widetilde{\omega}}(\widehat{\omega}) \ \ \forall\, \widehat{\omega} \in \Omega^s.
\end{aligned} \tag{7.5}
$$

By (P1), there exist an extension $(\Omega^t, \mathcal{F}', \mathbb{P}')$ of $(\Omega^t, \mathcal{F}^t_T, \mathbb{P})$ and $\Omega' \in \mathcal{F}'$ with $\mathbb{P}'(\Omega') = 1$ such that for any $\widetilde{\omega} \in \Omega'$, $\mathbb{P}^{s,\widetilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \widetilde{\omega})$. Since $Y^{t,\omega} \in \widehat{\mathbb{D}}(\mathbf{F}^t, \mathbb{P}) \subset \mathbb{D}^1(\mathbf{F}^t, \mathbb{P})$ by (3.2), we see from (2.6) that for all $\widetilde{\omega} \in \Omega^t$ except on some $\mathcal{N} \in \mathscr{N}^{\mathbb{P}}$, $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[(Y^{t,\omega}_*)^{s,\widetilde{\omega}}] = \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^t_s](\widetilde{\omega})$. Let A be the $\mathcal{F}^t_T$-measurable set containing $\mathcal{N}$ with $\mathbb{P}(A) = 0$. For any $\widetilde{\omega} \in \Omega' \cap A^c \in \mathcal{F}'$, (4.2) and (7.5) imply that

$$
\begin{aligned}
Y_s(\omega \otimes_t \widetilde{\omega}) \le \overline{Z}_s(\omega \otimes_t \widetilde{\omega}) &\le \sup_{\tau\in\mathcal{T}^s} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big] \le \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_*\big] \\
&\le \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\big(Y^{t,\omega}_*\big)^{s,\widetilde{\omega}}\Big] = \mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_* \big| \mathcal{F}^t_s\big](\widetilde{\omega}),
\end{aligned}
$$

so $\Omega' \cap A^c \subset \widetilde{\mathcal{A}} \stackrel{\triangle}{=} \{Y^{t,\omega}_s \le \overline{Z}^{t,\omega}_s \le \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^t_s]\}$. Remark 4.2 and Proposition 2.1(2) show that $\widetilde{\mathcal{A}} \in \mathcal{F}^t_s$, and it then follows that $\mathbb{P}(\widetilde{\mathcal{A}}) = \mathbb{P}'(\widetilde{\mathcal{A}}) \ge \mathbb{P}'(\Omega' \cap A^c) = 1$. To wit,

$$
Y^{t,\omega}_s \le \overline{Z}^{t,\omega}_s \le \mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_* \big| \mathcal{F}^t_s\big], \quad \mathbb{P}\text{-a.s.}, \tag{7.6}
$$

which leads to that $\mathbb{E}_{\mathbb{P}}[|\overline{Z}^{t,\omega}_s|] \le \mathbb{E}_{\mathbb{P}}[|Y^{t,\omega}_s| + \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^t_s]] \le 2\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_*] < \infty$. $\square$

Proof of Proposition 4.1. Fix $0 \le t \le s \le T$ and $\omega \in \Omega$. If $t = s$, Remark 4.2 and (2.3) imply that $\overline{Z}^{t,\omega}_t = \overline{Z}_t(\omega)$. Then (4.5) clearly holds. So we just assume $t < s$ and define

$$
\mathcal{Y}_r \stackrel{\triangle}{=} Y^{t,\omega}_r \quad \text{and} \quad \mathcal{Z}_r \stackrel{\triangle}{=} \overline{Z}^{t,\omega}_r \quad \forall\, r \in [t,T]. \tag{7.7}
$$

(1) *To show*

$$
\overline{Z}_t(\omega) \le \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \sup_{\tau\in\mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big], \tag{7.8}
$$

we shall paste the local approximating minimizers $\mathbb{P}_{\widetilde{\omega}}$ *of* $\overline{Z}^{t,\omega}_s(\widetilde{\omega})$ *according to* (P2) *and then make some estimations.*

Fix $\varepsilon > 0$ and let $\delta > 0$ such that $\rho_0(\delta) \vee \widehat{\rho}_0(\delta) \vee \rho_1(\delta) < \varepsilon/4$. Given $\widetilde{\omega} \in \Omega^t$, we can find a $\mathbb{P}_{\widetilde{\omega}} \in \mathcal{P}(s, \omega \otimes_t \widetilde{\omega})$ such that

$$
\overline{Z}_s(\omega \otimes_t \widetilde{\omega}) \ge \sup_{\tau\in\mathcal{T}^s} \mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big] - \varepsilon/4. \tag{7.9}
$$

Clearly, $O^s_\delta(\widetilde{\omega})$ is an open set of Ω^t. For any $\widetilde{\omega}' \in O^s_\delta(\widetilde{\omega})$, an analogy to (7.4) shows that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}'}_\tau\big] &\le \mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big] + \rho_0\big(\|\omega \otimes_t \widetilde{\omega}' - \omega \otimes_t \widetilde{\omega}\|_{0,s}\big) \\
&= \mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y^{s,\omega\otimes_t\widetilde{\omega}}_\tau\big] + \rho_0\big(\|\widetilde{\omega}' - \widetilde{\omega}\|_{t,s}\big) \quad \forall\, \tau \in \mathcal{T}^s.
\end{aligned}
$$

Taking supremum over $\tau \in \mathcal{T}^s$, we can deduce from (4.3) and (7.9) that

$$\begin{aligned}\sup_{\tau\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\big[Y_\tau^{s,\omega\otimes_t\widetilde{\omega}'}\big] &\le \overline{Z}_s\big(\omega\otimes_t\widetilde{\omega}\big)+\frac{1}{2}\varepsilon\\ &\le \overline{Z}_s(\omega\otimes_t\widetilde{\omega}')+\rho_1\big(\|\widetilde{\omega}'-\widetilde{\omega}\|_{t,s}\big)+\frac{1}{2}\varepsilon\\ &\le \mathcal{Z}_s(\widetilde{\omega}')+\frac{3}{4}\varepsilon \quad \forall\,\widetilde{\omega}'\in O^s_\delta(\widetilde{\omega}).\end{aligned} \tag{7.10}$$

Next, fix $\mathbb{P}\in\mathcal{P}(t,\omega)$ and $\lambda\in\mathbb{N}$. For $j=1,\dots,\lambda$, we set $\mathcal{A}_j \overset{\triangle}{=} (O^s_\delta(\widehat{\omega}^t_j)\backslash(\cup_{j'<j} O^s_\delta(\widehat{\omega}^t_{j'})))\in\mathcal{F}^t_s$ by (2.1) and set $\mathbb{P}_j \overset{\triangle}{=} \mathbb{P}_{\widehat{\omega}^t_j}$ (where $\widehat{\omega}^t_j$ is defined right after (2.1)). Let $\mathbb{P}_\lambda$ be the probability of $\mathcal{P}(t,\omega)$ in (P2) that corresponds to the partition $\{\mathcal{A}_j\}_{j=0}^\lambda$ and the probabilities $\{\mathbb{P}_j\}_{j=1}^\lambda$, where $\mathcal{A}_0 \overset{\triangle}{=} (\cup_{j=1}^\lambda\mathcal{A}_j)^c\in\mathcal{F}^t_s$. So

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_\lambda}[\xi]=\mathbb{E}_{\mathbb{P}}[\xi],\ \ \forall\,\xi\in L^1(\mathcal{F}^t_s,\mathbb{P}_\lambda)\cap L^1\big(\mathcal{F}^t_s,\mathbb{P}\big)\\ &\quad\text{and}\quad \mathbb{E}_{\mathbb{P}_\lambda}[\mathbf{1}_{\mathcal{A}_0}\xi]=\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\mathcal{A}_0}\xi]\quad \forall\,\xi\in L^1(\mathcal{F}^t_T,\mathbb{P}_\lambda)\cap L^1\big(\mathcal{F}^t_T,\mathbb{P}\big).\end{aligned} \tag{7.11}$$

Given $\tau\in\mathcal{T}^t$, one can deduce from (3.2), (3.3), (7.11), and (7.10) that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_\lambda}\big[\mathcal{Y}_\tau\big] &= \mathbb{E}_{\mathbb{P}_\lambda}\big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s}+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\mathcal{Y}_{\tau\vee s}\big]+\sum_{i=1}^{\lambda}\mathbb{E}_{\mathbb{P}_\lambda}\big[\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_j}Y^{t,\omega}_{\tau\vee s}\big]\\ &\le \mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s}+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\mathcal{Y}_{\tau\vee s}\big]\\ &\quad+\sum_{i=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\tau(\widetilde{\omega})\ge s\}\cap\{\widetilde{\omega}\in\mathcal{A}_j\}}\left(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]+\widehat{\rho}_0(\delta)\right)\right]\\ &\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}\cap\mathcal{A}_0^c}\mathcal{Z}_s\Big]+\varepsilon\\ &\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]+\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\mathcal{A}_0}\big(\mathcal{Y}_*+|\mathcal{Z}_s|\big)\Big]+\varepsilon.\end{aligned}$$

Taking supremum over $\tau\in\mathcal{T}^t$ yields that

$$\begin{aligned}\overline{Z}_t(\omega) &\le \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}_\lambda}\big[\mathcal{Y}_\tau\big] \le \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]\\ &\quad+\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\big(\underset{j=1}{\overset{\lambda}{\cup}}\mathcal{A}_j\big)^c}\big(\mathcal{Y}_*+|\mathcal{Z}_s|\big)\Big]+\varepsilon.\end{aligned} \tag{7.12}$$

Since $\cup_{j\in\mathbb{N}}\mathcal{A}_j=\cup_{j\in\mathbb{N}}O^s_\delta(\widehat{\omega}^t_j)=\Omega^t$ and since $\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_*+|\mathcal{Z}_s|]<\infty$ by (3.2) and Lemma 4.1, letting $\lambda\to\infty$ in (7.12), we can deduce from the dominated convergence theorem that $\overline{Z}_t(\omega)\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s]+\varepsilon$. Eventually, taking infimum over $\mathbb{P}\in\mathcal{P}(t,\omega)$ on the right-hand side and then letting $\varepsilon\to 0$, we obtain (7.8).

(2) *As to the reverse of* (7.8), *it suffices to show for a given* $\mathbb{P}\in\mathcal{P}(t,\omega)$ *that*

$$\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big]. \tag{7.13}$$

Let us start with the main idea of proving (7.13)*: Contrary to* (7.9), *we need upper bounds for* $\overline{Z}^{t,\omega}_s$ *this time. First note that* $\overline{Z}^{t,\omega}_s(\widetilde{\omega})\le\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}$

$[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}]$ $\forall\,\widetilde{\omega}\in\Omega^t$. *Given* $\zeta\in\mathcal{T}^s$, (2.6) *implies that*

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\Big]=\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi_s^t)}\big|\mathcal{F}_s^t\big](\widetilde{\omega})\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big](\widetilde{\omega}) \tag{7.14}$$

holds for any $\widetilde{\omega}\in\Omega^t$ *except on a* $\mathbb{P}$*-null set* $\mathcal{N}_\zeta$, *where* $\widehat{\tau}$ *is an optimal stopping time. Since* $\mathcal{T}^s$ *is an uncountable set, we cannot take supremum over* $\zeta\in\mathcal{T}^s$ *for* $\mathbb{P}$*-a.s.* $\widetilde{\omega}\in\Omega^t$ *in* (7.14) *to obtain*

$$\mathcal{Z}_s\le\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^{\mathbb{P}}\big],\quad \mathbb{P}\text{-a.s.} \tag{7.15}$$

To overcome this difficulty, we shall consider a "dense" countable subset Γ *of* $\mathcal{T}^s$ *in sense of* (7.16).

(2a) *Construction of* Γ*:* For any $n\in\mathbb{N}$, we set $\mathscr{D}_n\triangleq\big((s,T)\cap\{i2^{-n}\}_{i\in\mathbb{N}}\big)\cup\{T\}$ and $\mathscr{D}\triangleq\cup_{n\in\mathbb{N}}\mathscr{D}_n$. Given $q\in\mathscr{D}$, we simply denote the countable subset Θ_q^s of $\mathcal{F}_q^s$ by $\{O_j^q\}_{j\in\mathbb{N}}$ and define $\Upsilon_k^q\triangleq\{q\mathbf{1}_{\cup_{j\in I}O_j^q}+T\mathbf{1}_{\cap_{j\in I}(O_j^q)^c}:I\subset\{1,\dots,k\}\}\subset\mathcal{T}^s$ $\forall\,k\in\mathbb{N}$. For any $n,k\in\mathbb{N}$, we set $\Gamma_{n,k}\triangleq\{\wedge_{q\in\mathscr{D}_n}\tau_q:\tau_q\in\Upsilon_k^q\}\subset\mathcal{T}^s$. Then $\Gamma\triangleq\cup_{n,k\in\mathbb{N}}\Gamma_{n,k}$ is clearly a countable subset of $\mathcal{T}^s$.

Since the filtration $\mathbf{F}^{\mathbb{P}}$ is right-continuous, and since the process $\mathcal{Y}$ is right-continuous and left upper semi-continuous by Remark 3.1(2), the classic optimal stopping theory shows that $\operatorname{esssup}_{\tau\in\mathcal{T}_s^{\mathbb{P}}}\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\tau|\mathcal{F}_s^{\mathbb{P}}]$ admits an optimal stopping time $\widehat{\tau}\in\mathcal{T}_s^{\mathbb{P}}$, which is the first time after s the process $\mathcal{Y}$ meets the RCLL modification of its Snell envelope $\{\operatorname{esssup}_{\tau\in\mathcal{T}_r^{\mathbb{P}}}\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\tau|\mathcal{F}_r^{\mathbb{P}}]\}_{r\in[t,T]}$.

Fix $\varepsilon>0$. We claim that there exists a $\widehat{\tau}'\in\mathcal{T}_s^t$ such that

$$\mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\widehat{\tau}'}-\mathcal{Y}_{\widehat{\tau}}\big|\big]<\varepsilon/4. \tag{7.16}$$

To see this, let n be an integer ≥2. Given $i=1,\dots,n$, we set $s_i^n\triangleq s+\frac{i}{n}(T-s)$ and $A_i^n\triangleq\{s_{i-1}^n<\widehat{\tau}\le s_i^n\}\in\mathcal{F}_{s_i^n}^{\mathbb{P}}$ with $s_0^n=-1$. By, e.g., Problem 2.7.3 of [20], there exists an $(A')_i^n\in\mathcal{F}_{s_i^n}^t$ such that $A_i^n\,\Delta\,(A')_i^n\in\mathscr{N}^{\mathbb{P}}$. Define $(\mathcal{A}')_i^n\triangleq(A')_i^n\backslash\cup_{i'<i}(A')_{i'}^n\in\mathcal{F}_{s_i^n}^t$ and $\mathcal{A}'_n\triangleq\cup_{i=1}^n(\mathcal{A}')_i^n=\cup_{i=1}^n(A')_i^n\in\mathcal{F}_T^t$. Then $\tau_n\triangleq\sum_{i=1}^n\mathbf{1}_{A_i^n}\,s_i^n$ is a $\mathcal{T}_s^{\mathbb{P}}$-stopping time while $\tau'_n\triangleq\sum_{i=1}^n\mathbf{1}_{(\mathcal{A}')_i^n}\,s_i^n+\mathbf{1}_{(\mathcal{A}'_n)^c}T$ defines an $\mathcal{T}_s^t$-stopping time. Clearly, τ_n coincides with τ'_n over $\cup_{i=1}^n(A_i^n\cap(\mathcal{A}')_i^n)$, whose complement $\cup_{i=1}^n(A_i^n\backslash(\mathcal{A}')_i^n)$ is in fact of $\mathscr{N}^{\mathbb{P}}$ because for each $i\in\{1,\dots,n\}$

$$\begin{aligned}A_i^n\backslash(\mathcal{A}')_i^n&=A_i^n\cap\Big[\big((A')_i^n\big)^c\cup\Big(\underset{i'<i}{\cup}(A')_{i'}^n\Big)\Big]\\&=\big(A_i^n\backslash(A')_i^n\big)\cup\Big(\underset{i'<i}{\cup}\big((A')_{i'}^n\cap A_i^n\big)\Big)\\&\subset\big(A_i^n\Delta(A')_i^n\big)\cup\Big(\underset{i'<i}{\cup}\big((A')_{i'}^n\cap(A_{i'}^n)^c\big)\Big)\\&\subset\underset{i'\le i}{\cup}\big(A_{i'}^n\Delta(A')_{i'}^n\big)\in\mathscr{N}^{\mathbb{P}}.\end{aligned} \tag{7.17}$$

To wit, $\tau_n=\tau'_n$, $\mathbb{P}$-a.s. Since $\lim_{n\to\infty}\tau_n=\widehat{\tau}$ and since $\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_*]<\infty$ by (3.2), we can deduce from the right-continuity of the shifted process $\mathcal{Y}$

and the dominated convergence theorem that

$$\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\tau'_n}-\mathcal{Y}_{\widehat{\tau}}\big|\big]=\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\tau_n}-\mathcal{Y}_{\widehat{\tau}}\big|\big]=0. \tag{7.18}$$

So there exists a $N\in\mathbb{N}$ such that $\mathbb{E}_{\mathbb{P}}[|\mathcal{Y}_{\tau'_N}-\mathcal{Y}_{\widehat{\tau}}|]<\varepsilon/4$, i.e., (7.16) holds for $\widehat{\tau}'=\tau'_N$.

(2b) *In the next two steps, we will gradually demonstrate* (7.15).
Since $\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_*]<\infty$ and since $\zeta(\Pi^t_s)\in\mathcal{T}^t_s\subset\mathcal{T}^{\mathbb{P}}_s$ for any $\zeta\in\mathcal{T}^s$ by Lemma A.1, applying Lemma A.2(1) with $X=B^t$ show that except on an $\mathcal{N}\in\mathscr{N}^{\mathbb{P}}$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^t_s\big]&=\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^{\mathbb{P}}_s\big]\le\underset{\tau\in\mathcal{T}^{\mathbb{P}}_s}{\operatorname{esssup}}\,\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\tau}\big|\mathcal{F}^{\mathbb{P}}_s\big]\\&=\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^{\mathbb{P}}_s\big]=\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^t_s\big]\quad\forall\,\zeta\in\Gamma.\end{aligned}\tag{7.19}$$

Also in light of (2.6), there exists another $\widetilde{\mathcal{N}}\in\mathscr{N}^{\mathbb{P}}$ such that for any $\widetilde{\omega}\in\widetilde{\mathcal{N}}^c$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi^t_s)}\big|\mathcal{F}^t_s\big](\widetilde{\omega})&=\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\big(\mathcal{Y}_{\zeta(\Pi^t_s)}\big)^{s,\widetilde{\omega}}\Big]\\&=\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta}\Big]\quad\forall\,\zeta\in\Gamma,\end{aligned}\tag{7.20}$$

where we used the fact that for any $\widehat{\omega}\in\Omega^s$

$$\begin{aligned}\big(\mathcal{Y}_{\zeta(\Pi^t_s)}\big)^{s,\widetilde{\omega}}(\widehat{\omega})&=\mathcal{Y}_{\zeta(\Pi^t_s)}(\widetilde{\omega}\otimes_s\widehat{\omega})=Y\Big(\zeta\big(\Pi^t_s(\widetilde{\omega}\otimes_s\widehat{\omega})\big),\omega\otimes_t(\widetilde{\omega}\otimes_s\widehat{\omega})\Big)\\&=Y\big(\zeta(\widehat{\omega}),(\omega\otimes_t\widetilde{\omega})\otimes_s\widehat{\omega}\big)=Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta}(\widehat{\omega}).\end{aligned}$$

By (P1), there exist an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of $(\Omega^t,\mathcal{F}^t_T,\mathbb{P})$ and $\Omega'\in\mathcal{F}'$ with $\mathbb{P}'(\Omega')=1$ such that for any $\widetilde{\omega}\in\Omega'$, $\mathbb{P}^{s,\widetilde{\omega}}\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$. Let $\widehat{A}$ be the $\mathcal{F}^t_T$-measurable set containing $\mathcal{N}\cup\widetilde{\mathcal{N}}$ and with $\mathbb{P}(\widehat{A})=0$.

