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Optimal stopping for non-linear expectations—Part II[☆]

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Abstract

Relying on the stochastic analysis tools developed in Bayraktar and Yao (2011) [1], we solve the optimal stopping problems for non-linear expectations.

Keywords: Non-linear expectations; Optimal stopping; Snell envelope; Stability; g-expectations

1. Introduction

We build this paper on the results of [1] and analyze the optimal stopping problems for non-linear expectations. The background, literature review and the motivation for these problems are provided in the introduction section of [1]. The notation used in this paper is outlined in Section 1.1 of [1].

The rest of the paper is organized as follows: In Section 2 we solve a multi-prior optimal stopping problem for a collection $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ of non-linear expectations, in which *Nature* is in collaboration with the *Stopper*, and find an optimal stopping time in terms of the $\mathscr{E}$*-upper Snell envelope*. On the other hand, in Section 3 we solve the robust or the minimax optimization problem in terms of the $\mathscr{E}$*-lower Snell envelope*. In Section 4, we give some interpretations and remarks on our results. In Section 5, we consider the case when $\mathscr{E}$ is a certain collection of g-expectations. We see that in this framework, our assumptions on each $\mathcal{E}_i$, the stability condition and the uniform left/right-continuity conditions are naturally satisfied. We also determine an

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optimal prior $i^* \in \mathcal{I}$. Moreover, we show how the controller and stopper problem of [10] fits into our g-expectations framework. This lets us extend their result from bounded rewards to rewards satisfying linear growth. In this section, we also solve the optimal stopping problem for quadratic g-expectations. The proofs of our results are presented in Section 6.

2. Optimal stopping with multiple priors

In this section, we will solve an optimal stopping problem in which the objective of the stopper is to determine an optimal stopping time τ^* that satisfies

$$\sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho + H^i_\rho] = \sup_{i\in\mathcal{I}} \mathcal{E}_i[Y_{\tau^*} + H^i_{\tau^*}], \tag{2.1}$$

where $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ is a stable class of $\mathbf{F}$-expectations, Y is a primary reward process and H^i is a *model*-dependent cumulative reward process. (We will outline the assumptions on the reward processes below.) To find an optimal stopping time, we shall build a so-called "$\mathscr{E}$-upper Snell envelope" Z^0 of the reward process Y. Namely, Z^0 is the smallest RCLL $\mathbf{F}$-adapted process dominating Y such that $Z^0 + H^i$ is an $\widetilde{\mathcal{E}}_i$-supermartingale for any $i \in \mathcal{I}$. We will show under certain assumptions that the first time Z^0 meets Y is an optimal stopping time for (2.1).

We start by making some assumptions on the reward processes: Let $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ be a stable class of $\mathbf{F}$-expectations accompanied by a family $\mathscr{H} \triangleq \{H^i\}_{i\in\mathcal{I}}$ of right-continuous $\mathbf{F}$-adapted processes that satisfies:

(S1) For any $i \in \mathcal{I}$, $H^i_0 = 0$, a.s. and

$$H^i_{\nu,\rho} \triangleq H^i_\rho - H^i_\nu \in \mathrm{Dom}(\mathscr{E}), \quad \forall \nu, \rho \in \mathcal{S}_{0,T} \text{ with } \nu \le \rho, \text{ a.s.} \tag{2.2}$$

Moreover, if no member of $\mathscr{E}$ satisfies (2.5) of [1], then there exists a $j \in \mathcal{I}$ such that

$$\zeta^j \triangleq \operatorname*{esssup}_{s,t\in\mathcal{D}_T;\, s<t} H^j_{s,t} \in \mathrm{Dom}(\mathscr{E}). \tag{2.3}$$

(S2) There exists a $C_H < 0$ such that for any $i \in \mathcal{I}$, $\operatorname{essinf}_{s,t\in\mathcal{D}_T;s<t} H^i_{s,t} \ge C_H$, a.s.

(S3) For any $\nu \in \mathcal{S}_{0,T}$ and $i, j \in \mathcal{I}$, it holds for any $0 \le s < t \le T$ that $H^k_{s,t} = H^i_{\nu\wedge s,\nu\wedge t} + H^j_{\nu\vee s,\nu\vee t}$, a.s., where $k = k\big(i, j, \nu\big) \in \mathcal{I}$ is the index defined in Definition 3.2(2) of [1].

Remark 2.1. (1) For any $i \in \mathcal{I}$, (S2) and the right-continuity of H^i imply that except on a null set $N(i)$

$$H^i_{s,t} \ge C_H, \quad \text{for any } 0 \le s < t \le T, \quad \text{thus}$$

$$H^i_{\nu,\rho} \ge C_H, \quad \forall \nu, \rho \in \mathcal{S}_{0,T} \text{ with } \nu \le \rho, \text{ a.s.} \tag{2.4}$$

(2) If (2.3) is assumed for some $j \in \mathcal{I}$, the right-continuity of H^j and (2.4) imply that except on a null set N

$$C_H \le H^j_{s,t} \le \zeta^j, \quad \text{for any } 0 \le s < t \le T, \quad \text{thus}$$

$$C_H \le H^j_{\nu,\rho} \le \zeta^j, \quad \forall \nu, \rho \in \mathcal{S}_{0,T} \text{ with } \nu \le \rho, \text{ a.s.}$$

Then Lemma 3.2 of [1] implies that (2.2) holds for j. Hence we see that (2.3) is a stronger condition than (2.2).

(3) Since H^i, H^j and H^k are all right-continuous processes, (S3) is equivalent to the statement that a.s.

$$H^k_{s,t} = H^i_{\nu\wedge s,\nu\wedge t} + H^j_{\nu\vee s,\nu\vee t}, \quad \forall\, 0 \le s < t \le T. \tag{2.5}$$

Now we give an example of $\mathscr{H}$.

Lemma 2.1. *Let $\{h^i\}_{i\in\mathcal{I}}$ be a family of progressive processes satisfying the following assumptions:*

(h1) *For any $i \in \mathcal{I}$ and $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s., $\int_\nu^\rho h^i_t\,\mathrm{d}t \in \mathrm{Dom}(\mathscr{E})$. Moreover, if no member of $\mathscr{E}$ satisfies (2.5) of* [1], *we assume that there exists a $j \in \mathcal{I}$ such that $\int_0^T |h^j_t|\,\mathrm{d}t \in \mathrm{Dom}(\mathscr{E})$.*
(h2) *There exists a $c < 0$ such that for any $i \in \mathcal{I}$, $h^i_t \ge c$, $\mathrm{d}t \times \mathrm{d}P$-a.s.*
(h3) *For any $\nu \in \mathcal{S}_{0,T}$ and $i, j \in \mathcal{I}$, it holds for any $t \in [0, T]$ that $h^k_t = \mathbf{1}_{\{\nu\le t\}} h^j_t + \mathbf{1}_{\{\nu>t\}} h^i_t$, $\mathrm{d}t \times \mathrm{d}P$-a.s., where $k = k(i, j, \nu) \in \mathcal{I}$ is the index defined in Definition 3.2(2) of* [1].

Then $\left\{H^i_t \triangleq \int_0^t h^i_s \mathrm{d}s,\ t \in [0, T]\right\}_{i\in\mathcal{I}}$ is a family of right-continuous **F**-*adapted processes satisfying* (S1)–(S3).

Standing assumptions on Y in this section. Let Y be a right-continuous **F**-adapted process that satisfies:

(Y1) For any $\nu \in \mathcal{S}_{0,T}$, $Y_\nu \in \mathrm{Dom}(\mathscr{E})$.
(Y2) $\sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y^i_\rho] < \infty$, where $Y^i \triangleq \{Y_t + H^i_t\}_{t\in[0,T]}$. Moreover, if no member of $\mathscr{E}$ satisfies (2.5) of [1], then

$$\zeta_Y \triangleq \operatorname*{esssup}_{(i,\rho,t)\in\mathcal{I}\times\mathcal{S}_{0,T}\times\mathcal{D}_T} \widetilde{\mathcal{E}}_i[Y^i_\rho|\mathcal{F}_t] \in \mathrm{Dom}(\mathscr{E}). \tag{2.6}$$

(Y3) $\operatorname{essinf}_{t\in\mathcal{D}_T} Y_t \ge C_Y$, a.s. for some $C_Y < 0$.

Remark 2.2. (1) For any $i \in \mathcal{I}$, (A4) and (2.8) of [1] imply that $\mathcal{E}_i$ satisfies (2.5) of [1] if and only if $\widetilde{\mathcal{E}}_i$ satisfies the following statement: Let $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(\mathscr{E})$ be a sequence converging a.s. to some $\xi \in L^0(\mathcal{F}_T)$. If $\inf_{n\in\mathbb{N}} \xi_n \ge c$, a.s. for some $c \in \mathbb{R}$, then $\underline{\lim}_{n\to\infty} \widetilde{\mathcal{E}}_i[\xi_n] < \infty$ implies $\xi \in \mathrm{Dom}(\mathscr{E})$. The proof of this equivalence is similar to that of Corollary 2.2 of [1].
(2) It is clear that (2.6) implies $\sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i\left[Y^i_\rho\right] < \infty$.
(3) In light of (Y3) and the right-continuity of Y, it holds except on a null set N that

$$Y_t \ge C_Y, \quad \forall t \in [0, T], \quad \text{thus} \quad Y_\nu \ge C_Y, \quad \forall \nu \in \mathcal{S}_{0,T}. \tag{2.7}$$

Then for any $i \in \mathcal{I}$, Remark 2.1(1) implies that except on a null set $\widetilde{N}(i)$

$$Y^i_\nu = Y_\nu + H^i_\nu \ge C_* \triangleq C_Y + C_H, \quad \forall \nu \in \mathcal{S}_{0,T}. \tag{2.8}$$

The following lemma states that the supremum or infimum over a stable class of **F**-expectations can be approached by an increasing or decreasing sequence in the class.

Lemma 2.2. *Let $\nu \in \mathcal{S}_{0,T}$ and $\mathcal{U}$ be a non-empty subset of $\mathcal{S}_{\nu,T}$ such that*

$$\rho_1\mathbf{1}_A + \rho_2\mathbf{1}_{A^c} \in \mathcal{U}, \quad \forall \rho_1, \rho_2 \in \mathcal{U},\ \forall A \in \mathcal{F}_\nu.$$

Let $\{X(\rho)\}_{\rho\in\mathcal{U}} \subset \mathrm{Dom}(\mathscr{E})$ be a family of random variables, indexed by ρ, such that for any $\nu,\sigma \in \mathcal{U}$, $\mathbf{1}_{\{\nu=\sigma\}}X(\nu) = \mathbf{1}_{\{\nu=\sigma\}}X(\sigma)$, a.s., then for any stable subclass $\mathscr{E}' = \{\mathcal{E}_i\}_{i\in\mathcal{I}'}$ of $\mathscr{E}$, there exist two sequences $\{(i_n,\rho_n)\}_{n\in\mathbb{N}}$ and $\{(i'_n,\rho'_n)\}_{n\in\mathbb{N}}$ in $\mathcal{I}'\times\mathcal{U}$ such that

$$\operatorname*{esssup}_{(i,\rho)\in\mathcal{I}'\times\mathcal{U}} \widetilde{\mathcal{E}}_i\big[X(\rho)+H^i_{\nu,\rho}|\mathcal{F}_\nu\big] = \lim_{n\to\infty}\uparrow \widetilde{\mathcal{E}}_{i_n}\big[X(\rho_n)+H^{i_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big], \quad \text{a.s.}, \tag{2.9}$$

$$\operatorname*{essinf}_{(i,\rho)\in\mathcal{I}'\times\mathcal{U}} \widetilde{\mathcal{E}}_i\big[X(\rho)+H^i_{\nu,\rho}|\mathcal{F}_\nu\big] = \lim_{n\to\infty}\downarrow \widetilde{\mathcal{E}}_{i'_n}\big[X(\rho'_n)+H^{i'_n}_{\nu,\rho'_n}|\mathcal{F}_\nu\big], \quad \text{a.s.} \tag{2.10}$$

For any $\nu\in\mathcal{S}_{0,T}$ and $i\in\mathcal{I}$, let us define

$$Z(\nu) \triangleq \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}} \widetilde{\mathcal{E}}_i[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu] \in \mathcal{F}_\nu \quad \text{and} \quad Z^i(\nu) \triangleq Z(\nu) + H^i_\nu.$$

Clearly, taking $\rho=\nu$ above yields that

$$Y_\nu \le Z(\nu), \quad \text{a.s.} \tag{2.11}$$

The following two lemmas give the bounds on $Z(\nu), Z^i(\nu),\ i\in\mathcal{I}$, and show that they all belong to $\mathrm{Dom}(\mathscr{E})$.

Lemma 2.3. *For any $\nu\in\mathcal{S}_{0,T}$ and $i\in\mathcal{I}$*

$$Z(\nu) \ge C_* \quad \textit{and} \quad Z^i(\nu) \ge C_Y + 2C_H, \quad \text{a.s.} \tag{2.12}$$

Moreover, if no member of $\mathscr{E}$ satisfies (2.5) of [1]*, then we further have*

$$Z(\nu) \le \zeta_Y - C_H \quad \textit{and} \quad Z^i(\nu) \le \zeta_Y - C_H + H^i_\nu, \quad \text{a.s.}, \tag{2.13}$$

where $\zeta_Y - C_H$ and $\zeta_Y - C_H + H^i_\nu$ both belong to $\mathrm{Dom}(\mathscr{E})$.

Lemma 2.4. *For any $\nu\in\mathcal{S}_{0,T}$ and $i\in\mathcal{I}$, both $Z(\nu)$ and $Z^i(\nu)$ belong to* $\mathrm{Dom}(\mathscr{E})$.

In the next two propositions, we will see that the $\mathbf{F}$-adapted process $\{Z(t)\}_{t\in[0,T]}$ has an RCLL modification Z^0, and that both $\{Z^i(t)\}_{t\in[0,T]}$ and $Z^{i,0} \triangleq \{Z^0_t + H^i_t\}_{t\in[0,T]}$ are $\widetilde{\mathcal{E}}_i$-supermartingales for any $i\in\mathcal{I}$.

Proposition 2.1. *For any $\nu,\sigma\in\mathcal{S}_{0,T}$ and $\gamma\in\mathcal{S}_{\nu,T}$, we have*

$$Z(\nu) = Z(\sigma), \quad \text{a.s. on } \{\nu=\sigma\}, \tag{2.14}$$

$$\operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[Z(\gamma)+H^i_{\nu,\gamma}|\mathcal{F}_\nu] = \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}} \widetilde{\mathcal{E}}_i[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu] \le Z(\nu), \quad \text{a.s.} \tag{2.15}$$

Proposition 2.2. *Given $i\in\mathcal{I}$, for any $\nu,\rho\in\mathcal{S}_{0,T}$ with $\nu\le\rho$, a.s., we have*

$$\widetilde{\mathcal{E}}_i[Z^i(\rho)|\mathcal{F}_\nu] \le Z^i(\nu), \quad \text{a.s.} \tag{2.16}$$

In particular, $\{Z^i(t)\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale. Moreover, the process $\{Z(t)\}_{t\in[0,T]}$ admits an RCLL modification Z^0. The process $Z^{i,0} \triangleq \{Z^0_t + H^i_t\}_{t\in[0,T]}$ is also an $\widetilde{\mathcal{E}}_i$-supermartingale.

We call Z^0 the "$\mathscr{E}$-upper Snell envelope" of the reward process Y. From (2.11) and their right-continuity, we see that Z^0 dominates Y in the following sense:

Definition 2.1. We say that process X "dominates" process X' if $P\big(X_t \ge X'_t, \forall t \in [0,T]\big) = 1$.

Remark 2.3. (1) If X dominates X', then $X_\nu \ge X'_\nu$, a.s. for any $\nu \in \mathcal{S}_{0,T}$.
(2) Let X and X' be two right-continuous $\mathbf{F}$-adapted processes. If $P(X_t \ge X'_t) = 1$ holds for all t in a countable dense subset of $[0,T]$, then X dominates X'.

The following proposition demonstrates that Z^0 is the smallest RCLL $\mathbf{F}$-adapted process dominating Y such that $Z^{i,0}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale for any $i \in \mathcal{I}$.

Proposition 2.3. *The process Z^0 dominates the process Y. Moreover, for any $\nu \in \mathcal{S}_{0,T}$ and $i \in \mathcal{I}$, we have $Z^0_\nu, Z^{i,0}_\nu \in \mathrm{Dom}(\mathscr{E})$ and*

$$Z^0_\nu = Z(\nu), \qquad Z^{i,0}_\nu = Z^i(\nu), \quad a.s. \tag{2.17}$$

Furthermore, if X is another RCLL $\mathbf{F}$-adapted process dominating Y such that $X^i \triangleq \{X_t + H^i_t\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale for any $i \in \mathcal{I}$, then X also dominates Z^0.

As a consequence of Proposition 2.3 and (2.12), we have for any $\nu \in \mathcal{S}_{0,T}$ and $i \in \mathcal{I}$ that

$$Z^0_\nu \ge C_*, \qquad Z^{i,0}_\nu \ge C_Y + 2C_H, \quad \text{a.s.} \tag{2.18}$$

In what follows, we first give *approximately* optimal stopping times. This family of stopping times will be crucial in finding an optimal stopping time for (2.1).

Definition 2.2. For any $\delta \in (0,1)$ and $\nu \in \mathcal{S}_{0,T}$, we define

$$\tau_\delta(\nu) \triangleq \inf\big\{t \in [\nu,T] : Y_t \ge \delta Z^0_t + (1-\delta)(C_Y + 2C_H)\big\} \wedge T \in \mathcal{S}_{\nu,T}$$

and

$$J_\delta(\nu) \triangleq \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu].$$

Remark 2.4. (1) For any $\delta \in (0,1)$ and $\nu \in \mathcal{S}_{0,T}$, the right-continuity of Y and Z^0 implies that $\{\tau_\delta(t)\}_{t\in[0,T]}$ is also a right-continuous process. Moreover, since $Z^0_T = Z(T) = Y_T$, a.s., we can deduce from (Y3) that $Y_T > \delta Z^0_T + (1-\delta)(C_Y + 2C_H)$. Then the right-continuity of processes Y and Z^0 implies that

$$Y_{\tau_\delta(\nu)} \ge \delta Z^0_{\tau_\delta(\nu)} + (1-\delta)(C_Y + 2C_H), \quad \text{a.s.}$$

(2) For any $\nu \in \mathcal{S}_{0,T}$, we can deduce from (2.17) and (2.15) that

$$\begin{aligned} J_\delta(\nu) &= \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] \\ &= \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z(\tau_\delta(\nu)) + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] \le Z(\nu) = Z^0_\nu, \quad \text{a.s.} \end{aligned} \tag{2.19}$$

The following two results show that $\forall \delta \in (0,1)$, $\{J_\delta(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$ possesses similar properties to $\{Z(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$.

Lemma 2.5. *For any $\delta \in (0,1)$ and $\nu \in \mathcal{S}_{0,T}$, we have $J_\delta(\nu) \in \mathrm{Dom}(\mathscr{E})$. And for any $\sigma \in \mathcal{S}_{0,T}$, $J_\delta(\nu) = J_\delta(\sigma)$, a.s. on $\{\nu = \sigma\}$.*

Proposition 2.4. *Given $\delta \in (0,1)$, the following statements hold:*

(1) *For any $i \in \mathcal{I}$, $\{J^i_\delta(t) \triangleq J_\delta(t) + H^i_t\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale.*

(2) *$\{J_\delta(t)\}_{t\in[0,T]}$ admits an RCLL modification $J^{\delta,0}$ such that the process $J^{\delta,i,0} \triangleq \{J^{\delta,0}_t + H^i_t\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale for each $i \in \mathcal{I}$.*

(3) *For any $\nu \in \mathcal{S}_{0,T}$, $J^{\delta,0}_\nu \in \mathrm{Dom}(\mathscr{E})$ and $J^{\delta,0}_\nu = J_\delta(\nu)$, a.s.*

Fix $\nu \in \mathcal{S}_{0,T}$. The right-continuity of Z^0 and (2.18) imply that the stopping times $\tau_\delta(\nu)$ are increasing in δ. Therefore, we can define the limiting stopping time

$$\overline{\tau}(\nu) \triangleq \lim_{\delta\nearrow 1} \tau_\delta(\nu). \tag{2.20}$$

To show that $\overline{\tau}(0) \in \mathcal{S}_{0,T}$ is an optimal stopping time for (2.1), we need the family of processes $\{Y^i\}_{i\in\mathcal{I}}$ to be uniformly continuous from the left over the stable class $\mathscr{E}$.

Definition 2.3. The family $\{Y^i\}_{i\in\mathcal{I}}$ is called "$\mathscr{E}$-uniformly-left-continuous" if for any $\nu,\rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s. and for any sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\nu,T}$ increasing a.s. to ρ, we can find a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{k\to\infty} \operatorname*{esssup}_{i\in\mathcal{I}} \left| \widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\rho_{n_k}} + H^i_{\rho_{n_k}} |\mathcal{F}_\nu\Big] - \widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_\nu\big]\right| = 0, \quad \text{a.s.} \tag{2.21}$$

The next theorem shows that $\overline{\tau}(\nu)$ is not only the first time when process Z^0 meets the process Y after ν, but it is also an optimal stopping time after ν. The assumption that the elements of the stable class $\mathscr{E}$ are convex (see (3.1) of [1]) becomes crucial in the proof of this result.

Theorem 2.1. *Assume that $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-left-continuous". Then for each $\nu \in \mathcal{S}_{0,T}$, the stopping time $\overline{\tau}(\nu)$ defined by (2.20) satisfies*

$$\begin{aligned} Z(\nu) &= \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] = \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] \\ &= \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \textit{a.s.} \end{aligned} \tag{2.22}$$

for any $\rho \in \mathcal{S}_{\nu,\overline{\tau}(\nu)}$ and $\overline{\tau}(\nu) = \tau_1(\nu) \triangleq \inf\big\{t \in [\nu,T] : Z^0_t = Y_t\big\}$, a.s.

Taking $\nu = 0$ in (2.22), one can deduce from (2.8) of [1] that $\overline{\tau}(0) = \inf\big\{t \in [0,T] : Z^0_t = Y_t\big\}$ satisfies

$$\begin{aligned} \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho + H^i_\rho] &= \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i[Y_\rho + H^i_\rho] \\ &= Z(0) = \sup_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_{\overline{\tau}(0)} + H^i_{\overline{\tau}(0)}\big] = \sup_{i\in\mathcal{I}} \mathcal{E}_i\big[Y_{\overline{\tau}(0)} + H^i_{\overline{\tau}(0)}\big]. \end{aligned}$$

Therefore, we see that $\overline{\tau}(0)$, the first time the Snell envelope Z^0 meets the process Y after time $t = 0$, is an optimal stopping time for (2.1).

3. Robust optimal stopping

In this section we analyze the *robust* optimal stopping problem in which the stopper aims to find an optimal stopping time τ_* that satisfies

$$\sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y^i_\rho] = \inf_{i\in\mathcal{I}} \inf \mathcal{E}_i[Y^i_{\tau_*}], \tag{3.1}$$

where $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ is a stable class of $\mathbf{F}$-expectations and $Y^i = Y + H^i, i \in \mathcal{I}$, is the model-dependent reward process introduced in (3.1). (We will modify the assumptions on the reward processes shortly.) In order to find an optimal stopping time we construct the lower and the upper values of the optimal stopping problem at any stopping time $\nu \in \mathcal{S}_{0,T}$, i.e.,

$$\underline{V}(\nu) \triangleq \underset{\rho\in\mathcal{S}_{\nu,T}}{\text{esssup}}\Big(\underset{i\in\mathcal{I}}{\text{essinf}}\, \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\Big),$$

$$\overline{V}(\nu) \triangleq \underset{i\in\mathcal{I}}{\text{essinf}}\Big(\underset{\rho\in\mathcal{S}_{\nu,T}}{\text{esssup}}\, \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\Big).$$

With the so-called "$\mathscr{E}$-uniform-right-continuity" condition on $\{Y^i\}_{i\in\mathcal{I}}$, we can show that at any $\nu \in \mathcal{S}_{0,T}$, $\underline{V}(\nu)$ and $\overline{V}(\nu)$ coincide with each other (see Theorem 3.1). We denote the common value, the *value* of the robust optimal stopping problem, as $V(\nu)$ at ν. We will show that up to a stopping time $\underline{\tau}(0)$ (see Lemma 3.2), at which we have $V(\underline{\tau}(0)) = Y_{\underline{\tau}(0)}$, a.s., the stopped value process $\big\{V\big(\underline{\tau}(0) \wedge t\big)\big\}_{t\in[0,T]}$ admits an RCLL modification V^0. The main result in this section, Theorem 3.2, shows that the first time V^0 meets Y is an optimal stopping time for (3.1).

Standing assumptions on $\mathscr{H}$ and Y in this section. Let us introduce

$$R^i(\nu) \triangleq \underset{\rho\in\mathcal{S}_{\nu,T}}{\text{esssup}}\, \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \text{for any } i \in \mathcal{I} \text{ and } \nu \in \mathcal{S}_{0,T}.$$

To adapt the results that we obtained for the family $\{Z(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$ to each family $\{R^i(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$, $i \in \mathcal{I}$, we assume that $\mathscr{H} = \{H^i\}_{i\in\mathcal{I}}$ is a family of right-continuous $\mathbf{F}$-adapted processes satisfying (S2), (S3) and,

(S1′) For any $i \in \mathcal{I}$, $H^i_0 = 0$, a.s. and (2.2) holds. If $\mathcal{E}_i$ does not satisfy (2.5) of [1], then we assume that $\zeta^i = \text{esssup}_{s,t\in\mathcal{D}_T;s<t} H^i_{s,t} \in \text{Dom}(\mathscr{E})$.

On the other hand, we assume that Y is a right-continuous $\mathbf{F}$-adapted process that satisfies (Y1), (Y3) and

(Y2′) For any $i \in \mathcal{I}, \sup_{\rho\in\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i\big[Y^i_\rho\big] < \infty$. If $\mathcal{E}_i$ does not satisfy (2.5) of [1], then $\text{esssup}_{(\rho,t)\in\mathcal{S}_{0,T}\times\mathcal{D}_T} \widetilde{\mathcal{E}}_i[Y^i_\rho|\mathcal{F}_t] \in \text{Dom}(\mathscr{E})$.

We also assume that for any $i \in \mathcal{I}$, Y^i is "quasi-left-continuous" under $\widetilde{\mathcal{E}}_i$: for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s. and for any sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\nu,T}$ increasing a.s. to ρ, we can find a subsequence $\big\{n_k = n^{(i)}_k\big\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{n\to\infty} \widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\rho_{n_k}} + H^i_{\rho_{n_k}}|\mathcal{F}_\nu\Big] = \widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_\nu\big], \quad \text{a.s.} \tag{3.2}$$

Remark 3.1. (S1′) and (Y2′) are analogous to (S1) and (Y2) respectively while the quasi-left-continuity (3.2) is the counterpart of the $\mathscr{E}$-uniform-left-continuity (2.21). It is obvious that (S1′)

implies (S1) and that (2.21) gives rise to (3.2). Moreover, (2.6) implies (Y2′): In fact, for any $i \in \mathcal{I}$, one can deduce from (2.8) that

$$C_* \le \operatorname*{esssup}_{(\rho,t)\in\mathcal{S}_{0,T}\times\mathcal{D}_T} \widetilde{\mathcal{E}}_i[Y^i_\rho|\mathcal{F}_t] \le \operatorname*{esssup}_{(i,\rho,t)\in\mathcal{I}\times\mathcal{S}_{0,T}\times\mathcal{D}_T} \widetilde{\mathcal{E}}_i[Y^i_\rho|\mathcal{F}_t], \quad \text{a.s.}$$

Then Lemma 3.2 of [1] implies that $\operatorname{esssup}_{(\rho,t)\in\mathcal{S}_{0,T}\times\mathcal{D}_T}\widetilde{\mathcal{E}}_i[Y^i_\rho|\mathcal{F}_t] \in \operatorname{Dom}(\mathscr{E})$, and it follows that $\sup_{\rho\in\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i\left[Y^i_\rho\right] < \infty$. □

Fix $i \in \mathcal{I}$. Applying Lemma 2.4, (2.7), (2.4), (2.15), Propositions 2.2 and 2.3 and Theorem 2.1 to the family $\{R^i(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$, we obtain:

Proposition 3.1. (1) *For any $\nu \in \mathcal{S}_{0,T}$, $R^i(\nu)$ belongs to* $\operatorname{Dom}(\mathscr{E})$ *and satisfies*

$$C_Y \le Y_\nu \le \operatorname*{esssup}_{\rho\in\mathcal{S}_{\nu,T}} \widetilde{\mathcal{E}}_i[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu] = R^i(\nu), \quad \text{a.s.,} \quad \text{thus } C_* \le Y^i_\nu, \text{ a.s.} \tag{3.3}$$

(2) *For any $\nu, \sigma \in \mathcal{S}_{0,T}$ and $\gamma \in \mathcal{S}_{\nu,T}$, we have*

$$R^i(\nu) = R^i(\sigma), \quad \text{a.s. on } \{\nu = \sigma\}, \tag{3.4}$$

$$\widetilde{\mathcal{E}}_i[R^i(\gamma) + H^i_{\nu,\gamma}|\mathcal{F}_\nu] = \operatorname*{esssup}_{\rho\in\mathcal{S}_{\gamma,T}} \widetilde{\mathcal{E}}_i[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu] \le R^i(\nu), \quad \text{a.s.} \tag{3.5}$$

(3) *The process $\left\{R^i(t)\right\}_{t\in[0,T]}$ admits an RCLL modification $R^{i,0}$, called the "$\mathcal{E}_i$ Snell envelope", such that $\{R^{i,0}_t + H^i_t\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale and that for any $\nu \in \mathcal{S}_{0,T}$*

$$R^{i,0}_\nu = R^i(\nu), \quad \text{a.s.} \tag{3.6}$$

(4) *For any $\nu \in \mathcal{S}_{0,T}$, $\tau^i(\nu) \triangleq \inf\{t \in [\nu, T] : R^{i,0}_t = Y_t\}$ is an optimal stopping time with respect to $\mathscr{E}^i$ after time ν, i.e., for any $\gamma \in \mathcal{S}_{\nu,\tau^i(\nu)}$,*

$$\begin{aligned} R^i(\nu) &= \widetilde{\mathcal{E}}_i\left[Y_{\tau^i(\nu)} + H^i_{\nu,\tau^i(\nu)}|\mathcal{F}_\nu\right] = \widetilde{\mathcal{E}}_i\left[R^i(\tau^i(\nu)) + H^i_{\nu,\tau^i(\nu)}|\mathcal{F}_\nu\right] \\ &= \widetilde{\mathcal{E}}_i\left[R^i(\gamma) + H^i_{\nu,\gamma}|\mathcal{F}_\nu\right], \quad \text{a.s.} \end{aligned} \tag{3.7}$$

Corollary 3.1. *For any $\nu \in \mathcal{S}_{0,T}$, both $\underline{V}(\nu)$ and $\overline{V}(\nu)$ belong to* $\operatorname{Dom}(\mathscr{E})$.

Proof. Fix $(l, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$; for any $i \in \mathcal{I}$, (2.7), (2.4) and Proposition 2.7(5) of [1] imply that

$$\widetilde{\mathcal{E}}_i\left[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\right] \ge \widetilde{\mathcal{E}}_i\left[C_Y + C_H|\mathcal{F}_\nu\right] = C_*, \quad \text{a.s.}$$

Taking the essential infimum over $i \in \mathcal{I}$ on the left-hand side yields that

$$\begin{aligned} C_* &\le \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\left[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\right] \le \operatorname*{esssup}_{\rho\in\mathcal{S}_{\nu,T}}\left(\operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\left[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\right]\right) \\ &= \underline{V}(\nu) \le \overline{V}(\nu) = \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\nu) \le R^l(\nu), \quad \text{a.s.} \end{aligned}$$

Since $R^l(\nu) \in \operatorname{Dom}(\mathscr{E})$ by Proposition 3.1(1), a simple application of Lemma 3.2 of [1] proves the corollary. □

As we will see in the next lemma, since the stable class $\mathscr{E}$ is closed under pasting (see Definition 3.2(2) of [1]), $\overline{V}(\nu)$ can be approximated by a decreasing sequence that belongs to the family $\{R^i(\nu)\}_{i\in\mathcal{I}}$.

Lemma 3.1. *For any $\nu \in \mathcal{S}_{0,T}$, there exists a sequence $\{i_n\}_{n\in\mathbb{N}} \subset \mathcal{I}$ such that*

$$\overline{V}(\nu) = \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\nu) = \lim_{n\to\infty} \downarrow R^{i_n}(\nu), \quad a.s. \tag{3.8}$$

Thanks again to the stability of $\mathscr{E}$ under pasting, the infimum of the family $\{\tau^i(\nu)\}_{i\in\mathcal{I}}$ of optimal stopping times can be approached by a decreasing sequence in the family. As a result the infimum is also a stopping time.

Lemma 3.2. *For any $\nu \in \mathcal{S}_{0,T}$, there exists a sequence $\{i_n\}_{n\in\mathbb{N}} \subset \mathcal{I}$ such that*

$$\underline{\tau}(\nu) \triangleq \operatorname*{essinf}_{i\in\mathcal{I}} \tau^i(\nu) = \lim_{n\to\infty} \downarrow \tau^{i_n}(\nu), \quad a.s., \quad thus\ \underline{\tau}(\nu) \in \mathcal{S}_{\nu,T}.$$

The family of stopping times $\{\underline{\tau}(\nu)\}_{\nu\in\mathcal{S}_{0,T}}$ will play a critical role in this section. The next lemma basically shows that if $\widetilde{\mathcal{E}}_j$ and $\widetilde{\mathcal{E}}_k$ behave in the same way after some stopping time ν, then $R^{j,0}$ and $R^{k,0}$ are identical after ν:

Lemma 3.3. *For any $i, j \in \mathcal{I}$ and $\nu \in \mathcal{S}_{0,T}$, let $k = k(i, j, \nu) \in \mathcal{I}$ as in Definition 3.2 of* [1]. *For any $\sigma \in \mathcal{S}_{\nu,T}$, we have $R^{k,0}_\sigma = R^k(\sigma) = R^j(\sigma) = R^{j,0}_\sigma$, a.s.*

Proof. For any $\rho \in \mathcal{S}_{\sigma,T}$, applying Proposition 2.7(5) of [1] to $\widetilde{\mathcal{E}}_i$, we can deduce from (2.5) and (3.3) of [1] that

$$\begin{aligned}\widetilde{\mathcal{E}}_k\big[Y_\rho + H^k_{\sigma,\rho}|\mathcal{F}_\sigma\big] &= \widetilde{\mathcal{E}}_k\big[Y_\rho + H^j_{\sigma,\rho}|\mathcal{F}_\sigma\big] = \mathcal{E}^\nu_{i,j}\big[Y_\rho + H^j_{\sigma,\rho}|\mathcal{F}_\sigma\big]\\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\sigma,\rho}|\mathcal{F}_{\nu\vee\sigma}\big]|\mathcal{F}_\sigma\big]\\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\sigma,\rho}|\mathcal{F}_\sigma\big]|\mathcal{F}_\sigma\big] = \widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\sigma,\rho}|\mathcal{F}_\sigma\big], \quad \text{a.s.}\end{aligned}$$

Then (3.6) implies that

$$\begin{aligned}R^{k,0}_\sigma &= R^k(\sigma) = \operatorname*{esssup}_{\rho\in\mathcal{S}_{\sigma,T}} \widetilde{\mathcal{E}}_k\big[Y_\rho + H^k_{\sigma,\rho}|\mathcal{F}_\sigma\big] = \operatorname*{esssup}_{\rho\in\mathcal{S}_{\sigma,T}} \widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\sigma,\rho}|\mathcal{F}_\sigma\big]\\ &= R^j(\sigma) = R^{j,0}_\sigma, \quad \text{a.s.,}\end{aligned}$$

which proves the lemma. □

We now introduce the notion of the uniform-right-continuity of the family $\{Y^i\}_{i\in\mathcal{I}}$ over the stable class $\mathscr{E}$. With this assumption on the reward processes, we can show that at any $\nu \in \mathcal{S}_{0,T}$, $\underline{V}(\nu) = \overline{V}(\nu)$, a.s.; thus the robust optimal stopping problem has a value $V(\nu)$ at ν.

Definition 3.1. The family $\{Y^i\}_{i\in\mathcal{I}}$ is called "$\mathscr{E}$-uniformly-right-continuous" if for any $\nu \in \mathcal{S}_{0,T}$ and for any sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\nu,T}$ decreasing a.s. to ν, we can find a subsequence of $\{\nu_n\}_{n\in\mathbb{N}}$ (we still denote it by $\{\nu_n\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty} \operatorname{esssup}_{i\in\mathcal{I}} \big|\widetilde{\mathcal{E}}_i[Y^i_{\nu_n}|\mathcal{F}_\nu] - Y^i_\nu\big| = 0$, a.s.

Theorem 3.1. *Suppose that $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-right-continuous". Then for any $\nu \in \mathcal{S}_{0,T}$, we have*

$$\underline{V}(\nu) = \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_{\underline{\tau}(\nu)} + H^i_{\nu,\underline{\tau}(\nu)}|\mathcal{F}_\nu\big] = \overline{V}(\nu) \ge Y_\nu, \quad a.s. \tag{3.9}$$

We will denote the common value by $V(\nu)(=\underline{V}(\nu) = \overline{V}(\nu))$. Observe that $\underline{\tau}(0)$ is optimal for the robust optimal stopping problem in (3.1).

Standing assumption on Y for the rest of this section. We assume that the family of processes $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-right-continuous".

Proposition 3.2. *For any $\nu \in \mathcal{S}_{0,T}$, we have $V(\underline{\tau}(\nu)) = Y_{\underline{\tau}(\nu)}$, a.s.*

Note that $\underline{\tau}(\nu)$ may not be the first time after ν when the value of the robust optimal stopping problem is equal to the primary reward. Actually, since the process $\{V(t)\}_{t\in[0,T]}$ is not necessarily right-continuous, $\inf\{t \in [\nu, T] \mid V(t) = Y_t\}$ may not even be a stopping time. We will address this issue in the next two results.

Proposition 3.3. *Given $i \in \mathcal{I}$, for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s., we have*

$$\operatorname*{essinf}_{k\in\mathcal{I}} \widetilde{\mathcal{E}}_k[V^k(\rho)|\mathcal{F}_\nu] \le V^i(\nu), \quad a.s., \tag{3.10}$$

where $V^i(\nu) \triangleq V(\nu) + H^i_\nu \in \mathrm{Dom}(\mathscr{E})$. Moreover if $\rho \le \underline{\tau}(\nu)$, a.s., then

$$\widetilde{\mathcal{E}}_i[V^i(\rho)|\mathcal{F}_\nu] \ge V^i(\nu), \quad a.s. \tag{3.11}$$

In particular, the stopped process $\left\{V^i\big(\underline{\tau}(0) \wedge t\big)\right\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-submartingale.

Now we show that the stopped value process $\left\{V\big(\underline{\tau}(0) \wedge t\big)\right\}_{t\in[0,T]}$ admits an RCLL modification V^0. As a result, the first time when the process V^0 meets the process Y after time $t = 0$ is an optimal stopping time of the robust optimal stopping problem.

Theorem 3.2. *Assume that for some $i' \in \mathcal{I}$, $\zeta^{i'} = \operatorname{esssup}_{s,t\in\mathcal{D}_T; s<t} H^{i'}_{s,t} \in \mathrm{Dom}(\mathscr{E})$ and that there exists a concave* **F***-expectation $\mathcal{E}'$ (not necessarily in $\mathscr{E}$) satisfying* (H0) *and* (H1) *such that*

$$\mathrm{Dom}(\mathcal{E}') \supset \{-\xi : \xi \in \mathrm{Dom}(\mathscr{E})\} \quad \text{and} \quad \text{for every } \widetilde{\mathcal{E}}_{i'}\text{-submartingale } X, \\ -X \text{ is an } \mathcal{E}'\text{-supermartingale}. \tag{3.12}$$

We also assume that for any $\rho \in \mathcal{S}_{0,T}$, there exists a $j = j(\rho) \in \mathcal{I}$ such that $\operatorname{esssup}_{t\in\mathcal{D}_T} \widetilde{\mathcal{E}}_j [Y^j_\rho|\mathcal{F}_t] \in \mathrm{Dom}(\mathscr{E})$.

(1) *Then the stopped value process $\left\{V\big(\underline{\tau}(0)\wedge t\big)\right\}_{t\in[0,T]}$ admits an RCLL modification V^0 (called the "$\mathscr{E}$-lower Snell envelope" of Y) such that for any $\nu \in \mathcal{S}_{0,T}$*

$$V^0_\nu = V(\underline{\tau}(0) \wedge \nu), \quad a.s. \tag{3.13}$$

(2) *Consequently,*

$$\tau_V \triangleq \inf\{t \in [0, T] : V^0_t = Y_t\} \tag{3.14}$$

is a stopping time. In fact, it is an optimal stopping time of (3.1).

4. Remarks on Sections 2 and 3

Remark 1. Let $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ be a stable class of **F**-expectations. For any $\xi \in \mathrm{Dom}(\mathscr{E})$ and $\nu \in \mathcal{S}_{0,T}$, we define

$$\overline{\mathscr{E}}[\xi|\mathcal{F}_\nu] \triangleq \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_\nu] \quad \text{and} \quad \underline{\mathscr{E}}[\xi|\mathcal{F}_\nu] \triangleq \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_\nu]$$

as the maximal and minimal expectation of ξ over $\mathscr{E}$ at the stopping time ν. It is worth pointing out that $\overline{\mathscr{E}}$ is not an $\mathcal{F}$-expectation on $\mathrm{Dom}(\mathscr{E})$ simply because $\overline{\mathscr{E}}[\xi|\mathcal{F}_t]$ may not belong to $\mathrm{Dom}(\mathscr{E})$ for some $\xi \in \mathrm{Dom}(\mathscr{E})$ and $t \in [0, T]$. On the other hand, we will see in Example 4.1 that neither $\overline{\mathscr{E}}$ nor $\underline{\mathscr{E}}$ satisfies strict monotonicity. Moreover, as we shall see in the same example, $\overline{\mathscr{E}}$ does not satisfy (H2) while $\underline{\mathscr{E}}$ does not satisfy (H1); thus we do not have a dominated convergence theorem for either $\overline{\mathscr{E}}$ or $\underline{\mathscr{E}}$. Note also that $\underline{\mathscr{E}}$ may not even be convex.