Now, fix $\widetilde{\omega}\in\Omega'\cap\widehat{A}^c$ and set $\mathcal{T}^{s,\widetilde{\omega}}_r\overset{\triangle}{=}\mathcal{T}^{\mathbb{P}^{s,\widetilde{\omega}}}_r$, $r\in[s,T]$. Analogous to $\widehat{\tau}$, the first time $\zeta_{\widetilde{\omega}}\in\mathcal{T}^{s,\widetilde{\omega}}=\mathcal{T}^{\mathbb{P}^{s,\widetilde{\omega}}}$ when the process $Y^{s,\omega\otimes_t\widetilde{\omega}}$ meets the RCLL modification of its Snell envelope $\{\operatorname{esssup}_{\zeta\in\mathcal{T}^{s,\widetilde{\omega}}_r}\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta}|\mathcal{F}^{\mathbb{P}^{s,\widetilde{\omega}}}_r]\}_{r\in[s,T]}$ is an optimal stopping time for $\sup_{\zeta\in\mathcal{T}^{s,\widetilde{\omega}}}\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta}]$. Similar to (7.16), there exists a $\zeta'_{\widetilde{\omega}}\in\mathcal{T}^s$ such that

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\Big|Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta'_{\widetilde{\omega}}}-Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta_{\widetilde{\omega}}}\Big|\Big]<\varepsilon/4. \tag{7.21}$$

(2c) *Next, we will approximate $\zeta'_{\widetilde{\omega}}$ by a sequence $\{\zeta^n\}_{n\in\mathbb{N}}$ in Γ:* As $\mathbb{P}^{s,\widetilde{\omega}}\in\mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$, (3.2) shows that $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[Y^{s,\omega\otimes_t\widetilde{\omega}}_*]<\infty$. So there exists a $\delta>0$ such that

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[\mathbf{1}_A Y^{s,\omega\otimes_t\widetilde{\omega}}_*\big]<\varepsilon/4\quad\text{for any } A\in\mathcal{F}^s_T \text{ with } \mathbb{P}^{s,\widetilde{\omega}}(A)<\delta. \tag{7.22}$$

Given $n\in\mathbb{N}$ and $i\in\{\lfloor 2^n s\rfloor,\dots,\lfloor 2^n T\rfloor\}$, let $q^n_i\overset{\triangle}{=}\frac{i+1}{2^n}\wedge T\in\mathscr{D}_n$ and $\widetilde{A}^n_i\overset{\triangle}{=}\{\frac{i}{2^n}\le\zeta'_{\widetilde{\omega}}<\frac{i+1}{2^n}\}\in\mathcal{F}^s_{q^n_i}$. We can find a subsequence $\{O^{n,i}_\ell\}_{\ell\in\mathbb{N}}$ of $\Theta^s_{q^n_i}=\{O^{q^n_i}_j\}_{j\in\mathbb{N}}$ such that

$$\widetilde{A}^n_i\subset\underset{\ell\in\mathbb{N}}{\cup}O^{n,i}_\ell\quad\text{and}\quad\mathbb{P}^{s,\widetilde{\omega}}(\widetilde{A}^n_i)>\mathbb{P}^{s,\widetilde{\omega}}\Big(\underset{\ell\in\mathbb{N}}{\cup}O^{n,i}_\ell\Big)-\frac{\delta}{\lfloor 2^n T\rfloor^2}. \tag{7.23}$$

(See Lemma A.7 of [6] for details.) Moreover, there exists an $\ell_i^n \in \mathbb{N}$ such that

$$\mathbb{P}^{s,\widetilde{\omega}}\big(\mathcal{O}_i^n\big) > \mathbb{P}^{s,\widetilde{\omega}}\Big(\underset{\ell\in\mathbb{N}}{\cup} O_\ell^{n,i}\Big) - \frac{\delta}{\lfloor 2^n T\rfloor^2} \tag{7.24}$$

with $\mathcal{O}_i^n \triangleq \cup_{\ell=1}^{\ell_i^n} O_\ell^{n,i} \in \mathcal{F}^s_{q_i^n}$. Clearly, $\zeta_i^n \triangleq q_i^n \mathbf{1}_{\mathcal{O}_i^n} + T\mathbf{1}_{(\mathcal{O}_i^n)^c} \in \Upsilon^{q_i^n}_{k_i^n}$ for some $k_i^n \in \mathbb{N}$. Set $\widehat{\mathcal{O}}_i^n \triangleq \mathcal{O}_i^n \backslash \cup_{i'=\lfloor 2^n s\rfloor}^{i-1} \mathcal{O}_{i'}^n \in \mathcal{F}^s_{q_i^n}$. An analogy to (7.17) shows that $\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n = \widetilde{A}_i^n \cap [(\mathcal{O}_i^n)^c \cup (\cup_{i'=\lfloor 2^n s\rfloor}^{i-1} \mathcal{O}_{i'}^n)] \subset ((\cup_{\ell\in\mathbb{N}} O_\ell^{n,i})\backslash \mathcal{O}_i^n) \cup (\cup_{i'=\lfloor 2^n s\rfloor}^{i-1} (\mathcal{O}_{i'}^n \cap (\widetilde{A}_{i'}^n)^c))$. It then follows from (7.23) and (7.24) that

$$\begin{aligned}\mathbb{P}^{s,\widetilde{\omega}}(\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n) &\le \mathbb{P}^{s,\widetilde{\omega}}\left(\Big(\underset{\ell\in\mathbb{N}}{\cup} O_\ell^{n,i}\Big)\backslash \mathcal{O}_i^n\right)\\ &\quad + \sum_{i'=\lfloor 2^n s\rfloor}^{i-1} \mathbb{P}^{s,\widetilde{\omega}}\left(\Big(\underset{\ell\in\mathbb{N}}{\cup} O_\ell^{n,i'}\Big)\backslash \widetilde{A}_{i'}^n\right) < \frac{i\delta}{\lfloor 2^n T\rfloor^2} \le \frac{\delta}{\lfloor 2^n T\rfloor}.\end{aligned} \tag{7.25}$$

Set $\widehat{\mathcal{O}}_n \triangleq \cup_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} \widehat{\mathcal{O}}_i^n = \cup_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} \mathcal{O}_i^n$ and $k_n \triangleq \max\{k_i^n : i = \lfloor 2^n s\rfloor, \dots, \lfloor 2^n T\rfloor\}$, we see that $\zeta^n \triangleq \wedge_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} \zeta_i^n = \sum_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} q_i^n \mathbf{1}_{\widehat{\mathcal{O}}_i^n} + \mathbf{1}_{\widehat{\mathcal{O}}_n^c} T$ is a stopping time of Γ_{n,k_n}, which equals to $\widehat{\zeta}^n \triangleq \sum_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} q_i^n \mathbf{1}_{\widetilde{A}_i^n} \in \mathcal{T}^s$ over $\mathcal{A}_n \triangleq \cup_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} (\widetilde{A}_i^n \cap \widehat{\mathcal{O}}_i^n) \in \mathcal{F}_T^s$. As $\cup_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} \widetilde{A}_i^n = \Omega^s$, (7.25) implies that $\mathbb{P}^{s,\widetilde{\omega}}(\mathcal{A}_n^c) = \mathbb{P}^{s,\widetilde{\omega}}(\cup_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} (\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n)) = \sum_{i=\lfloor 2^n s\rfloor}^{\lfloor 2^n T\rfloor} \mathbb{P}^{s,\widetilde{\omega}}(\widetilde{A}_i^n \backslash \widehat{\mathcal{O}}_i^n) < \delta$. It then follows from (7.22) that $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[|Y^{s,\omega\otimes_t\widetilde{\omega}}_{\widehat{\zeta}^n} - Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta^n}|] = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[\mathbf{1}_{\mathcal{A}_n^c}|Y^{s,\omega\otimes_t\widetilde{\omega}}_{\widehat{\zeta}^n} - Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta^n}|] \le 2\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[\mathbf{1}_{\mathcal{A}_n^c} Y_*^{s,\omega\otimes_t\widetilde{\omega}}] < \varepsilon/2$, which together with (7.19) and (7.20) shows that $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\widehat{\zeta}^n}] < \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta^n}] + \varepsilon/2 \le \mathbb{E}_{\mathbb{P}}[\mathcal{Y}_{\widehat{\tau}}|\mathcal{F}_s^t](\widetilde{\omega}) + \varepsilon/2$. Since $\lim_{n\to\infty}\!\downarrow \widehat{\zeta}^n = \zeta'_{\widetilde{\omega}}$ and since $\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}[Y_*^{s,\omega\otimes_t\widetilde{\omega}}] < \infty$, letting $n\to\infty$, we can deduce from (7.21), the right-continuity of the shifted process $Y^{s,\omega\otimes_t\widetilde{\omega}}$ and the dominated convergence theorem that

$$\begin{aligned}\mathcal{Z}_s(\widetilde{\omega}) = \overline{Z}_s(\omega\otimes_t\widetilde{\omega}) &\le \sup_{\zeta\in\mathcal{T}^s} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big]\\ &\le \sup_{\zeta\in\mathcal{T}^{s,\widetilde{\omega}}} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_\zeta^{s,\omega\otimes_t\widetilde{\omega}}\big] = \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\zeta_{\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}\big]\\ &\le \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\zeta'_{\widetilde{\omega}}}^{s,\omega\otimes_t\widetilde{\omega}}\big] + \varepsilon/4 = \lim_{n\to\infty}\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\big[Y_{\widehat{\zeta}^n}^{s,\omega\otimes_t\widetilde{\omega}}\big]\\ &\quad + \varepsilon/4 \le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big](\widetilde{\omega}) + \frac{3}{4}\varepsilon \quad \forall\, \widetilde{\omega}\in\Omega'\cap\widehat{A}^c.\end{aligned}$$

Since $\mathcal{Z}_s \in \mathcal{F}_s^t$ by Remark 4.2 and Proposition 2.1(2), an analogy to (7.6) yields that

$$\mathcal{Z}_s \le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big] + \frac{3}{4}\varepsilon, \quad \mathbb{P}\text{-a.s.} \tag{7.26}$$

If sending ε to 0 and applying Lemma A.2(1) with $X = B^t$ now, we will immediately obtain (7.15).

(2d) Given $\tau \in \mathcal{T}^t$, let $\overline{\tau} \stackrel{\triangle}{=} \mathbf{1}_{\{\tau<s\}}\tau + \mathbf{1}_{\{\tau\ge s\}}\widehat{\tau}' \in \mathcal{T}^t$. We can deduce from (7.26) and (7.16) that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big]\\
&\quad\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s} + \mathbf{1}_{\{\tau\ge s\}}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^t_s\big]\Big] + \frac{3}{4}\varepsilon\\
&\quad= \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s} + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}^t_s\big]\Big] + \frac{3}{4}\varepsilon\\
&\quad= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Y}_{\widehat{\tau}}\Big] + \frac{3}{4}\varepsilon \le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Y}_{\widehat{\tau}'}\Big] + \varepsilon\\
&\quad= \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\overline{\tau}}\big] + \varepsilon \le \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big] + \varepsilon.
\end{aligned}$$

Taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand side and then letting $\varepsilon \to 0$ yields (7.13). So we proved the proposition. ∎

Proof of Proposition 4.2.

(1) Fix $\omega \in \Omega$. Letting $0 \le t < s \le T$ such that $\sup_{r\in[t,s]}|\omega(r) - \omega(t)| \le T$. We shall show that

$$\big|\overline{Z}_s(\omega) - \overline{Z}_t(\omega)\big| \le 2\rho_\alpha(\delta_{t,s}), \tag{7.27}$$

where $\alpha \stackrel{\triangle}{=} 1 + \|\omega\|_{0,T}$ and $\delta_{t,s} \stackrel{\triangle}{=} (s-t) \vee \sup_{r\in[t,s]}|\omega(r) - \omega(t)| \le T$. Given $\varepsilon > 0$, there exists a $\mathbb{P} = \mathbb{P}(t,\omega,\varepsilon) \in \mathcal{P}(t,\omega)$ such that

$$\begin{aligned}
\overline{Z}_t(\omega) &\ge \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_\tau\big] - \varepsilon\\
&\ge \sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}Y^{t,\omega}_\tau + \mathbf{1}_{\{\tau\ge s\}}\overline{Z}^{t,\omega}_s\Big] - \varepsilon\\
&\ge \mathbb{E}_{\mathbb{P}}\Big[\overline{Z}^{t,\omega}_s\Big] - \varepsilon,
\end{aligned} \tag{7.28}$$

where we used (7.13) in the second inequality and took $\tau = s$ in the last inequality. In light of (4.3)

$$\begin{aligned}
\big|\overline{Z}_s(\omega) - \overline{Z}^{t,\omega}_s(\widetilde{\omega})\big| &= \big|\overline{Z}_s(\omega) - \overline{Z}(s, \omega\otimes_t\widetilde{\omega})\big| \le \rho_1\big(\|\omega - \omega\otimes_t\widetilde{\omega}\|_{0,s}\big)\\
&= \rho_1\left(\sup_{r\in[t,s]}\big|\widetilde{\omega}(r) + \omega(t) - \omega(r)\big|\right)\\
&\le \rho_1\left(\sup_{r\in[t,s]}\big|\widetilde{\omega}(r)\big| + \sup_{r\in[t,s]}\big|\omega(r) - \omega(t)\big|\right)\\
&\le \rho_1\left(\sup_{r\in[t,(t+\delta_{t,s})\wedge T]}\big|B^t_r(\widetilde{\omega})\big| + \delta_{t,s}\right) \quad \forall\, \widetilde{\omega} \in \Omega^t.
\end{aligned} \tag{7.29}$$

Since $\|\omega\|_{0,t} \le \|\omega\|_{0,T} < \alpha$, (7.28) and (4.4) imply that

$$\begin{aligned}
\overline{Z}_s(\omega) - \overline{Z}_t(\omega) &\le \mathbb{E}_{\mathbb{P}}\Big[\overline{Z}_s(\omega) - \overline{Z}^{t,\omega}_s\Big] + \varepsilon\\
&\le \mathbb{E}_{\mathbb{P}}\left[\rho_1\left(\delta_{t,s} + \sup_{r\in[t,(t+\delta_{t,s})\wedge T]}\big|B^t_r\big|\right)\right] + \varepsilon \le \rho_\alpha(\delta_{t,s}) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields that

$$\overline{Z}_s(\omega) - \overline{Z}_t(\omega) \le \rho_\alpha(\delta_{t,s}). \tag{7.30}$$

On the other hand, let $\widehat{\mathbb{P}}$ be an arbitrary probability in $\mathcal{P}(t,\omega)$. Applying Proposition 4.1 yields that

$$\overline{Z}_t(\omega) - \overline{Z}_s(\omega) \le \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{\{\tau < s\}} Y^{t,\omega}_\tau + \mathbf{1}_{\{\tau \ge s\}} \overline{Z}^{t,\omega}_s\Big] - \overline{Z}_s(\omega). \tag{7.31}$$

For any $\tau \in \mathcal{T}^t$ and $\widetilde{\omega} \in \{\tau < s\}$, (3.1) shows that

$$\begin{aligned} Y^{t,\omega}_\tau(\widetilde{\omega}) - Y^{t,\omega}_s(\widetilde{\omega}) &= Y\big(\tau(\widetilde{\omega}), \omega \otimes_t \widetilde{\omega}\big) - Y\big(s, \omega \otimes_t \widetilde{\omega}\big) \\ &\le \rho_0\Big(\mathbf{d}_\infty\big((\tau(\widetilde{\omega}), \omega \otimes_t \widetilde{\omega}), (s, \omega \otimes_t \widetilde{\omega})\big)\Big) \\ &\le \rho_0\left((s-t) + \sup_{r \in [t,T]} \big|\widetilde{\omega}\big(r \wedge \tau(\widetilde{\omega})\big) - \widetilde{\omega}\big(r \wedge s\big)\big|\right) \\ &\le \rho_1\left((s-t) + 2\sup_{r \in [t,s]} \big|B^t_r(\widetilde{\omega})\big|\right). \end{aligned}$$

Plugging this into (7.31), we can deduce from (4.4), (4.2), and (7.29) that

$$\begin{aligned} \overline{Z}_t(\omega) - \overline{Z}_s(\omega) &\le \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\widehat{\mathbb{P}}}\left[\mathbf{1}_{\{\tau < s\}} \rho_1\left((s-t) + 2\sup_{r \in [t,s]} \big|B^t_r\big|\right)\right. \\ &\qquad \left. + \mathbf{1}_{\{\tau < s\}} Y^{t,\omega}_s + \mathbf{1}_{\{\tau \ge s\}} \overline{Z}^{t,\omega}_s - \overline{Z}_s(\omega)\right] \\ &\le \rho_\alpha(s-t) + \mathbb{E}_{\widehat{\mathbb{P}}}\Big[\overline{Z}^{t,\omega}_s - \overline{Z}_s(\omega)\Big] \le 2\rho_\alpha(\delta_{t,s}), \end{aligned}$$

which together with (7.30) proves (7.27). As $\lim_{t \nearrow s \downarrow} \delta_{t,s} = \lim_{s \searrow t \downarrow} \delta_{t,s} = 0$, the continuity of $\overline{Z}$ easily follows.

(2) Let $(t,\omega) \in [0,T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t,\omega)$. As $\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_*] < \infty$ by (3.2), using (7.6) and applying Lemma A.2(1) with $X = B^t$ show that for any $s \in [t,T]$, $\overline{Z}^{t,\omega}_s \le \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^t_s] = \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^{\mathbb{P}}_s]$, $\mathbb{P}$-a.s. Then by the continuity of process $\overline{Z}$ and the right continuity of process $\{\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^{\mathbb{P}}_s]\}_{s \in [t,T]}$, it holds $\mathbb{P}$-a.s. that $\overline{Z}^{t,\omega}_s \le \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^{\mathbb{P}}_s]$ for any $s \in [t,T]$. It follows that $\overline{Z}^{t,\omega}_* \le \sup_{s \in [t,T]} \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^{\mathbb{P}}_s]$, $\mathbb{P}$-a.s. Applying Doob's martingale inequality and Jensen's inequality and using the convexity of ϕ yield that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\Big[\overline{Z}^{t,\omega}_*\Big] &\le \frac{e}{e-1}\Big(1 + \sup_{s \in [t,T]} \mathbb{E}_{\mathbb{P}}\big[\phi\big(\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_* | \mathcal{F}^{\mathbb{P}}_s]\big)\big]\Big) \\ &\le \frac{e}{e-1}\Big(1 + \sup_{s \in [t,T]} \mathbb{E}_{\mathbb{P}}\big[\mathbb{E}_{\mathbb{P}}\big[\phi(Y^{t,\omega}_*)\big|\mathcal{F}^{\mathbb{P}}_s\big]\big]\Big) \\ &= \frac{e}{e-1}\Big(1 + \mathbb{E}_{\mathbb{P}}\big[\phi(Y^{t,\omega}_*)\big]\Big) < \infty. \qquad \square \end{aligned}$$

Proof of Proposition 4.3. When $t = T$, (4.6) trivially holds as an equality. So let us fix $(t,\omega) \in [0,T) \times \Omega$ and still define $\mathcal{Y}$, $\mathcal{Z}$ as in (7.7). For (4.6), it suffices to show for a given $\mathbb{P} \in \mathcal{P}(t,\omega)$ that

$$\sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau < \nu\}} \mathcal{Y}_\tau + \mathbf{1}_{\{\tau \ge \nu\}} \mathcal{Z}_\nu\Big] \le \sup_{\tau \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big]. \tag{7.32}$$

Fix $\varepsilon > 0$, $\nu, \tau \in \mathcal{T}^t$, and $n \in \mathbb{N}$. We define $\tau_n \stackrel{\triangle}{=} \mathbf{1}_{\{\tau \le t_1^n\}} t_1^n + \sum_{i=2}^n \mathbf{1}_{\{t_{i-1}^n < \tau \le t_i^n\}} t_i^n \in \mathcal{T}^t$. Let k be an integer ≥ 2. For $i = 1, \dots, k$, applying (7.15) with $s = t_i^k \stackrel{\triangle}{=} t + \frac{i}{k}(T-t)$ yields that

$$\mathcal{Z}_{t_i^k} \le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\tau_i^k} \big| \mathcal{F}^{\mathbb{P}}_{t_i^k}\big], \quad \mathbb{P}-a.s., \tag{7.33}$$

where $\tau_i^k \in \mathcal{T}^{\mathbb{P}}_{t_i^k}$ is the optimal stopping time for $\operatorname{esssup}_{\tau \in \mathcal{T}^{\mathbb{P}}_{t_i^k}} \mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\tau | \mathcal{F}^{\mathbb{P}}_{t_i^k}]$. Similar to (7.16), we can find a $\widehat{\tau}_i^k \in \mathcal{T}^t_{t_i^k}$ such that

$$\mathbb{E}_{\mathbb{P}}\big[\big|\mathcal{Y}_{\widehat{\tau}_i^k} - \mathcal{Y}_{\tau_i^k}\big|\big] < \varepsilon/k. \tag{7.34}$$

Define $\nu_k \stackrel{\triangle}{=} \mathbf{1}_{\{\nu \le t_1^k\}} t_1^k + \sum_{i=2}^k \mathbf{1}_{\{t_{i-1}^k < \nu \le t_i^k\}} t_i^k \in \mathcal{T}^t$ and $\overline{\tau}_k^n \stackrel{\triangle}{=} \sum_{i=1}^k \mathbf{1}_{A_i^k} (\mathbf{1}_{\{\tau_n < t_i^k\}} \tau_n + \mathbf{1}_{\{\tau_n \ge t_i^k\}} \widehat{\tau}_i^k) \in \mathcal{T}^t$, where $A_i^k \stackrel{\triangle}{=} \{\nu_k = t_i^k\} \in \mathcal{F}^t_{t_i^k}$. We can deduce from (7.33) and (7.34) that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau_n < \nu_k\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n \ge \nu_k\}} \mathcal{Z}_{\nu_k}\Big] \\
&\quad \le \sum_{i=1}^k \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{A_i^k}\Big(\mathbf{1}_{\{\tau_n < t_i^k\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n \ge t_i^k\}} \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\tau_i^k} \big| \mathcal{F}^{\mathbb{P}}_{t_i^k}\big]\Big)\Big] \\
&\quad = \sum_{i=1}^k \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{A_i^k}\big(\mathbf{1}_{\{\tau_n < t_i^k\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n \ge t_i^k\}} \mathcal{Y}_{\tau_i^k}\big) \big| \mathcal{F}^{\mathbb{P}}_{t_i^k}\big]\Big] \\
&\quad = \sum_{i=1}^k \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{A_i^k}\big(\mathbf{1}_{\{\tau_n < t_i^k\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n \ge t_i^k\}} \mathcal{Y}_{\tau_i^k}\big)\Big] \\
&\quad < \sum_{i=1}^k \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{A_i^k}\big(\mathbf{1}_{\{\tau_n < t_i^k\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n \ge t_i^k\}} \mathcal{Y}_{\widehat{\tau}_i^k}\big)\Big] + \varepsilon \\
&\quad = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\overline{\tau}_k^n}\big] + \varepsilon \le \sup_{\zeta \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\zeta\big] + \varepsilon.
\end{aligned} \tag{7.35}$$

Since $\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_* + \mathcal{Z}_*] < \infty$ by (3.2) and Proposition 4.2, letting $k \to \infty$ in (7.35), we can deduce from the continuity of $\overline{Z}$ and the dominated convergence theorem that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau_n \le \nu\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n > \nu\}} \mathcal{Z}_\nu\Big] &= \lim_{k \to \infty} \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau_n < \nu_k\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n \ge \nu_k\}} \mathcal{Z}_{\nu_k}\Big] \\
&\le \sup_{\zeta \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\zeta\big] + \varepsilon.
\end{aligned}$$