Our results in Sections 2 and 3 can be interpreted as a first step in extending the results for the optimal stopping problem $\sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho]$, in which $\mathcal{E}_i$ $(i\in\mathcal{I})$ is an **F**-expectation satisfying positive convexity and the assumptions (H1)–(H3), to optimal stopping problems for other non-linear expectations, such as $\overline{\mathscr{E}}$ and $\underline{\mathscr{E}}$, which may fail to satisfy these assumptions.

Example 4.1. Consider the probability space $\big([0,\infty),\mathscr{B}[0,\infty),\mathbf{F}=\{\mathcal{F}_t\}_{t\ge 0},P\big)$ to be a filtered probability space in which P is defined by $P(A)\triangleq\int_A e^{-x}dx$, $\forall A\in\mathscr{B}[0,\infty)$. We assume that the filtration **F** satisfies the usual hypothesis. Let $\mathcal{P}$ denote the set of all probability measures equivalent to P. For any $n\in\mathbb{N}$, we define a $P_n\in\mathcal{P}$ by $P_n(A)\triangleq n\int_A e^{-nx}dx$, $\forall A\in\mathscr{B}[0,\infty)$. As discussed in Example 3.1 of [1], $\mathscr{E}=\{E_Q\}_{Q\in\mathcal{P}}$ is a stable class. For any $h>0$, one can deduce that

$$\begin{aligned}1&=\sup_{Q\in\mathcal{P}}E_Q[1]\ge\overline{\mathscr{E}}\big[\mathbf{1}_{[0,h]}\big]=\sup_{Q\in\mathcal{P}}E_Q\big[\mathbf{1}_{[0,h]}\big]\\&\ge\sup_{n\in\mathbb{N}}E_{P_n}\big[\mathbf{1}_{[0,h]}\big]=\sup_{n\in\mathbb{N}}P_n[0,h]=\lim_{n\in\mathbb{N}}\big(1-e^{-nh}\big)=1,\end{aligned}$$

where we used the fact that $\widetilde{E}_Q=E_Q$ for any $Q\in\mathcal{P}$ since $E_Q[\xi|\mathcal{F}_\cdot]$ is an RCLL process for any $\xi\in L^1\big([0,\infty),\mathscr{B}[0,\infty),P\big)$. Hence, we have $\overline{\mathscr{E}}\big[\mathbf{1}_{[0,h]}\big]=1$, $\forall h>0$, which implies that $\overline{\mathscr{E}}$ does not satisfy strict monotonicity.

Moreover, $\overline{\mathscr{E}}$ does not satisfy (H2). For $\xi=0$, $\eta=1$ and $A_n=\left[0,\frac{1}{n}\right]$, $n\in\mathbb{N}$, it follows that

$$\lim_{n\to\infty}\downarrow\overline{\mathscr{E}}[\xi+\mathbf{1}_{A_n}\eta]=\lim_{n\to\infty}\overline{\mathscr{E}}[\mathbf{1}_{[0,\frac{1}{n}]}]=1\neq 0=\sup_{Q\in\mathcal{P}}\widetilde{E}_Q[0]=\overline{\mathscr{E}}[0]=\overline{\mathscr{E}}[\xi].$$

On the other hand, it is simple to see that $\underline{\mathscr{E}}[\mathbf{1}_{[h,\infty)}]=0$ for any $h>0$, which means that $\underline{\mathscr{E}}$ does not satisfy strict monotonicity either. Furthermore, $\underline{\mathscr{E}}$ does not satisfy (H1). For $\xi=1$ and $A_n=\left[\frac{1}{n},\infty\right)$, $n\in\mathbb{N}$, we have that

$$\lim_{n\to\infty}\uparrow\underline{\mathscr{E}}[\mathbf{1}_{A_n}\xi]=\lim_{n\to\infty}\underline{\mathscr{E}}[\mathbf{1}_{[\frac{1}{n},\infty)}]=0\neq 1=\inf_{Q\in\mathcal{P}}\widetilde{E}_Q[1]=\underline{\mathscr{E}}[1]=\underline{\mathscr{E}}[\xi].\quad\square$$

Although it does not satisfy strict monotonicity, $\underline{\mathscr{E}}$ is almost an **F**-expectation on $\mathrm{Dom}(\mathscr{E})$ as we will see next.

Proposition 4.1. *For any $t\in[0,T]$, $\underline{\mathscr{E}}[\cdot|\mathcal{F}_t]$ is an operator from* $\mathrm{Dom}(\mathscr{E})$ *to* $\mathrm{Dom}_t(\mathscr{E})\triangleq\mathrm{Dom}(\mathscr{E})\cap L^0(\mathcal{F}_t)$. *Moreover, the family of operators $\big\{\underline{\mathscr{E}}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ satisfies* (A2)–(A4) *as well as*

$$\underline{\mathscr{E}}[\xi|\mathcal{F}_t]\le\underline{\mathscr{E}}[\eta|\mathcal{F}_t],\quad a.s.\quad\text{for any }\xi,\ \eta\in\mathrm{Dom}(\mathscr{E})\text{ with }\xi\le\eta,\ a.s.\tag{4.1}$$

Remark 2. We have found that the first time $\overline{\tau}(0)$ when the Snell envelope Z^0 meets the process Y is an optimal stopping time for (2.1) while the first time τ_V when the process V^0 meets the process Y is an optimal stopping time for (3.1). It is natural to ask whether $\overline{\tau}(0)$ (resp. τ_V) is the minimal optimal stopping time of (2.1) (resp. (3.1)). This answer is affirmative when $\mathscr{E}$ is a singleton. Let $\mathcal{E}$ be a positively-convex **F**-expectation satisfying (H1)–(H3) and let Y be a right-continuous **F**-adapted process satisfying (Y1), (Y3) and the following:

$$\sup_{\rho\in\mathcal{S}_{0,T}}\mathcal{E}\left[Y_\rho\right]<\infty;\quad\text{if }\mathscr{E}\text{ does not satisfy (2.5) of [1], then}$$

$$\operatorname*{esssup}_{(\rho,t)\in\mathcal{S}_{0,T}\times\mathcal{D}_T}\widetilde{\mathcal{E}}[Y_\rho|\mathcal{F}_t]\in\mathrm{Dom}^{\#}(\mathcal{E}).$$

(Note that we have here merged the cumulative reward process H into the primary reward process Y.) If $\tau \in \mathcal{S}_{0,T}$ is an optimal stopping time for (2.1), i.e. $\sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}[Y_\rho] = \mathcal{E}[Y_\tau]$, Proposition 2.2 and (2.17) imply that

$$\begin{aligned}\sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}[Y_\rho] &= \sup_{\rho\in\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}[Y_\rho] = Z(0) \geq \widetilde{\mathcal{E}}[Z(\tau)] = \widetilde{\mathcal{E}}\big[Z^0_\tau\big] = \mathcal{E}\big[Z^0_\tau\big] \geq \mathcal{E}[Y_\tau]\\ &= \sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}[Y_\rho],\end{aligned}$$

and thus $\mathcal{E}[Z^0_\tau] = \mathcal{E}[Y_\tau]$. The second part of (A1) then implies that $Z^0_\tau = Y_\tau$, a.s. Hence $\overline{\tau}(0) \leq \tau$, a.s., which means that $\overline{\tau}(0)$ is the minimal stopping time for (2.1).

However, this is not the case in general. Let $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ be a stable class of $\mathbf{F}$-expectations and let Y be a right-continuous $\mathbf{F}$-adapted process satisfying (Y1)–(Y3). We take $H^i \equiv 0$ for any $i \in \mathcal{I}$. If $\tau \in \mathcal{S}_{0,T}$ is an optimal stopping time for (2.1), i.e. $\sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho] = \sup_{i\in\mathcal{I}} \mathcal{E}_i[Y_\tau]$, (2.15) and (2.17) then imply that

$$\begin{aligned}\sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho] &= \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i[Y_\rho] = Z(0) \geq \sup_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[Z(\tau)] = \sup_{i\in\mathcal{I}} \mathcal{E}_i[Z(\tau)]\\ &= \sup_{i\in\mathcal{I}} \mathcal{E}_i\big[Z^0_\tau\big] \geq \sup_{i\in\mathcal{I}} \mathcal{E}_i[Y_\tau] = \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho],\end{aligned}$$

and thus $\overline{\mathscr{E}}\big[Z^0_\tau\big] = \sup_{i\in\mathcal{I}} \mathcal{E}_i\big[Z^0_\tau\big] = \sup_{i\in\mathcal{I}} \mathcal{E}_i[Y_\tau] = \overline{\mathscr{E}}[Y_\tau]$. However, this may not imply that $Z^0_\tau = Y_\tau$, a.s. since $\overline{\mathscr{E}}$ does not satisfy strict monotonicity as we have seen in Example 4.1.

Now we further assume that Y satisfies (Y2′); if $\tau' \in \mathcal{S}_{0,T}$ is an optimal stopping time for (3.1), i.e. $\sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y_\rho] = \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y_{\tau'}]$, (3.10) and Theorem 3.1 imply that

$$\begin{aligned}\sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y_\rho] &= \sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[Y_\rho] = \underline{V}(0) = V(0) \geq \inf_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[V(\tau')] = \inf_{i\in\mathcal{I}} \mathcal{E}_i[V(\tau')]\\ &\geq \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y_{\tau'}] = \sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y_\rho],\end{aligned}$$

and thus $\underline{\mathscr{E}}[V(\tau')] = \inf_{i\in\mathcal{I}} \mathcal{E}_i[V(\tau')] = \inf_{i\in\mathcal{I}} \mathcal{E}_i[Y_{\tau'}] = \underline{\mathscr{E}}[Y_{\tau'}]$. However, this may not imply that $V(\tau') = Y_{\tau'}$, a.s. since $\underline{\mathscr{E}}$ does not satisfy strict monotonicity, which we have also seen in Example 4.1. (If $V(\tau')$ were a.s. equal to $Y_{\tau'}$, for any $i \in \mathcal{I}$, applying (2.14) to singleton $\{\mathcal{E}_i\}$, we would deduce from (3.13) and Lemma 3.3 of [1] that

$$\begin{aligned}V^0_{\tau'\wedge\tau_V} &= V(\tau'\wedge\tau_V) = \overline{V}(\tau'\wedge\tau_V)\\ &= \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\tau'\wedge\tau_V) = \operatorname*{essinf}_{i\in\mathcal{I}}\big(\mathbf{1}_{\{\tau'\leq\tau_V\}} R^i(\tau') + \mathbf{1}_{\{\tau'>\tau_V\}} R^i(\tau_V)\big)\\ &= \mathbf{1}_{\{\tau'\leq\tau_V\}} \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\tau') + \mathbf{1}_{\{\tau'>\tau_V\}} \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\tau_V)\\ &= \mathbf{1}_{\{\tau'\leq\tau_V\}} \overline{V}(\tau') + \mathbf{1}_{\{\tau'>\tau_V\}} \overline{V}(\tau_V)\\ &= \mathbf{1}_{\{\tau'\leq\tau_V\}} V(\tau') + \mathbf{1}_{\{\tau'>\tau_V\}} V(\tau_V) = \mathbf{1}_{\{\tau'\leq\tau_V\}} V(\tau') + \mathbf{1}_{\{\tau'>\tau_V\}} V^0_{\tau_V}\\ &= \mathbf{1}_{\{\tau'\leq\tau_V\}} Y_{\tau'} + \mathbf{1}_{\{\tau'>\tau_V\}} Y_{\tau_V} = Y_{\tau'\wedge\tau_V}, \quad \text{a.s.,}\end{aligned}$$

which would further imply that $\tau_V = \tau'\wedge\tau_V$, a.s., and thus $\tau_V \leq \tau'$, a.s.)

5. Applications

In this section, we take a d-dimensional Brownian motion B on the probability space $(\Omega, \mathcal{F}, P)$ and consider the Brownian filtration generated by it:

$$\mathbf{F} = \left\{\mathcal{F}_t \triangleq \sigma\Big(\sigma\big(B_s; s \in [0,t]\big) \cup \mathscr{N}\Big)\right\}_{t\in[0,T]},$$
$$\text{where } \mathscr{N} \text{ collects all } P\text{-null sets in } \mathcal{F}. \tag{5.1}$$

We also let $\mathscr{P}$ denote the predictable σ-algebra with respect to $\mathbf{F}$.

5.1. *Lipschitz g-expectations*

Suppose that a "generator" function $g = g(t, \omega, z) : [0, T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfies

$$\begin{cases} \text{(i) } g(t,\omega,0) = 0, \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.} \\ \text{(ii) } g \text{ is Lipschitz in } z \text{ for some } K_g > 0: \text{ it holds } \mathrm{d}t \times \mathrm{d}P\text{-a.s. that} \\ |g(t,\omega,z_1) - g(t,\omega,z_2)| \le K_g|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^d. \end{cases} \tag{5.2}$$

For any $\xi \in L^2(\mathcal{F}_T)$, it is well-known from [12] that the backward stochastic differential equation (BSDE)

$$\Gamma_t = \xi + \int_t^T g(s, \Theta_s)\mathrm{d}s - \int_t^T \Theta_s \mathrm{d}B_s, \quad t \in [0, T] \tag{5.3}$$

admits a unique solution $\left(\Gamma^{\xi,g}, \Theta^{\xi,g}\right) \in \mathbb{C}^2_{\mathbf{F}}([0,T]) \times \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ (for convenience, we will denote (5.3) by BSDE(ξ, g) in the sequel), based on which [13] introduced the so-called "g-expectation" of ξ by

$$\mathcal{E}_g[\xi|\mathcal{F}_t] \triangleq \Gamma^{\xi,g}_t, \quad t \in [0, T]. \tag{5.4}$$

To show that the g-expectation $\mathcal{E}_g$ is an $\mathbf{F}$-expectation with domain $\mathrm{Dom}(\mathcal{E}_g) = L^2(\mathcal{F}_T)$, we first note that $L^2(\mathcal{F}_T) \in \widetilde{\mathscr{D}}_T$. The (strict) Comparison Theorem for BSDEs (see e.g. [13, Theorem 35.3]) then shows that (A1) holds for the family of operators $\left\{\mathcal{E}_g[\cdot|\mathcal{F}_t] : L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)\right\}_{t\in[0,T]}$, while the uniqueness of the solution $(\Gamma^{\xi,g}, \Theta^{\xi,g})$ to the BSDE(ξ, g) implies that the family $\left\{\mathcal{E}_g[\cdot|\mathcal{F}_t]\right\}_{t\in[0,T]}$ satisfies (A2)–(A4) (see e.g. [13, Lemma 36.6] and [5, Lemma 2.1]). Therefore, $\mathcal{E}_g$ is an $\mathbf{F}$-expectation with domain $\mathrm{Dom}(\mathcal{E}_g) = L^2(\mathcal{F}_T)$.

Moreover, the generator g characterizes $\mathcal{E}_g$ in the following ways:

(1) Theorem 3.2 of [8] (see also Proposition 10 of [16]) shows that $\mathcal{E}_g[\cdot|\mathcal{F}_t]$ is a convex (resp. concave) operator on $L^2(\mathcal{F}_T)$ for any $t \in [0, T]$ if and only if the generator g is *convex* (resp. *concave*) in z, i.e., it holds $\mathrm{d}t \times \mathrm{d}P$-a.s. that

$$g(t, \lambda z_1 + (1-\lambda)z_2) \le (\text{resp. } \ge)\lambda g(t, z_1) + (1-\lambda)g(t, z_2),$$
$$\forall \lambda \in (0,1),\ \forall z_1, z_2 \in \mathbb{R}^d. \tag{5.5}$$

(2) Let $\tilde{g}$ be another generator satisfying (5.2). If it holds $\mathrm{d}t \times \mathrm{d}P$-a.s. that $g(t,z) \ge \tilde{g}(t,z)$ for any $z \in \mathbb{R}^d$, then the Comparison Theorem for BSDEs (see e.g. [6]) shows that for any $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0, T]$

$$\mathcal{E}_g[\xi|\mathcal{F}_t] \ge \mathcal{E}_{\tilde{g}}[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{5.6}$$

In light of Theorem 4.1 of [3], the reverse statement also holds given that almost surely, the mapping $t \to g(t,z)$ is continuous for any $z \in \mathbb{R}^d$.

(3) $g^-(t,\omega,z) \triangleq -g(t,\omega,-z), (t,\omega,z) \in [0,T] \times \Omega \times \mathbb{R}^d$ also satisfies (5.2). Its corresponding g-expectation $\mathcal{E}_{g^-}$ relates to $\mathcal{E}_g$ in that for any $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0,T]$

$$\mathcal{E}_{g^-}[\xi|\mathcal{F}_t] = -\mathcal{E}_g[-\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{5.7}$$

(In fact, multiplying both sides of BSDE$(-\xi, g)$ by -1 makes $\left(-\Gamma^{-\xi,g}, -\Theta^{-\xi,g}\right)$ solve the BSDE(ξ, g^-).)

To show that the g-expectation $\mathcal{E}_g$ satisfies (H0)–(H3), we need two basic inequalities that it satisfies.

Lemma 5.1. *Let g be a generator satisfying (5.2).*

(1) *For any $\xi \in L^2(\mathcal{F}_T)$, we have*

$$\left\| \sup_{t\in[0,T]} \left|\mathcal{E}_g[\xi|\mathcal{F}_t]\right| \right\|_{L^2(\mathcal{F}_T)} + \left\| \Theta^{\xi,g} \right\|_{L^2_{\mathbf{F}}([0,T];\mathbb{R}^d)} \le C e^{(K_g+K_g^2)T} \|\xi\|_{L^2(\mathcal{F}_T)},$$

where C is a universal constant independent of ξ and g.

(2) *For any $\mu \ge K_g$ and $\xi, \eta \in L^2(\mathcal{F}_T)$, it holds a.s. that*

$$\left|\mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta|\mathcal{F}_t]\right| \le \mathcal{E}_{g_\mu}\left[|\xi-\eta|\,|\mathcal{F}_t\right], \quad \forall t \in [0,T],$$

where the generator g_μ is defined by $g_\mu(z) \triangleq \mu|z|, z \in \mathbb{R}^d$.

Proof. A simple application of [3, Proposition 2.2] yields (1). On the other hand, (2) is a mere generalization of [14, Proposition 3.7, inequality (60)] obtained by taking into account the continuity of processes $\mathcal{E}_g[\xi|\mathcal{F}_\cdot]$ and $\mathcal{E}_{g_\mu}[\xi|\mathcal{F}_\cdot]$ for any $\xi \in L^2(\mathcal{F}_T)$. □

Proposition 5.1. *Let g be a generator satisfying (5.2). Then $\mathcal{E}_g$ satisfies* (H0)–(H3).

Remark 5.1. Since $\mathcal{E}_g[\xi|\mathcal{F}_\cdot]$ is a continuous process for any $\xi \in L^2(\mathcal{F}_T)$, we see from (2.6) of [1] that $\widetilde{\mathcal{E}}_g[\cdot|\mathcal{F}_\nu]$ is just a restriction of $\mathcal{E}_g[\cdot|\mathcal{F}_\nu]$ to $L^{2,\#}(\mathcal{F}_T) \triangleq \{\xi \in L^2(\mathcal{F}_T) : \xi \ge c$, a.s. for some $c = c(\xi) \in \mathbb{R}\}$ for any $\nu \in \mathcal{S}_{0,T}$.

Thanks to Proposition 5.1, all results on **F**-expectations $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ in Section 2 of [1] are applicable to g-expectations. In the following example we deliver the promise that we made in Remark 2.7 of [1]. This example indicates that for some g-expectations, $\underline{\lim}_{n\to\infty}\mathcal{E}_g[\xi_n] < \infty$ is not a sufficient condition for $\lim_{n\to\infty}\xi_n \in \mathrm{Dom}^+(\mathcal{E}_g) = L^{2,+}(\mathcal{F}_T) \triangleq \{\xi \in L^2(\mathcal{F}_T) : \xi \ge 0,\ \text{a.s.}\}$ given that $\{\xi_n\}_{n\in\mathbb{N}}$ is an a.s. convergent sequence in $\mathrm{Dom}^+(\mathcal{E}_g)$.

Example 5.1. Consider a probability space $([0,1], \mathscr{B}[0,1], \lambda)$, where λ is the Lebesgue measure on $[0,1]$. We define a generator $\tilde{g}(z) \triangleq -|z|, z \in \mathbb{R}^d$. For any $n \in \mathbb{N}$, it is clear that the random variable $\left\{\xi_n(\omega) \triangleq \omega^{-\frac{1}{2}+\frac{1}{n+2}}\right\}_{\omega\in[0,1]} \in L^{2,+}(\mathcal{F}_T) = \mathrm{Dom}^+(\tilde{g})$. Proposition 2.2(2) of [1] then implies that

$$\begin{aligned} 0 = \mathcal{E}_{\tilde{g}}[0] \le \mathcal{E}_{\tilde{g}}[\xi_n] = \Gamma_0^{\xi_n,\tilde{g}} &= \xi_n - \int_0^T |\Theta_s^{\xi_n,\tilde{g}}|\mathrm{d}s - \int_0^T \Theta_s^{\xi_n,\tilde{g}}\mathrm{d}B_s \\ &\le \xi_n - \int_0^T \Theta_s^{\xi_n,\tilde{g}}\mathrm{d}B_s. \end{aligned}$$

Taking the expected value of the above inequality yields that

$$0 \le \mathcal{E}_{\tilde{g}}[\xi_n] \le E\Big[\xi_n - \int_0^T \Theta_s^{\xi_n,\tilde{g}} \mathrm{d}B_s\Big] = E[\xi_n] = \int_0^1 \omega^{-\frac{1}{2}+\frac{1}{n+2}} \mathrm{d}\omega = \frac{1}{\frac{1}{2}+\frac{1}{n+2}} < 2. \quad (5.8)$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ is an increasing sequence, we can deduce from (A1) and (5.8) that $0 \le \lim_{n\to\infty} \uparrow \mathcal{E}_{\tilde{g}}[\xi_n] \le 2$. However, $\lim_{n\to\infty} \uparrow \xi_n = \big\{\omega^{-\frac{1}{2}}\big\}_{\omega\in[0,1]}$ does not belong to $L^{2,+}(\mathcal{F}_T) = \mathrm{Dom}^+(\tilde{g})$. $\square$

Like in Proposition 3.1 of [1], pasting two g-expectations at any stopping time generates another g-expectation.

Proposition 5.2. *Let g_1, g_2 be two generators satisfying (5.2) with Lipschitz coefficients K_1 and K_2 respectively. For any $\nu \in \mathcal{S}_{0,T}$, we define the pasting of $\mathcal{E}_{g_1}, \mathcal{E}_{g_2}$ at ν to be the following continuous* **F***-adapted process*

$$\mathcal{E}^{\nu}_{g_1,g_2}[\xi|\mathcal{F}_t] \triangleq \mathbf{1}_{\{\nu\le t\}}\mathcal{E}_{g_2}[\xi|\mathcal{F}_t] + \mathbf{1}_{\{\nu>t\}}\mathcal{E}_{g_1}\big[\mathcal{E}_{g_2}[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big], \quad \forall t \in [0,T] \quad (5.9)$$

for any $\xi \in L^2(\mathcal{F}_T)$. Then $\mathcal{E}^{\nu}_{g_1,g_2}$ is exactly the g-expectation $\mathcal{E}_{g^\nu}$ with

$$g^\nu(t,\omega,z) \triangleq \mathbf{1}_{\{\nu(\omega)\le t\}} g_2(t,\omega,z) + \mathbf{1}_{\{\nu(\omega)>t\}} g_1(t,\omega,z),$$
$$(t,\omega,z) \in [0,T]\times\Omega\times\mathbb{R}^d, \quad (5.10)$$

which is a generator satisfying (5.2) with the Lipschitz coefficient $K_1 \vee K_2$.

Fix $M > 0$; we denote by $\mathscr{G}_M$ the collection of all convex generators g satisfying (5.2) with Lipschitz coefficient $K_g \le M$. Proposition 5.2 shows that the family of convex g-expectations $\mathscr{E}_M \triangleq \{\mathcal{E}_g\}_{g\in\mathscr{G}_M}$ is closed under the pasting (5.9). To wit, $\mathscr{E}_M$ is a stable class of g-expectations in the sense of Definition 3.2 of [1]. In what follows we let $\mathscr{G}'$ be a non-empty subset of $\mathscr{G}_M$ such that $\mathscr{E}' \triangleq \{\mathcal{E}_g\}_{g\in\mathscr{G}'}$ is closed under pasting. Now we make the following assumptions on the reward processes:

Standing assumptions on the reward processes in this subsection. Let Y be a continuous **F**-adapted process with

$$\zeta'_Y \triangleq \Big(\operatorname*{esssup}_{t\in\mathcal{D}_T} Y_t\Big)^+ \in L^2(\mathcal{F}_T) \quad (5.11)$$

and satisfying (Y3). Moreover, for any $g \in \mathscr{G}'$, the model-dependent cumulative reward process is in the form of

$$H_t^g \triangleq \int_0^t h_s^g \mathrm{d}s, \quad \forall t \in [0,T],$$

where $\{h_t^g, t\in[0,T]\}_{g\in\mathscr{G}'}$ is a family of predictable processes that satisfy:

($\tilde{\mathrm{h}}$1) There exists a $c' < 0$ such that for any $g \in \mathscr{G}'$, $h_t^g \ge c'$, $\mathrm{d}t \times \mathrm{d}P$-a.s.

($\tilde{\mathrm{h}}$2) The random variable $\omega \mapsto \int_0^T h'(t,\omega)\,\mathrm{d}t$ belongs to $L^2(\mathcal{F}_T)$ with $h'(t,\omega) \triangleq \big(\operatorname{esssup}_{g\in\mathscr{G}'} h_t^g(\omega)\big)^+$ (the essential supremum is taken with respect to the product measure space $([0,T]\times\Omega, \mathscr{P}, \lambda\times P)$, where λ denotes the Lebesgue measure on $[0,T]$).

($\tilde{\text{h}}$3) For any $\nu \in \mathcal{S}_{0,T}$ and $g_1, g_2 \in \mathscr{G}'$, with g^ν defined in (5.10), it holds for any $0 \le s < t \le T$ that

$$h_t^{g^\nu} = \mathbf{1}_{\{\nu \le t\}} h_t^{g_2} + \mathbf{1}_{\{\nu > t\}} h_t^{g_1}, \quad dt \times dP\text{-a.s.}$$

Then the triple $\left(\mathscr{E}', \mathscr{H}' \triangleq \{H^g\}_{g \in \mathscr{G}'}, Y\right)$ satisfies all the conditions stated in Sections 2 and 3. Thus we can carry through the optimal stopping theory developed for $\mathbf{F}$-expectations to $(\mathscr{E}', \mathscr{H}', Y)$ as we will see next.

Theorem 5.1. *The stable class $\mathscr{E}'$ satisfies (3.12), the family of processes $\mathscr{H}'$ satisfies* (S1′) *(and thus* (S1)*; see Remark 3.1),* (S2) *and* (S3)*, while the process Y satisfies* (Y1)*, (2.6) (and thus* (Y2′)*, again by Remark 3.1) and* (Y3)*. Moreover, the family of processes $\left\{Y_t^g \triangleq Y_t + H_t^g, t \in [0,T]\right\}_{g \in \mathscr{G}'}$ is both "$\mathscr{E}'$-uniformly-left-continuous" (and thus satisfies (3.2); see also Remark 3.1) and "$\mathscr{E}'$-uniformly-right-continuous".*

5.2. *Existence of an optimal prior in (2.1) for g-expectations*

For certain collections of g-expectations, we can even determine an optimal generator $g*$ in the following sense:

$$\mathcal{E}_{g*}[Y^{g*}_{\overline{\tau}(0)}] = \sup_{g \in \mathcal{G}} \mathcal{E}_g\left[Y^g_{\overline{\tau}(0)}\right] = \sup_{(g,\rho) \in \mathcal{G} \times \mathcal{S}_{0,T}} \mathcal{E}_g[Y^g_\rho],$$

where the optimal stopping time $\overline{\tau}(0)$ is defined as in Theorem 2.1.

Let S be a separable metric space with metric $|\cdot|_S$ such that S is a countable union of non-empty compact subsets. We denote by $\mathfrak{S}$ the Borel σ-algebra of S and take $\mathcal{H}^0_{\mathbf{F}}([0,T]; S)$ as the space of admissible control strategies. For any $U \in \mathcal{H}^0_{\mathbf{F}}([0,T]; S)$, we define the generator

$$g_U(t, \omega, z) \triangleq g^o(t, \omega, z, U_t(\omega)), \tag{5.12}$$

where the function $g^o(t, \omega, z, u) : [0,T] \times \Omega \times \mathbb{R}^d \times S \mapsto \mathbb{R}$ satisfies:

(g^{o}1) g^o is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathfrak{S}/\mathscr{B}(\mathbb{R})$-measurable.
(g^{o}2) It holds $dt \times dP$-a.s. that $g^o(t, \omega, 0, u) = 0$ for any $u \in S$.
(g^{o}3) g^o is Lipschitz in z: For some $K_o > 0$, it holds $dt \times dP$-a.s. that

$$|g^o(t, \omega, z_1, u) - g^o(t, \omega, z_2, u)| \le K_o |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^d, \ \forall u \in S.$$

(g^{o}4) g^o is convex in z: It holds $dt \times dP$-a.s. that

$$g^o(t, \omega, \lambda z_1 + (1-\lambda) z_2, u) \le \lambda g^o(t, \omega, z_1, u) + (1-\lambda) g^o(t, \omega, z_2, u),$$
$$\forall \lambda \in (0,1), \ \forall z_1, z_2 \in \mathbb{R}^d, \ \forall u \in S.$$

Now fix a non-empty subset $\mathfrak{U}$ of $\mathcal{H}^0_{\mathbf{F}}([0,T]; S)$ that preserves "pasting", i.e., for any $\nu \in \mathcal{S}_{0,T}$ and $U^1, U^2 \in \mathfrak{U}$,

$$U_t^\nu(\omega) \triangleq \mathbf{1}_{\{\nu(\omega) \le t\}} U_t^2(\omega) + \mathbf{1}_{\{\nu(\omega) > t\}} U_t^1(\omega), \quad (t, \omega) \in [0,T] \times \Omega, \tag{5.13}$$

also belongs to $\mathfrak{U}$. Then it is easy to check that $\{\mathcal{E}_{g_U}\}_{U \in \mathfrak{U}} \subset \mathscr{E}_{K_o}$ forms a stable class of g-expectations.

Let Y still be a continuous $\mathbf{F}$-adapted process satisfying (5.11) and (Y3). For any $U \in \mathfrak{U}$, assume that the model-dependent reward process has a density which is given by

$$h_t^U(\omega) \triangleq h(t, \omega, U_t(\omega)), \quad (t, \omega) \in [0,T] \times \Omega,$$

where $h(t, \omega, u) : [0, T] \times \Omega \times S \mapsto \mathbb{R}$ is a $\mathscr{P} \otimes \mathfrak{S}/\mathscr{B}(\mathbb{R})$-measurable function satisfying the following assumptions:

($\hat{\text{h}}$1) For some $c < 0$, it holds $\mathrm{d}t \times \mathrm{d}P$-a.s. that $h(t, \omega, u) \geq c$ for any $u \in S$.

($\hat{\text{h}}$2) The random variable $\omega \mapsto \int_0^T \hat{h}(t, \omega)\mathrm{d}t$ belongs to $L^2(\mathcal{F}_T)$ with $\hat{h}(t, \omega) \triangleq \big(\operatorname{esssup}_{U \in \mathfrak{U}} h_t^U(\omega)\big)^+$ (the essential supremum is taken with respect to the product measure space $([0, T] \times \Omega, \mathscr{P}, \lambda \times P)$, where λ denotes the Lebesgue measure on $[0, T]$).

It is easy to see that $\{h_t^U, t \in [0, T]\}_{U \in \mathfrak{U}}$ is a family of predictable processes satisfying ($\tilde{\text{h}}$1)–($\tilde{\text{h}}$3). Hence, we can apply the optimal stopping theory developed for **F**-expectations to the triple $\big(\{\mathcal{E}_{g_U}\}_{U \in \mathfrak{U}}, \{h^U\}_{U \in \mathfrak{U}}, Y\big)$ thanks to Theorem 5.1. Now let us construct a so-called *Hamiltonian* function

$$H(t, \omega, z, u) \triangleq g^o(t, \omega, z, u) + h(t, \omega, u), \quad (t, \omega, z, u) \in [0, T] \times \Omega \times \mathbb{R}^d \times S.$$

We assume that for any $(t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d$, there exists a $u = u^*(t, \omega, z) \in S$ such that

$$\sup_{u \in S} H(t, \omega, z, u) = H\big(t, \omega, z, u^*(t, \omega, z)\big). \tag{5.14}$$

(This is valid, for example, when the metric space S is compact and the mapping $u \mapsto H(t, \omega, z, u)$ is continuous.) Then it can be shown (see [2, Lemma 1] or [7, Lemma 16.34]) that the mapping $u^* : [0, T] \times \Omega \times \mathbb{R}^d \mapsto S$ can be selected to be $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^d)/\mathfrak{S}$-measurable.

The following theorem is the main result of this subsection.

Theorem 5.2. *There exists a $U^* \in \mathfrak{U}$ such that $\sup_{(U,\rho) \in \mathfrak{U} \times \mathcal{S}_{0,T}} \mathcal{E}_{g_U}\big[Y_\rho^U\big] = \mathcal{E}_{g_{U^*}}\big[Y_{\overline{\tau}(0)}^{U^*}\big]$, where the stopping time $\overline{\tau}(0)$ is as in Theorem 2.1.*

5.3. *The cooperative game of Karatzas and Zamfirescu [2006] revisited*

In this subsection, we apply results of the last subsection to extend those of [10]. Let us first recall their setting:

- Consider the canonical space $(\Omega, \mathcal{F}) = \big(C([0, T]; \mathbb{R}^d), \mathscr{B}(C([0, T]; \mathbb{R}^d))\big)$ endowed with Wiener measure P, under which the coordinate mapping process $B(t, \omega) = \omega(t), t \in [0, T]$ becomes a standard d-dimensional Brownian motion. We still take the filtration **F** generated by the Brownian motion B (see (5.1)) and let $\mathscr{P}$ denote the predictable σ-algebra with respect to **F**.
- It is well-known (see e.g. [7, Theorem 14.6]) that given a $x \in \mathbb{R}^d$, there exists a pathwise unique, strong solution $X(\cdot)$ of the stochastic equation

$$X(t) = x + \int_0^t \sigma(s, X)\mathrm{d}B_s, \quad t \in [0, T],$$

where the diffusion term $\sigma(t, \omega)$ is a $\mathbb{R}^{d \times d}$-valued predictable process satisfying:

(σ1) $\int_0^T |\sigma(t, \vec{0})|^2 \mathrm{d}t < \infty$ and $\sigma(t, \omega)$ is non-singular for any $(t, \omega) \in [0, T] \times \Omega$.

(σ2) There exists a $K > 0$ such that for any $\omega, \widetilde{\omega} \in \Omega$ and $t \in [0, T]$

$$\|\sigma^{-1}(t, \omega)\| \leq K \quad \text{and} \quad \big|\sigma_{ij}(t, \omega) - \sigma_{ij}(t, \widetilde{\omega})\big| \leq K\|\omega - \widetilde{\omega}\|_t^*, \quad \forall 1 \leq i, j \leq n, \tag{5.15}$$

where $\|\omega\|_t^* \triangleq \sup_{s \in [0,t]} |\omega(s)|$.

- Let $f(t,\omega,u) : [0,T]\times\Omega\times S\mapsto\mathbb{R}^d$ be a $\mathscr{P}\otimes\mathfrak{S}/\mathscr{B}(\mathbb{R}^d)$-measurable function such that:
  (f1) For any $u\in S$, the mapping $(t,\omega)\mapsto f(t,\omega,u)$ is predictable (i.e. $\mathscr{P}$-measurable).
  (f2) With the same K as in (5.15), $|f(t,\omega,u)| \le K\big(1+\|\omega\|_t^*\big)$ for any $(t,\omega,u)\in[0,T]\times\Omega\times S$.

  For any $U\in\widetilde{\mathfrak{U}}\triangleq\mathcal{H}^0_{\mathbf{F}}([0,T];S)$, [10, page 166] shows that

$$\frac{\mathrm{d}P_U}{\mathrm{d}P}\triangleq\exp\left\{\int_0^T\langle\sigma^{-1}(t,X)f(t,X,U_t),\mathrm{d}B_t\rangle-\frac{1}{2}\int_0^T|\sigma^{-1}(t,X)f(t,X,U_t)|^2\mathrm{d}t\right\}$$

defines a probability measure P_U. The objective of [10] is to find an optimal stopping time $\tau^*\in\mathcal{S}_{0,T}$ and an optimal control strategy $U^*\in\widetilde{\mathfrak{U}}$ that maximizes the expected reward $E_U\big[\varphi(X(\rho))+\int_0^\rho h(s,X,U_s)\mathrm{d}s\big]$ over $(\rho,U)\in\mathcal{S}_{0,T}\times\widetilde{\mathfrak{U}}$. Here $\varphi:\mathbb{R}^d\mapsto\mathbb{R}$ is a bounded continuous function, and $h(t,\omega,u):[0,T]\times\Omega\times S\mapsto\mathbb{R}$ is a $\mathscr{P}\otimes\mathfrak{S}/\mathscr{B}(\mathbb{R})$-measurable function such that $|h(t,\omega,u)|\le K$ for any $(t,\omega,u)\in[0,T]\times\Omega\times S$ (with the same K as appears in (5.15)).

Corollary 8 of [10] shows that under (f2), the process

$$\widetilde{Z}(t)\triangleq\operatorname*{esssup}_{(U,\rho)\in\widetilde{\mathfrak{U}}\times\mathcal{S}_{t,T}}E_U\left[\varphi\big(X(\rho)\big)+\int_t^\rho h(s,X,U_s)\mathrm{d}s|\mathcal{F}_t\right],\quad t\in[0,T]$$

admits an RCLL modification $\widetilde{Z}^0$, and that the first time that processes $\widetilde{Z}^0$ and $\big\{\varphi\big(X(t)\big)\big\}_{t\in[0,T]}$ meet with each other, i.e. $\overline{\tau}(0)\triangleq\inf\big\{t\in[0,T]\mid\widetilde{Z}^0_t=\varphi\big(X(t)\big)\big\}$, is an optimal stopping time. That is,

$$\begin{aligned}&\sup_{(U,\rho)\in\widetilde{\mathfrak{U}}\times\mathcal{S}_{0,T}}E_U\left[\varphi\big(X(\rho)\big)+\int_0^\rho h(s,X,U_s)\mathrm{d}s\right]\\&=\sup_{U\in\widetilde{\mathfrak{U}}}E_U\left[\varphi\big(X(\overline{\tau}(0))\big)+\int_0^{\overline{\tau}(0)}h(s,X,U_s)\mathrm{d}s\right].\end{aligned}\tag{5.16}$$

Moreover, if there is a function $u^*:[0,T]\times\Omega\times\mathbb{R}^d\to S$ such that for any $(t,\omega,z)\in[0,T]\times\Omega\times\mathbb{R}^d$

$$\sup_{u\in S}\widetilde{H}(t,\omega,z,u)=\widetilde{H}\big(t,\omega,z,u^*(t,\omega,z)\big)\tag{5.17}$$

with $\widetilde{H}(t,\omega,z,u)\triangleq\big\langle\sigma^{-1}(t,\omega)f(t,\omega,u),z\big\rangle+h(t,\omega,u)$ (u^* can be chosen to be $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)/\mathfrak{S}$-measurable), then there exists an optimal control strategy $U^*\in\widetilde{\mathfrak{U}}$ (see Section 8 of [10]) such that

$$\begin{aligned}&\sup_{(U,\rho)\in\widetilde{\mathfrak{U}}\times\mathcal{S}_{0,T}}E_U\left[\varphi\big(X(\rho)\big)+\int_0^\rho h(s,X,U_s)\mathrm{d}s\right]\\&=E_{U^*}\left[\varphi\big(X(\overline{\tau}(0))\big)+\int_0^{\overline{\tau}(0)}h(s,X,U^*_s)\mathrm{d}s\right].\end{aligned}\tag{5.18}$$

In the main result of this subsection, we will show that the assumption of [10] that φ and h are bounded from above by constants can be relaxed and replaced by linear-growth conditions. This comes, however, at the cost of strengthening the assumption stated in (f2).

Proposition 5.3. *With the same K as in (5.15), we assume that*

$$-K \le \varphi(x) \le K|x|, \quad \forall x \in \mathbb{R}^d \tag{5.19}$$

and that for a.e. $t \in [0,T]$

$$|f(t,\omega,u)| \le K \quad \text{and} \quad -K \le h(t,\omega,u) \le K\|\omega\|_T^*, \quad \forall (\omega,u) \in \Omega \times S. \tag{5.20}$$

Then $\{\widetilde{Z}(t)\}_{t\in[0,T]}$ has an RCLL modification $\widetilde{Z}^0$ such that the first time $\overline{\tau}(0)$ when processes $\widetilde{Z}^0$ and $\{\varphi(X(t))\}_{t\in[0,T]}$ meet is an optimal stopping time; i.e., $\overline{\tau}(0)$ satisfies (5.16). Moreover, if there exists a measurable mapping $u^* : [0,T] \times \Omega \times \mathbb{R}^d \mapsto S$ satisfying (5.17), then there is an optimal control strategy $U^* \in \widetilde{\mathfrak{U}}$ such that (5.18) holds.

5.4. Quadratic g-expectations

Now we consider a *quadratic* generator $\hat{g} = \hat{g}(t,\omega,z) : [0,T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}$ that satisfies

$$\begin{cases} \text{(i) } \hat{g}(t,\omega,0) = 0, \quad \mathrm{d}t \times \mathrm{d}P\text{-a.s.} \\ \text{(ii) For some } \kappa > 0, \text{ it holds } \mathrm{d}t \times \mathrm{d}P\text{-a.s. that } \left|\dfrac{\partial \hat{g}}{\partial z}(t,\omega,z)\right| \le \kappa(1+|z|), \\ \qquad \forall z \in \mathbb{R}^d. \\ \text{(iii) } \hat{g} \text{ is convex in } z \text{ in the sense of (5.5).} \end{cases} \tag{5.21}$$

Note that under (ii), (i) is equivalent to the following statement: It holds $\mathrm{d}t \times \mathrm{d}P$-a.s. that

$$|\hat{g}(t,\omega,z)| \le \kappa\left(|z| + \frac{1}{2}|z|^2\right), \quad \forall z \in \mathbb{R}^d. \tag{5.22}$$

In fact, it is clear that (5.22) implies (i). Conversely, for $\mathrm{d}t \times \mathrm{d}P$-a.s.$(t,\omega) \in [0,T] \times \Omega$, one can deduce that for any $z \in \mathbb{R}^d$, $|\hat{g}(t,\omega,z)| = \big|\hat{g}(t,\omega,z) - \hat{g}(t,\omega,0)\big| = \left|\int_0^1 \frac{\partial \hat{g}}{\partial z}(t,\lambda z) z \mathrm{d}\lambda\right| \le \kappa \int_0^1 (1+\lambda|z|)|z|\mathrm{d}\lambda = \kappa\big(|z| + \frac{1}{2}|z|^2\big)$.