As $n \to \infty$, the right continuity of Y and the dominated convergence theorem imply that

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau < \nu\}} \mathcal{Y}_\tau + \mathbf{1}_{\{\tau \ge \nu\}} \mathcal{Z}_\nu\Big] = \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau_n \le \nu\}} \mathcal{Y}_{\tau_n} + \mathbf{1}_{\{\tau_n > \nu\}} \mathcal{Z}_\nu\Big] \le \sup_{\zeta \in \mathcal{T}^t} \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\zeta\big] + \varepsilon.$$

Taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand side and then letting $\varepsilon \to 0$ yields (7.32). $\square$

7.3. Proofs of the results in section 5.

Proof of Remark 5.1. Let $(t,\omega)\in[0,T]\times\Omega$. As Y_τ is $\mathcal{F}_T$-measurable, Lemma 2.1(1) shows that $(Y_\tau)^{t,\omega}$ is in turn $\mathcal{F}^t_T$-measurable. Since $Y_{\tau\wedge t}\in\mathcal{F}_t$, we can deduce from (2.3) that

$$\begin{aligned}\big|(Y_\tau)^{t,\omega}(\widetilde{\omega})\big| &= \big|Y\big(\tau(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\big|\\ &\le \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})<t\}}\big|Y_{\tau\wedge t}(\omega\otimes_t\widetilde{\omega})\big|+\mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})\ge t\}}Y^{t,\omega}_*(\widetilde{\omega})\\ &= \mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})<t\}}\big|Y_{\tau\wedge t}(\omega)\big|+\mathbf{1}_{\{\tau(\omega\otimes_t\widetilde{\omega})\ge t\}}Y^{t,\omega}_*(\widetilde{\omega})\quad \forall\,\widetilde{\omega}\in\Omega^t.\end{aligned}$$

For any $\mathbb{P}\in\mathcal{P}(t,\omega)$, it then follows from (3.2) that $\mathbb{E}_\mathbb{P}[|(Y_\tau)^{t,\omega}|]\le|Y_{\tau\wedge t}(\omega)|+\mathbb{E}_\mathbb{P}[Y^{t,\omega}_*]<\infty$. Thus, $Y_\tau\in\mathscr{L}_t$. Similarly, one can deduce from Remark 4.2 and Proposition 4.2 that $\overline{Z}_\tau\in\mathscr{L}_t$. □

Proof of Theorem 5.1. When $t=T$, (5.2) clearly holds. So let us fix $(t,\omega)\in[0,T)\times\Omega$ and $\nu\in\mathcal{T}_t$. We still define $\mathcal{Y}$ and $\mathcal{Z}$ as in (7.7). By Corollary 2.1, $\nu^{t,\omega}\in\mathcal{T}^t$. Taking $\tau=\nu=\nu^{t,\omega}$ in (4.6) yields that

$$\begin{aligned}\overline{Z}_t(\omega)&\ge\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_\mathbb{P}\Big[\mathbf{1}_{\{\tau<\nu^{t,\omega}\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau\ge\nu^{t,\omega}\}}\mathcal{Z}_{\nu^{t,\omega}}\Big]\\ &\ge\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_\mathbb{P}\big[\mathcal{Z}_{\nu^{t,\omega}}\big]=\underline{\mathscr{E}}_t\big[\overline{Z}_\nu\big](\omega),\end{aligned}\tag{7.36}$$

which shows that $\overline{Z}$ is an $\underline{\mathscr{E}}$-supermartingale.

Next, let us show the $\underline{\mathscr{E}}$-martingality of $\overline{Z}^*$: If $\widehat{t}\overset{\triangle}{=}\tau^*(\omega)\le t$, i.e., $\omega\in\{\tau^*=\widehat{t}\}\in\mathcal{F}_{\widehat{t}}\subset\mathcal{F}_t$, Lemma 2.1 implies that $\omega\otimes_t\Omega^t\subset\{\tau^*=\widehat{t}\}$. Then for any $(s,\widetilde{\omega})\in[t,T]\times\Omega^t$, we have $\nu(\omega\otimes_t\widetilde{\omega})\ge t\ge\widehat{t}=\tau^*(\omega\otimes_t\widetilde{\omega})$. Applying (2.3) to $\overline{Z}_{\widehat{t}}\in\mathcal{F}_{\widehat{t}}\subset\mathcal{F}_t$ yields that $(\overline{Z}^*_\nu)^{t,\omega}(\widetilde{\omega})=\overline{Z}_{\nu\wedge\tau^*}(\omega\otimes_t\widetilde{\omega})=\overline{Z}(\nu(\omega\otimes_t\widetilde{\omega})\wedge\tau^*(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega})=\overline{Z}(\widehat{t},\omega\otimes_t\widetilde{\omega})=\overline{Z}(\widehat{t},\omega)$. It follows that

$$\begin{aligned}\underline{\mathscr{E}}_t\big[\overline{Z}^*_\nu\big](\omega)&=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_\mathbb{P}\Big[\big(\overline{Z}^*_\nu\big)^{t,\omega}\Big]=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}_\mathbb{P}\big[\overline{Z}(\widehat{t},\omega)\big]\\ &=\overline{Z}\big(\widehat{t},\omega\big)=\overline{Z}\big(t\wedge\tau^*(\omega),\omega\big)=\overline{Z}^*_t(\omega).\end{aligned}\tag{7.37}$$

We now suppose $\tau^*(\omega)>t$, i.e., $\omega\in\{\tau^*>t\}\in\mathcal{F}_t$. Lemma 2.1 again shows that

$$\omega\otimes_t\Omega^t\subset\{\tau^*>t\}.\tag{7.38}$$

By Corollary 2.1, $(\tau^*)^{t,\omega}\in\mathcal{T}^t$. Similar to (7.36), taking $\tau=\nu^{t,\omega}\wedge(\tau^*)^{t,\omega}=(\nu\wedge\tau^*)^{t,\omega}$ in (4.6) yields that

$$\overline{Z}^*_t(\omega)=\overline{Z}\big(t\wedge\tau^*(\omega),\omega\big)=\overline{Z}(t,\omega)\ge\underline{\mathscr{E}}_t\big[\overline{Z}_{\nu\wedge\tau^*}\big](\omega)=\underline{\mathscr{E}}_t\big[\overline{Z}^*_\nu\big](\omega).\tag{7.39}$$

The demonstration of

$$\overline{Z}^*_t(\omega)\le\underline{\mathscr{E}}_t\big[\overline{Z}^*_\nu\big](\omega)\tag{7.40}$$

in the case of $\tau^*(\omega)>t$ is relatively lengthy. We split it into several steps. The main idea is the following: We approximate τ^* by the hitting time $\tau^n\overset{\triangle}{=}\inf\{s\in[0,T]\colon\overline{Z}_s\le Y_s+1/n\}$ and then approximate the corresponding shifted stopping time $\zeta^n\overset{\triangle}{=}(\nu\wedge(\tau^n\vee t))^{t,\omega}$ by stopping time ζ^n_k that takes finite values $t^k_i\overset{\triangle}{=}t+\frac{i}{k}(T-t)$, $i=1,\dots,k$. We will paste in accordance with (P2) *the local approximating minimizers $\mathbb{P}^i_{\widetilde{\omega}}$ of $\mathcal{Z}_{t^k_i}(\widetilde{\omega})$ over the set $\{\zeta^n_k=t^k_i\}$ backwardly to get a probability $\mathbb{P}_1\in\mathcal{P}(t,\omega)$ that* *satisfies* $\mathbb{E}_{\mathbb{P}_1}[\mathcal{Y}_\tau|\mathcal{F}^{\mathbb{P}_1}_{\zeta^n_k}] \le \mathcal{Z}_{\zeta^n_k} + \varepsilon$ *for all stopping times* τ*. Taking essential supremum over* τ*'s shows that*

$$\mathscr{Z}^{\mathbb{P}_1}_{\zeta^n_k} \le \mathcal{Z}_{\zeta^n_k} + \varepsilon, \tag{7.41}$$

where $\mathscr{Z}^{\mathbb{P}_1}$ *denotes the Snell envelope of* $\mathcal{Y}$ *under the single probability* $\mathbb{P}_1$*. By the martingale property of* $\mathscr{Z}^{\mathbb{P}_1}$*,*

$$\overline{Z}_t(\omega) \le \mathscr{Z}^{\mathbb{P}_1}_t \le \mathbb{E}_{\mathbb{P}_1}\Big[\mathscr{Z}^{\mathbb{P}_1}_{\zeta^n_k \wedge \tau_{\mathbb{P}_1}}\Big], \tag{7.42}$$

where $\tau_{\mathbb{P}_1}$ *is the optimal stopping time for* $\mathscr{Z}^{\mathbb{P}_1}$*. As the first time* $\mathscr{Z}^{\mathbb{P}_1}$ *meets* $\mathcal{Y}$*,* $\tau_{\mathbb{P}_1} \ge (\tau^*)^{t,\omega}$*. Since* $\tau^* = \lim_{n\to\infty}\uparrow \tau^n$ *and* $\lim_{k\to\infty}\zeta^n_k = \zeta^n$*, for* n,k *large enough we have* $\tau_{\mathbb{P}_1} \ge \zeta^n_k$ *except for a tiny probability. Then combining* (7.42) *with* (7.41) *and applying a series of estimations yield that* $\overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}_1}[\mathcal{Z}_{\zeta^n_k}] + \varepsilon \le \mathbb{E}_{\mathbb{P}}[\mathcal{Z}_{\zeta^n_k}] + \varepsilon$*. Finally, letting* $k,n\to\infty$*,* $\varepsilon\to 0$ *and taking infimum over* $\mathbb{P}\in\mathcal{P}(t,\omega)$ *leads to* (7.40)*.*

(a) *In the first step, we paste the local approximating minimizers* $\mathbb{P}^i_{\widetilde{\omega}}$ *of* $\mathcal{Z}_{t^k_i}(\widetilde{\omega})$ *over the set* $\{\zeta^n_k = t^k_i\}$ *backwardly.*

Fix $\mathbb{P}\in\mathcal{P}(t,\omega)$, $\varepsilon\in(0,1)$ and $\alpha,n,k,\lambda\in\mathbb{N}$ with $k\ge 2$. We let $\{\omega^\alpha_j\}_{j\in\mathbb{N}}$ be a subsequence of $\{\widehat{\omega}^t_j\}_{j\in\mathbb{N}}$ in $O_\alpha(\mathbf{0}^t)$ and define an $\mathbf{F}$-stopping time $\tau^n \overset{\triangle}{=} \inf\{s\in[0,T] : \overline{Z}_s \le Y_s + 1/n\}$. By Corollary 2.1 and (7.38), both $\zeta^n \overset{\triangle}{=} (\nu\wedge(\tau^n\vee t))^{t,\omega}$ and $\zeta^* \overset{\triangle}{=} (\tau^*)^{t,\omega}$ are $\mathcal{T}^t$-stopping times. We set $t_i = t^k_i \overset{\triangle}{=} t + \frac{i}{k}(T-t)$ for $i=1,\dots,k$ and define $\zeta^n_k \overset{\triangle}{=} \mathbf{1}_{\{\zeta^n\le t_1\}}t_1 + \sum_{i=2}^k \mathbf{1}_{\{t_{i-1}<\zeta^n\le t_i\}}t_i \in \mathcal{T}^t$.

There exists a $\delta>0$ such that $\rho_0(\delta)\vee\widehat{\rho}_0(\delta)\vee\rho_1(\delta) < \varepsilon/4$. Given $(i,j)\in\{1,\dots,k\}\times\{1,\dots,\lambda\}$, we set $\mathcal{A}^i_j \overset{\triangle}{=} \{\zeta^n_k = t_i\}\cap(O^{t_i}_\delta(\omega^\alpha_j)\backslash\cup_{j'<j}O^{t_i}_\delta(\omega^\alpha_{j'}))\in\mathcal{F}^t_{t_i}$ by (2.1). There exists a $\mathbb{P}^i_j\in\mathcal{P}(t_i,\omega\otimes_t\omega^\alpha_j)$ such that $\overline{Z}_{t_i}(\omega\otimes_t\omega^\alpha_j) \ge \sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}[Y^{t_i,\omega\otimes_t\omega^\alpha_j}_\tau] - \varepsilon/4$. For any $\widetilde{\omega}\in\mathcal{A}^i_j$ with $\mathcal{A}^i_j\ne\emptyset$, one can deduce from (3.1) and (4.3) that

$$\begin{aligned}
\sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[\mathcal{Y}^{t_i,\widetilde{\omega}}_\tau\Big] &= \sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\tau\Big]\\
&\le \sup_{\tau\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\Big[Y^{t_i,\omega\otimes_t\omega^\alpha_j}_\tau\Big] + \rho_0\big(\big\|\widetilde{\omega}-\omega^\alpha_j\big\|_{t,t_i}\big)\\
&\le \overline{Z}_{t_i}(\omega\otimes_t\omega^\alpha_j) + \frac{\varepsilon}{4} + \rho_0\big(\big\|\widetilde{\omega}-\omega^\alpha_j\big\|_{t,t_i}\big)\\
&< \overline{Z}_{t_i}(\omega\otimes_t\widetilde{\omega}) + \rho_1\big(\big\|\widetilde{\omega}-\omega^\alpha_j\big\|_{t,t_i}\big) + \frac{1}{2}\varepsilon < \mathcal{Z}_{t_i}(\widetilde{\omega}) + \frac{3}{4}\varepsilon,
\end{aligned} \tag{7.43}$$

where we used in the first inequality the fact that for any $\tau\in\mathcal{T}^{t_i}$ and $\widehat{\omega}\in\Omega^{t_i}$

$$\begin{aligned}
&\Big|Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\tau(\widehat{\omega}) - Y^{t_i,\omega\otimes_t\omega^\alpha_j}_\tau(\widehat{\omega})\Big|\\
&\quad= \Big|Y\big(\tau(\widehat{\omega}),(\omega\otimes_t\widetilde{\omega})\otimes_{t_i}\widehat{\omega}\big) - Y\big(\tau(\widehat{\omega}),(\omega\otimes_t\omega^\alpha_j)\otimes_{t_i}\widehat{\omega}\big)\Big|\\
&\quad\le \rho_0\Big(\big\|(\omega\otimes_t\widetilde{\omega})\otimes_{t_i}\widehat{\omega} - (\omega\otimes_t\omega^\alpha_j)\otimes_{t_i}\widehat{\omega}\big\|_{0,\tau(\widehat{\omega})}\Big) = \rho_0\big(\big\|\widetilde{\omega}-\omega^\alpha_j\big\|_{t,t_i}\big).
\end{aligned}$$

Setting $\mathbb{P}^\lambda_k \overset{\triangle}{=} \mathbb{P}$, we recursively pick up $\mathbb{P}^\lambda_i$, $i=k-1,\dots,1$ from $\mathcal{P}(t,\omega)$ such that (P2) holds for $(s,\widehat{\mathbb{P}},\mathbb{P},\{\mathcal{A}_j\}_{j=0}^\lambda,\{\mathbb{P}_j\}_{j=1}^\lambda) = (t_i,\mathbb{P}^\lambda_i,\mathbb{P}^\lambda_{i+1},\{\mathcal{A}^i_j\}_{j=0}^\lambda,$

$\{\mathbb{P}^i_j\}_{j=1}^{\lambda})$ with $\mathcal{A}^i_0 \stackrel{\triangle}{=} (\cup_{j=1}^{\lambda}\mathcal{A}^i_j)^c$. In particular

$$\begin{aligned}&\sup_{\tau\in\mathcal{T}^t_{t_i}} \mathbb{E}_{\mathbb{P}^\lambda_i}\big[\mathbf{1}_{A\cap\mathcal{A}^i_j} Y^{t,\omega}_\tau\big]\\ &\qquad \le \mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}^i_j\}}\left(\sup_{\zeta\in\mathcal{T}^{t_i}} \mathbb{E}_{\mathbb{P}^i_j}\big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\zeta\big]+\widehat{\rho}_0(\delta)\right)\right]\\ &\qquad\quad \forall\, j=1,\dots,\lambda,\ \ \forall\, A\in\mathcal{F}^t_{t_i}.\end{aligned}\tag{7.44}$$

Similar to (7.11), we have

$$\mathbb{E}_{\mathbb{P}^\lambda_i}[\xi]=\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}[\xi] \qquad \forall\,\xi\in L^1(\mathcal{F}^t_{t_i},\mathbb{P}^\lambda_i)\cap L^1\big(\mathcal{F}^t_{t_i},\mathbb{P}^\lambda_{i+1}\big)\tag{7.45}$$

$$\text{and } \mathbb{E}_{\mathbb{P}^\lambda_i}[\mathbf{1}_{\mathcal{A}^i_0}\xi]=\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}[\mathbf{1}_{\mathcal{A}^i_0}\xi] \qquad \forall\,\xi\in L^1(\mathcal{F}^t_{T},\mathbb{P}^\lambda_i)\cap L^1\big(\mathcal{F}^t_{T},\mathbb{P}^\lambda_{i+1}\big).\tag{7.46}$$

(b) *Now, let us consider the Snell envelope* $Z^{\mathbb{P}^\lambda_1}$ *of* $\mathcal{Y}$ *under* $\mathbb{P}^\lambda_1$, *i.e.*, $Z^{\mathbb{P}^\lambda_1}_s \stackrel{\triangle}{=} \operatorname{esssup}_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_s} \mathbb{E}_{\mathbb{P}^\lambda_1}[\mathcal{Y}_\tau|\mathcal{F}^{\mathbb{P}^\lambda_1}_s]$, $s\in[t,T]$.

Since the filtration $\mathbf{F}^{\mathbb{P}^\lambda_1}$ is right-continuous, and since the process $\mathcal{Y}$ is right-continuous and left upper semicontinuous by Remark 3.1 (2), the classic optimal stopping theory shows that $Z^{\mathbb{P}^\lambda_1}$ admits an RCLL modification $\{\mathscr{Z}^{\mathbb{P}^\lambda_1}_s\}_{s\in[t,T]}$ such that for any $s\in[t,T]$, $\tau^s_{\mathbb{P}^\lambda_1}\stackrel{\triangle}{=}\inf\{r\in[s,T]:\mathscr{Z}^{\mathbb{P}^\lambda_1}_r=\mathcal{Y}_r\}\in\mathcal{T}^{\mathbb{P}^\lambda_1}_s$ is an optimal stopping time for $\operatorname{esssup}_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_s}\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathcal{Y}_\tau|\mathcal{F}^{\mathbb{P}^\lambda_1}_s]$. Simply denoting $\tau^t_{\mathbb{P}^\lambda_1}$ by τ_λ, we also know that $\mathscr{Z}^{\mathbb{P}^\lambda_1}$ (resp., $\{\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\tau_\lambda\wedge s}\}_{s\in[t,T]}$) is a supermartingale (resp., martingale) with respect to $(\mathbf{F}^{\mathbb{P}^\lambda_1},\mathbb{P}^\lambda_1)$. It follows from optional sampling theorem that

$$\begin{aligned}\overline{Z}_t(\omega)&=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_\tau\big]\le\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\mathcal{Y}_\tau\big]\\ &\le\sup_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}}\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\mathcal{Y}_\tau\big]=Z^{\mathbb{P}^\lambda_1}_t=\mathscr{Z}^{\mathbb{P}^\lambda_1}_t=\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}\Big].\end{aligned}\tag{7.47}$$

Moreover, for any $s\in[t,T]$, applying (7.15) with $\mathbb{P}=\mathbb{P}^\lambda_1$ yields that $\mathcal{Z}_s\le\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathcal{Y}_{\tau^s_{\mathbb{P}^\lambda_1}}\big|\mathcal{F}^{\mathbb{P}^\lambda_1}_s]=\operatorname{esssup}_{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_s}\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathcal{Y}_\tau|\mathcal{F}^{\mathbb{P}^\lambda_1}_s]=Z^{\mathbb{P}^\lambda_1}_s=\mathscr{Z}^{\mathbb{P}^\lambda_1}_s$, $\mathbb{P}^\lambda_1$-a.s. By the continuity of $\overline{Z}$ and the right continuity of $\mathscr{Z}^{\mathbb{P}^\lambda_1}$, it holds for $\mathbb{P}^\lambda_1$-a.s. $\widetilde{\omega}\in\Omega^t$ that $\mathcal{Z}_s(\widetilde{\omega})\le\mathscr{Z}^{\mathbb{P}^\lambda_1}_s(\widetilde{\omega})$ for any $s\in[t,T]$. Since $\tau^*(\omega\otimes_t\widetilde{\omega})>t$ by (7.38), one can deduce that

$$\begin{aligned}\zeta^*(\widetilde{\omega})=\tau^*(\omega\otimes_t\widetilde{\omega})&=\inf\{s\in[0,T]:\overline{Z}_s(\omega\otimes_t\widetilde{\omega})=Y_s(\omega\otimes_t\widetilde{\omega})\}\\ &=\inf\{s\in[t,T]:\overline{Z}_s(\omega\otimes_t\widetilde{\omega})=Y_s(\omega\otimes_t\widetilde{\omega})\}\\ &=\inf\{s\in[t,T]:\mathcal{Z}_s(\widetilde{\omega})=\mathcal{Y}_s(\widetilde{\omega})\}\\ &\le\inf\{s\in[t,T]:\mathscr{Z}^{\mathbb{P}^\lambda_1}_s(\widetilde{\omega})=\mathcal{Y}_s(\widetilde{\omega})\}=\tau_\lambda(\widetilde{\omega}).\end{aligned}\tag{7.48}$$

Next, let us use (7.43)–(7.46) *to show that*

$$\mathbf{1}_{\underset{i=1}{\overset{k-1}{\cup}}(\mathcal{A}^i_0)^c}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k}\le\mathbf{1}_{\underset{i=1}{\overset{k-1}{\cup}}(\mathcal{A}^i_0)^c}\big(\mathcal{Z}_{\zeta^n_k}+\varepsilon\big),\quad \mathbb{P}^\lambda_1-a.s.\tag{7.49}$$