For any $\xi \in L^e(\mathcal{F}_T)$, [4, Corollary 6] (where we take $f = g$, and thus $\alpha(t) \equiv \frac{\kappa}{2}$ and $(\beta,\gamma) = (0, 2\kappa)$) shows that the quadratic BSDE$(\xi,\hat{g})$ admits a unique solution $\big(\Gamma^{\xi,\hat{g}}, \Theta^{\xi,\hat{g}}\big) \in \mathbb{C}^e_{\mathbf{F}}([0,T]) \times M_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Hence we can correspondingly define the "quadratic" g-expectation of ξ by $\mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_t] \triangleq \Gamma_t^{\xi,\hat{g}}, \forall t \in [0,T]$.

To show that the quadratic g-expectation $\mathcal{E}_{\hat{g}}$ is an $\mathbf{F}$-expectation with domain $\mathrm{Dom}(\mathcal{E}_{\hat{g}}) = L^e(\mathcal{F}_T)$, we first note that $L^e(\mathcal{F}_T) \in \widetilde{\mathscr{D}}_T$ (clearly, $L^e(\mathcal{F}_T)$ satisfies (D1) and (D3) and $\mathbb{R} \subset L^e(\mathcal{F}_T)$. For any $\xi,\eta \in L^e(\mathcal{F}_T)$, $A \in \mathcal{F}_T$ and $\lambda > 0$, we have $E\big[\mathrm{e}^{\lambda|\mathbf{1}_A\xi|}\big] \le E\big[\mathrm{e}^{\lambda|\xi|}\big] < \infty$ and $E\big[\mathrm{e}^{\lambda|\xi+\eta|}\big] \le E\big[\mathrm{e}^{\lambda|\xi|}\mathrm{e}^{\lambda|\eta|}\big] \le \frac{1}{2}E\big[\mathrm{e}^{2\lambda|\xi|}\big] + \frac{1}{2}E\big[\mathrm{e}^{2\lambda|\eta|}\big] < \infty$, and thus (D2) also holds for $L^e(\mathcal{F}_T)$). Like for the Lipschitz g-expectation case, the uniqueness of the solution $(\Gamma^{\xi,\hat{g}}, \Theta^{\xi,\hat{g}})$ to the quadratic BSDE$(\xi,\hat{g})$ implies that the family of operators $\big\{\mathcal{E}_{\hat{g}}[\cdot|\mathcal{F}_t] : L^e(\mathcal{F}_T) \mapsto L^e(\mathcal{F}_t)\big\}_{t\in[0,T]}$ satisfies (A2)–(A4) (cf. [13, Lemma 36.6] and [5, Lemma 2.1]), while a comparison theorem for quadratic BSDEs (see e.g. [4, Theorem 5]) and the following proposition show that (A1) also holds for the family $\big\{\mathcal{E}_{\hat{g}}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$.

Proposition 5.4. *Let $\hat{g}$ satisfy (5.21). For any $\xi^1,\xi^2 \in L^e(\mathcal{F}_T)$, if $\xi^1 \ge \xi^2$, a.s., then it holds a.s. that*

$$\Gamma_t^{\xi_1,\hat{g}} \ge \Gamma_t^{\xi_2,\hat{g}}, \quad \forall t \in [0,T]. \tag{5.23}$$

Moreover, if $\Gamma_\nu^{\xi_1,\hat{g}} = \Gamma_\nu^{\xi_2,\hat{g}}$, a.s. for some $\nu \in \mathcal{S}_{0,T}$, then $\xi^1 = \xi^2$, a.s.

Therefore, the quadratic g-expectation $\mathcal{E}_{\hat{g}}$ is an $\mathbf{F}$-expectation with domain $\mathrm{Dom}(\mathcal{E}_{\hat{g}}) = L^e(\mathcal{F}_T)$. Like for the Lipschitz g-expectation case, the convexity (5.21)(iii) of $\hat{g}$ as well as Theorem 5 of [4] determine that $\mathcal{E}_{\hat{g}}[\cdot|\mathcal{F}_t]$ is a convex operator on $L^e(\mathcal{F}_T)$ for any $t \in [0, T]$. Hence, $\mathcal{E}_{\hat{g}}$ satisfies (H0) thanks to Lemma 3.1 of [1]. To see $\mathcal{E}_{\hat{g}}$ also satisfying (H1)–(H3), we need the following stability result.

Lemma 5.2. *If $\xi_n \to \xi$, a.s., and $E\big[e^{\lambda|\xi|}\big] + \sup_{n\in\mathbb{N}} E\big[e^{\lambda|\xi_n|}\big] < \infty$ for any $\lambda > 0$, then*

$$\lim_{n\to\infty} E\left[\sup_{t\in[0,T]} \left|\mathcal{E}_{\hat{g}}[\xi_n|\mathcal{F}_t] - \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_t]\right|\right] = 0. \tag{5.24}$$

Proof. Taking $f_n \equiv g$ and $f = g$ in Proposition 7 of [4] yields that

$$\lim_{n\to\infty} E\left[\exp\left\{p \sup_{t\in[0,T]} \left|\mathcal{E}_{\hat{g}}[\xi_n|\mathcal{F}_t] - \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_t]\right|\right\}\right] = 0, \quad \forall p \geq 1.$$

Then (5.24) follows since $E[\sup_{t\in[0,T]} |\mathcal{E}_{\hat{g}}[\xi_n|\mathcal{F}_t] - \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_t]|] \leq E[\exp\{\sup_{t\in[0,T]} |\mathcal{E}_{\hat{g}}[\xi_n|\mathcal{F}_t] - \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_t]|\}]$ for any $n \in \mathbb{N}$. $\square$

Proposition 5.5. *Let $\hat{g}$ satisfy (5.21). Then the quadratic g-expectation $\mathcal{E}_{\hat{g}}$ satisfies* (H0)–(H3).

Like for Remark 5.1, since $\mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_\cdot]$ is a continuous process for any $\xi \in L^e(\mathcal{F}_T)$, we see from (2.6) of [1] that $\widetilde{\mathcal{E}}_{\hat{g}}[\cdot|\mathcal{F}_\nu]$ is just a restriction of $\mathcal{E}_{\hat{g}}[\cdot|\mathcal{F}_\nu]$ to $L^{e,\#}(\mathcal{F}_T) \triangleq \{\xi \in L^e(\mathcal{F}_T) : \xi \geq c$, a.s. for some $c \in \mathbb{R}\}$ for any $\nu \in \mathcal{S}_{0,T}$. Therefore, all results on $\mathbf{F}$-expectations $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ in Section 2 of [1] work for quadratic g-expectations.

The next result, which shows the existence of an optimal stopping time for a quadratic g-expectation, is the main result of this subsection.

Theorem 5.3. *Let $\hat{g}$ satisfy (5.21). For any right-continuous $\mathbf{F}$-adapted process Y with $\zeta'_Y \triangleq \big(\operatorname{esssup}_{t\in\mathcal{D}_T} Y_t\big)^+ \in L^e(\mathcal{F}_T)$ and satisfying* (Y3)*, we have $\sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}_{\hat{g}}[Y_\rho] = \mathcal{E}_{\hat{g}}[Y_{\overline{\tau}(0)}]$, where $\overline{\tau}(0)$ is as in Theorem 2.1.*

6. Proofs

Proof of Lemma 2.1. For any $i \in \mathcal{I}$, it is clear that $H^i_0 = 0$ and that (2.2) directly follows from (h1). For any $s, t \in \mathcal{D}_T$ with $s < t$, we can deduce from (h2) that

$$H^i_{s,t} = \int_s^t h^i_r \mathrm{d}r \geq c \int_s^t \mathrm{d}s \geq cT, \quad \text{a.s.,} \tag{6.1}$$

which implies that $\operatorname{essinf}_{s,t\in\mathcal{D}_T;s<t} H^i_{s,t} \geq cT$, a.s. Thus (S2) holds with $C_H = cT$.

If no member of $\mathscr{E}$ satisfies (2.5) of [1], then $\int_0^T |h^j_t|\,\mathrm{d}t \in \mathrm{Dom}(\mathscr{E})$ for some $j \in \mathcal{I}$ is assumed. For any $s, t \in \mathcal{D}_T$ with $s < t$, we can deduce from (6.1) and (h2) that $C_H \leq H^j_{s,t} \leq \int_s^t |h^j_r|\mathrm{d}r \leq \int_0^T |h^j_r|\mathrm{d}r$, a.s., which implies that $C_H \leq \operatorname{esssup}_{s,t\in\mathcal{D}_T;s<t} H^j_{s,t} \leq \int_0^T |h^j_r|\mathrm{d}r$ a.s. Then Lemma 3.2 of [1] shows that $\operatorname{esssup}_{s,t\in\mathcal{D}_T;s<t} H^j_{s,t} \in \mathrm{Dom}(\mathscr{E})$, i.e. (2.3). Moreover, we can derive (S3) directly from (h3). $\square$

Proof of Lemma 2.2. For any $i, j \in \mathcal{I}'$ and $\rho_1, \rho_2 \in \mathcal{U}$, we consider the event

$$A \triangleq \Big\{\widetilde{\mathcal{E}}_i\big[X(\rho_1) + H^i_{\nu,\rho_1}|\mathcal{F}_\nu\big] \le \widetilde{\mathcal{E}}_j\big[X(\rho_2) + H^j_{\nu,\rho_2}|\mathcal{F}_\nu\big]\Big\} \in \mathcal{F}_\nu,$$

and define stopping times $\rho \triangleq \rho_2 \mathbf{1}_A + \rho_1 \mathbf{1}_{A^c} \in \mathcal{U}$ and $\nu(A) \triangleq \nu \mathbf{1}_A + T\mathbf{1}_{A^c} \in \mathcal{S}_{\nu,T}$. Since $\mathscr{E}' = \{\mathcal{E}_i\}_{i\in\mathcal{I}'}$ is a stable subclass of $\mathscr{E}$, Definition 3.2 of [1] assures the existence of $k = k\big(i, j, \nu(A)\big) \in \mathcal{I}'$ such that $\widetilde{\mathcal{E}}_k = \mathcal{E}^{\nu(A)}_{i,j}$. We can deduce from Proposition 2.7(5) of [1] and (3.3) of [1] that for any $\xi \in \mathrm{Dom}(\mathscr{E})$

$$\begin{aligned}\widetilde{\mathcal{E}}_k[\xi|\mathcal{F}_\nu] &= \mathcal{E}^{\nu(A)}_{i,j}[\xi|\mathcal{F}_\nu] = \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu(A)\vee\nu}]|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[\mathbf{1}_A\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_T]|\mathcal{F}_\nu\big]\\ &= \widetilde{\mathcal{E}}_i\big[\mathbf{1}_A\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu] + \mathbf{1}_{A^c}\xi|\mathcal{F}_\nu\big] = \mathbf{1}_A\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_\nu], \quad \text{a.s.} \end{aligned} \tag{6.2}$$

Moreover, (2.5) implies that $H^k_{\nu,\rho} = H^i_{\nu(A)\wedge\nu,\nu(A)\wedge\rho} + H^j_{\nu(A)\vee\nu,\nu(A)\vee\rho} = \mathbf{1}_{A^c}H^i_{\nu,\rho_1} + \mathbf{1}_A H^j_{\nu,\rho_2}$, a.s. Then applying Proposition 2.7(2) of [1], we see from (6.2) that

$$\begin{aligned}\widetilde{\mathcal{E}}_k\big[X(\rho) + H^k_{\nu,\rho}|\mathcal{F}_\nu\big] &= \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[X(\rho) + H^k_{\nu,\rho}|\mathcal{F}_\nu\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[X(\rho) + H^k_{\nu,\rho}|\mathcal{F}_\nu\big]\\ &= \widetilde{\mathcal{E}}_j\big[\mathbf{1}_A X(\rho_2) + \mathbf{1}_A H^j_{\nu,\rho_2}|\mathcal{F}_\nu\big] + \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{A^c}X(\rho_1) + \mathbf{1}_{A^c}H^i_{\nu,\rho_1}|\mathcal{F}_\nu\big]\\ &= \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[X(\rho_2) + H^j_{\nu,\rho_2}|\mathcal{F}_\nu\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[X(\rho_1) + H^i_{\nu,\rho_1}|\mathcal{F}_\nu\big]\\ &= \widetilde{\mathcal{E}}_i\big[X(\rho_1) + H^i_{\nu,\rho_1}|\mathcal{F}_\nu\big] \vee \widetilde{\mathcal{E}}_j\big[X(\rho_2) + H^j_{\nu,\rho_2}|\mathcal{F}_\nu\big], \quad \text{a.s.}\end{aligned} \tag{6.3}$$

Similarly, taking $\rho' \triangleq \rho_1\mathbf{1}_A + \rho_2\mathbf{1}_{A^c}$ and $k' = k\big(i, j, \nu(A^c)\big)$, we obtain

$$\widetilde{\mathcal{E}}_{k'}\big[X(\rho') + H^{k'}_{\nu,\rho'}|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[X(\rho_1) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \wedge \widetilde{\mathcal{E}}_j\big[X(\rho_2) + H^j_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \text{a.s.}$$

Hence, the family $\Big\{\widetilde{\mathcal{E}}_i\big[X(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\Big\}_{(i,\rho)\in\mathcal{I}'\times\mathcal{U}}$ is closed under pairwise maximization and pairwise minimization. Thanks to [11, Proposition VI-**1**-1], we can find two sequences $\{(i_n, \rho_n)\}_{n\in\mathbb{N}}$ and $\big\{(i'_n, \rho'_n)\big\}_{n\in\mathbb{N}}$ in $\mathcal{I}' \times \mathcal{U}$ such that (2.9) and (2.10) hold. □

Proof of Lemma 2.3. We fix $\nu \in \mathcal{S}_{0,T}$. For any $(i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$, (2.7), (2.4) and Proposition 2.7(5) of [1] show that $\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \ge \widetilde{\mathcal{E}}_i[C_Y + C_H|\mathcal{F}_\nu] = C_*$, a.s. Taking the essential supremum over $(i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$ gives

$$Z(\nu) = \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \ge C_*, \quad \text{a.s.}$$

Then for any $i \in \mathcal{I}$, (2.4) implies that $Z^i(\nu) = Z(\nu) + H^i_\nu \ge C_* + C_H = C_Y + 2C_H$, a.s.

If no member of $\mathscr{E}$ satisfies (2.5) of [1] (and thus (2.6) is assumed), then for any $(i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$, it holds a.s. that $\widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_t\big] \le \zeta_Y$ for any $t \in \mathcal{D}_T$. Since $\widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_\cdot\big]$ is an RCLL process, it holds except on a null set $N = N(i, \rho)$ that

$$\widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_t\big] \le \zeta_Y, \quad \forall t \in [0, T], \quad \text{thus} \quad \widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_\nu\big] \le \zeta_Y.$$

Moreover, Proposition 2.7(3) of [1] and (2.4) imply that

$$\zeta_Y \ge \widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] + H^i_\nu \ge \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] + C_H, \quad \text{a.s.}$$

Taking the essential supremum over $(i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$ yields that

$$Z(\nu) = \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \le \zeta_Y - C_H, \quad \text{a.s.}$$

where $\zeta_Y - C_H \in \mathrm{Dom}(\mathscr{E})$ thanks to (2.6) and (D2). Hence, for any $i \in \mathcal{I}$, we have $Z^i(\nu) \le \zeta_Y - C_H + H^i_\nu$, a.s. And (2.2) together with (D2) implies that $\zeta_Y - C_H + H^i_\nu \in \mathrm{Dom}(\mathscr{E})$. □

Proof of Lemma 2.4. If no member of $\mathscr{E}$ satisfies (2.5) of [1], then we see from Lemma 2.3 that $C_* \le Z(\nu) \le \zeta_Y - C_H$, a.s., and that $\zeta_Y - C_H \in \mathrm{Dom}(\mathscr{E})$. Hence $Z(\nu) \in \mathrm{Dom}(\mathscr{E})$ thanks to Lemma 3.2 of [1].

On the other hand, if $\mathcal{E}_j$ satisfies (2.5) of [1] for some $j \in \mathcal{I}$, letting $(X, \mathcal{I}', \mathcal{U}) = (Y, \mathcal{I}, \mathcal{S}_{\nu,T})$ in Lemma 2.2, we can find a sequence $\{(i_n, \rho_n)\}_{n\in\mathbb{N}}$ in $\mathcal{I} \times \mathcal{S}_{\nu,T}$ such that

$$Z(\nu) = \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] = \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big], \quad \text{a.s.}$$

For any $n \in \mathbb{N}$, it follows from Definition 3.2 of [1] that there exists $k_n = k(j, i_n, \nu) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n} = \mathcal{E}^\nu_{j,i_n}$. Applying Proposition 2.7(3) of [1] to $\widetilde{\mathcal{E}}_{k_n}$, we can deduce from (2.4), (3.3) of [1] and (2.5) that

$$\begin{aligned}\widetilde{\mathcal{E}}_{k_n}\big[Y^{k_n}_{\rho_n}\big] - C_H &= \widetilde{\mathcal{E}}_{k_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n} + H^{k_n}_\nu - C_H\big] \ge \widetilde{\mathcal{E}}_{k_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n}\big] \\ &= \mathcal{E}^\nu_{j,i_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n}\big] = \widetilde{\mathcal{E}}_j\Big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big]\Big] \\ &= \widetilde{\mathcal{E}}_j\Big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big]\Big],\end{aligned}$$

which together with (Y2) shows that

$$\varliminf_{n\to\infty} \mathcal{E}_j\Big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big]\Big] \le \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i\big[Y^i_\rho\big] - C_H < \infty.$$

For any $n \in \mathbb{N}$, (2.7), (2.4) and Proposition 2.7(5) of [1] imply that $\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big] \ge \widetilde{\mathcal{E}}_{i_n}[C_*|\mathcal{F}_\nu] = C_*$, a.s. Therefore, we can deduce from Remark 2.2(1) that

$$Z(\nu) = \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big] \in \mathrm{Dom}(\mathscr{E}).$$

For any $i \in \mathcal{I}$, (2.2) and (D2) imply that $Z^i(\nu) = Z(\nu) + H^i_\nu \in \mathrm{Dom}(\mathscr{E})$. □

Proof of Proposition 2.1. To see (2.14), we first note that the event $A \triangleq \{\nu = \sigma\}$ belongs to $\mathcal{F}_{\nu\wedge\sigma}$ thanks to [9, Lemma 1.2.16]. For any $i \in \mathcal{I}$ and $\rho \in \mathcal{S}_{\nu,T}$, we define $\rho(A) \triangleq \rho\mathbf{1}_A + T\mathbf{1}_{A^c}$, which clearly belongs to $\mathcal{S}_{\sigma,T}$. Proposition 2.7(2)–(3) of [1] then implies that

$$\begin{aligned}\mathbf{1}_A\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] &= \mathbf{1}_A\Big(\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_\rho|\mathcal{F}_\nu\big] - H^i_\nu\Big) = \mathbf{1}_A\Big(\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_\rho|\mathcal{F}_\sigma\big] - H^i_\sigma\Big) \\ &= \mathbf{1}_A\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\sigma,\rho}|\mathcal{F}_\sigma\big] \\ &= \widetilde{\mathcal{E}}_i\big[\mathbf{1}_A\big(Y_{\rho(A)} + H^i_{\sigma,\rho(A)}\big)|\mathcal{F}_\sigma\big] = \mathbf{1}_A\widetilde{\mathcal{E}}_i\big[Y_{\rho(A)} + H^i_{\sigma,\rho(A)}|\mathcal{F}_\sigma\big] \\ &\le \mathbf{1}_A \operatorname*{esssup}_{(i,\gamma)\in\mathcal{I}\times\mathcal{S}_{\sigma,T}} \widetilde{\mathcal{E}}_i\big[Y_\gamma + H^i_{\sigma,\gamma}|\mathcal{F}_\sigma\big] = \mathbf{1}_A Z(\sigma), \quad \text{a.s.}\end{aligned}$$

Taking the essential supremum of the left-hand side over $(i, \rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$ and applying Lemma 3.3(2) of [1], we obtain

$$\begin{aligned}\mathbf{1}_A Z(\nu) &= \mathbf{1}_A \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \\ &= \operatorname*{esssup}_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}} \Big(\mathbf{1}_A\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\Big) \le \mathbf{1}_A Z(\sigma), \quad \text{a.s.}\end{aligned}$$

Reversing the roles of ν and σ, we obtain (2.14).

As regards (2.15), since $\mathcal{S}_{\gamma,T} \subset \mathcal{S}_{\nu,T}$, it is clear that

$$\underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \le \underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\nu,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] = Z(\nu), \quad \text{a.s.}$$

Letting $(X,\nu,\mathcal{I}',\mathcal{U}) = (Y,\gamma,\mathcal{I},\mathcal{S}_{\gamma,T})$ in Lemma 2.2, we can find a sequence $\{(i_n,\rho_n)\}_{n\in\mathbb{N}}$ in $\mathcal{I}\times\mathcal{S}_{\gamma,T}$ such that

$$Z(\gamma) = \underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\gamma,\rho}|\mathcal{F}_\gamma\big] = \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\gamma,\rho_n}|\mathcal{F}_\gamma\big], \quad \text{a.s.}$$

Now fix $j\in\mathcal{I}$. For any $n\in\mathbb{N}$, it follows from Definition 3.2 of [1] that there exists a $k_n = k(j,i_n,\gamma)\in\mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n} = \mathcal{E}^\gamma_{j,i_n}$. Applying Proposition 2.7(3) of [1] to $\widetilde{\mathcal{E}}_{i_n}$, we can deduce from (3.3) of [1], (2.5) that

$$\begin{aligned}
\underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] &\ge \widetilde{\mathcal{E}}_{k_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big] = \mathcal{E}^\gamma_{j,i_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n}|\mathcal{F}_\nu\big] \\
&= \widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{k_n}_{\nu,\rho_n}|\mathcal{F}_\gamma\big]|\mathcal{F}_\nu\big] \\
&= \widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{k_n}_{\gamma,\rho_n}|\mathcal{F}_\gamma\big] + H^{k_n}_{\nu,\gamma}|\mathcal{F}_\nu\big] \\
&= \widetilde{\mathcal{E}}_j\Big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\gamma,\rho_n}|\mathcal{F}_\gamma\big] + H^j_{\nu,\gamma}|\mathcal{F}_\nu\Big], \quad \text{a.s.}
\end{aligned} \tag{6.4}$$

For any $n\in\mathbb{N}$, Proposition 2.7(5) of [1], (2.7) and (2.4) show that

$$C_Y + 2C_H = \widetilde{\mathcal{E}}_{i_n}\big[C_*|\mathcal{F}_\gamma\big] + C_H \le \widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\gamma,\rho_n}|\mathcal{F}_\gamma\big] + H^j_{\nu,\gamma} \le Z(\gamma) + H^j_{\nu,\gamma}, \quad \text{a.s.,}$$

where $Z(\gamma) + H^j_{\nu,\gamma} \in \text{Dom}(\mathscr{E})$ due to Lemma 2.4, (2.2) and (D2). Then Proposition 2.9 of [1] and (6.4) imply that

$$\begin{aligned}
\widetilde{\mathcal{E}}_j\big[Z(\gamma) + H^j_{\nu,\gamma}|\mathcal{F}_\nu\big] &= \lim_{n\to\infty} \widetilde{\mathcal{E}}_j\Big[\widetilde{\mathcal{E}}_{i_n}\big[Y_{\rho_n} + H^{i_n}_{\gamma,\rho_n}|\mathcal{F}_\gamma\big] + H^j_{\nu,\gamma}|\mathcal{F}_\nu\Big] \\
&\le \underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \text{a.s.}
\end{aligned}$$

Taking the essential supremum of the left-hand side over $j\in\mathcal{I}$, we obtain

$$\underset{j\in\mathcal{I}}{\text{esssup}}\, \widetilde{\mathcal{E}}_j\big[Z(\gamma) + H^j_{\nu,\gamma}|\mathcal{F}_\nu\big] \le \underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \text{a.s.} \tag{6.5}$$

On the other hand, for any $i\in\mathcal{I}$ and $\rho\in\mathcal{S}_{\gamma,T}$, Corollary 2.3 of [1] and Proposition 2.7(3) of [1] imply that

$$\begin{aligned}
\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\gamma,\rho}|\mathcal{F}_\gamma\big] + H^i_{\nu,\gamma}|\mathcal{F}_\nu\big] \\
&\le \widetilde{\mathcal{E}}_i\big[Z(\gamma) + H^i_{\nu,\gamma}|\mathcal{F}_\nu\big] \le \underset{i\in\mathcal{I}}{\text{esssup}}\, \widetilde{\mathcal{E}}_i\big[Z(\gamma) + H^i_{\nu,\gamma}|\mathcal{F}_\nu\big], \quad \text{a.s.}
\end{aligned}$$

Taking the essential supremum of the left-hand side over $(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}$ yields that

$$\underset{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{\gamma,T}}{\text{esssup}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \le \underset{i\in\mathcal{I}}{\text{esssup}}\, \widetilde{\mathcal{E}}_i\big[Z(\gamma) + H^i_{\nu,\gamma}|\mathcal{F}_\nu\big], \quad \text{a.s.,}$$

which together with (6.5) proves (2.15). □

Proof of Proposition 2.2. For any $i \in \mathcal{I}, \nu \in \mathcal{S}_{0,T}$ and $\gamma \in \mathcal{S}_{\nu,T}$, Proposition 2.7(3) of [1], (2.15) imply that

$$\widetilde{\mathcal{E}}_i\big[Z^i(\rho)|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[Z(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] + H^i_\nu \le \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] + H^i_\nu$$
$$\le Z(\nu) + H^i_\nu = Z^i(\nu), \quad \text{a.s.},$$

which implies that $\{Z^i(t)\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale. Proposition 2.6 of [1], Theorem 2.3 of [1] and (2.12) then show that $\Big\{Z^{i,+}_t \triangleq \underline{\lim}_{n\to\infty} Z^i\big(q^+_n(t)\big)\Big\}_{t\in[0,T]}$ defines an RCLL process. Moreover, (2.12) implies that

$$\operatorname*{essinf}_{t\in[0,T]} Z^i(t) \ge C_Y + 2C_H, \quad \text{a.s.} \tag{6.6}$$

If $\mathcal{E}_j$ satisfies (2.5) of [1] for some $j \in \mathcal{I}$, Corollary 2.2 of [1] and (6.6) imply that

$$Z^{j,+}_\nu \in \mathrm{Dom}^\#(\mathcal{E}_j) = \mathrm{Dom}(\mathscr{E}), \quad \forall \nu \in \mathcal{S}_{0,T}, \tag{6.7}$$

and that $Z^{j,+}$ is an RCLL $\widetilde{\mathcal{E}}_j$-supermartingale such that for any

$$t \in [0,T],\ Z^{j,+}_t \le Z^j(t), \text{ a.s.} \tag{6.8}$$

Otherwise, if no member of $\mathscr{E}$ satisfies (2.5) of [1], we suppose that (2.3) holds for some $j \in \mathcal{I}$. Then Lemma 2.3 and (2.3) imply that for any $t \in \mathcal{D}_T$, $C_Y + 2C_H \le Z^j(t) = Z(t) + H^j_t \le \zeta_Y - C_H + \zeta^j$, a.s. Taking the essential supremum of $Z^j(t)$ over $t \in \mathcal{D}_T$ yields that

$$C_Y + 2C_H \le \operatorname*{esssup}_{t\in\mathcal{D}_T} Z^j(t) \le \zeta_Y - C_H + \zeta^j, \quad \text{a.s.},$$

where $\zeta_Y - C_H + \zeta^j \in \mathrm{Dom}(\mathscr{E})$ thanks to (2.6), (2.3) and (D2). Hence Lemma 3.2 of [1] implies that $\operatorname{esssup}_{t\in\mathcal{D}_T} Z^j(t) \in \mathrm{Dom}(\mathscr{E}) = \mathrm{Dom}^\#(\mathcal{E}_j)$. Applying Corollary 2.2 of [1] and (6.6) again yields (6.7) and (6.8).

To see that $Z^{j,+}$ is a modification of $\{Z^j(t)\}_{t\in[0,T]}$, it suffices to show that for any $t \in [0,T]$, $Z^{j,+}_t \ge Z^j(t)$, a.s. Fix $t \in [0,T]$. For any $(i,\nu) \in \mathcal{I} \times \mathcal{S}_{t,T}$, Definition 3.2 of [1] assures that there exists a $k = k(j,i,t) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_k = \mathcal{E}^t_{j,i}$. (S1) and (2.5) imply that

$$H^k_t = H^k_{0,t} = H^j_{0,t} = H^j_t, \quad \text{and} \quad H^k_{t,\nu} = H^i_{t,\nu}, \quad \text{a.s.} \tag{6.9}$$

For any $n \in \mathbb{N}$, we set $t_n \triangleq q^+_n(t)$ and define $\nu_n \triangleq (\nu + 2^{-n}) \wedge T \in \mathcal{S}_{t,T}$. Let $m \ge n$; it is clear that $t_m \le t_n \le \nu_n$, a.s. Then Proposition 2.7(3) of [1] implies that

$$\widetilde{\mathcal{E}}_k\big[Y^k_{\nu_n}|\mathcal{F}_{t_m}\big] = \widetilde{\mathcal{E}}_k\big[Y_{\nu_n} + H^k_{t_m,\nu_n}|\mathcal{F}_{t_m}\big] + H^k_{t_m} \le Z(t_m) + H^k_{t_m}$$
$$= Z^j(t_m) + H^k_{t_m} - H^j_{t_m}, \quad \text{a.s.}$$

As $m \to \infty$, (6.9) as well as the right-continuity of the processes $\widetilde{\mathcal{E}}_k\big[Y^k_{\nu_n}|\mathcal{F}_\cdot\big]$, H^k and H^j imply that

$$\widetilde{\mathcal{E}}_k\big[Y^k_{\nu_n}|\mathcal{F}_t\big] = \lim_{m\to\infty} \widetilde{\mathcal{E}}_k\big[Y^k_{\nu_n}|\mathcal{F}_{t_m}\big] \le \varliminf_{m\to\infty} Z^j(t_m) + H^k_t - H^j_t$$
$$= \varliminf_{m\to\infty} Z^j(t_m) = Z^{j,+}_t, \quad \text{a.s.}$$

Since $\lim_{n\to\infty} \downarrow \nu_n = \nu$, a.s., the right-continuity of the process Y^k implies that $Y^k_{\nu_n}$ converges a.s. to Y^k_ν, which belongs to $\mathrm{Dom}(\mathscr{E})$ due to assumption (Y1) and (2.2). Then (2.8) and Theorem 2.1 of [1] imply that

$$\widetilde{\mathcal{E}}_k\big[Y^k_\nu|\mathcal{F}_t\big] \le \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_k\big[Y^k_{\nu_n}|\mathcal{F}_t\big] \le Z^{j,+}_t, \quad \text{a.s.}$$

We can deduce from Proposition 2.7(5), (3) of [1] and (3.3) of [1] and (6.9) that

$$\begin{aligned} Z^{j,+}_t &\ge \widetilde{\mathcal{E}}_k\big[Y^k_\nu|\mathcal{F}_t\big] = \mathcal{E}^t_{j,i}\big[Y^k_\nu|\mathcal{F}_t\big] = \widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_i\big[Y^k_\nu|\mathcal{F}_t\big]|\mathcal{F}_t\big] \\ &= \widetilde{\mathcal{E}}_i\big[Y^k_\nu|\mathcal{F}_t\big] = \widetilde{\mathcal{E}}_i\big[Y_\nu + H^i_{t,\nu}|\mathcal{F}_t\big] + H^j_t, \quad \text{a.s.} \end{aligned} \tag{6.10}$$

Letting (i,ν) run throughout $\mathcal{I}\times\mathcal{S}_{t,T}$ yields that

$$Z^{j,+}_t \ge \operatorname*{esssup}_{(i,\nu)\in\mathcal{I}\times\mathcal{S}_{t,T}} \widetilde{\mathcal{E}}_i\big[Y_\nu + H^i_{t,\nu}|\mathcal{F}_t\big] + H^j_t = Z(t) + H^j_t = Z^j(t), \quad \text{a.s.,}$$

which implies that $Z^{j,+}$ is an RCLL modification of $\big\{Z^j(t)\big\}_{t\in[0,T]}$. Correspondingly, $Z^0 \triangleq \big\{Z^{j,+}_t - H^j_t\big\}_{t\in[0,T]}$ is an RCLL modification of $\{Z(t)\}_{t\in[0,T]}$. Moreover, for any $i \in \mathcal{I}$, $Z^{i,0} \triangleq \big\{Z^0_t + H^i_t\big\}_{t\in[0,T]}$ defines an RCLL modification of $\big\{Z^i(t)\big\}_{t\in[0,T]}$; thus it is an $\widetilde{\mathcal{E}}_i$-supermartingale. $\square$

Proof of Proposition 2.3. For any $t \in [0,T]$, we know from (2.11) and Proposition 2.2 that $Y_t \le Z(t) = Z^0_t$, a.s. Since the processes Y and Z^0 are both right-continuous, it follows from Remark 2.3(2) that Z^0 dominates Y.

If $\nu \in \mathcal{S}^F_{0,T}$ takes values in a finite set $\{t_1 < \cdots < t_n\}$, for any $\alpha \in \{1\cdots n\}$, we can deduce from (2.14) that

$$\mathbf{1}_{\{\nu=t_\alpha\}} Z(\nu) = \mathbf{1}_{\{\nu=t_\alpha\}} Z(t_\alpha) = \mathbf{1}_{\{\nu=t_\alpha\}} Z^0_{t_\alpha} = \mathbf{1}_{\{\nu=t_\alpha\}} Z^0_\nu, \quad \text{a.s.}$$

Summing the above expression over α, we obtain

$$Z^0_\nu = Z(\nu), \quad \text{a.s.} \tag{6.11}$$

For general stopping time $\nu \in \mathcal{S}_{0,T}$, we let $\{\nu_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{S}^F_{0,T}$ such that $\lim_{n\to\infty} \downarrow \nu_n = \nu$, a.s. Thus for any $i \in \mathcal{I}$, the right-continuity of the process $Z^{i,0}$ shows that

$$Z^{i,0}_\nu = \lim_{n\to\infty} Z^{i,0}_{\nu_n}, \quad \text{a.s.} \tag{6.12}$$

For any $n \in \mathbb{N}$, (6.11) and (2.12) imply that

$$Z^{i,0}_{\nu_n} = Z^i(\nu_n) \ge C_Y + 2C_H, \quad \text{a.s.} \tag{6.13}$$

If $\mathcal{E}_j$ satisfies (2.5) of [1] for some $j \in \mathcal{I}$, we can deduce from (2.16) and (Y2) that

$$\widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}\big] = \widetilde{\mathcal{E}}_j\big[Z^j(\nu_n)\big] \le Z^j(0) = Z(0) = \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_\rho\big] < \infty,$$

and thus $\varliminf_{n\to\infty}\widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}\big] < \infty$. Then Remark 2.2(1) implies that $Z^{j,0}_\nu \in \mathrm{Dom}(\mathscr{E})$.

On the other hand, if no member of $\mathscr{E}$ satisfies (2.5) of [1], we suppose that (2.3) holds for some $j \in \mathcal{I}$. In light of Proposition 2.2 and Lemma 2.3, it holds a.s. that

$$C_Y + 2C_H \le Z^{j,0}_t = Z^0_t + H^j_t = Z(t) + H^j_t \le \zeta_Y - C_H + \zeta^j, \quad \forall t \in \mathcal{D}_T,$$

where $\zeta_Y - C_H + \zeta^j \in \text{Dom}(\mathscr{E})$. Since $Z^{j,0}$ is an RCLL process, it holds except on a null set N that

$$\begin{aligned} & C_Y + 2C_H \le Z^{j,0}_t \le \zeta_Y - C_H + \zeta^j, \quad \forall t \in [0,T], \quad \text{thus} \\ & C_Y + 2C_H \le Z^{j,0}_\nu \le \zeta_Y - C_H + \zeta^j. \end{aligned} \tag{6.14}$$

Lemma 3.2 of [1] then implies that $Z^{j,0}_\nu \in \text{Dom}(\mathscr{E})$. In both cases, we have seen that $Z^{j,0}_\nu \in \text{Dom}(\mathscr{E})$ for some $j \in \mathcal{I}$.

Since $Z^{j,0}$ is an RCLL $\widetilde{\mathcal{E}}_j$-supermartingale by Proposition 2.2, (6.13) and Theorem 2.4 of [1] imply that $\widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}|\mathcal{F}_{\nu_{n+1}}\big] \le Z^{j,0}_{\nu_{n+1}}$, a.s. for any $n \in \mathbb{N}$. Corollary 2.3 of [1] and Theorem 2.4 of [1] again show that

$$\widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}|\mathcal{F}_{\nu_{n+1}}\big]|\mathcal{F}_\nu\big] \le \widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_{n+1}}|\mathcal{F}_\nu\big] \le Z^{j,0}_\nu, \quad \text{a.s.,} \tag{6.15}$$

which implies that $\lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}|\mathcal{F}_\nu\big] \le Z^{j,0}_\nu$, a.s. On the other hand, using (6.12) and (6.13), we can deduce from Proposition 2.7(5) of [1] and Theorem 2.1 of [1] that

$$Z^{j,0}_\nu = \widetilde{\mathcal{E}}_j\big[Z^{j,0}_\nu|\mathcal{F}_\nu\big] \le \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}|\mathcal{F}_\nu\big] \le Z^{j,0}_\nu, \quad \text{a.s.}$$

Then (6.11) and (2.16) imply that

$$\begin{aligned} Z^{j,0}_\nu &= \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j\big[Z^{j,0}_{\nu_n}|\mathcal{F}_\nu\big] \\ &= \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j\big[Z^j(\nu_n)|\mathcal{F}_\nu\big] \le Z^j(\nu), \quad \text{a.s.,} \quad \text{thus } Z^0_\nu \le Z(\nu) \text{ a.s.} \end{aligned} \tag{6.16}$$

For any $(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$ and $n \in \mathbb{N}$, we define $\rho_n \triangleq \rho \vee \nu_n \in \mathcal{S}_{\nu_n,T}$. Proposition 2.7(3) of [1] implies that

$$\widetilde{\mathcal{E}}_i\big[Y^i_{\rho_n}|\mathcal{F}_{\nu_n}\big] = \widetilde{\mathcal{E}}_i\big[Y_{\rho_n} + H^i_{\nu_n,\rho_n}|\mathcal{F}_{\nu_n}\big] + H^i_{\nu_n} \le Z(\nu_n) + H^i_{\nu_n} = Z^i(\nu_n), \quad \text{a.s.}$$

Taking $\widetilde{\mathcal{E}}_i\big[\cdot|\mathcal{F}_\nu\big]$ on both sides, we see from Corollary 2.3 of [1] that $\widetilde{\mathcal{E}}_i\big[Y^i_{\rho_n}|\mathcal{F}_\nu\big] \le \widetilde{\mathcal{E}}_i\big[Z^i(\nu_n)|\mathcal{F}_\nu\big]$, a.s. It is easy to see that $\lim_{n\to\infty} \downarrow \rho_n = \rho$, a.s. Using the right-continuity of processes Y and H^i, we can deduce from (2.8), Proposition 2.8 of [1] and (6.16) that

$$\widetilde{\mathcal{E}}_i\big[Y^i_\rho|\mathcal{F}_\nu\big] \le \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_i\big[Y^i_{\rho_n}|\mathcal{F}_\nu\big] \le \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_i\big[Z^i(\nu_n)|\mathcal{F}_\nu\big] = Z^{i,0}_\nu, \quad \text{a.s.}$$

Then subtracting H^i_ν from both sides and taking the essential supremum over $(i,\rho) \in \mathcal{I} \times \mathcal{S}_{\nu,T}$ yields that $Z(\nu) \le Z^0_\nu$, a.s., which together with (6.16) shows that $Z^0_\nu = Z(\nu)$, a.s. Hence $Z^0_\nu \in \text{Dom}(\mathscr{E})$ by Lemma 2.4. For any $i \in \mathcal{I}$,

$$Z^{i,0}_\nu = Z^0_\nu + H^i_\nu = Z(\nu) + H^i_\nu = Z^i(\nu), \quad \text{a.s.,}$$

and thus $Z^{i,0}_\nu \in \text{Dom}(\mathscr{E})$; thanks to Lemma 2.4 once again, (2.17) is proved.