To see this, we let $(i,j) \in \{1,\dots,k-1\} \times \{1,\dots,\lambda\}$, $\tau \in \mathcal{T}^t_{t_i}$ and $A \in \mathcal{F}^t_{t_i}$. Since $\mathcal{A}^i_j \subset \mathcal{A}^{i'}_0$ for $i' \in \{1,\dots,k-1\}\backslash\{i\}$, we can deduce from (7.46), (3.2), (7.44), (7.43), (7.45), and Proposition 4.2 that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\mathcal{Y}_\tau\Big] &= \cdots = \mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\mathcal{Y}_\tau\Big]\\
&\le \mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}^i_j\}}\left(\sup_{\zeta\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}^i_j}\big[Y^{t_i,\omega\otimes_t\widetilde{\omega}}_\zeta\big]+\widehat{\rho}_0(\delta)\right)\right]\\
&\le \mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big)\Big] = \mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big)\Big]\\
&= \cdots = \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{A\cap\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big)\Big],
\end{aligned}$$

where we used the fact that $\mathcal{Z}_{t_i} \in \mathcal{F}^t_{t_i}$ by Remark 4.2 and Proposition 2.1(1). Letting A vary over $\mathcal{F}^t_{t_i}$ and applying Lemma A.2(1) with $(\mathbb{P}, X) = (\mathbb{P}^\lambda_1, B^t)$ yields that

$$\text{(7.50)}\quad \mathbf{1}_{\mathcal{A}^i_j}\big(\mathcal{Z}_{t_i}+\varepsilon\big) \ge \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\big|\mathcal{F}^t_{t_i}\Big] = \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big], \quad \mathbb{P}^\lambda_1 - a.s.$$

For any $\tau \in \mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}$, similar to (7.18), one can find a sequence $\{\tau^i_\ell\}_{\ell\in\mathbb{N}}$ of $\mathcal{T}^t_{t_i}$ such that $\lim_{\ell\to\infty}\mathbb{E}_{\mathbb{P}^\lambda_1}[|\mathcal{Y}_{\tau^i_\ell}-\mathcal{Y}_\tau|] = 0$. Then $\{\tau^i_\ell\}_{\ell\in\mathbb{N}}$ in turn has a subsequence (we still denote it by $\{\tau^i_\ell\}_{\ell\in\mathbb{N}}$) such that $\lim_{\ell\to\infty}\mathcal{Y}_{\tau^i_\ell} = \mathcal{Y}_\tau$, $\mathbb{P}^\lambda_1$-a.s. As $\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathcal{Y}_*] < \infty$ by (3.2), a conditional-expectation version of the dominated convergence theorem and (7.50) imply that $\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}] = \lim_{\ell\to\infty}\mathbb{E}_{\mathbb{P}^\lambda_1}[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_{\tau^i_\ell}|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}] \le \mathbf{1}_{\mathcal{A}^i_j}(\mathcal{Z}_{t_i}+\varepsilon)$, $\mathbb{P}^\lambda_1$–a.s. Since $\mathcal{A}^i_j \in \mathcal{F}^t_{t_i}$, it follows that

$$\begin{aligned}
\mathbf{1}_{\mathcal{A}^i_j}\mathscr{Z}^{\mathbb{P}^\lambda_1}_{\zeta^n_k} = \mathbf{1}_{\mathcal{A}^i_j}Z^{\mathbb{P}^\lambda_1}_{t_i} &= \underset{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}}{\operatorname{esssup}}\,\mathbf{1}_{\mathcal{A}^i_j}\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big]\\
&= \underset{\tau\in\mathcal{T}^{\mathbb{P}^\lambda_1}_{t_i}}{\operatorname{esssup}}\,\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_\tau\Big|\mathcal{F}^{\mathbb{P}^\lambda_1}_{t_i}\Big] \le \mathbf{1}_{\mathcal{A}^i_j}\big(\mathcal{Z}_{\zeta^n_k}+\varepsilon\big), \quad \mathbb{P}^\lambda_1 - a.s.
\end{aligned}$$

Summing them up over $j \in \{1,\dots,\lambda\}$ and then over $i \in \{1,\dots,k-1\}$ yields (7.49).

(c) *In this step, we will use* (7.47) *and* (7.49) *to show*

$$\text{(7.51)}\qquad \overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\mathscr{A}_\lambda}\mathcal{Z}_{\zeta^n_k} + \mathbf{1}_{\mathscr{A}^c_\lambda}\mathcal{Y}_{\tau_\lambda}\Big] + \varepsilon,$$

where $\mathscr{A}_\lambda \triangleq \{\zeta^n_k \le \zeta^*\} \cap (\cup_{i=1}^{k-1}(\mathcal{A}^i_0)^c) = \{\zeta^n_k \le \zeta^*\} \cap (\cup_{i=1}^{k-1}\cup_{j=1}^{\lambda}\mathcal{A}^i_j)$.

We first claim that $\mathscr{A}_\lambda \in \mathcal{F}^t_{\zeta^n_k\wedge\zeta^*} \cap \mathcal{F}^{\mathbb{P}^\lambda_1}_{\zeta^n_k\wedge\tau_\lambda}$. To see this claim, we set an auxiliary set $\widehat{\mathscr{A}_\lambda} \triangleq \{\zeta^n_k \le \tau_\lambda\} \cap (\cup_{i=1}^{k-1}(\mathcal{A}^i_0)^c)$. Given $s \in [t,T]$, if $s < t_1$, then $\mathscr{A}_\lambda \cap \{\zeta^n_k\wedge\zeta^* \le s\} = \mathscr{A}_\lambda \cap \{\zeta^n_k \le s\} = \emptyset$ and $\widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k\wedge\tau_\lambda \le s\} = \widehat{\mathscr{A}_\lambda} \cap \{\zeta^n_k \le s\} = \emptyset$. Otherwise, let k' be the largest integer from $\{1,\dots,k-1\}$ such that

$t_{k'} \le s$. Since $(\mathcal{A}_0^i)^c = \cup_{j=1}^{\lambda} \mathcal{A}_j^i \subset \{\zeta_k^n = t_i\}$ for $i = 1, \dots, k-1$,

$$\begin{aligned} \mathscr{A}_\lambda \cap \{\zeta_k^n \wedge \zeta^* \le s\} &= \mathscr{A}_\lambda \cap \{\zeta_k^n \le s\} \\ &= \{\zeta_k^n \le \zeta^*\} \cap \left(\bigcup_{i=1}^{k'} (\mathcal{A}_0^i)^c \right) \cap \{\zeta_k^n \le s\} \\ \text{and} \quad \widehat{\mathscr{A}_\lambda} \cap \{\zeta_k^n \wedge \tau_\lambda \le s\} &= \widehat{\mathscr{A}_\lambda} \cap \{\zeta_k^n \le s\} \\ &= \{\zeta_k^n \le \tau_\lambda\} \cap \left(\bigcup_{i=1}^{k'} (\mathcal{A}_0^i)^c \right) \cap \{\zeta_k^n \le s\}. \end{aligned}$$

Clearly, $\cup_{i=1}^{k'} (\mathcal{A}_0^i)^c \in \mathcal{F}_{t_{k'}}^t \subset \mathcal{F}_s^t \subset \mathcal{F}_s^{\mathbb{P}_1^\lambda}$. As $\{\zeta_k^n \le \zeta^*\} \in \mathcal{F}_{\zeta_k^n \wedge \zeta^*}^t \subset \mathcal{F}_{\zeta_k^n}^t$ and $\{\zeta_k^n \le \tau_\lambda\} \in \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda} \subset \mathcal{F}_{\zeta_k^n}^{\mathbb{P}_1^\lambda}$, we also have $\{\zeta_k^n \le \zeta^*\} \cap \{\zeta_k^n \le s\} \in \mathcal{F}_s^t$ and $\{\zeta_k^n \le \tau_\lambda\} \cap \{\zeta_k^n \le s\} \in \mathcal{F}_s^{\mathbb{P}_1^\lambda}$. It follows that $\mathscr{A}_\lambda \cap \{\zeta_k^n \wedge \zeta^* \le s\} \in \mathcal{F}_s^t$ and $\widehat{\mathscr{A}_\lambda} \cap \{\zeta_k^n \wedge \tau_\lambda \le s\} \in \mathcal{F}_s^{\mathbb{P}_1^\lambda}$. Hence $\mathscr{A}_\lambda \in \mathcal{F}_{\zeta_k^n \wedge \zeta^*}^t$ and $\widehat{\mathscr{A}_\lambda} \in \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}$.

By (7.48), $\mathcal{N} \stackrel{\triangle}{=} \{\zeta^* > \tau_\lambda\} \in \mathscr{N}^{\mathbb{P}_1^\lambda}$. Since $\mathscr{A}_\lambda \cap \mathcal{N}^c \subset \{\zeta_k^n \le \tau_\lambda\}$ and since $\{\zeta_k^n \le \zeta^* \wedge \tau_\lambda\} \in \mathcal{F}_{\zeta_k^n \wedge \zeta^* \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda} \subset \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}$, one can deduce that

$$\begin{aligned} \mathscr{A}_\lambda \cap \mathcal{N}^c = \mathscr{A}_\lambda \cap \{\zeta_k^n \le \tau_\lambda\} \cap \mathcal{N}^c &= \{\zeta_k^n \le \zeta^* \wedge \tau_\lambda\} \cap \left(\bigcup_{i=1}^{k-1} (\mathcal{A}_0^i)^c \right) \cap \mathcal{N}^c \\ &= \{\zeta_k^n \le \zeta^* \wedge \tau_\lambda\} \cap \widehat{\mathscr{A}_\lambda} \cap \mathcal{N}^c \in \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}. \end{aligned}$$

As $\mathscr{A}_\lambda \cap \mathcal{N} \in \mathscr{N}^{\mathbb{P}_1^\lambda}$, we see that $\mathscr{A}_\lambda \in \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}$.

Since $\{\mathscr{Z}_{\tau_\lambda \wedge s}^{\mathbb{P}_1^\lambda}\}_{s \in [t,T]}$ is a martingale with respect to $(\mathbf{F}^{\mathbb{P}_1^\lambda}, \mathbb{P}_1^\lambda)$, it follows from optional sampling theorem that $\mathbf{1}_{\mathscr{A}_\lambda^c} \mathscr{Z}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda} = \mathbf{1}_{\mathscr{A}_\lambda^c} \mathbb{E}_{\mathbb{P}_1^\lambda}[\mathscr{Z}_{\tau_\lambda}^{\mathbb{P}_1^\lambda} | \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}] = \mathbb{E}_{\mathbb{P}_1^\lambda}[\mathbf{1}_{\mathscr{A}_\lambda^c} \mathscr{Z}_{\tau_\lambda}^{\mathbb{P}_1^\lambda} | \mathcal{F}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}]$, $\mathbb{P}_1^\lambda$-a.s. Taking expectation $\mathbb{E}_{\mathbb{P}_1^\lambda}$ yields that

$$\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\mathscr{A}_\lambda^c} \mathscr{Z}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}\Big] = \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\mathscr{A}_\lambda^c} \mathscr{Z}_{\tau_\lambda}^{\mathbb{P}_1^\lambda}\Big] = \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\mathscr{A}_\lambda^c} \mathcal{Y}_{\tau_\lambda}\Big]. \tag{7.52}$$

Since $\zeta_k^n \le \tau_\lambda$ holds $\mathbb{P}_1^\lambda$-a.s. on $\mathscr{A}_\lambda$ by (7.48), we can deduce from (7.47), (7.52), and (7.49) that

$$\begin{aligned} \overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathscr{Z}_{\zeta_k^n \wedge \tau_\lambda}^{\mathbb{P}_1^\lambda}\Big] &= \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\mathscr{A}_\lambda} \mathscr{Z}_{\zeta_k^n}^{\mathbb{P}_1^\lambda} + \mathbf{1}_{\mathscr{A}_\lambda^c} \mathcal{Y}_{\tau_\lambda}\Big] \\ &\le \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\mathscr{A}_\lambda} \mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{\mathscr{A}_\lambda^c} \mathcal{Y}_{\tau_\lambda}\Big] + \varepsilon. \end{aligned}$$

(d) *In the next step, we replace* $\mathbb{E}_{\mathbb{P}_1^\lambda}[\mathbf{1}_{\mathscr{A}_\lambda} \mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{\mathscr{A}_\lambda^c} \mathcal{Y}_{\tau_\lambda}]$ *on the right-hand side of* (7.51) *by an expectation under* $\mathbb{P}$.

For $i = 1, \dots, k-1$, as $\mathscr{A}_\lambda \in \mathcal{F}_{\zeta_k^n \wedge \zeta^*}^t \subset \mathcal{F}_{\zeta_k^n}^t$, one has $\mathscr{A}_\lambda^i \stackrel{\triangle}{=} \mathscr{A}_\lambda \cap \{\zeta_k^n = t_i\} = \{\zeta_k^n \le \zeta^*\} \cap (\mathcal{A}_0^i)^c \in \mathcal{F}_{t_i}^t$. By (7.46), (7.45), Remark 4.2, and Proposition 4.2, $\mathbb{E}_{\mathbb{P}_1^\lambda}[\mathbf{1}_{\mathscr{A}_\lambda^i} \mathcal{Z}_{t_i}] = \cdots = \mathbb{E}_{\mathbb{P}_i^\lambda}[\mathbf{1}_{\mathscr{A}_\lambda^i} \mathcal{Z}_{t_i}] = \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}[\mathbf{1}_{\mathscr{A}_\lambda^i} \mathcal{Z}_{t_i}] = \cdots = \mathbb{E}_{\mathbb{P}_k^\lambda}[\mathbf{1}_{\mathscr{A}_\lambda^i} \mathcal{Z}_{t_i}] = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\mathscr{A}_\lambda^i} \mathcal{Z}_{t_i}]$. Their sum over $i \in \{1, \dots, k-1\}$ is

$$\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\mathscr{A}_\lambda} \mathcal{Z}_{\zeta_k^n}\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\mathscr{A}_\lambda} \mathcal{Z}_{\zeta_k^n}\Big]. \tag{7.53}$$

Since $\overline{Z}_T(\omega') = \inf_{\mathbb{P}\in\mathcal{P}(T,\omega')} \mathbb{E}_{\mathbb{P}}[Y_T^{T,\omega'}] = \inf_{\mathbb{P}\in\mathcal{P}(T,\omega')} \mathbb{E}_{\mathbb{P}}[Y(T,\omega')] = Y(T,\omega')$ for all $\omega' \in \Omega$, (7.48) implies that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_{\tau_\lambda}\Big] &= \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_T\Big]\\ &= \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Z}_T\Big] = \mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n}\Big].\end{aligned} \tag{7.54}$$

As $\{T=\zeta_k^n \le \zeta^*\} \subset \{\zeta_k^n = T\} \subset \cap_{i=1}^{k-1}\mathcal{A}_0^i$, one can deduce from (7.46) and Proposition 4.2 again that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n}\Big] &= \mathbb{E}_{\mathbb{P}_2^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n}\Big]\\ &= \cdots = \mathbb{E}_{\mathbb{P}_k^\lambda}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n}\Big] = \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n}\Big]\end{aligned} \tag{7.55}$$

and similarly that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\left(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}_0^i\right)\setminus\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_{\tau_\lambda}\Big] &= \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\left(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}_0^i\right)\setminus\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_{\tau_\lambda}\Big]\\ &\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\left(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}_0^i\right)\setminus\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_*\Big].\end{aligned} \tag{7.56}$$

Similar to (7.18), one can find a sequence $\{\tau_\lambda^\ell\}_{\ell\in\mathbb{N}}$ of $\mathcal{T}^t$ such that $\lim_{\ell\to\infty}\mathbb{E}_{\mathbb{P}_1^\lambda}[|\mathcal{Y}_{\tau_\lambda^\ell} - \mathcal{Y}_{\tau_\lambda}|] = 0$. Let $\ell\in\mathbb{N}$ and $(i,j)\in\{1,\dots,k-1\}\times\{1,\dots,\lambda\}$. Since $\{\zeta^* < \zeta_k^n\} \in \mathcal{F}^t_{\zeta^*\wedge\zeta_k^n} \subset \mathcal{F}^t_{\zeta_k^n}$, we have $\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i = \{\zeta^*<\zeta_k^n\}\cap\{\zeta_k^n = t_i\}\cap\mathcal{A}_j^i \in \mathcal{F}^t_{t_i}$. We can deduce from (3.2) and (7.44)–(7.46) that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}\Big]\\ &\quad= \cdots = \mathbb{E}_{\mathbb{P}_i^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}\Big]\\ &\quad= \mathbb{E}_{\mathbb{P}_i^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i\cap\{\tau_\lambda^\ell\le t_i\}}\mathcal{Y}_{\tau_\lambda^\ell\wedge t_i} + \mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i\cap\{\tau_\lambda^\ell> t_i\}}\mathcal{Y}_{\tau_\lambda^\ell\vee t_i}\Big]\\ &\quad\le \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i\cap\{\tau_\lambda^\ell\le t_i\}}\mathcal{Y}_{\tau_\lambda^\ell\wedge t_i}\Big]\\ &\qquad+ \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}\Bigg[\mathbf{1}_{\{\zeta^*(\widetilde{\omega})<\zeta_k^n(\widetilde{\omega})\}}\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_j^i\}}\mathbf{1}_{\{\tau_\lambda^\ell(\widetilde{\omega})>t_i\}}\\ &\qquad\qquad\Bigg(\sup_{\zeta\in\mathcal{T}^{t_i}}\mathbb{E}_{\mathbb{P}_j^i}\Big[Y_\zeta^{t_i,\omega\otimes_t\widetilde{\omega}}\Big] + \widehat{\rho}_0(\delta)\Bigg)\Bigg].\end{aligned} \tag{7.57}$$

If $M \stackrel{\triangle}{=} \sup_{(t,\omega')\in[0,T]\times\Omega} Y_t(\omega') < \infty$, it follows that

$$\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}\Big] \le \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+M^+)\Big]. \tag{7.58}$$

Suppose otherwise that $M=\infty$. The right continuity of process Y and Proposition 2.1 (1) imply that $\xi_i \stackrel{\triangle}{=} \sup_{r\in[t,t_i]}|\mathcal{Y}_r| = (\sup_{r\in\mathbb{Q}\cap[t,t_i)}|\mathcal{Y}_r|)\vee|\mathcal{Y}_{t_i}|$ is $\mathcal{F}^t_{t_i}$-measurable. For any $\zeta\in\mathcal{T}^{t_i}$, $\widetilde{\omega}\in\Omega^t$, and $\widehat{\omega}\in\Omega^{t_i}$, since $\widehat{t} \stackrel{\triangle}{=} \zeta(\widehat{\omega}) \ge t_i$ and since $Y_r(\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})) = Y_r(\omega)$ for any $r\in[0,t]$ by (2.3) again, (5.1) implies that

$$
\begin{aligned}
Y_\zeta^{t_i,\omega\otimes_t\widetilde{\omega}}(\widehat{\omega}) &= Y\big(\widehat{t},\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big) \le Y\big(t_i,\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)+L\\
&\quad +\phi\left(\sup_{r\in[0,t_i]}\big|Y\big(r,\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)\big|\right)+\rho_1\left(\sup_{r\in[t_i,\widehat{t}]}\big|\widehat{\omega}(r)\big|\right)\\
&= \mathcal{Y}\big(t_i,\widetilde{\omega}\otimes_{t_i}\widehat{\omega}\big)+L+\phi\left(\sup_{r\in[0,t]}\big|Y\big(r,\omega\big)\big|\right)\\
&\quad \vee\phi\left(\sup_{r\in[t,t_i]}\big|\mathcal{Y}\big(r,\widetilde{\omega}\otimes_{t_i}\widehat{\omega}\big)\big|\right)+\rho_1\left(\sup_{r\in[t_i,\widehat{t}]}\big|B_r^{t_i}(\widehat{\omega})\big|\right)\\
&\le L+\xi_i(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})+\phi\left(\sup_{r\in[0,t]}\big|Y_r(\omega)\big|\right)\\
&\quad +\phi\big(\xi_i(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)+\rho_1\left(\sup_{r\in[t_i,T]}\big|B_r^{t_i}(\widehat{\omega})\big|\right)\\
&= L+\xi_i(\widetilde{\omega})+\phi\left(\sup_{r\in[0,t]}\big|Y_r(\omega)\big|\right)+\phi\big(\xi_i(\widetilde{\omega})\big)\\
&\quad +\rho_1\left(\sup_{r\in[t_i,T]}\big|B_r^{t_i}(\widehat{\omega})\big|\right).
\end{aligned}
$$

Remark 3.1(1) implies that (see Lemma A.8 of [6] for details)

$$
Y_*(\omega)=\sup_{r\in[0,T]}\big|Y_r(\omega)\big|<\infty. \tag{7.59}
$$

Since $\|\omega\otimes_t\omega_j^\alpha\|_{0,t_i}\le\|\omega\|_{0,t}+\|\omega_j^\alpha\|_{t,t_i}\le\|\omega\|_{0,t}+\|\omega_j^\alpha\|_{t,T}<\|\omega\|_{0,t}+\alpha\triangleq\alpha'$, (4.4) shows that $\mathbb{E}_{\mathbb{P}_j^i}[Y_\zeta^{t_i,\omega\otimes_t\widetilde{\omega}}]\le\widetilde{L}+\mathcal{Y}_*+\phi(\mathcal{Y}_*)+\rho_{\alpha'}(T-t_i)$, where $\widetilde{L}\triangleq L+\phi(\sup_{r\in[0,t]}|Y_r(\omega)|)<\infty$. Plugging this into (7.57) yields that $\mathbb{E}_{\mathbb{P}_1^\lambda}[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}]\le\mathbb{E}_{\mathbb{P}_{i+1}^\lambda}[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\widetilde{L}+\mathcal{Y}_*+\phi(\mathcal{Y}_*)+\rho_{\alpha'}(T-t_i))]$, which together with (7.58), (7.46), and (3.2) shows that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}\mathcal{Y}_{\tau_\lambda^\ell}\Big] &\le \mathbb{E}_{\mathbb{P}_{i+1}^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\eta_{\alpha'})\Big]\\
&=\cdots=\mathbb{E}_{\mathbb{P}_k^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\eta_{\alpha'})\Big]\\
&=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\mathcal{A}_j^i}(1+\eta_{\alpha'})\Big]
\end{aligned}
$$

for $\eta_{\alpha'}\triangleq\mathbf{1}_{\{M<\infty\}}M^++\mathbf{1}_{\{M=\infty\}}(\widetilde{L}+\mathcal{Y}_*+\phi(\mathcal{Y}_*)+\rho_{\alpha'}(T))$. Summing them up over $j\in\{1,\ldots,\lambda\}$ and then over $i\in\{1,\ldots,k-1\}$ gives that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\left(\bigcup_{i=1}^{k-1}(\mathcal{A}_0^i)^c\right)}\mathcal{Y}_{\tau_\lambda}\Big]\\
&\quad\le\mathbb{E}_{\mathbb{P}_1^\lambda}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\left(\bigcup_{i=1}^{k-1}(\mathcal{A}_0^i)^c\right)}\mathcal{Y}_{\tau_\lambda^\ell}\Big]+\mathbb{E}_{\mathbb{P}_1^\lambda}\big[\big|\mathcal{Y}_{\tau_\lambda}-\mathcal{Y}_{\tau_\lambda^\ell}\big|\big]\\
&\quad=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\left(\bigcup_{i=1}^{k-1}(\mathcal{A}_0^i)^c\right)}(1+\eta_{\alpha'})\Big]+\mathbb{E}_{\mathbb{P}_1^\lambda}\big[\big|\mathcal{Y}_{\tau_\lambda}-\mathcal{Y}_{\tau_\lambda^\ell}\big|\big].
\end{aligned}
$$

As $\ell \to \infty$, we obtain $\mathbb{E}_{\mathbb{P}_1^\lambda}[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap(\cup_{i=1}^{k-1}(\mathcal{A}_0^i)^c)}\mathcal{Y}_{\tau_\lambda}] \le \mathbb{E}_{\mathbb{P}}[\mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap(\cup_{i=1}^{k-1}(\mathcal{A}_0^i)^c)}(1+\eta_{\alpha'})]$. Putting this and (7.53)–(7.56) back into (7.51) yields that

$$\overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}}\Bigg[\left(\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\left(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\right)} + \mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\right)\mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{\left(\overset{k-1}{\underset{i=1}{\cap}}\mathcal{A}_0^i\right)\setminus\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_* + \mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap\left(\overset{k-1}{\underset{i=1}{\cup}}(\mathcal{A}_0^i)^c\right)}(1+\eta_{\alpha'})\Bigg] + \varepsilon. \tag{7.60}$$

(e) *In the last step, we will gradually send the parameters λ, k, n, α to ∞ to obtain* (7.40).