Now let X be another RCLL $\mathbf{F}$-adapted process dominating Y such that $X^i \triangleq \{X_t + H^i_t\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale for any $i \in \mathcal{I}$. We fix $t \in [0,T]$. For any $i \in \mathcal{I}$ and $\nu \in \mathcal{S}_{t,T}$, we let $\{\nu_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{S}^F_{t,T}$ such that $\lim_{n\to\infty} \downarrow \nu_n = \nu$, a.s. For any $n \in \mathbb{N}$, since X^i dominates Y^i, Remark 2.3(1) shows that $X^i_{\nu_n} \ge Y^i_{\nu_n}$, a.s. Then (A4), Proposition 2.6 of [1] and Theorem 2.4 of [1] imply that

$$\widetilde{\mathcal{E}}_i\big[Y_{\nu_n} + H^i_{t,\nu_n}|\mathcal{F}_t\big] = \widetilde{\mathcal{E}}_i\big[Y^i_{\nu_n}|\mathcal{F}_t\big] - H^i_t \le \widetilde{\mathcal{E}}_i\big[X^i_{\nu_n}|\mathcal{F}_t\big] - H^i_t \le X^i_t - H^i_t = X_t, \quad \text{a.s.}$$

The right-continuity of the processes Y and H^i shows that $Y_\nu + H^i_{t,\nu} = \lim_{n\to\infty}\big(Y_{\nu_n} + H^i_{t,\nu_n}\big)$, a.s.; thus it follows from (2.7), (2.4) and Proposition 2.8 of [1] that

$$\widetilde{\mathcal{E}}_i\big[Y_\nu + H^i_{t,\nu}|\mathcal{F}_t\big] \le \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_i\big[Y_{\nu_n} + H^i_{t,\nu_n}|\mathcal{F}_t\big] \le X_t, \quad \text{a.s.}$$

Taking the essential supremum of the left-hand side over $(i,\nu) \in \mathcal{I} \times \mathcal{S}_{t,T}$, we can deduce from Proposition 2.2 that

$$Z^0_t = Z(t) = \operatorname*{esssup}_{(i,\nu)\in\mathcal{I}\times\mathcal{S}_{t,T}} \widetilde{\mathcal{E}}_i\big[Y_\nu + H^i_{t,\nu}|\mathcal{F}_t\big] \le X_t, \quad \text{a.s.}$$

Since both Z^0 and X are RCLL processes, Remark 2.3(2) once again shows that X dominates Z^0. □

Proof of Lemma 2.5. For any $i \in \mathcal{I}$, (2.18), (2.4), as well as Proposition 2.7(5) of [1], imply that

$$\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] \ge \widetilde{\mathcal{E}}_i\big[C_* + C_H|\mathcal{F}_\nu\big] = C_Y + 2C_H, \quad \text{a.s.}$$

Taking the essential supremum of the left-hand side over $i \in \mathcal{I}$, we can deduce from (2.19) that

$$C_Y + 2C_H \le \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] = J_\delta(\nu) \le Z(\nu), \quad \text{a.s.} \tag{6.17}$$

Then Lemma 3.2 of [1] implies that $J_\delta(\nu) \in \mathrm{Dom}(\mathscr{E})$. Let σ be another stopping time in $\mathcal{S}_{0,T}$. By (2.17) and (2.14),

$$\begin{aligned}\mathbf{1}_{\{\tau_\delta(\nu)=\tau_\delta(\sigma)\}} Z^0_{\tau_\delta(\nu)} &= \mathbf{1}_{\{\tau_\delta(\nu)=\tau_\delta(\sigma)\}} Z(\tau_\delta(\nu)) = \mathbf{1}_{\{\tau_\delta(\nu)=\tau_\delta(\sigma)\}} Z(\tau_\delta(\sigma))\\ &= \mathbf{1}_{\{\tau_\delta(\nu)=\tau_\delta(\sigma)\}} Z^0_{\tau_\delta(\sigma)}, \quad \text{a.s.}\end{aligned}$$

Since $\{\nu=\sigma\} \subset \{\tau_\delta(\nu)=\tau_\delta(\sigma)\}$, multiplying by $\mathbf{1}_{\{\nu=\sigma\}}$ on both sides yields that $\mathbf{1}_{\{\nu=\sigma\}} Z^0_{\tau_\delta(\nu)} = \mathbf{1}_{\{\nu=\sigma\}} Z^0_{\tau_\delta(\sigma)}$, a.s. For any $i \in \mathcal{I}$, applying Proposition 2.7(2) of [1] and recalling how $\widetilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\nu]$ and $\widetilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\sigma]$ are defined in (2.6) of [1], we obtain

$$\begin{aligned}\mathbf{1}_{\{\nu=\sigma\}}\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] &= \mathbf{1}_{\{\nu=\sigma\}}\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\sigma\big]\\ &= \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\nu=\sigma\}} Z^0_{\tau_\delta(\sigma)} + \mathbf{1}_{\{\nu=\sigma\}} H^i_{\sigma,\tau_\delta(\sigma)}|\mathcal{F}_\sigma\big]\\ &= \mathbf{1}_{\{\nu=\sigma\}}\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\sigma)} + H^i_{\sigma,\tau_\delta(\sigma)}|\mathcal{F}_\sigma\big], \quad \text{a.s.,}\end{aligned}$$

where we use the fact that $\{\nu=\sigma\} \in \mathcal{F}_{\nu\wedge\sigma}$ thanks to [9, Lemma 1.2.16]. Taking the essential supremum of both sides over $i \in \mathcal{I}$, we can deduce from Lemma 3.3(2) of [1] that a.s.

$$\begin{aligned}\mathbf{1}_{\{\nu=\sigma\}} J_\delta(\nu) &= \operatorname*{esssup}_{i\in\mathcal{I}} \mathbf{1}_{\{\nu=\sigma\}}\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big]\\ &= \operatorname*{esssup}_{i\in\mathcal{I}} \mathbf{1}_{\{\nu=\sigma\}}\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(\sigma)} + H^i_{\sigma,\tau_\delta(\sigma)}|\mathcal{F}_\sigma\big] = \mathbf{1}_{\{\nu=\sigma\}} J_\delta(\sigma). \quad \square\end{aligned}$$

Proof of Proposition 2.4.

Proof of (1). We fix $i \in \mathcal{I}$ and $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s. Taking $(\nu, \mathcal{I}', \mathcal{U}) = \big(\rho, \mathcal{I}, \{\tau_\delta(\rho)\}\big)$ and $X\big(\tau_\delta(\rho)\big) = Z^0_{\tau_\delta(\rho)}$ in Lemma 2.2, we can find a sequence $\{j_n\}_{n=1}^\infty$ in $\mathcal{I}$ such that

$$J_\delta(\rho) = \operatorname*{esssup}_{j\in\mathcal{I}} \widetilde{\mathcal{E}}_j[Z^0_{\tau_\delta(\rho)} + H^j_{\rho,\tau_\delta(\rho)}|\mathcal{F}_\rho] = \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_{j_n}[Z^0_{\tau_\delta(\rho)} + H^{j_n}_{\rho,\tau_\delta(\rho)}|\mathcal{F}_\rho], \quad \text{a.s.}$$

For any $n \in \mathbb{N}$, it follows from Definition 3.2 of [1] that there exists a $k_n = k(i, j_n, \rho) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n} = \mathcal{E}^{\rho}_{i,j_n}$. Applying Proposition 2.7(3) of [1] to $\widetilde{\mathcal{E}}_{j_n}$, we can deduce from (3.3) of [1] and (2.5) that

$$\begin{aligned}\widetilde{\mathcal{E}}_{k_n}\big[Z^0_{\tau_\delta(\rho)} + H^{k_n}_{\nu,\tau_\delta(\rho)}|\mathcal{F}_\nu\big] &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_{j_n}\big[Z^0_{\tau_\delta(\rho)} + H^{k_n}_{\nu,\tau_\delta(\rho)}|\mathcal{F}_\rho\big]|\mathcal{F}_\nu\big]\\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_{j_n}\big[Z^0_{\tau_\delta(\rho)} + H^{j_n}_{\rho,\tau_\delta(\rho)}|\mathcal{F}_\rho\big] + H^i_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \text{a.s.} \end{aligned} \tag{6.18}$$

Since $\nu \le \rho$, a.s., we see that $\tau_\delta(\nu) \le \tau_\delta(\rho)$, a.s. Due to (2.17) and (2.15), we have that

$$\begin{aligned}\widetilde{\mathcal{E}}_{k_n}\big[Z^0_{\tau_\delta(\rho)} + H^{k_n}_{\tau_\delta(\nu),\tau_\delta(\rho)}|\mathcal{F}_{\tau_\delta(\nu)}\big] &\le \operatorname*{esssup}_{j\in\mathcal{I}} \widetilde{\mathcal{E}}_j\big[Z(\tau_\delta(\rho)) + H^j_{\tau_\delta(\nu),\tau_\delta(\rho)}|\mathcal{F}_{\tau_\delta(\nu)}\big]\\ &\le Z(\tau_\delta(\nu)) = Z^0_{\tau_\delta(\nu)}, \quad \text{a.s.}\end{aligned}$$

Then using Corollary 2.3 of [1] and applying Proposition 2.7(3) of [1], (1) of [1] to $\widetilde{\mathcal{E}}_{k_n}$, we obtain

$$\begin{aligned}\widetilde{\mathcal{E}}_{k_n}\big[Z^0_{\tau_\delta(\rho)} + H^{k_n}_{\nu,\tau_\delta(\rho)}|\mathcal{F}_\nu\big] &= \widetilde{\mathcal{E}}_{k_n}\big[\widetilde{\mathcal{E}}_{k_n}\big[Z^0_{\tau_\delta(\rho)} + H^{k_n}_{\tau_\delta(\nu),\tau_\delta(\rho)}|\mathcal{F}_{\tau_\delta(\nu)}\big] + H^{k_n}_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big]\\ &\le \widetilde{\mathcal{E}}_{k_n}\big[Z^0_{\tau_\delta(\nu)} + H^{k_n}_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] \le \operatorname*{esssup}_{j\in\mathcal{I}} \widetilde{\mathcal{E}}_j\big[Z^0_{\tau_\delta(\nu)} + H^j_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big]\\ &= J_\delta(\nu), \quad \text{a.s.}\end{aligned} \tag{6.19}$$

For any $n \in \mathbb{N}$, we see from (2.18), (2.4) and Proposition 2.7(5) of [1] that

$$\widetilde{\mathcal{E}}_{j_n}\big[Z^0_{\tau_\delta(\rho)} + H^{j_n}_{\rho,\tau_\delta(\rho)}|\mathcal{F}_\rho\big] + H^i_{\nu,\rho} \ge \widetilde{\mathcal{E}}_{j_n}\big[C_Y + 2C_H|\mathcal{F}_\rho\big] + C_H = C_Y + 3C_H, \quad \text{a.s.}$$

Then Proposition 2.8 of [1], and (6.18) and (6.19) imply that a.s.

$$\begin{aligned}\widetilde{\mathcal{E}}_i\big[J_\delta(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] &\le \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_{j_n}\big[Z^0_{\tau_\delta(\rho)} + H^{j_n}_{\rho,\tau_\delta(\rho)}|\mathcal{F}_\rho\big] + H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\\ &= \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_{k_n}\big[Z^0_{\tau_\delta(\rho)} + H^{k_n}_{\nu,\tau_\delta(\rho)}|\mathcal{F}_\nu\big] \le J_\delta(\nu),\end{aligned}$$

where we used the fact that $J_\delta(\rho) + H^i_{\nu,\rho} \in \mathrm{Dom}(\mathscr{E})$, established thanks to Lemma 2.5, (2.2) and (D2). Also, $J^i_\delta(\rho) \triangleq J_\delta(\rho) + H^i_\rho \in \mathrm{Dom}(\mathscr{E})$. A simple application of Proposition 2.7(3) of [1] yields that

$$\widetilde{\mathcal{E}}_i\big[J^i_\delta(\rho)|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[J_\delta(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] + H^i_\nu \le J_\delta(\nu) + H^i_\nu = J^i_\delta(\nu), \quad \text{a.s.} \tag{6.20}$$

In particular, for any $0 \le s < t \le T$, $\widetilde{\mathcal{E}}_i\big[J^i_\delta(t)|\mathcal{F}_s\big] \le J^i_\delta(s)$, a.s. Hence, $\big\{J^i_\delta(t)\big\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale.

Proof of (2). Proposition 2.6 of [1] and Theorem 2.3 of [1] then show that $\Big\{J^{\delta,i,+}_t \triangleq \varliminf_{n\to\infty} J^i_\delta\big(q^+_n(t)\big)\Big\}_{t\in[0,T]}$ defines an RCLL process. For any $i \in \mathcal{I}$ and $\nu \in \mathcal{S}_{0,T}$, (6.17) and (2.4) imply that

$$J^i_\delta(\nu) = J_\delta(\nu) + H^i_\nu \ge C_Y + 3C_H, \quad \text{a.s.} \tag{6.21}$$

which implies that $\operatorname{essinf}_{t\in[0,T]} J^i_\delta(t) \ge C_Y + 3C_H$, a.s. Like in the discussion of (6.7) and (6.8), one can show by means of two cases that for some $j \in \mathcal{I}$, $J^{\delta,j,+}_\nu \in \mathrm{Dom}(\mathscr{E})$ for any $\nu \in \mathcal{S}_{0,T}$, and $J^{\delta,j,+}$ is an RCLL $\widetilde{\mathcal{E}}_j$-supermartingale such that for any $t \in [0, T]$

$$J^{\delta,j,+}_t \le J^j_\delta(t), \quad \text{a.s.} \tag{6.22}$$

To see the reverse of (6.22), we fix $t \in [0, T]$. For any $i \in \mathcal{I}$, Definition 3.2 of [1] assures that there exists a $k = k(j, i, t) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_k = \mathcal{E}^t_{j,i}$. Moreover, (S1) and (2.5) imply that

$$H^k_t = H^k_{0,t} = H^j_{0,t} = H^j_t, \quad \text{and} \quad H^k_{t,\tau_\delta(t)} = H^i_{t,\tau_\delta(t)}, \quad \text{a.s.} \tag{6.23}$$

For any $n \in \mathbb{N}$, we set $t_n \triangleq q^+_n(t)$. Let $m \ge n$; thus $t_m \le t_n$. Then (2.17), Corollary 2.3 of [1], and Proposition 2.7(3) of [1] as well as (2.15) imply that

$$\begin{aligned}
\widetilde{\mathcal{E}}_k\big[Z^{k,0}_{\tau_\delta(t_n)}|\mathcal{F}_{t_m}\big] &= \widetilde{\mathcal{E}}_k\big[Z^k(\tau_\delta(t_n))|\mathcal{F}_{t_m}\big] = \widetilde{\mathcal{E}}_k\big[\widetilde{\mathcal{E}}_k\big[Z^k(\tau_\delta(t_n))|\mathcal{F}_{\tau_\delta(t_m)}\big]|\mathcal{F}_{t_m}\big] \\
&= \widetilde{\mathcal{E}}_k\big[\widetilde{\mathcal{E}}_k\big[Z(\tau_\delta(t_n)) + H^k_{\tau_\delta(t_m),\tau_\delta(t_n)}|\mathcal{F}_{\tau_\delta(t_m)}\big] + H^k_{t_m,\tau_\delta(t_m)}|\mathcal{F}_{t_m}\big] + H^k_{t_m} \\
&\le \widetilde{\mathcal{E}}_k\big[\operatorname*{esssup}_{l\in\mathcal{I}} \widetilde{\mathcal{E}}_l\big[Z(\tau_\delta(t_n)) + H^l_{\tau_\delta(t_m),\tau_\delta(t_n)}|\mathcal{F}_{\tau_\delta(t_m)}\big] + H^k_{t_m,\tau_\delta(t_m)}|\mathcal{F}_{t_m}\big] + H^k_{t_m} \\
&\le \widetilde{\mathcal{E}}_k\big[Z(\tau_\delta(t_m)) + H^k_{t_m,\tau_\delta(t_m)}|\mathcal{F}_{t_m}\big] + H^k_{t_m} \\
&\le \operatorname*{esssup}_{l\in\mathcal{I}} \widetilde{\mathcal{E}}_l\big[Z^0_{\tau_\delta(t_m)} + H^l_{t_m,\tau_\delta(t_m)}|\mathcal{F}_{t_m}\big] + H^k_{t_m} \\
&= J_\delta(t_m) + H^k_{t_m} = J^j_\delta(t_m) + H^k_{t_m} - H^j_{t_m}, \quad \text{a.s.}
\end{aligned}$$

As $m \to \infty$, (6.23), as well as the right-continuity of the processes $\widetilde{\mathcal{E}}_k\big[Z^{k,0}_{\tau_\delta(t_n)}|\mathcal{F}_\cdot\big]$, H^k and H^j, implies that

$$\begin{aligned}
\widetilde{\mathcal{E}}_k\big[Z^{k,0}_{\tau_\delta(t_n)}|\mathcal{F}_t\big] &= \lim_{m\to\infty} \widetilde{\mathcal{E}}_k\big[Z^{k,0}_{\tau_\delta(t_n)}|\mathcal{F}_{t_m}\big] \le \varliminf_{m\to\infty} J^j_\delta(t_m) + H^k_t - H^j_t \\
&= \varliminf_{m\to\infty} J^j_\delta(t_m) = J^{\delta,j,+}_t, \quad \text{a.s.}
\end{aligned}$$

Since $\lim_{n\to\infty} \downarrow \tau_\delta(t_n) = \tau_\delta(t)$ a.s., the right-continuity of process $Z^{k,0}$, (2.18) and Theorem 2.1 of [1] imply that

$$\widetilde{\mathcal{E}}_k\big[Z^{k,0}_{\tau_\delta(t)}|\mathcal{F}_t\big] \le \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_k\big[Z^{k,0}_{\tau_\delta(t_n)}|\mathcal{F}_t\big] \le J^{\delta,j,+}_t, \quad \text{a.s.}$$

Then like for (6.10), one can deduce from (3.3) of [1] and (6.23) that $J^{\delta,j,+}_t \ge \widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(t)} + H^i_{t,\tau_\delta(t)}|\mathcal{F}_t\big] + H^j_t$, a.s. Taking the essential supremum of the right-hand side over $i \in \mathcal{I}$ yields that $J^{\delta,j,+}_t \ge J^j_\delta(t)$, a.s., which together with (6.22) implies that $J^{\delta,j,+}$ is an RCLL modification of $\big\{J^j_\delta(t)\big\}_{t\in[0,T]}$. Then we can draw similar conclusions to those at the end of proof of Proposition 2.2.

Proof of (3). Like for (6.11), we can deduce from Lemma 2.5 that for any $\nu \in \mathcal{S}^F_{0,T}$

$$J^{\delta,0}_\nu = J_\delta(\nu), \quad \text{a.s.} \tag{6.24}$$

For a general stopping time $\nu \in \mathcal{S}_{0,T}$, we let $\{\nu_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{S}^F_{0,T}$ such that $\lim_{n\to\infty} \downarrow \nu_n = \nu$, a.s. Thus for any $i \in \mathcal{I}$, the right-continuity of the process $J^{\delta,i,0}$ shows that

$$J^{\delta,i,0}_\nu = \lim_{n\to\infty} J^{\delta,i,0}_{\nu_n}, \quad \text{a.s.} \tag{6.25}$$

In light of (6.24) and (6.21), it holds a.s. that $J^{\delta,i,0}_t = J^i_\delta(t) \ge C_Y + 3C_H$, $\forall t \in \mathcal{D}_T$. Since $J^{\delta,i,0}$ is an RCLL process, it holds except on a null set N that

$$J^{\delta,i,0}_t \ge C_Y + 3C_H, \quad \forall t \in [0, T], \quad \text{thus} \quad J^{\delta,i,0}_\sigma \ge C_Y + 3C_H, \quad \forall \sigma \in \mathcal{S}_{0,T}. \tag{6.26}$$

Like in the discussion in Proof of Proposition 2.3, one can show by means of two cases that $J^{\delta,j,0}_\nu \in \mathrm{Dom}(\mathscr{E})$ for some $j \in \mathcal{I}$. And like in the arguments used in (6.15) through (6.16) (with (6.24)–(6.26) replacing (6.11)–(6.13) respectively, and with (6.20) replacing (2.16)), we can deduce that

$$J^{\delta,j,0}_\nu = \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j\big[J^{\delta,j,0}_{\nu_n}|\mathcal{F}_\nu\big]$$
$$= \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j\big[J^j_\delta(\nu_n)|\mathcal{F}_\nu\big] \le J^j_\delta(\nu), \quad \text{a.s.} \quad \text{thus, } J^{\delta,0}_\nu \le J_\delta(\nu), \text{ a.s.} \tag{6.27}$$

The right-continuity of the process $J^{\delta,0}$, and (6.24) and (6.17) show that $J^{\delta,0}_\nu = \lim_{n\to\infty} J^{\delta,0}_{\nu_n} = \lim_{n\to\infty} J_\delta(\nu_n) \ge C_Y + 2C_H$, a.s. Lemma 2.5 and Lemma 3.2 of [1] thus imply that $J^{\delta,0}_\nu \in \mathrm{Dom}(\mathscr{E})$.

For any $i \in \mathcal{I}$ and $n \in \mathbb{N}$, since $\nu \le \nu_n \le \tau_\delta(\nu_n)$, a.s., Corollary 2.3 of [1], and (6.27) and (6.20) imply that

$$\widetilde{\mathcal{E}}_i\big[J^{\delta,i,0}_{\tau_\delta(\nu_n)}|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[J^{\delta,i,0}_{\tau_\delta(\nu_n)}|\mathcal{F}_{\nu_n}\big]|\mathcal{F}_\nu\big] \le \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[J^i_\delta\big(\tau_\delta(\nu_n)\big)|\mathcal{F}_{\nu_n}\big]|\mathcal{F}_\nu\big]$$
$$\le \widetilde{\mathcal{E}}_i\big[J^i_\delta(\nu_n)|\mathcal{F}_\nu\big], \quad \text{a.s.}$$

It is easy to see that $\lim_{n\to\infty} \downarrow \tau_\delta(\nu_n) = \tau_\delta(\nu)$, a.s. Using the right-continuity of the process $J^{\delta,i,0}$, we can deduce from (6.26), Proposition 2.8 of [1] and (6.27) that

$$\widetilde{\mathcal{E}}_i\big[J^{\delta,i,0}_{\tau_\delta(\nu)}|\mathcal{F}_\nu\big] \le \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_i\big[J^{\delta,i,0}_{\tau_\delta(\nu_n)}|\mathcal{F}_\nu\big] \le \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_i\big[J^i_\delta(\nu_n)|\mathcal{F}_\nu\big] = J^{\delta,i,0}_\nu, \quad \text{a.s.}$$

Then subtracting H^i_ν from both sides and taking the essential supremum over $i \in \mathcal{I}$ yields that

$$J_\delta(\nu) = \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[J^{\delta,0}_{\tau_\delta(\nu)} + H^i_{\nu,\tau_\delta(\nu)}|\mathcal{F}_\nu\big] \le J^{\delta,0}_\nu, \quad \text{a.s.},$$

which together with (6.27) shows that $J^{\delta,0}_\nu = J_\delta(\nu)$, a.s. □

Proof of Theorem 2.1. We first show that for any $\delta \in (0,1)$ and $\nu \in \mathcal{S}_{0,T}$

$$J_\delta(\nu) = Z^0_\nu = Z(\nu), \quad \text{a.s.} \tag{6.28}$$

Fix $i \in \mathcal{I}$. Lemma 3.1 of [1] indicates that $\widetilde{\mathcal{E}}_i$ is a convex $\mathbf{F}$-expectation on $\mathrm{Dom}(\mathscr{E})$. Since $Z^{i,0}$ and $J^{\delta,i,0}$ are both $\widetilde{\mathcal{E}}_i$-supermartingales, we can deduce that for any $0 \le s < t \le T$,

$$\widetilde{\mathcal{E}}_i\big[\delta Z^0_t + (1-\delta)J^{\delta,0}_t + H^i_t|\mathcal{F}_s\big] = \widetilde{\mathcal{E}}_i\big[\delta Z^{i,0}_t + (1-\delta)J^{\delta,i,0}_t|\mathcal{F}_s\big]$$
$$\le \delta\widetilde{\mathcal{E}}_i\big[Z^{i,0}_t|\mathcal{F}_s\big] + (1-\delta)\widetilde{\mathcal{E}}_i\big[J^{\delta,i,0}_t|\mathcal{F}_s\big]$$
$$\le \delta Z^{i,0}_s + (1-\delta)J^{\delta,i,0}_s = \delta Z^0_s + (1-\delta)J^{\delta,0}_s + H^i_s, \quad \text{a.s.},$$

which shows that $\left\{\delta Z^0_t + (1-\delta)J^{\delta,0}_t + H^i_t\right\}_{t\in[0,T]}$ is an RCLL $\widetilde{\mathcal{E}}_i$-supermartingale.