Let $A_{n,k}^\alpha \overset{\triangle}{=} \cup_{\lambda\in\mathbb{N}}\cup_{i=1}^{k-1}(\mathcal{A}_0^i)^c$ and $\mathfrak{O}_\delta^\alpha \overset{\triangle}{=} \cup_{j\in\mathbb{N}} O_\delta(\omega_j^\alpha)$. Since $O_\delta(\omega_j^\alpha) \subset O_\delta^{t_i}(\omega_j^\alpha)$ for $(i,j) \in \{1,\dots,k-1\}\times\mathbb{N}$, one can deduce that

$$\begin{aligned}
A_{n,k}^\alpha &= \overset{k-1}{\underset{i=1}{\cup}}\ \underset{\lambda\in\mathbb{N}}{\cup}(\mathcal{A}_0^i)^c = \overset{k-1}{\underset{i=1}{\cup}}\ \underset{j\in\mathbb{N}}{\cup}\mathcal{A}_j^i = \overset{k-1}{\underset{i=1}{\cup}}\left(\{\zeta_k^n = t_i\}\cap\left(\underset{j\in\mathbb{N}}{\cup}O_\delta^{t_i}(\omega_j^\alpha)\right)\right)\\
&\subset \overset{k-1}{\underset{i=1}{\cup}}\{\zeta_k^n = t_i\} = \{\zeta_k^n < T\} \quad \text{and}\\
A_{n,k}^\alpha &= \overset{k-1}{\underset{i=1}{\cup}}\left(\{\zeta_k^n = t_i\}\cap\left(\underset{j\in\mathbb{N}}{\cup}O_\delta^{t_i}(\omega_j^\alpha)\right)\right) \supset \overset{k-1}{\underset{i=1}{\cup}}\left(\{\zeta_k^n = t_i\}\cap\mathfrak{O}_\delta^\alpha\right)\\
&= \left(\overset{k-1}{\underset{i=1}{\cup}}\{\zeta_k^n = t_i\}\right)\cap\mathfrak{O}_\delta^\alpha = \{\zeta_k^n < T\}\cap\mathfrak{O}_\delta^\alpha.
\end{aligned}$$

As $\mathbb{E}_{\mathbb{P}}[\mathcal{Z}_* + \eta_{\alpha'} + \mathcal{Y}_*] < \infty$ by (3.2) and Proposition 4.2, letting $\lambda \to \infty$ in (7.60) and applying the dominated convergence theorem yields that

$$\begin{aligned}
\overline{Z}_t(\omega) &\le \mathbb{E}_{\mathbb{P}}\Big[\big(\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap A_{n,k}^\alpha} + \mathbf{1}_{\{T=\zeta_k^n\le\zeta^*\}}\big)\mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{(A_{n,k}^\alpha)^c\setminus\{T=\zeta_k^n\le\zeta^*\}}\mathcal{Y}_*\\
&\qquad + \mathbf{1}_{\{\zeta^*<\zeta_k^n\}\cap A_{n,k}^\alpha}(1+\eta_{\alpha'})\Big] + \varepsilon\\
&\le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}}\mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}\mathcal{Z}_* + \mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c\cup\{T=\zeta_k^n>\zeta^*\}}\mathcal{Y}_*\\
&\qquad + \mathbf{1}_{\{\zeta^*<\zeta_k^n\}}(1+\eta_{\alpha'})\Big] + \varepsilon,
\end{aligned} \tag{7.61}$$

where we used the fact that

$$\begin{aligned}
\mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap A_{n,k}^\alpha}\mathcal{Z}_{\zeta_k^n} &= \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n} - \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap(\{\zeta_k^n<T\}\setminus A_{n,k}^\alpha)}\mathcal{Z}_{\zeta_k^n}\\
&\le \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}\cap(\mathfrak{O}_\delta^\alpha)^c}\mathcal{Z}_*\\
&\le \mathbf{1}_{\{\zeta_k^n\le\zeta^*\}\cap\{\zeta_k^n<T\}}\mathcal{Z}_{\zeta_k^n} + \mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}\mathcal{Z}_*.
\end{aligned}$$

Since $\lim_{k\to\infty}\zeta_k^n = \zeta^n \le (\tau^n\vee t)^{t,\omega} < (\tau^*)^{t,\omega} = \zeta^* \le T$ by (7.38), letting $k\to\infty$ in (7.61), using the continuity of $\overline{Z}$ (Proposition 4.2), and applying the dominated convergence theorem again yields that

$$\begin{aligned}
\overline{Z}_t(\omega) &\le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{\zeta^n} + \mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}(\mathcal{Z}_* + \mathcal{Y}_*)\big] + \varepsilon\\
&= \mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{(\nu\wedge(\tau^n\vee t))^{t,\omega}} + \mathbf{1}_{(\mathfrak{O}_\delta^\alpha)^c}(\mathcal{Z}_* + \mathcal{Y}_*)\big] + \varepsilon.
\end{aligned}$$

Since $\tau^* = \lim_{n\to\infty}\!\uparrow\tau^n$ and $\cup_{\alpha\in\mathbb{N}}\mathfrak{O}_\delta^\alpha = \Omega^t$, letting $n\to\infty$, letting $\alpha\to\infty$, and then letting $\varepsilon\to 0$, we can deduce from the continuity of $\overline{Z}$, the dominated convergence theorem, and (7.38) that

$$\overline{Z}_t^*(\omega) = \overline{Z}_t(\omega) \le \mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{(\nu\wedge(\tau^*\vee t))^{t,\omega}}\big] = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Z}_{(\nu\wedge\tau^*)^{t,\omega}}\big] = \mathbb{E}_{\mathbb{P}}\Big[\big(\overline{Z}_\nu^*\big)^{t,\omega}\Big],$$

where we used the fact that for any $\widetilde{\omega} \in \Omega^t$

$$\mathcal{Z}_{(\nu\wedge\tau^*)^{t,\omega}}(\widetilde{\omega}) = \overline{Z}^{t,\omega}\big((\nu\wedge\tau^*)^{t,\omega}(\widetilde{\omega}),\widetilde{\omega}\big) = \overline{Z}\big((\nu\wedge\tau^*)(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$$
$$= \big(\overline{Z}_{\nu\wedge\tau^*}\big)(\omega\otimes_t\widetilde{\omega}) = \big(\overline{Z}^*_\nu\big)(\omega\otimes_t\widetilde{\omega}) = \big(\overline{Z}^*_\nu\big)^{t,\omega}(\widetilde{\omega}).$$

Eventually, taking infimum over $\mathbb{P} \in \mathcal{P}(t,\omega)$ yields (7.40), which together with (7.39) and (7.37) shows that $\overline{Z}^*$ is an $\underline{\mathscr{E}}$-martingale. In particular, taking $(t,\omega,\nu) = (0,\mathbf{0},T)$ yields that

$$\inf_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_\mathbb{P}\big[Y_\tau\big] = \overline{Z}_0 = \underline{\mathscr{E}}_0\big[\,\overline{Z}^*_T\big] = \underline{\mathscr{E}}_0\big[\overline{Z}_{\tau^*}\big] = \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}\big[\,\overline{Z}_{\tau^*}\big]$$
$$= \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}\big[\,Y_{\tau^*}\big] \le \sup_{\tau\in\mathcal{T}}\inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}[Y_\tau] \le \inf_{\mathbb{P}\in\mathcal{P}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_\mathbb{P}\big[Y_\tau\big]. \qquad \square$$

7.4. Proofs of the results in section 6.

Proof of Proposition 6.2. Fix $(t,\omega) \in [0,T]\times\Omega$ and $\mu \in \mathcal{U}_t$. Let us set $(\mathbb{P},\mathfrak{p},\mathcal{X}) = (\mathbb{P}^{t,\omega,\mu},\mathfrak{p}^{t,\omega,\mu},X^{t,\omega,\mu})$. Similar to (3.6) of [28] and Lemma 2.2 of [35] (or see [6] for details), one can find an $\mathbf{F}^t$-progressively measurable process $\mathcal{W} = W^{t,\omega,\mu}$ such that for all $\widetilde{\omega} \in \Omega^t$ except on a $\mathbb{P}^t_0$-null set $\mathcal{N}_\mathcal{X}$

$$B^t_s(\widetilde{\omega}) = \mathcal{W}_s\big(\mathcal{X}(\widetilde{\omega})\big) \quad \forall\, s \in [t,T], \tag{7.62}$$

which implies that for any $\widetilde{\omega} \in \mathcal{N}^c_\mathcal{X}$, $\mathcal{X}_s(\widetilde{\omega}) = \mathcal{X}_s(\mathcal{W}(\mathcal{X}(\widetilde{\omega})))$ $\forall\, s \in [t,T]$. It follows that for any $\widetilde{\omega}' \in A_\mathcal{X} \stackrel{\triangle}{=} \{\widetilde{\omega}' \in \Omega^t : \exists\,\widetilde{\omega} \in \mathcal{N}^c_\mathcal{X}$ such that $\widetilde{\omega}' = \mathcal{X}(\widetilde{\omega})\} = \{\widetilde{\omega}' \in \Omega^t : \mathcal{N}^c_\mathcal{X} \cap \mathcal{X}^{-1}(\widetilde{\omega}') \neq \emptyset\}$, one has

$$B^t_s(\widetilde{\omega}') = \mathcal{X}_s\big(\mathcal{W}(\widetilde{\omega}')\big) \quad \forall\, s \in [t,T]. \tag{7.63}$$

As $A^c_\mathcal{X} = \{\widetilde{\omega}' \in \Omega^t : \mathcal{X}^{-1}(\widetilde{\omega}') \subset \mathcal{N}_\mathcal{X}\}$, we see that $\mathcal{X}^{-1}(A^c_\mathcal{X}) \subset \mathcal{N}_\mathcal{X}$, i.e., $\mathcal{X}^{-1}(A^c_\mathcal{X}) \in \overline{\mathscr{N}}^t \subset \overline{\mathcal{F}}^t_T$. So $A^c_\mathcal{X} \in \mathcal{G}^\mathcal{X}_T = \{A \subset \Omega^t : \mathcal{X}^{-1}(A) \in \overline{\mathcal{F}}^t_T\}$ with $\mathfrak{p}(A^c_\mathcal{X}) = \mathbb{P}^t_0(\mathcal{X}^{-1}(A^c_\mathcal{X})) = 0$, namely, $A^c_\mathcal{X}$ is a $\mathfrak{p}$-null set. (It is worth pointing out that $A^c_\mathcal{X}$ may not belong to $\mathcal{F}^t_T$ though $\mathcal{X}^{-1}(A^c_\mathcal{X}) \in \overline{\mathcal{F}}^t_T$. In general, the inverse conclusion of (6.4) may not be true.) Since

$$A_\mathcal{X} = \{\widetilde{\omega}' \in \Omega^t : \exists\,\widetilde{\omega} \in \mathcal{N}^c_\mathcal{X} \text{ such that } \widetilde{\omega}' = \mathcal{X}(\widetilde{\omega})\} \subset \{\widetilde{\omega}' \in \Omega^t : \mathcal{W}_\cdot(\widetilde{\omega}') \in \Omega^t\} \tag{7.64}$$

by (7.62), the process $\mathcal{W}$ has $\mathfrak{p}$-a.s. continuous paths starting from 0.

One can also deduce from (7.62) that the distribution of $\mathcal{W}$ under $\mathfrak{p}$ is identical to that of B^t under $\mathbb{P}^t_0$. To wit, $\mathcal{W}$ is a Brownian motion on Ω^t under $\mathfrak{p}$. Then the corresponding augmented Brownian filtration

$$\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s \stackrel{\triangle}{=} \sigma\Big(\mathcal{F}^\mathcal{W}_s \cup \mathscr{N}^{\mathcal{W},\mathfrak{p}}\Big), \quad s \in [t,T] \tag{7.65}$$

is right-continuous, where $\mathscr{N}^{\mathcal{W},\mathfrak{p}} \stackrel{\triangle}{=} \{\mathcal{N}' \subset \Omega^t : \mathcal{N}' \subset A$ for some $A \in \mathcal{F}^\mathcal{W}_T$ with $\mathfrak{p}(A) = 0\}$.

To demonstrate the right-continuity of $\mathbf{F}^\mathbb{P}$, we first show that of $\mathbf{F}^{\mathcal{W},\mathbb{P}}$: Since $\mathcal{F}^\mathcal{W}_T \subset \mathcal{F}^t_T$ by the $\mathbf{F}^t$-adaptedness of $\mathcal{W}$, we see from Lemma A.3(1) that $\mathscr{N}^{\mathcal{W},\mathfrak{p}} = \{\mathcal{N}' \subset \Omega^t : \mathcal{N}' \subset A$ for some $A \in \mathcal{F}^\mathcal{W}_T$ with $\mathbb{P}(A) = 0\} \subset \{\mathcal{N}' \subset \Omega^t : \mathcal{N}' \subset A$ for some $A \in \mathcal{F}^t_T$ with $\mathbb{P}(A) = 0\} = \mathscr{N}^\mathbb{P}$. It follows that $\sigma(\widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s \cup \mathscr{N}^\mathbb{P}) = \sigma(\mathcal{F}^\mathcal{W}_s \cup \mathscr{N}^\mathbb{P}) = \mathcal{F}^{\mathcal{W},\mathbb{P}}_s$ $\forall\, s \in [t,T]$. Similar to Problem 2.7.3 of [20], one can show that

$$\mathcal{F}^{\mathcal{W},\mathbb{P}}_s = \Big\{A \subset \Omega^t : A\Delta\widetilde{A} \in \mathscr{N}^\mathbb{P} \text{ for some } \widetilde{A} \in \widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_s\Big\} \quad \forall\, s \in [t,T]. \tag{7.66}$$

Let $s \in [t,T)$ and $A \in \mathcal{F}^{\mathcal{W},\mathbb{P}}_{s+} \stackrel{\triangle}{=} \cap_{s'\in(s,T]}\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s'}$. For any $n \geq n_s \stackrel{\triangle}{=} \lceil \frac{1}{T-s} \rceil$, as $A \in \mathcal{F}^{\mathcal{W},\mathbb{P}}_{s+1/n}$, there exists $A_n \in \widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_{s+1/n}$ such that $A\Delta A_n \in \mathscr{N}^{\mathbb{P}}$. By (7.65), $\widetilde{A} \stackrel{\triangle}{=} \cap_{n\geq n_s}\cup_{i\geq n}A_i \in \widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_{s+} = \widetilde{\mathcal{F}}^{\mathcal{W},\mathfrak{p}}_{s}$. Since $\widetilde{A}\backslash A \subset \cap_{n\geq n_s}\cup_{i\geq n}(A_i\backslash A) \subset \cap_{n\geq n_s}\cup_{i\geq n}(A\Delta A_i)$ and since $A\backslash\widetilde{A} = \cup_{n\geq n_s}\cap_{i\geq n}(A\backslash A_i) \subset \cup_{n\geq n_s}\cap_{i\geq n}(A\Delta A_i)$, we see that $A\Delta\widetilde{A} \subset \cup_{n\geq n_s}(A\Delta A_n) \in \mathscr{N}^{\mathbb{P}}$, namely, $A \in \mathcal{F}^{\mathcal{W},\mathbb{P}}_{s}$ by (7.66). So $\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s+} = \mathcal{F}^{\mathcal{W},\mathbb{P}}_{s}$, which shows that $\{\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s}\}_{s\in[t,T]}$ is also a right-continuous filtration.

It remains to show that $\mathbf{F}^{\mathbb{P}}$ is exactly $\mathbf{F}^{\mathcal{W},\mathbb{P}}$ and thus $\mathbb{P} \in \mathfrak{P}_t$: Fix $s \in [t,T]$. Since $\mathcal{W}$ is $\mathbf{F}^t$-adapted, it is clear that $\mathcal{F}^{\mathcal{W},\mathbb{P}}_{s} = \sigma(\mathcal{F}^{\mathcal{W}}_{s} \cup \mathscr{N}^{\mathbb{P}}) \subset \sigma(\mathcal{F}^{t}_{s} \cup \mathscr{N}^{\mathbb{P}}) = \mathcal{F}^{\mathbb{P}}_{s}$. To see the reverse inclusion, let $r \in [t,s]$ and $\mathcal{E} \in \mathscr{B}(\mathbb{R}^d)$. We can deduce from (7.62) that $\{\widetilde{\omega} \in \Omega^t : B^t_r(\widetilde{\omega}) \in \mathcal{E}\}\,\Delta\,\{\widetilde{\omega} \in \Omega^t : \mathcal{W}_r(\mathcal{X}(\widetilde{\omega})) \in \mathcal{E}\} \subset \mathcal{N}_{\mathcal{X}} \in \overline{\mathscr{N}}^t$, which shows that $(B^t_r)^{-1}(\mathcal{E}) \in \widehat{\Lambda}_s \stackrel{\triangle}{=} \{A \subset \Omega^t : A\Delta\widetilde{A} \in \overline{\mathscr{N}}^t \text{ for some } \widetilde{A} \in \mathcal{X}^{-1}(\mathcal{F}^{\mathcal{W}}_{s})\}$. As $\mathcal{X}^{-1}(\mathcal{F}^{\mathcal{W}}_{s})$ is a σ-field of Ω^t, similar to Problem 2.7.3 of [20], $\widehat{\Lambda}_s$ forms a σ-field of Ω^t. Then $\mathcal{F}^t_s \subset \widehat{\Lambda}_s$. Clearly, $\overline{\mathscr{N}}^t \subset \widehat{\Lambda}_s$, so we further have $\overline{\mathcal{F}}^t_s \subset \widehat{\Lambda}_s$. For any $A \in \mathcal{F}^{\mathbb{P}}_{s}$, Lemma A.3(1) shows that $\mathcal{X}^{-1}(A) \in \overline{\mathcal{F}}^t_s \subset \widehat{\Lambda}_s$, i.e., for some $\widetilde{A} \in \mathcal{F}^{\mathcal{W}}_{s} \subset \mathcal{F}^t_s$, one has $\mathcal{X}^{-1}(A\,\Delta\,\widetilde{A}) = (\mathcal{X}^{-1}(A))\Delta(\mathcal{X}^{-1}(\widetilde{A})) \in \overline{\mathscr{N}}^t$. As $A\,\Delta\,\widetilde{A} \in \mathcal{F}^{\mathbb{P}}_{s} \subset \mathcal{F}^{\mathbb{P}}_{T}$, applying Lemma A.3(1) again yields that $\mathbb{P}(A\,\Delta\,\widetilde{A}) = \mathfrak{p}(A\,\Delta\,\widetilde{A}) = \mathbb{P}^t_0(\mathcal{X}^{-1}(A\,\Delta\,\widetilde{A})) = 0$, i.e., $A\,\Delta\,\widetilde{A} \in \mathscr{N}^{\mathbb{P}}$. It follows that $A = \widetilde{A}\,\Delta\,(A\,\Delta\,\widetilde{A}) \in \mathcal{F}^{\mathcal{W},\mathbb{P}}_{s}$. Hence, $\mathcal{F}^{\mathbb{P}}_{s} = \mathcal{F}^{\mathcal{W},\mathbb{P}}_{s}$. $\square$

Proof of Lemma 6.1. Fix $(t,\omega) \in [0,T] \times \Omega$ and $\mu \in \mathcal{U}_t$. We set $(\mathcal{X},\mathbb{P}) = (X^{t,\omega,\mu},\mathbb{P}^{t,\omega,\mu})$. Given $\widetilde{\omega} \in \Omega^t$, (3.1) shows

$$\begin{aligned}\big|Y^{t,\mathbf{0}}_r(\mathcal{X}(\widetilde{\omega})) - Y_r(\mathbf{0})\big| &= \big|Y_r(\mathbf{0}\otimes_t \mathcal{X}(\widetilde{\omega})) - Y_r(\mathbf{0})\big| \leq \rho_0\big(\|\mathbf{0}\otimes_t \mathcal{X}(\widetilde{\omega})\|_{0,r}\big)\\ &\leq \kappa\big(1+\|\mathcal{X}(\widetilde{\omega})\|^{\varpi}_{t,r}\big) \ \ \forall\, r \in [t,T].\end{aligned}$$

It follows that $Y^{t,\mathbf{0}}_*(\mathcal{X}(\widetilde{\omega})) = \sup_{r\in[t,T]}|Y^{t,\mathbf{0}}_r(\mathcal{X}(\widetilde{\omega}))| \leq \kappa(1+\|\mathcal{X}(\widetilde{\omega})\|^{\varpi}_{t,T}) + \mathfrak{m}_Y$, where $\mathfrak{m}_Y \stackrel{\triangle}{=} \sup_{r\in[t,T]}|Y_r(\mathbf{0})| < \infty$ by (7.59). Since $\phi(x) = x\ln^+(x) \leq x^2$ for all $x \in [0,\infty)$, (2.2) implies that

$$\begin{aligned}\phi\big(Y^{t,\mathbf{0}}_*(\mathcal{X}(\widetilde{\omega}))\big) &\leq 4\phi\big(\kappa\big(1+\|\mathcal{X}(\widetilde{\omega})\|^{\varpi}_{t,T}\big)\big) + 4\phi(\mathfrak{m}_Y) + \phi(4)\\ &\leq 8\kappa^2\big(1+\|\mathcal{X}(\widetilde{\omega})\|^{2\varpi}_{t,T}\big) + 4\phi(\mathfrak{m}_Y) + \phi(4).\end{aligned}$$

Then we can deduce from (6.2) that $\mathbb{E}_{\mathbb{P}}[\phi(Y^{t,\mathbf{0}}_*)] = \mathbb{E}_t[\phi(Y^{t,\mathbf{0}}_*(\mathcal{X}))] \leq 8\kappa^2(1+\varphi_{2\varpi}(\|\omega\|_{0,t})\,T^{\varpi}) + 4\phi(\mathfrak{m}_Y) + \phi(4) < \infty$. Namely, $Y^{t,\mathbf{0}} \in \widehat{\mathbb{D}}(\mathbf{F}^t,\mathbb{P})$, which together with Proposition 6.2 shows that $\mathbb{P} = \mathbb{P}^{t,\omega,\mu} \in \mathfrak{P}^Y_t$. $\square$

Proof of Proposition 6.3. Fix $0 \leq t < s \leq T$, $\omega \in \Omega$ and $\mu \in \mathcal{U}_t$. We will denote $(\mathbb{P}^{t,\omega,\mu},\mathfrak{p}^{t,\omega,\mu},X^{t,\omega,\mu},W^{t,\omega,\mu})$ by $(\mathbb{P},\mathfrak{p},\mathcal{X},\mathcal{W})$. For any $r \in [t,T]$, (6.4) and Lemma A.3(2) show that $\mathfrak{F}_r \stackrel{\triangle}{=} \sigma(\mathcal{F}^t_r \cup \mathscr{N}^{\mathfrak{p}}) \subset \mathcal{G}^{\mathcal{X}}_r$.

Let $A_{\mathcal{X}}$ as defined in (7.63). As $A^c_{\mathcal{X}} \in \mathscr{N}^{\mathfrak{p}}$, we see from the $\mathbf{F}^t$-adaptedness of $\mathcal{W}$ and (7.64) that the process $\widetilde{\mathcal{W}}_r(\widetilde{\omega}) \stackrel{\triangle}{=} \mathbf{1}_{\{\widetilde{\omega}\in A_{\mathcal{X}}\}}\mathcal{W}_r(\widetilde{\omega})$ for all $(r,\widetilde{\omega}) \in [t,T]\times\Omega^t$ is adapted to the filtration $\{\mathfrak{F}_r\}_{r\in[t,T]}$ and all its paths belong to Ω^t. Given $r \in [t,T]$, for any $r' \in [t,r]$ and $\mathcal{E} \in \mathscr{B}(\mathbb{R}^d)$, an analogy to (6.3) shows that $\widetilde{\mathcal{W}}^{-1}((B^t_{r'})^{-1}(\mathcal{E})) = \{\widetilde{\omega} \in \Omega^t : \widetilde{\mathcal{W}}(\widetilde{\omega}) \in (B^t_r)^{-1}(\mathcal{E})\} = \{\widetilde{\omega} \in \Omega^t : \widetilde{\mathcal{W}}_{r'}(\widetilde{\omega}) \in \mathcal{E}\} \in \mathcal{F}^{\widetilde{\mathcal{W}}}_r$. Thus, $(B^t_{r'})^{-1}(\mathcal{E}) \in \Lambda_r \stackrel{\triangle}{=} \{A \subset \Omega^t : \widetilde{\mathcal{W}}^{-1}(A) \in \mathcal{F}^{\widetilde{\mathcal{W}}}_r\}$, which is clearly a σ-field of Ω^t. It follows that $\mathcal{F}^t_r \subset \Lambda_r$, i.e.,

$$\widetilde{\mathcal{W}}^{-1}(A) \in \mathcal{F}^{\widetilde{\mathcal{W}}}_r \subset \mathfrak{F}_r \quad \forall\, A \in \mathcal{F}^t_r,\ \ \forall\, r\in[t,T]. \tag{7.67}$$

For any $\widetilde{\omega} \in \mathcal{N}_{\mathcal{X}}^c$, set $\widetilde{\omega}' \stackrel{\triangle}{=} \mathcal{X}(\widetilde{\omega})$. As $\widetilde{\omega} \in \mathcal{X}^{-1}(\widetilde{\omega}') \cap \mathcal{N}_{\mathcal{X}}^c$, we see that $\widetilde{\omega}' = \mathcal{X}(\widetilde{\omega}) \in A_{\mathcal{X}}$. Then (7.62) shows that

$$\widetilde{\omega} = B^t(\widetilde{\omega}) = \mathcal{W}\big(\mathcal{X}(\widetilde{\omega})\big) = \widetilde{\mathcal{W}}\big(\mathcal{X}(\widetilde{\omega})\big) \quad \forall\, \widetilde{\omega} \in \mathcal{N}_{\mathcal{X}}^c. \tag{7.68}$$

Given $\mathcal{N}' \in \overline{\mathscr{N}}^t$, there exists an $A \in \mathcal{F}_T^t$ with $\mathbb{P}_0^t(A) = 0$ such that $\mathcal{N}' \subset A$. Since $\widetilde{\mathcal{W}}^{-1}(A) \in \mathfrak{F}_T \subset \mathcal{G}_T^{\mathcal{X}}$ by (7.67), one can deduce from (7.68) that $\mathfrak{p}(\widetilde{\mathcal{W}}^{-1}(A)) = \mathbb{P}_0^t(\mathcal{X}^{-1}(\widetilde{\mathcal{W}}^{-1}(A))) = \mathbb{P}_0^t\{\widetilde{\mathcal{W}}(\mathcal{X}) \in A\} = \mathbb{P}_0^t(A) = 0$, which implies that $\widetilde{\mathcal{W}}^{-1}(A) \in \mathscr{N}^{\mathfrak{p}}$ and thus $\widetilde{\mathcal{W}}^{-1}(\mathcal{N}') \in \mathscr{N}^{\mathfrak{p}}$. Hence, it holds for any $r \in [t,T]$ that $\overline{\mathscr{N}}^t \in \widetilde{\Lambda}_r \stackrel{\triangle}{=} \{A' \subset \Omega^t : \widetilde{\mathcal{W}}^{-1}(A') \in \mathfrak{F}_r\}$. Clearly $\widetilde{\Lambda}_r$ is a σ-field of Ω^t, and then we see from (7.67) that $\overline{\mathcal{F}}_r^t \subset \widetilde{\Lambda}_r$, i.e.,

$$\widetilde{\mathcal{W}}^{-1}(A') \in \mathfrak{F}_r \quad \forall\, A' \in \overline{\mathcal{F}}_r^t,\ \forall\, r \in [t,T]. \tag{7.69}$$

(1) *Using similar arguments to those that lead to* (3.8) *of* [28], *we can deduce from* (7.62) *and* (7.63) *that* $\mathfrak{p}$*-a.s.* $\widetilde{\omega} \in \Omega^t$, $\mathbb{P}^{s,\widetilde{\omega}} = \mathbb{P}^{s,\omega\otimes_t\widetilde{\omega},\mu^{s,\mathcal{W}(\widetilde{\omega})}} \in \mathcal{P}(s,\omega\otimes_t\widetilde{\omega})$, *and thus the probability class* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies* (P1); *see* [6] *for details.*

(2) *We next show that the probability class* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies* (P2). *Given* $\delta \in \mathbb{Q}_+$ *and* $\lambda \in \mathbb{N}$, *let* $\{\mathcal{A}_j\}_{j=0}^{\lambda}$ *be a* $\mathcal{F}_s^t$*-partition of* Ω^t *such that for* $j = 1,\ldots,\lambda$, $\mathcal{A}_j \subset O_\delta^s(\widetilde{\omega}_j)$ *for some* $\widetilde{\omega}_j \in \Omega^t$, *and let* $\{\mu^j\}_{j=1}^{\lambda} \subset \mathcal{U}_s$. *We will paste these* $\mathcal{U}_s$*-controls* $\{\mu^j\}_{j=1}^{\lambda}$ *with the given* $\mathcal{U}_t$*-control* μ *to form a new* $\mathcal{U}_t$*-control* $\widehat{\mu}$; *see* (7.71) *below. Then we will use the uniqueness of controlled SDE* (6.1), *the continuity* (3.1) *of* Y, *and the estimates* (6.2) *of* $X^{t,\omega,\mu}$ *to show that* $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ *satisfies the conditions* (P2)(i)–(ii).

(2a) We first claim that

$$\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j \in \overline{\mathscr{N}}^t \quad \forall\, j = 0,\ldots,\lambda. \tag{7.70}$$

Given $j = 1,\ldots,\lambda$, (6.4) shows that $\mathcal{A}_j^{\mathcal{X}} \stackrel{\triangle}{=} \mathcal{X}^{-1}(\mathcal{A}_j) \in \overline{\mathcal{F}}_s^t$. So there exists an $A_j \in \mathcal{F}_s^t$ such that $\mathcal{A}_j^{\mathcal{X}} \,\Delta\, A_j \in \overline{\mathscr{N}}^t$ (see, e.g., Problem 2.7.3 of [20]). Set $\widetilde{A}_j \stackrel{\triangle}{=} A_j \backslash \cup_{j'<j} A_{j'} \in \mathcal{F}_s^t$. As $\{\mathcal{A}_j^{\mathcal{X}}\}_{j=0}^{\lambda}$ is a partition of Ω^t with $\mathcal{A}_0^{\mathcal{X}} \stackrel{\triangle}{=} \mathcal{X}^{-1}(\mathcal{A}_0) \in \overline{\mathcal{F}}_s^t$, an analogy to (7.17) shows that $\mathcal{A}_j^{\mathcal{X}} \backslash \widetilde{A}_j \subset \cup_{j'\le j}(\mathcal{A}_{j'}^{\mathcal{X}} \Delta A_{j'}) \in \overline{\mathscr{N}}^t$. Also, it is clear that $\widetilde{A}_j \backslash \mathcal{A}_j^{\mathcal{X}} \subset A_j \backslash \mathcal{A}_j^{\mathcal{X}} \subset \mathcal{A}_j^{\mathcal{X}} \,\Delta\, A_j \in \overline{\mathscr{N}}^t$. Thus, $\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j \in \overline{\mathscr{N}}^t$. On the other hand, let $\widetilde{A}_0 \stackrel{\triangle}{=} (\cup_{j=1}^{\lambda} \widetilde{A}_j)^c \in \mathcal{F}_s^t$. As $\mathcal{A}_0^{\mathcal{X}} = (\cup_{j=1}^{\lambda} \mathcal{A}_j^{\mathcal{X}})^c$, one can deduce that

$$\begin{aligned}
\widetilde{A}_0 \backslash \mathcal{A}_0^{\mathcal{X}} &= \widetilde{A}_0 \cap \left(\bigcup_{j=1}^{\lambda} \mathcal{A}_j^{\mathcal{X}}\right) = \bigcup_{j=1}^{\lambda} \big(\widetilde{A}_0 \cap \mathcal{A}_j^{\mathcal{X}}\big) \\
&\subset \bigcup_{j=1}^{\lambda} \big(\widetilde{A}_j^c \cap \mathcal{A}_j^{\mathcal{X}}\big) \subset \bigcup_{j=1}^{\lambda} \big(\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j\big) \in \overline{\mathscr{N}}^t \\
\text{and } \mathcal{A}_0^{\mathcal{X}} \backslash \widetilde{A}_0 &= \mathcal{A}_0^{\mathcal{X}} \cap \left(\bigcup_{j=1}^{\lambda} \widetilde{A}_j\right) = \bigcup_{j=1}^{\lambda} \big(\mathcal{A}_0^{\mathcal{X}} \cap \widetilde{A}_j\big) \\
&\subset \bigcup_{j=1}^{\lambda} \big((\mathcal{A}_j^{\mathcal{X}})^c \cap \widetilde{A}_j\big) \subset \bigcup_{j=1}^{\lambda} \big(\mathcal{A}_j^{\mathcal{X}} \Delta \widetilde{A}_j\big) \in \overline{\mathscr{N}}^t.
\end{aligned}$$

So $\mathcal{A}_0^{\mathcal{X}} \Delta \widetilde{A}_0 \in \overline{\mathscr{N}}^t$, proving the claim (7.70).

One can easily deduce from $\mathbf{F}^t$-progressive measurability of μ and $\mathbf{F}^s$-progressive measurability of μ^j that

$$\begin{aligned}\widehat{\mu}_r(\widetilde{\omega}) &\stackrel{\triangle}{=} \mathbf{1}_{\{r\in[t,s)\}}\mu_r(\widetilde{\omega})\\ &\quad + \mathbf{1}_{\{r\in[s,T]\}}\left(\mathbf{1}_{\{\widetilde{\omega}\in\widetilde{A}_0\}}\mu_r(\widetilde{\omega}) + \sum_{j=1}^{\lambda}\mathbf{1}_{\{\widetilde{\omega}\in\widetilde{A}_j\}}\mu_r^j(\Pi_s^t(\widetilde{\omega}))\right) \\ &\qquad \forall\,(r,\widetilde{\omega})\in[t,T]\times\Omega^t\end{aligned} \tag{7.71}$$

defines an $\mathbf{F}^t$-progressively measurable process, and thus $\widehat{\mu}\in\mathcal{U}_t$ (see [6] for details). Let $(r,\widetilde{\omega})\in[s,T]\times\widetilde{A}_j$ for some $j=0,\dots,\lambda$. For any $\widehat{\omega}\in\Omega^s$, since $\widetilde{\omega}\otimes_s\widehat{\omega}\in\widetilde{A}_j$ by Lemma 2.1, we see from (7.71) that

$$\begin{aligned}\widehat{\mu}_r^{s,\widetilde{\omega}}(\widehat{\omega}) &= \widehat{\mu}_r\big(\widetilde{\omega}\otimes_s\widehat{\omega}\big)\\ &= \begin{cases}\mu_r\big(\widetilde{\omega}\otimes_s\widehat{\omega}\big)=\mu_r^{s,\widetilde{\omega}}(\widehat{\omega}) & \text{if } j=0;\\ \mu_r^j\big(\Pi_s^t(\widetilde{\omega}\otimes_s\widehat{\omega})\big)=\mu_r^j(\widehat{\omega}) & \text{if } j=1,\dots,\lambda.\end{cases}\end{aligned} \tag{7.72}$$

(2b) *Next, we use the uniqueness of controlled SDE* (6.1) *to show that* $\widehat{\mu}=\mu$ *implies* $\widehat{\mathcal{X}}\stackrel{\triangle}{=}X^{t,\omega,\widehat{\mu}}=\mathcal{X}$ *over* $([t,s]\times\Omega^t)\cup([s,T]\times\widetilde{A}_0)$. *It follows that* $\widehat{\mathbb{P}}\stackrel{\triangle}{=}\mathbb{P}^{t,\omega,\widehat{\mu}}$ *satisfies* (P2)(i) *and the first part of* (P2)(ii).

Since $\{X_r^{t,\omega,\mu}\}_{r\in[t,s]}$ and $\{X_r^{t,\omega,\widehat{\mu}}\}_{r\in[t,s]}$ satisfy the same SDE, $X_r = \int_t^r b^{t,\omega}(r',X,\mu_{r'})\,dr' + \int_t^r \mu_{r'}\,dB_{r'}^t$, $r\in[t,s]$, the uniqueness of solution to such a SDE shows that except on an $\widehat{\mathcal{N}}\in\overline{\mathscr{N}}^t$

$$\mathcal{X}_r = X_r^{t,\omega,\mu} = X_r^{t,\omega,\widehat{\mu}} = \widehat{\mathcal{X}}_r \quad \forall\, r\in[t,s]. \tag{7.73}$$

Given $A\in\mathcal{F}_s^t$, we claim that $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c\cap(\widehat{\mathcal{X}}^{-1}(A))^c=\emptyset$: Without loss of generality, assume that $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c$ is not empty and contains some $\widetilde{\omega}$. By (7.73) and Lemma 2.1, $\widehat{\mathcal{X}}(\widetilde{\omega})\in\mathcal{X}(\widetilde{\omega})\otimes_s\Omega^s\subset A$, i.e., $\widetilde{\omega}\in\widehat{\mathcal{X}}^{-1}(A)$. So $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c\subset\widehat{\mathcal{X}}^{-1}(A)$, which shows that $\mathcal{X}^{-1}(A)\cap\widehat{\mathcal{N}}^c\cap(\widehat{\mathcal{X}}^{-1}(A))^c=\emptyset$, proving the claim. It then follows that $\mathcal{X}^{-1}(A)\cap(\widehat{\mathcal{X}}^{-1}(A))^c\subset\widehat{\mathcal{N}}$. Exchanging the role of $\mathcal{X}^{-1}(A)$ and $\widehat{\mathcal{X}}^{-1}(A)$ gives that $\widehat{\mathcal{X}}^{-1}(A)\cap(\mathcal{X}^{-1}(A))^c\subset\widehat{\mathcal{N}}$. Hence,

$$\mathcal{X}^{-1}(A)\Delta\widehat{\mathcal{X}}^{-1}(A)\in\overline{\mathscr{N}}^t \quad \forall\, A\in\mathcal{F}_s^t. \tag{7.74}$$

Multiplying $\mathbf{1}_{\widetilde{A}_0}$ to the SDE (6.1) for $\mathcal{X}=X^{t,\omega,\mu}$ and $\widehat{\mathcal{X}}=X^{t,\omega,\widehat{\mu}}$ over period $[s,T]$ yields that for any $r\in[s,T]$

$$\begin{aligned}\mathbf{1}_{\widetilde{A}_0}(\mathcal{X}_r-\mathcal{X}_s) &= \int_s^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',\mathbf{1}_{\widetilde{A}_0}\mathcal{X},\mu_{r'})\,dr' + \int_s^r \mathbf{1}_{\widetilde{A}_0}\mu_{r'}\,dB_{r'}^t \quad \text{and}\\ \mathbf{1}_{\widetilde{A}_0}(\widehat{\mathcal{X}}_r-\widehat{\mathcal{X}}_s) &= \int_s^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',\mathbf{1}_{\widetilde{A}_0}\widehat{\mathcal{X}},\widehat{\mu}_{r'})\,dr' + \int_s^r \mathbf{1}_{\widetilde{A}_0}\widehat{\mu}_{r'}\,dB_{r'}^t\\ &= \int_s^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r',\mathbf{1}_{\widetilde{A}_0}\widehat{\mathcal{X}},\mu_{r'})\,dr' + \int_s^r \mathbf{1}_{\widetilde{A}_0}\mu_{r'}\,dB_{r'}^t.\end{aligned}$$

By (7.73), $\{\mathbf{1}_{\widetilde{A}_0}\mathcal{X}_r\}_{r\in[s,T]}$ and $\{\mathbf{1}_{\widetilde{A}_0}\widehat{\mathcal{X}}_r\}_{r\in[s,T]}$ satisfy the same SDE:

$$X'_r = \mathbf{1}_{\widetilde{A}_0}\mathcal{X}_s + \int_t^r \mathbf{1}_{\widetilde{A}_0} b^{t,\omega}(r', X', \mu_{r'})\,dr' + \int_t^r \mathbf{1}_{\widetilde{A}_0}\mu_{r'}\,dB^t_{r'}, \quad r\in[s,T].$$

Similar to (6.1), this SDE admits a unique solution. So it holds $\mathbb{P}^t_0$-a.s. on $\widetilde{A}_0$ that

$$\mathcal{X}_r = \widehat{\mathcal{X}}_r \quad \forall\, r\in[s,T]. \tag{7.75}$$

Let $j = 1,\dots,\lambda$. Proposition 6.1, (7.73), and (7.72) show that for all $\widetilde{\omega}\in\widetilde{A}_j$ except on an $\mathcal{N}_j \in \overline{\mathscr{N}}^t$

$$\widehat{\mathcal{X}}^{s,\widetilde{\omega}} = X^{s,\omega\otimes_t\widehat{\mathcal{X}}(\widetilde{\omega}),\widehat{\mu}^{s,\widetilde{\omega}}} + \widehat{\mathcal{X}}_s(\widetilde{\omega}) = X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j} + \mathcal{X}_s(\widetilde{\omega}), \tag{7.76}$$

where we used the fact that $X^{s,\omega\otimes_t\widehat{\mathcal{X}}(\widetilde{\omega}),\widehat{\mu}^{s,\widetilde{\omega}}}$ depends only on $\omega\otimes_t \widehat{\mathcal{X}}(\widetilde{\omega})|_{[0,s]}$. Lemma 2.4, an analogy to (3.8) of [28], and the continuity of $\mathcal{X}$ imply that for all $\widetilde{\omega}\in\Omega^t$ except on an $\widehat{\mathcal{N}}'\in\overline{\mathscr{N}}^t$

$$\begin{aligned} &\widehat{\mathcal{N}}^{s,\widetilde{\omega}} \in \overline{\mathscr{N}}^s \\ &\quad\text{and}\quad \mathbb{P}^s_0\big\{\widehat{\omega}\in\Omega^s : \mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) = \mathcal{X}_r(\widetilde{\omega})\ \ \forall\, r\in[t,s]\big\} = 1. \end{aligned} \tag{7.77}$$

Set $\widetilde{\mathcal{N}}_j \overset{\triangle}{=} \mathcal{N}_j \cup \widehat{\mathcal{N}}' \in \overline{\mathscr{N}}^t$. Given $\widetilde{\omega}\in\widetilde{A}_j\cap\widetilde{\mathcal{N}}^c_j$, since $\{\widehat{\omega}\in\Omega^s : \mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) \neq \widehat{\mathcal{X}}_r(\widetilde{\omega}\otimes_s\widehat{\omega})$ for some $r\in[t,s]\} = \{\widehat{\omega}\in\Omega^s : \widetilde{\omega}\otimes_s\widehat{\omega}\in\widehat{\mathcal{N}}\} = \widehat{\mathcal{N}}^{s,\widetilde{\omega}} \in \overline{\mathscr{N}}^s$, we can deduce from (7.76) and (7.77) that for $\mathbb{P}^s_0$-a.s. $\widehat{\omega}\in\Omega^s$

$$\begin{aligned} \widehat{\mathcal{X}}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) &= \mathbf{1}_{\{r\in[t,s)\}}\mathcal{X}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) \\ &\quad + \mathbf{1}_{\{r\in[s,T]\}}\big(X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}_r(\widehat{\omega}) + \mathcal{X}_s(\widetilde{\omega})\big) \\ &= \mathbf{1}_{\{r\in[t,s)\}}\mathcal{X}_r(\widetilde{\omega}) + \mathbf{1}_{\{r\in[s,T]\}}\big(X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}_r(\widehat{\omega}) + \mathcal{X}_s(\widetilde{\omega})\big) \\ &= \big(\mathcal{X}(\widetilde{\omega})\otimes_s X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}(\widehat{\omega})\big)(r) \quad \forall\, r\in[t,T]. \end{aligned} \tag{7.78}$$

For any $A\in\mathcal{F}^t_T$, applying (7.74) with $A=\mathcal{A}_0$, we can deduce from (7.70), (7.73), and (7.75) that

$$\begin{aligned} \widehat{\mathbb{P}}(A\cap\mathcal{A}_0) &= \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A)\cap\widehat{\mathcal{X}}^{-1}(\mathcal{A}_0)\big) = \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A)\cap\mathcal{X}^{-1}(\mathcal{A}_0)\big) \\ &= \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A)\cap\widetilde{A}_0\big) = \mathbb{P}^t_0\big\{\widetilde{\omega}\in\widetilde{A}_0 : \widehat{\mathcal{X}}(\widetilde{\omega})\in A\big\} \\ &= \mathbb{P}^t_0\big\{\widetilde{\omega}\in\widetilde{A}_0 : \mathcal{X}(\widetilde{\omega})\in A\big\} = \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A)\cap\widetilde{A}_0\big) \\ &= \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A)\cap\mathcal{X}^{-1}(\mathcal{A}_0)\big) = \mathbb{P}(A\cap\mathcal{A}_0). \end{aligned}$$

On the other hand, for any $A\in\mathcal{F}^t_s$ and $j=1,\dots,\lambda$, applying (7.74) with $A = A\cap\mathcal{A}_j$ yields that

$$\widehat{\mathbb{P}}(A\cap\mathcal{A}_j) = \mathbb{P}^t_0\big(\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)\big) = \mathbb{P}^t_0\big(\mathcal{X}^{-1}(A\cap\mathcal{A}_j)\big) = \mathbb{P}(A\cap\mathcal{A}_j).$$

(2c) *Now, we will use the continuity* (3.1) *of* Y *and the estimates* (6.2) *of* $X^{t,\omega,\mu}$ *to verify* (3.3) *for* $\widehat{\mathbb{P}}$.

Given $j = 1, \dots, \lambda$, we set $(\mathbb{P}_j, \mathfrak{p}_j, \mathcal{X}^j, \mathcal{W}^j) \stackrel{\triangle}{=} (\mathbb{P}^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j}, \mathfrak{p}^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j}, X^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j}, W^{s,\omega\otimes_t\widetilde{\omega}_j,\mu^j})$. Let us fix $A \in \mathcal{F}^t_s$. By Proposition 2.1(2) and Remark 3.1(1), the shifted process $\mathcal{Y}_r \stackrel{\triangle}{=} Y^{t,\omega}_r$, $r \in [t,T]$ as defined in (7.7) is $\mathbf{F}^t$-adapted and its paths are all RCLL. Then (6.4) implies that $\mathcal{Y}(\widehat{\mathcal{X}})$ is an $\overline{\mathbf{F}}^t$-adapted process whose paths are all RCLL. Applying Lemma A.2(3) with $(\mathbb{P}, X) = (\mathbb{P}^t_0, B^t)$ shows that $\mathcal{Y}(\widehat{\mathcal{X}})$ has an $(\mathbf{F}^t, \mathbb{P}^t_0)$-version $\mathscr{Y}$, which is $\mathbf{F}^t$-progressively measurable process with $\mathcal{N}_Y \stackrel{\triangle}{=} \{\widetilde{\omega} \in \Omega^t : \mathscr{Y}_r(\widetilde{\omega}) \neq \mathcal{Y}_r(\widehat{\mathcal{X}}(\widetilde{\omega}))$ for some $r \in [t,T]\} \in \overline{\mathscr{N}}^t$. By Lemma 2.4, it holds for all $\widetilde{\omega} \in \Omega^t$ except on an $\widetilde{\mathcal{N}}_Y \in \overline{\mathscr{N}}^t$ that $\mathcal{N}^{s,\widetilde{\omega}}_Y \in \overline{\mathscr{N}}^s$.

Fix $\tau \in \mathcal{T}^t_s$ and set $\widehat{\tau} = \tau(\widehat{\mathcal{X}})$. For any $r \in [s,T]$, since $A_r \stackrel{\triangle}{=} \{\tau \le r\} \in \mathcal{F}^t_r$, (6.4) shows that

$$\begin{aligned}\{\widehat{\tau} \le r\} &= \big\{\widetilde{\omega} \in \Omega^t : \tau\big(\widehat{\mathcal{X}}(\widetilde{\omega})\big) \le r\big\} = \{\widetilde{\omega} \in \Omega^t : \widehat{\mathcal{X}}(\widetilde{\omega}) \in A_r\} \\ &= \widehat{\mathcal{X}}^{-1}(A_r) \in \overline{\mathcal{F}}^t_r, \text{ namely, } \widehat{\tau} \in \overline{\mathcal{T}}^t_s.\end{aligned}$$

For any $\widetilde{\omega} \in \mathcal{N}^c_Y$, we have

$$\mathscr{Y}(r,\widetilde{\omega}) = \mathcal{Y}\big(r, \widehat{\mathcal{X}}(\widetilde{\omega})\big) \quad \forall\, r \in [t,T]. \tag{7.79}$$

In particular, taking $r = \widehat{\tau}(\widetilde{\omega})$ gives that $\mathscr{Y}_{\widehat{\tau}}(\widetilde{\omega}) = \mathscr{Y}(\widehat{\tau}(\widetilde{\omega}), \widetilde{\omega}) = \mathcal{Y}(\widehat{\tau}(\widetilde{\omega}), \widehat{\mathcal{X}}(\widetilde{\omega})) = \mathcal{Y}(\tau(\widehat{\mathcal{X}}(\widetilde{\omega})), \widehat{\mathcal{X}}(\widetilde{\omega})) = \mathcal{Y}_\tau(\widehat{\mathcal{X}}(\widetilde{\omega}))$. So

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j} Y^{t,\omega}_\tau\big] = \mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}\mathcal{Y}_\tau\big] &= \mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathcal{Y}_\tau\big(\widehat{\mathcal{X}}\big)\Big] \\ &= \mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathscr{Y}_{\widehat{\tau}}\Big].\end{aligned} \tag{7.80}$$

Also, one can deduce from (7.79), Lemma 6.1, and (3.2) that

$$\mathbb{E}_t[\mathscr{Y}_*] = \mathbb{E}_t\big[\mathcal{Y}_*\big(\widehat{\mathcal{X}}\big)\big] = \mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathcal{Y}_*\big] = \mathbb{E}_{\widehat{\mathbb{P}}}\big[Y^{t,\omega}_*\big] < \infty. \tag{7.81}$$

Given $\varepsilon > 0$, similar to (7.16), there exists $\widehat{\tau}' \in \mathcal{T}^t_s$ such that $\mathbb{E}_t[|\mathscr{Y}_{\widehat{\tau}'} - \mathscr{Y}_{\widehat{\tau}}|] < \frac{\varepsilon}{2}$. Since $\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j) \in \overline{\mathcal{F}}^t_s$ by (6.4) and since $\mathscr{Y}_{\widehat{\tau}'} \in L^1(\mathcal{F}^t_T, \mathbb{P}^t_0)$ by (7.81), applying Lemma A.2(1) and (2.6) with $(\mathbb{P}, X, \xi) = (\mathbb{P}^t_0, B^t, \mathscr{Y}_{\widehat{\tau}'})$ as well as using (7.74) with $A = A\cap\mathcal{A}_j$, we can deduce from (7.80), Lemma 2.3, and (7.70) that

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j} Y^{t,\omega}_\tau\big] &\le \mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathscr{Y}_{\widehat{\tau}'}\Big] + \frac{\varepsilon}{2} \\ &= \mathbb{E}_t\Big[\mathbf{1}_{\widehat{\mathcal{X}}^{-1}(A\cap\mathcal{A}_j)}\mathbb{E}_t\big[\mathscr{Y}_{\widehat{\tau}'}\big|\overline{\mathcal{F}}^t_s\big]\Big] + \frac{\varepsilon}{2} \\ &= \mathbb{E}_t\Big[\mathbf{1}_{\mathcal{X}^{-1}(A\cap\mathcal{A}_j)}\mathbb{E}_t\big[\mathscr{Y}_{\widehat{\tau}'}\big|\mathcal{F}^t_s\big]\Big] + \frac{\varepsilon}{2} \\ &= \mathbb{E}_t\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A)\cap\mathcal{A}^{\mathcal{X}}_j\}}\mathbb{E}_s\big[(\mathscr{Y}_{\widehat{\tau}'})^{s,\widetilde{\omega}}\big]\Big] + \frac{\varepsilon}{2} \\ &= \mathbb{E}_t\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A)\cap\widetilde{A}_j\}}\mathbb{E}_s\big[(\mathscr{Y}_{\widehat{\tau}'})^{s,\widetilde{\omega}}\big]\Big] + \frac{\varepsilon}{2}.\end{aligned} \tag{7.82}$$

Let $\widetilde{\omega} \in \mathcal{A}^{\mathcal{X}}_j \cap \widetilde{A}_j \cap \widetilde{\mathcal{N}}^c_j \cap \widetilde{\mathcal{N}}^c_Y$. Corollary 2.1 shows that $\zeta_{\widetilde{\omega}} \stackrel{\triangle}{=} (\widehat{\tau}')^{s,\widetilde{\omega}} \in \mathcal{T}^s$. And one has

$$\begin{aligned}&\big\{\widehat{\omega} \in \Omega^s : \mathscr{Y}_r(\widetilde{\omega}\otimes_s\widehat{\omega}) \neq \mathcal{Y}_r\big(\widehat{\mathcal{X}}(\widetilde{\omega}\otimes_s\widehat{\omega})\big) \text{ for some } r \in [t,T]\big\} \\ &\qquad = \{\widehat{\omega} \in \Omega^s : \widetilde{\omega}\otimes_s\widehat{\omega} \in \mathcal{N}_Y\} = \mathcal{N}^{s,\widetilde{\omega}}_Y \in \overline{\mathscr{N}}^s.\end{aligned} \tag{7.83}$$

Similar to (7.62), it holds for all $\widehat{\omega} \in \Omega^s$ except on a $\mathbb{P}_0^s$-null set $\mathcal{N}_{\mathcal{X}^j}$ that

$$B_r^s(\widehat{\omega}) = \mathcal{W}_r^j\big(\mathcal{X}^j(\widehat{\omega})\big) \quad \forall\, r \in [s,T]. \tag{7.84}$$

Set $A_{\mathcal{X}^j} \stackrel{\triangle}{=} \{\widehat{\omega}' \in \Omega^s : \mathcal{N}^c_{\mathcal{X}^j}(\mathcal{X}^j)^{-1}(\widehat{\omega}') \neq \emptyset\}$ and $\mathfrak{F}_r^j \stackrel{\triangle}{=} \sigma(\mathcal{F}_r^s \cup \mathscr{N}^{\mathfrak{p}_j}) \subset \mathcal{G}_r^{\mathcal{X}^j}$ for all $r \in [s,T]$. Similar to $\widetilde{\mathcal{W}}$, the process $\widetilde{\mathcal{W}}_r^j(\widehat{\omega}) \stackrel{\triangle}{=} \mathbf{1}_{\{\widehat{\omega} \in A_{\mathcal{X}^j}\}}\mathcal{W}_r^j(\widehat{\omega})$ for all $(r,\widehat{\omega}) \in [s,T]\times\Omega^s$ is adapted to the filtration $\{\mathfrak{F}_r^j\}_{r\in[s,T]}$ and all its paths belong to Ω^s.

For any $\widehat{\omega} \in \Omega^s$ except $\mathcal{N}_Y^{s,\widetilde{\omega}} \cup \mathcal{N}_{\mathcal{X}^j} \in \overline{\mathscr{N}}^s$, similar to (7.68), we see that $\mathcal{X}_j(\widehat{\omega}) \in A_{\mathcal{X}^j}$, and can deduce from (7.84) that $\widehat{\omega} = B^s(\widehat{\omega}) = \mathcal{W}^j(\mathcal{X}^j(\widehat{\omega})) = \widetilde{\mathcal{W}}^j(\mathcal{X}^j(\widehat{\omega}))$. Then (7.83), (7.78), and (3.1) imply that

$$\begin{aligned}
(\mathscr{Y}_{\widehat{\tau}'})^{s,\widetilde{\omega}}(\widehat{\omega}) &= \mathscr{Y}\big(\widehat{\tau}'(\widetilde{\omega}\otimes_s\widehat{\omega}), \widetilde{\omega}\otimes_s\widehat{\omega}\big) = \mathcal{Y}\big(\zeta_{\widetilde{\omega}}(\widehat{\omega}), \widehat{\mathcal{X}}(\widetilde{\omega}\otimes_s\widehat{\omega})\big)\\
&= Y\Big(\widehat{\zeta}_{\widetilde{\omega}}\big(\mathcal{X}^j(\widehat{\omega})\big), \omega\otimes_t\big(\mathcal{X}(\widetilde{\omega})\otimes_s X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}(\widehat{\omega})\big)\Big)\\
&\le Y\big(\widehat{\zeta}_{\widetilde{\omega}}\big(\mathcal{X}^j(\widehat{\omega})\big), \omega\otimes_t\big(\mathcal{X}(\widetilde{\omega})\otimes_s \mathcal{X}^j(\widehat{\omega})\big)\big) + \rho_0\big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\big)\\
&= Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}_{\widetilde{\omega}}}\big(\mathcal{X}^j(\widehat{\omega})\big) + \rho_0\big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\big)\\
&\le Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}_{\widetilde{\omega}}}\big(\mathcal{X}^j(\widehat{\omega})\big) + \mathbf{1}_{\left\{\Delta X^j_{\widetilde{\omega}}(\widehat{\omega}) \le \delta^{1/2}\right\}}\rho_0\big(\delta^{1/2}\big)\\
&\quad + \mathbf{1}_{\left\{\Delta X^j_{\widetilde{\omega}}(\widehat{\omega}) > \delta^{1/2}\right\}}\kappa\delta^{-1/2}\Big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega}) + \big(\Delta X^j_{\widetilde{\omega}}(\widehat{\omega})\big)^{\varpi+1}\Big),
\end{aligned} \tag{7.85}$$

where $\widehat{\zeta}_{\widetilde{\omega}}(\widehat{\omega}') \stackrel{\triangle}{=} \zeta_{\widetilde{\omega}}(\widetilde{\mathcal{W}}^j(\widehat{\omega}'))$ for all $\widehat{\omega}' \in \Omega^s$ and $\Delta X^j_{\widetilde{\omega}}(\widehat{\omega}) \stackrel{\triangle}{=} \|X^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega}),\mu^j}(\widehat{\omega}) - \mathcal{X}^j(\widehat{\omega})\|_{s,T}$.

For any $r \in [s,T]$, as $\widetilde{A}_r \stackrel{\triangle}{=} \{\zeta_{\widetilde{\omega}} \le r\} \in \mathcal{F}_r^s$, an analogy to (7.67) shows that $\{\widehat{\zeta}_{\widetilde{\omega}} \le r\} = \{\widehat{\omega} \in \Omega^s : \widetilde{\mathcal{W}}^j(\widehat{\omega}) \in \widetilde{A}_r\} = (\widetilde{\mathcal{W}}^j)^{-1}(\widetilde{A}_r) \in \mathfrak{F}_r^j$. So $\widehat{\zeta}_{\widetilde{\omega}}$ is a stopping time with respect to the filtration $\{\mathfrak{F}_r^j\}_{r\in[s,T]}$. Similar to (7.16),

$$\mathbb{E}_{\mathfrak{p}_j}\Big[\big|Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}'_{\widetilde{\omega}}} - Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}_{\widetilde{\omega}}}\big|\Big] < \frac{\varepsilon}{2} \quad \text{for some } \widehat{\zeta}'_{\widetilde{\omega}} \in \mathcal{T}^s. \tag{7.86}$$

As $\widetilde{\omega} \in \mathcal{A}_j^{\mathcal{X}} = \mathcal{X}^{-1}(\mathcal{A}_j)$, i.e., $\mathcal{X}(\widetilde{\omega}) \in \mathcal{A}_j \subset O^s_\delta(\widetilde{\omega}_j)$, we see that $\|\omega\otimes_t\mathcal{X}(\widetilde{\omega}) - \omega\otimes_t\widetilde{\omega}_j\|_{0,s} = \|\mathcal{X}(\widetilde{\omega}) - \widetilde{\omega}_j\|_{t,s} < \delta$. It then follows from (7.85) and (6.2) that

$$\begin{aligned}
\mathbb{E}_s\big[(\mathscr{Y}_{\widehat{\tau}'})^{s,\widetilde{\omega}}\big] &\le \mathbb{E}_s\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}_{\widetilde{\omega}}}\big(\mathcal{X}^j\big)\Big] + \rho_0\big(\delta^{1/2}\big)\\
&\quad + \kappa\delta^{-1/2}\big(C_1T\|\omega\otimes_t\mathcal{X}(\widetilde{\omega}) - \omega\otimes_t\widetilde{\omega}_j\|_{0,s}\\
&\qquad\qquad + C_{\varpi+1}T^{\varpi+1}\|\omega\otimes_t\mathcal{X}(\widetilde{\omega}) - \omega\otimes_t\widetilde{\omega}_j\|^{\varpi+1}_{0,s}\big)\\
&\le \mathbb{E}_{\mathfrak{p}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}_{\widetilde{\omega}}}\Big] + \rho_0\big(\delta^{1/2}\big)\\
&\quad + \kappa\big(C_1T\delta^{1/2} + C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2}\big)\\
&\le \mathbb{E}_{\mathfrak{p}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}'_{\widetilde{\omega}}}\Big] + \widehat{\rho}_0(\delta) + \frac{\varepsilon}{2},
\end{aligned} \tag{7.87}$$

where $\widehat{\rho}(\delta) \stackrel{\triangle}{=} \rho_0(\delta^{1/2}) + \kappa(C_1T\delta^{1/2} + C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2})$. Since $\widehat{\zeta}'_{\widetilde{\omega}} \in \mathcal{T}^s$, the $\mathbf{F}$-adaptedness of Y and Proposition 2.1(2) show that $Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}'_{\widetilde{\omega}}} \in$

$\mathcal{F}^s_T$, and thus

$$\mathbb{E}_{\mathfrak{p}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}'_{\widetilde{\omega}}}\Big]=\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\widehat{\zeta}'_{\widetilde{\omega}}}\Big]\le\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta}\Big]. \tag{7.88}$$

Then plugging (7.87) into (7.82), we can deduce from (7.70) and Lemma A.3(1) that

$$\begin{aligned}
&\mathbb{E}_{\widehat{\mathbb{P}}}\big[\mathbf{1}_{A\cap\mathcal{A}_j}Y^{t,\omega}_\tau\big]\\
&\quad\le\mathbb{E}_t\left[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{X}^{-1}(A)\cap\mathcal{X}^{-1}(\mathcal{A}_j)\}}\left(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\mathcal{X}(\widetilde{\omega})}_{\zeta}\Big]+\widehat{\rho}_0(\delta)\right)\right]+\varepsilon\\
&\quad=\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\left(\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}\Big[Y^{s,\omega\otimes_t\widetilde{\omega}}_{\zeta}\Big]+\widehat{\rho}_0(\delta)\right)\right]+\varepsilon,
\end{aligned}$$

where we used the fact that the mapping $\widetilde{\omega}\to\sup_{\zeta\in\mathcal{T}^s}\mathbb{E}_{\mathbb{P}_j}[Y^{s,\omega\otimes_t\widetilde{\omega}}_\zeta]$ is continuous by Remark 3.3(1). Letting $\varepsilon\to0$ and taking supremum over $\tau\in\mathcal{T}^t_s$, we see that the inequality (3.3) holds.

(3) Let $\omega'\in\Omega$. We set $(\mathcal{X}',\mathbb{P}')=(X^{t,\omega',\mu},\mathbb{P}^{t,\omega',\mu})$ and $\delta\overset{\triangle}{=}\|\omega'-\omega\|_{0,t}$. For any $\widetilde{\omega}\in\Omega^t$, define $\Delta X(\widetilde{\omega})\overset{\triangle}{=}\|\mathcal{X}'(\widetilde{\omega})-\mathcal{X}(\widetilde{\omega})\|_{t,T}$. Similar to (7.85), we can deduce from (3.1) that for any $r\in[t,T]$

$$\begin{aligned}
&Y\big(r,\omega'\otimes_t\mathcal{X}'(\widetilde{\omega})\big)-Y\big(r,\omega\otimes_t\mathcal{X}(\widetilde{\omega})\big)\\
&\quad\le\rho_0\big(\|\omega'\otimes_t\mathcal{X}'(\widetilde{\omega})-\omega\otimes_t\mathcal{X}(\widetilde{\omega})\|_{0,r}\big)\\
&\quad\le\rho_0\big(\|\omega'-\omega\|_{0,t}+\|\mathcal{X}'(\widetilde{\omega})-\mathcal{X}(\widetilde{\omega})\|_{t,r}\big)\\
&\quad\le\rho_0\big(\delta+\Delta X(\widetilde{\omega})\big)\le\mathbf{1}_{\{\Delta X(\widetilde{\omega})\le\delta^{1/2}\}}\rho_0\big(\delta+\delta^{1/2}\big)\\
&\qquad+\mathbf{1}_{\{\Delta X(\widetilde{\omega})>\delta^{1/2}\}}\kappa\delta^{-1/2}\big((1+2^{\varpi-1}\delta^{\varpi})\Delta X(\widetilde{\omega})+2^{\varpi-1}(\Delta X(\widetilde{\omega}))^{\varpi+1}\big).
\end{aligned}$$

Given $\tau\in\mathcal{T}^t$, it follows from (6.2) that

$$\begin{aligned}
&\mathbb{E}_t\Big[Y\big(\tau(\mathcal{X}'),\omega'\otimes_t\mathcal{X}'\big)-Y\big(\tau(\mathcal{X}'),\omega\otimes_t\mathcal{X}\big)\Big]\\
&\quad\le\rho_0(\delta+\delta^{1/2})+\kappa\big(1+2^{\varpi-1}\delta^{\varpi}\big)C_1T\delta^{1/2}\\
&\qquad+\kappa2^{\varpi-1}C_{\varpi+1}T^{\varpi+1}\delta^{\varpi+1/2}\overset{\triangle}{=}\rho_1(\delta).
\end{aligned}$$

Clearly, ρ_1 is a modulus of continuity function greater than ρ_0. Then (7.68) implies that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}'}\big[Y^{t,\omega'}_\tau\big]&=\mathbb{E}_t\big[Y^{t,\omega'}\big(\tau(\mathcal{X}'),\mathcal{X}'\big)\big]=\mathbb{E}_t\Big[Y\big(\tau(\mathcal{X}'),\omega'\otimes_t\mathcal{X}'\big)\Big]\\
&\le\mathbb{E}_t\Big[Y\big(\tau\big(\mathcal{X}'\big),\omega\otimes_t\mathcal{X}\big)\big)\Big]+\rho_1(\delta)\\
&=\mathbb{E}_t\Big[Y\Big(\tau\big(\mathcal{X}'\big(\widetilde{\mathcal{W}}(\mathcal{X})\big)\big),\omega\otimes_t\mathcal{X}\Big)\Big]+\rho_1(\delta)\\
&=\mathbb{E}_t\Big[Y\big(\zeta(\mathcal{X}),\omega\otimes_t\mathcal{X}\big)\Big]+\rho_1(\delta)=\mathbb{E}_{\mathfrak{p}}\big[Y^{t,\omega}_\zeta\big]+\rho_1(\delta),
\end{aligned}\tag{7.89}$$

where $\zeta\overset{\triangle}{=}\tau(\mathcal{X}'(\widetilde{\mathcal{W}}))$. For any $r\in[t,T]$, as $\widehat{A}_r\overset{\triangle}{=}\{\tau\le r\}\in\mathcal{F}^t_r$, (6.4) shows that $(\mathcal{X}')^{-1}(\widehat{A}_r)\in\overline{\mathcal{F}}^t_r$. By (7.69), $\{\zeta\le r\}=\{\widetilde{\omega}\in\Omega^t:\mathcal{X}'(\widetilde{\mathcal{W}}(\widetilde{\omega}))\in\widehat{A}_r\}=$

$\widetilde{\mathcal{W}}^{-1}((\mathcal{X}')^{-1}(\widehat{A}_r)) \in \mathfrak{F}_r$. So ζ is a stopping time with respect to the filtration $\{\mathfrak{F}_r\}_{r\in[s,T]}$. Given $\varepsilon > 0$, similar to (7.86) and (7.88), there exists a $\zeta' \in \mathcal{T}^t$ such that $\mathbb{E}_{\mathfrak{p}}[|Y^{t,\omega}_{\zeta'} - Y^{t,\omega}_{\zeta}|] < \varepsilon$ and $\mathbb{E}_{\mathfrak{p}}[Y^{t,\omega}_{\zeta'}] = \mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_{\zeta'}] \le \sup_{\tau'\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}[Y^{t,\omega}_{\tau'}]$, which together with (7.89) shows that

$$\mathbb{E}_{\mathbb{P}'}\big[Y^{t,\omega'}_{\tau}\big] \le \mathbb{E}_{\mathfrak{p}}\big[Y^{t,\omega}_{\zeta}\big] + \rho_1(\delta) \le \sup_{\tau'\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_{\tau'}\big] + \rho_1(\delta) + \varepsilon.$$

Letting $\varepsilon \to 0$, taking supremum over $\tau \in \mathcal{T}^t$ on the left-hand side, and then taking infimum over $\mu \in \mathcal{U}_t$ yields that

$$\begin{aligned}\overline{Z}_t(\omega') &= \inf_{\mu\in\mathcal{U}_t}\sup_{\tau\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}'}\big[Y^{t,\omega'}_{\tau}\big] \le \inf_{\mu\in\mathcal{U}_t}\sup_{\tau'\in\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[Y^{t,\omega}_{\tau'}\big] + \rho_1\big(\|\omega'-\omega\|_{0,t}\big)\\ &= \overline{Z}_t(\omega) + \rho_1\big(\|\omega'-\omega\|_{0,t}\big).\end{aligned}$$

Exchanging the roles of ω' and ω shows that $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies Assumption 4.1.

(4) There exists a constant $\widetilde{C}_\varpi$ depending on ϖ and T such that $\rho_1(\delta) \le \kappa\widetilde{C}_\varpi(1+\delta^{\varpi+1/2})$ for all $\delta > 0$. Let $\alpha > \|\omega\|_{0,t}$ and $\delta \in (0,T]$. We can deduce from (6.2) that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\rho_1\left(\delta + 2\sup_{r\in[t,(t+\delta)\wedge T]}\big|B^t_r\big|\right)\right]\\ &\quad= \mathbb{E}_t\left[\rho_1\left(\delta + 2\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|\right)\right] \le \rho_1(\delta+2\delta^{1/4})\\ &\qquad+ \kappa\widetilde{C}_\varpi\mathbb{E}_t\left[\mathbf{1}_{\left\{\sup\limits_{r\in[t,(t+\delta)\wedge T]}|\mathcal{X}_r|>\delta^{1/4}\right\}}\left(1+2^{\varpi-1/2}\delta^{\varpi+1/2}\right.\right.\\ &\qquad\qquad\left.\left.+2^{2\varpi}\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|^{\varpi+1/2}\right)\right]\\ &\quad\le \rho_1(\delta+2\delta^{1/4}) + \kappa\widetilde{C}_\varpi\delta^{-1/4}\mathbb{E}_t\left[\big(1+2^{\varpi-1/2}\delta^{\varpi+1/2}\big)\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|\right.\\ &\qquad\qquad\left.+2^{2\varpi}\sup_{r\in[t,(t+\delta)\wedge T]}\big|\mathcal{X}_r\big|^{\varpi+3/2}\Big)\right]\\ &\quad\le \rho_1(\delta+2\delta^{1/4}) + \kappa\widetilde{C}_\varpi\big(1+2^{\varpi-1/2}\delta^{\varpi+1/2}\big)\varphi_1(\alpha)\,\delta^{1/4}\\ &\qquad+\kappa\widetilde{C}_\varpi 2^{2\varpi}\varphi_{\varpi+\frac{3}{2}}(\alpha)\,\delta^{\varpi/2+1/2} \stackrel{\triangle}{=} \rho_\alpha(\delta).\end{aligned}$$

Clearly, ρ_α is a modulus of continuity function. Hence, $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[0,T]\times\Omega}$ satisfies Assumption 4.2. $\square$

Appendix A. Technical lemmata.

LEMMA A.1. *Let $0 \le t \le s \le S \le T < \infty$. The mapping $\Pi^{t,T}_{s,S}$ is continuous (under the uniform norms) and is $\mathcal{F}^{t,T}_r/\mathcal{F}^{s,S}_r$-measurable for any $r \in [s,S]$. The law of $\Pi^{t,T}_{s,S}$ under $\mathbb{P}^{t,T}_0$ is exactly $\mathbb{P}^{s,S}_0$, i.e.,*

$$\mathbb{P}^{t,T}_0\Big(\big(\Pi^{t,T}_{s,S}\big)^{-1}(A)\Big) = \mathbb{P}^{s,S}_0(A) \quad \textit{for all } A \in \mathcal{F}^{s,S}_S. \tag{A.1}$$

It also holds for any $r \in [s,S]$ and $\tau \in \mathcal{T}^{s,S}_r$ that $\tau(\Pi^{t,T}_{s,S}) \in \mathcal{T}^{t,T}_r$.

Proof. For simplicity, let us denote $\Pi^{t,T}_{s,S}$ by Π.

(1) We first show the continuity of Π. Let A be an open subset of $\Omega^{s,S}$. Given $\omega \in \Pi^{-1}(A)$, since $\Pi(\omega) \in A$, there exist a $\delta > 0$ such that $O_\delta(\Pi(\omega)) = \{\widetilde{\omega} \in \Omega^{s,S} : \|\widetilde{\omega} - \Pi(\omega)\|_{s,S} < \delta\} \subset A$. For any $\omega' \in O_{\delta/2}(\omega)$, one can deduce that

$$\big\|\Pi(\omega') - \Pi(\omega)\big\|_{s,S} \le \big|\omega'(s) - \omega(s)\big| + \big\|\omega' - \omega\big\|_{s,S} \le 2\|\omega' - \omega\|_{t,T} < \delta,$$

which shows that $\Pi(\omega') \in O_\delta(\Pi(\omega)) \subset A$ or $\omega' \in \Pi^{-1}(A)$. Hence, $\Pi^{-1}(A)$ is an open subset of $\Omega^{t,T}$.

Let $r \in [s,S]$. For any $s' \in [s,r]$ and $\mathcal{E} \in \mathscr{B}(\mathbb{R}^d)$, one can deduce that

$$\begin{aligned}\Pi^{-1}\Big(\big(B^{s,S}_{s'}\big)^{-1}(\mathcal{E})\Big) &= \Big\{\omega \in \Omega^{t,T} : B^{s,S}_{s'}\big(\Pi(\omega)\big) \in \mathcal{E}\Big\}\\ &= \big\{\omega \in \Omega^{t,T} : \omega(s') - \omega(s) \in \mathcal{E}\big\}\\ &= (B^{t,T}_{s'} - B^{t,T}_s)^{-1}(\mathcal{E}) \in \mathcal{F}^{t,T}_r.\end{aligned}$$

Thus all the generating sets of $\mathcal{F}^{s,S}_r$ belong to $\Lambda \stackrel{\triangle}{=} \{A \subset \Omega^{s,S} : \Pi^{-1}(A) \in \mathcal{F}^{t,T}_r\}$, which is clearly a σ-field of $\Omega^{s,S}$. It follows that $\mathcal{F}^{s,S}_r \subset \Lambda$, i.e., $\Pi^{-1}(A) \in \mathcal{F}^{t,T}_r$ for any $A \in \mathcal{F}^{s,S}_r$.

(2) As a Brownian motion on $(\Omega^{t,T}, \mathscr{B}(\Omega^{t,T}))$ under $\mathbb{P}^{t,T}_0$, $B^{t,T}$ has independent and stationary increments with standard normal $\mathbb{P}^{t,T}_0$-distribution. Then one can easily deduce that $B^{s,S}$ also has independent and stationary increments with standard normal distribution under $\widetilde{\mathbb{P}} \stackrel{\triangle}{=} \mathbb{P}^{t,T}_0 \circ \Pi^{-1}$ (see [6] for details), which shows that $B^{s,S}$ is a Brownian motion on $(\Omega^{s,S}, \mathscr{B}(\Omega^{s,S}))$ under $\widetilde{\mathbb{P}}$. Since the Wiener measure on $(\Omega^{s,S}, \mathscr{B}(\Omega^{s,S}))$ is unique (see, e.g., Proposition I.3.3 of [31]), we have (A.1).

(3) Now, let $r \in [s,S]$ and $\tau \in \mathcal{T}^{s,S}_r$. For any $r' \in [r,S]$, as $\widetilde{A} \stackrel{\triangle}{=} \{\widetilde{\omega} \in \Omega^{s,S} : \tau(\widetilde{\omega}) \le r'\} \in \mathcal{F}^{s,S}_{r'}$, one can deduce that $\{\omega \in \Omega^{t,T} : \tau(\Pi^{t,T}_{s,S}(\omega)) \le r'\} = \{\omega \in \Omega^{t,T} : \Pi^{t,T}_{s,S}(\omega) \in \widetilde{A}\} = (\Pi^{t,T}_{s,S})^{-1}(\widetilde{A}) \in \mathcal{F}^{t,T}_{r'}$. So $\tau(\Pi^{t,T}_{s,S}) \in \mathcal{T}^{t,T}_r$. ∎

LEMMA A.2. *Given $t \in [0,T]$ and $\widetilde{d}, \widetilde{d}' \in \mathbb{N}$, let $\mathbb{P}$ be a probability on $(\Omega^t, \mathscr{B}(\Omega^t))$ and let $\{X_s\}_{s\in[t,T]}$ be an $\mathbb{R}^{\widetilde{d}}$-valued, $\mathbf{F}^{\mathbb{P}}$-adapted process.*

(1) *For any $s \in [t,T]$ and any $\mathbb{R}^{\widetilde{d}'}$-valued, $\mathcal{F}^{X,\mathbb{P}}_T$-measurable random variable ξ with $\mathbb{E}_{\mathbb{P}}[|\xi|] < \infty$, $\mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}^{X,\mathbb{P}}_s] = \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}^X_s]$, $\mathbb{P}$-a.s.*

(2) *For any $s \in [t,T]$ and any $\mathbb{R}^{\widetilde{d}'}$-valued, $\mathcal{F}^{X,\mathbb{P}}_s$-measurable random variable ξ, there exists an $\mathbb{R}^{\widetilde{d}'}$-valued, $\mathcal{F}^X_s$-measurable random variable $\widetilde{\xi}$ such that $\widetilde{\xi} = \xi$, $\mathbb{P}$-a.s.*

(3) *For any $\mathbb{R}^{\widetilde{d}'}$-valued, $\mathbf{F}^{X,\mathbb{P}}$-adapted process $\{K_s\}_{s\in[t,T]}$ with $\mathbb{P}$-a.s. right-continuous paths, there exists an $\mathbb{R}^{\widetilde{d}'}$-valued, $\mathbf{F}^X$-progressively measurable process $\{\widetilde{K}_s\}_{s\in[t,T]}$ such that $\{\omega \in \Omega^t : \widetilde{K}_s(\omega) \ne K_s(\omega)$ for some $s \in [t,T]\} \in \mathscr{N}^{\mathbb{P}}$. We call $\widetilde{K}$ the $(\mathbf{F}^X, \mathbb{P})$-version of K.*

Proof. This result is similar to Lemma 2.4 of [36]. We refer the interested readers to their proof; see also [6]. ∎

LEMMA A.3.

Let $(t,\omega) \in [0,T] \times \Omega$ and let μ be a $\mathcal{U}_t$-control considered in section 6.

(1) *It holds for any $s \in [t,T]$ that $\mathcal{F}^{\mathbb{P}^{t,\omega,\mu}}_s \subset \mathcal{G}^{X^{t,\omega,\mu}}_s$, and $\mathfrak{p}^{t,\omega,\mu}$ coincides with $\mathbb{P}^{t,\omega,\mu}$ on $\mathcal{F}^{\mathbb{P}^{t,\omega,\mu}}_T$.*

(2) *The σ-field $\mathcal{G}_T^{X^{t,\omega,\mu}}$ is complete under $\mathfrak{p}^{t,\omega,\mu}$, and $\mathscr{N}^{\mathbb{P}^{t,\omega,\mu}} \subset \mathscr{N}^{\mathfrak{p}^{t,\omega,\mu}} \stackrel{\triangle}{=} \{A \in \mathcal{G}_T^{X^{t,\omega,\mu}} : \mathfrak{p}^{t,\omega,\mu}(A) = 0\} \subset \mathcal{G}_s^{X^{t,\omega,\mu}}$ holds for any $s \in [t,T]$.*

Proof.

(1) Set $\vartheta = (t,\omega,\mu)$ and let $s \in [t,T]$. For any $\mathcal{N} \in \mathscr{N}^{\mathbb{P}^\vartheta}$, there exists an $A \in \mathcal{F}_T^t$ with $\mathbb{P}^\vartheta(A) = 0$ such that $\mathcal{N} \subset A$. By (6.4), $(X^\vartheta)^{-1}(A) \in \overline{\mathcal{F}}_T^t$ and thus $\mathbb{P}_0^t((X^\vartheta)^{-1}(A)) = \mathbb{P}^\vartheta(A) = 0$. Then, as a subset of $(X^\vartheta)^{-1}(A)$,

$$\big(X^\vartheta\big)^{-1}\big(\mathcal{N}\big) \in \overline{\mathscr{N}}^t \subset \overline{\mathcal{F}}_s^t. \tag{A.2}$$

So $\mathscr{N}^{\mathbb{P}^\vartheta} \subset \mathcal{G}_s^{X^\vartheta}$, which already contains $\mathcal{F}_s^t$ by (6.4). It follows that $\mathcal{F}_s^{\mathbb{P}^\vartheta} \subset \mathcal{G}_s^{X^\vartheta}$.

Given $A \in \mathcal{F}_T^{\mathbb{P}^\vartheta} \subset \mathcal{G}_T^{X^\vartheta}$, we know (see, e.g., Proposition 11.4 of [33]) that $A = \widetilde{A} \cup \mathcal{N}$ for some $\widetilde{A} \in \mathcal{F}_T^t$ and $\mathcal{N} \in \mathscr{N}^{\mathbb{P}^\vartheta}$. Since $(X^\vartheta)^{-1}(\widetilde{A}) \in \overline{\mathcal{F}}_T^t$ by (6.4) and since $(X^\vartheta)^{-1}(\mathcal{N}) \in \overline{\mathscr{N}}^t$ by (A.2), one can deduce that

$$\begin{aligned}\mathfrak{p}^\vartheta(A) &= \mathbb{P}_0^t\big((X^\vartheta)^{-1}(A)\big) = \mathbb{P}_0^t\Big((X^\vartheta)^{-1}\big(\widetilde{A}\big) \cup (X^\vartheta)^{-1}\big(\mathcal{N}\big)\Big)\\ &= \mathbb{P}_0^t\Big((X^\vartheta)^{-1}\big(\widetilde{A}\big)\Big) = \mathbb{P}^\vartheta\big(\widetilde{A}\big) = \mathbb{P}^\vartheta(A).\end{aligned}$$

(2) Let $\mathfrak{N} \subset A$ for some $A \in \mathcal{G}_T^{X^\vartheta}$ with $\mathfrak{p}^\vartheta(A) = 0$. As $(X^\vartheta)^{-1}(\mathfrak{N}) \subset (X^\vartheta)^{-1}(A) \in \overline{\mathcal{F}}_T^t$ and $0 = \mathfrak{p}^\vartheta(A) = \mathbb{P}_0^t((X^\vartheta)^{-1}(A))$, we see that

$$(X^\vartheta)^{-1}(\mathfrak{N}) \in \overline{\mathscr{N}}^t. \tag{A.3}$$

In particular, $\mathfrak{N} \in \mathcal{G}_T^{X^\vartheta}$, so the σ-field $\mathcal{G}_T^{X^\vartheta}$ is complete under $\mathfrak{p}^\vartheta$. Then it easily follows from part (1) that $\mathscr{N}^{\mathbb{P}^\vartheta} = \{A \in \mathcal{F}_T^{\mathbb{P}^\vartheta} : \mathbb{P}^\vartheta(A) = 0\} = \{A \in \mathcal{F}_T^{\mathbb{P}^\vartheta} : \mathfrak{p}^\vartheta(A) = 0\} \subset \{A \in \mathcal{G}_T^{X^\vartheta} : \mathfrak{p}^\vartheta(A) = 0\} = \mathscr{N}^{\mathfrak{p}^\vartheta}$. Moreover, taking $\mathfrak{N} = A$ for any $A \in \mathcal{G}_T^{X^\vartheta}$ with $\mathfrak{p}^\vartheta(A) = 0$ in (A.3) shows that $\mathscr{N}^{\mathfrak{p}^\vartheta} \subset \mathcal{G}_s^{X^\vartheta}$ for all $s \in [t,T]$. ∎

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