Now we fix $t \in [0,T]$ and define $A \triangleq \{\tau_\delta(t) = t\} \in \mathcal{F}_t$. Using Proposition 2.4(3), and Lemma 3.3(2) of [1] as well as applying Proposition 2.7(2), (5) of [1] to each $\widetilde{\mathcal{E}}_i$, we obtain that a.s.

$$\mathbf{1}_A J^{\delta,0}_t = \mathbf{1}_A J_\delta(t) = \mathbf{1}_A \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Z^0_{\tau_\delta(t)} + H^i_{t,\tau_\delta(t)}|\mathcal{F}_t\big]$$
$$= \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[\mathbf{1}_A\big(Z^0_{\tau_\delta(t)} + H^i_{t,\tau_\delta(t)}\big)|\mathcal{F}_t\big]$$
$$= \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[\mathbf{1}_A Z^0_t|\mathcal{F}_t\big] = \mathbf{1}_A Z^0_t.$$

Then (2.17) and (2.11) imply that

$$\mathbf{1}_A\big(\delta Z^0_t + (1-\delta)J^{\delta,0}_t\big) = \mathbf{1}_A Z^0_t = \mathbf{1}_A Z(t) \ge \mathbf{1}_A Y_t, \quad \text{a.s.} \tag{6.29}$$

Moreover, we see from the definition of $\tau_\delta(t)$ that for any $\omega \in A^c$

$$Y_s(\omega) < \delta Z^0_s(\omega) + (1-\delta)(C_Y + 2C_H), \quad \forall s \in \big[t, \tau_\delta\big(t\big)(\omega)\big). \tag{6.30}$$

Since both Z^0 and Y are right-continuous processes, (6.30) and (6.17) imply that

$$Y_t \le \delta Z^0_t + (1-\delta)(C_Y + 2C_H) \le \delta Z^0_t + (1-\delta)J^{\delta,0}_t \quad \text{a.s. on } A^c,$$

which in conjunction with (6.29) and Remark 2.3(2) shows that the RCLL process $\delta Z^0 + (1-\delta)J^{\delta,0}$ dominates Y, and thus dominates Z^0 thanks to Proposition 2.3. It follows that $J^{\delta,0}$ also dominates Z^0. Then for any $\nu \in \mathcal{S}_{0,T}$, Proposition 2.4(3), Remark 2.3(1) and (2.17) imply that $J_\delta(\nu) = J^{\delta,0}_\nu \ge Z^0_\nu = Z(\nu)$, a.s. The reverse inequality comes from (2.19). This proves (6.28).

Next, we fix $\nu \in \mathcal{S}_{0,T}$ and set $\delta^n = \frac{n-1}{n}$, $n \in \mathbb{N}$. It is clear that the sequence $\big\{\tau_{\delta^n}(\nu)\big\}_{n\in\mathbb{N}}$ increases a.s. to $\overline{\tau}(\nu)$. Since the family of processes $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-left-continuous", we can find a subsequence $\{\delta^{n_k}\}_{k\in\mathbb{N}}$ of $\{\delta^n\}_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \operatorname*{esssup}_{i\in\mathcal{I}} \left| \widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] - \widetilde{\mathcal{E}}_i\big[Y^i_{\overline{\tau}(\nu)}|\mathcal{F}_\nu\big]\right| = 0, \quad \text{a.s.} \tag{6.31}$$

For any $i \in \mathcal{I}$ and $k \in \mathbb{N}$, Remark 2.4(1) implies that $Y_{\tau_{\delta^{n_k}}(\nu)} \ge \delta^{n_k} Z^0_{\tau_{\delta^{n_k}}(\nu)} + \big(1-\delta^{n_k}\big)(C_Y + 2C_H)$, a.s. Hence Proposition(3) of [1] shows that

$$\begin{aligned}
&\widetilde{\mathcal{E}}_i\Big[Z^0_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\nu,\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] + \frac{1}{n_k-1}(C_Y+2C_H)\\
&\quad \le \widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] - H^i_\nu\\
&\quad = \widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] - \widetilde{\mathcal{E}}_i\Big[Y^i_{\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big] + \widetilde{\mathcal{E}}_i\Big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big]\\
&\quad \le \operatorname*{esssup}_{i\in\mathcal{I}} \left|\widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] - \widetilde{\mathcal{E}}_i\Big[Y^i_{\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big]\right|\\
&\qquad + \widetilde{\mathcal{E}}_i\Big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big]\\
&\quad \le \operatorname*{esssup}_{i\in\mathcal{I}} \left|\widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] - \widetilde{\mathcal{E}}_i\Big[Y^i_{\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big]\right|\\
&\qquad + \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\Big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big], \quad \text{a.s.}
\end{aligned} \tag{6.32}$$

Taking the esssup of the left-hand side over $\mathcal{I}$, we see from (6.28) that

$$\begin{aligned}
&\operatorname*{esssup}_{i\in\mathcal{I}} \left|\widetilde{\mathcal{E}}_i\Big[\frac{n_k}{n_k-1} Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] - \widetilde{\mathcal{E}}_i\Big[Y^i_{\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big]\right|\\
&\qquad + \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\Big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\Big]\\
&\quad \ge J_{\delta^{n_k}}(\nu) + \frac{1}{n_k-1}(C_Y+2C_H) = Z(\nu) + \frac{1}{n_k-1}(C_Y+2C_H), \quad \text{a.s..}
\end{aligned}$$

As $k \to \infty$, (6.31), (2.11) and (2.15) imply that

$$Z(\nu) \le \underset{i\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_i\big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] \le \underset{i\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] \le Z(\nu), \quad \text{a.s.},$$

which shows that

$$Z(\nu) = \underset{i\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_i\big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] = \underset{i\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big], \quad \text{a.s.} \tag{6.33}$$

Now we fix $\rho \in \mathcal{S}_{\nu,\overline{\tau}(\nu)}$. For any $i \in \mathcal{I}$, Corollary 2.3 of [1] and (2.16) show that

$$\widetilde{\mathcal{E}}_i\big[Z^i(\overline{\tau}(\nu))|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[Z^i(\overline{\tau}(\nu))|\mathcal{F}_\rho\big]|\mathcal{F}_\nu\big] \le \widetilde{\mathcal{E}}_i\big[Z^i(\rho)|\mathcal{F}_\nu\big], \quad \text{a.s.}$$

Then Proposition 2.7(3) of [1] implies that

$$\begin{aligned}\widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] &= \widetilde{\mathcal{E}}_i\big[Z^i(\overline{\tau}(\nu))|\mathcal{F}_\nu\big] - H^i_\nu \le \widetilde{\mathcal{E}}_i\big[Z^i(\rho)|\mathcal{F}_\nu\big] - H^i_\nu \\ &= \widetilde{\mathcal{E}}_i\big[Z(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big], \quad \text{a.s.}\end{aligned}$$

Taking the essential supremum of both sides over $\mathcal{I}$, we can deduce from (2.15) that

$$\underset{i\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] \le \underset{i\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_i\big[Z(\rho) + H^i_{\nu,\rho}|\mathcal{F}_\nu\big] \le Z(\nu), \quad \text{a.s.},$$

which together with (6.33) proves (2.22).

Finally, we will prove that $\overline{\tau}(\nu) = \tau_1(\nu)$. For any $i \in \mathcal{I}$ and $k \in \mathbb{N}$, (2.17), (2.15), and Proposition 2.7(3) of [1] as well as Corollary 2.3 of [1] imply that

$$\begin{aligned}&\widetilde{\mathcal{E}}_i\big[Z^0_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\nu,\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_i\big[Z(\tau_{\delta^{n_k}}(\nu)) + H^i_{\nu,\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\big] \\ &\ge \widetilde{\mathcal{E}}_i\Big[\underset{j\in\mathcal{I}}{\operatorname{esssup}}\, \widetilde{\mathcal{E}}_j\big[Z(\overline{\tau}(\nu)) + H^j_{\tau_{\delta^{n_k}}(\nu),\overline{\tau}(\nu)}|\mathcal{F}_{\tau_{\delta^{n_k}}(\nu)}\big] + H^i_{\nu,\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\Big] \\ &\ge \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\tau_{\delta^{n_k}}(\nu),\overline{\tau}(\nu)}|\mathcal{F}_{\tau_{\delta^{n_k}}(\nu)}\big] + H^i_{\nu,\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_{\tau_{\delta^{n_k}}(\nu)}\big]|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big], \quad \text{a.s.},\end{aligned}$$

which together with (6.32) shows that

$$\begin{aligned}&\underset{i\in\mathcal{I}}{\operatorname{esssup}}\left|\widetilde{\mathcal{E}}_i\left[\frac{n_k}{n_k-1}Y_{\tau_{\delta^{n_k}}(\nu)} + H^i_{\tau_{\delta^{n_k}}(\nu)}|\mathcal{F}_\nu\right] - \widetilde{\mathcal{E}}_i\left[Y^i_{\overline{\tau}(\nu)}|\mathcal{F}_\nu\right]\right| + \widetilde{\mathcal{E}}_i\left[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\right] \\ &\ge \widetilde{\mathcal{E}}_i\left[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\right] + \frac{1}{n_k-1}(C_Y + 2C_H), \quad \text{a.s.}\end{aligned}$$

As $k \to \infty$, (6.31) implies that

$$\widetilde{\mathcal{E}}_i\big[Y_{\overline{\tau}(\nu)} + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big] \ge \widetilde{\mathcal{E}}_i\big[Z(\overline{\tau}(\nu)) + H^i_{\nu,\overline{\tau}(\nu)}|\mathcal{F}_\nu\big], \quad \text{a.s.} \tag{6.34}$$

The reverse inequality follows easily from (2.11); thus (6.34) is in fact an equality. Then the second part of Proposition 2.7(1) of [1] and (2.17) imply that $Y_{\overline{\tau}(\nu)} = Z(\overline{\tau}(\nu)) = Z^0_{\overline{\tau}(\nu)}$, a.s., which shows that $\inf\{t \in [\nu, T] : Z^0_t = Y_t\} \le \overline{\tau}(\nu)$, a.s. For any $\delta \in (0, 1)$, since $\{t \in [\nu, T] : Z^0_t = Y_t\} \subset \{t \in [\nu, T] : Y_t \ge \delta Z^0_t + (1 - \delta)(C_Y + 2C_H)\}$, one can deduce that

$$\overline{\tau}(\nu) \ge \inf\{t \in [\nu, T] : Z^0_t = Y_t\} \\ \ge \inf\{t \in [\nu, T] : Y_t \ge \delta Z^0_t + (1 - \delta)(C_Y + 2C_H)\} \wedge T = \tau_\delta(\nu), \quad \text{a.s.}$$

Letting $\delta \to 1$ yields that

$$\overline{\tau}(\nu) \ge \inf\{t \in [\nu, T] : Z^0_t = Y_t\} \ge \lim_{\delta\to 1} \tau_\delta(\nu) = \overline{\tau}(\nu), \quad \text{a.s.},$$

which implies that $\overline{\tau}(\nu) = \inf\{t \in [\nu, T] : Z^0_t = Y_t\}$, a.s. □

Definition 6.1. A family $\{\xi_i\}_{i\in\mathcal{I}} \subset L^0(\mathcal{F}_T)$ is said to be directed downwards if for any $i, j \in \mathcal{I}$, there exists a $k \in \mathcal{I}$ such that $\xi_k \le \xi_i \wedge \xi_j$, a.s.

Proof of Lemma 3.1. In light of [11, Proposition VI-**1**-1], it suffices to show that the family $\{R^i(\nu)\}_{i\in\mathcal{I}}$ is directed downwards. To see this, we define the event $A \triangleq \{R^i(\nu) \ge R^j(\nu)\}$ and the stopping times

$$\rho \triangleq \tau^j(\nu)\mathbf{1}_A + \tau^i(\nu)\mathbf{1}_{A^c} \in \mathcal{S}_{\nu,T} \quad \text{and} \quad \nu(A) \triangleq \nu\mathbf{1}_A + T\mathbf{1}_{A^c} \in \mathcal{S}_{\nu,T}.$$

By Definition 3.2 of [1], there exists a $k = k\big(i, j, \nu(A)\big) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_k = \mathcal{E}^{\nu(A)}_{i,j}$. It follows from (2.5) that

$$H^k_{\nu,\tau^k(\nu)} = H^i_{\nu(A)\wedge\nu,\nu(A)\wedge\tau^k(\nu)} + H^j_{\nu(A)\vee\nu,\nu(A)\vee\tau^k(\nu)} = \mathbf{1}_{A^c} H^i_{\nu,\tau^k(\nu)} + \mathbf{1}_A H^j_{\nu,\tau^k(\nu)}, \quad \text{a.s.}$$

Like for (6.3) and (6.2), we can deduce from Proposition 2.7(2) of [1] and (3.7) that a.s.

$$\begin{aligned} R^k(\nu) &\ge \widetilde{\mathcal{E}}_k\big[Y_\rho + H^k_{\nu,\rho}|\mathcal{F}_\nu\big] = \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[Y_{\tau^j(\nu)} + H^j_{\nu,\tau^j(\nu)}|\mathcal{F}_\nu\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[Y_{\tau^i(\nu)} + H^i_{\nu,\tau^i(\nu)}|\mathcal{F}_\nu\big] \\ &= \mathbf{1}_A R^j(\nu) + \mathbf{1}_{A^c} R^i(\nu) \\ &\ge \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[Y_{\tau^k(\nu)} + H^j_{\nu,\tau^k(\nu)}|\mathcal{F}_\nu\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[Y_{\tau^k(\nu)} + H^i_{\nu,\tau^k(\nu)}|\mathcal{F}_\nu\big] \\ &= \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[Y_{\tau^k(\nu)} + H^k_{\nu,\tau^k(\nu)}|\mathcal{F}_\nu\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[Y_{\tau^k(\nu)} + H^k_{\nu,\tau^k(\nu)}|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_k\big[Y_{\tau^k(\nu)} + H^k_{\nu,\tau^k(\nu)}|\mathcal{F}_\nu\big] = R^k(\nu), \end{aligned}$$

which shows that $R^k(\nu) = \mathbf{1}_A R^j(\nu) + \mathbf{1}_{A^c} R^i(\nu) = R^i(\nu) \wedge R^j(\nu)$, a.s. In light of the basic properties of the essential infimum (e.g., [11, Proposition VI-**1**-1]), we can find a sequence $\{i_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that (3.8) holds. □

Proof of Lemma 3.2. As in the proof of Lemma 3.1, it suffices to show that the family $\{\tau^i(\nu)\}_{i\in\mathcal{I}}$ is directed downwards. To see this, we define the stopping time $\sigma \triangleq \tau^i(\nu) \wedge \tau^j(\nu) \in \mathcal{S}_{\nu,T}$, and the event $A \triangleq \{R^{i,0}_\sigma \ge R^{j,0}_\sigma\} \in \mathcal{F}_\sigma$ as well as the stopping time $\sigma(A) \triangleq \sigma\mathbf{1}_A + T\mathbf{1}_{A^c} \in \mathcal{S}_{\sigma,T}$. By Definition 3.2 of [1], there exists a $k = k\big(i, j, \sigma(A)\big) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_k = \mathcal{E}^{\sigma(A)}_{i,j}$. Fix $t \in [0, T]$; it follows from (2.5) that for any $\rho \in \mathcal{S}_{\sigma\vee t,T}$

$$H^k_{\sigma\vee t,\rho} = H^i_{\sigma(A)\wedge(\sigma\vee t),\sigma(A)\wedge\rho} + H^j_{\sigma(A)\vee(\sigma\vee t),\sigma(A)\vee\rho} = \mathbf{1}_{A^c} H^i_{\sigma\vee t,\rho} + \mathbf{1}_A H^j_{\sigma\vee t,\rho}, \quad \text{a.s.}$$

Like for (6.2), one can deduce from Proposition 2.7(2) of [1] that

$$\begin{aligned} \widetilde{\mathcal{E}}_k\big[Y_\rho + H^k_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] &= \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[Y_\rho + H^k_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^k_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] \\ &= \widetilde{\mathcal{E}}_j\big[\mathbf{1}_A Y_\rho + \mathbf{1}_A H^j_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] + \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{A^c} Y_\rho + \mathbf{1}_{A^c} H^i_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] \\ &= \mathbf{1}_A\widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big], \quad \text{a.s.} \end{aligned}$$

Then applying Proposition 2.7(3) of [1], and Lemma 3.3(2) of [1], as well as (3.6), we obtain

$$\begin{aligned}R^{k,0}_{\sigma\vee t} &= R^k(\sigma\vee t) = \operatorname*{esssup}_{\rho\in\mathcal{S}_{\sigma\vee t,T}} \widetilde{\mathcal{E}}_k\big[Y_\rho + H^k_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big]\\ &= \mathbf{1}_A \operatorname*{esssup}_{\rho\in\mathcal{S}_{\sigma\vee t,T}} \widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big] + \mathbf{1}_{A^c} \operatorname*{esssup}_{\rho\in\mathcal{S}_{\sigma\vee t,T}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\sigma\vee t,\rho}|\mathcal{F}_{\sigma\vee t}\big]\\ &= \mathbf{1}_A R^j(\sigma\vee t) + \mathbf{1}_{A^c} R^i(\sigma\vee t) = \mathbf{1}_A R^{j,0}_{\sigma\vee t} + \mathbf{1}_{A^c} R^{i,0}_{\sigma\vee t}, \quad \text{a.s.}\end{aligned}$$

Since $R^{i,0}$, $R^{j,0}$ and $R^{k,0}$ are all RCLL processes, it holds a.s. that $R^{k,0}_{\sigma\vee t} = \mathbf{1}_A R^{j,0}_{\sigma\vee t} + \mathbf{1}_{A^c} R^{i,0}_{\sigma\vee t}$ for any $t\in[0,T]$, which further implies that

$$\begin{aligned}\tau^k(\nu) &= \inf\Big\{t\in[\nu,T] : R^{k,0}_t = Y_t\Big\} \le \inf\Big\{t\in[\sigma,T] : R^{k,0}_t = Y_t\Big\}\\ &= \mathbf{1}_A \inf\Big\{t\in[\sigma,T] : R^{j,0}_t = Y_t\Big\} + \mathbf{1}_{A^c} \inf\Big\{t\in[\sigma,T] : R^{i,0}_t = Y_t\Big\}, \quad \text{a.s.} \qquad (6.35)\end{aligned}$$

Since $R^{i,0}_{\tau^i(\nu)} = Y_{\tau^i(\nu)}$, $R^{j,0}_{\tau^j(\nu)} = Y_{\tau^j(\nu)}$, a.s. and since $\sigma = \tau^i(\nu)\wedge\tau^j(\nu)$, it holds a.s. that Y_σ is equal either to $R^{i,0}_\sigma$ or to $R^{j,0}_\sigma$. Then the definition of the set A shows that $R^{j,0}_\sigma = Y_\sigma$ a.s. on A and that $R^{i,0}_\sigma = Y_\sigma$ a.s. on A^c, both of which further imply that

$$\begin{aligned}&\mathbf{1}_A \inf\Big\{t\in[\sigma,T] : R^{j,0}_t = Y_t\Big\} = \sigma\mathbf{1}_A \quad \text{and}\\ &\mathbf{1}_{A^c} \inf\Big\{t\in[\sigma,T] : R^{i,0}_t = Y_t\Big\} = \sigma\mathbf{1}_{A^c}, \quad \text{a.s.}\end{aligned}$$

Hence, we see from (6.35) that $\tau^k(\nu)\le\sigma = \tau^i(\nu)\wedge\tau^j(\nu)$, a.s. Thanks to the basic properties of the essential infimum (e.g., [11, Proposition VI-**1**-1]), we can find a sequence $\{i_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$\underline{\tau}(\nu) = \operatorname*{essinf}_{i\in\mathcal{I}} \tau^i(\nu) = \lim_{n\to\infty}\downarrow \tau^{i_n}(\nu), \quad \text{a.s.}$$

The limit $\lim_{n\to\infty}\downarrow\tau^{i_n}(\nu)$ is also a stopping time; thus we have $\underline{\tau}(\nu)\in\mathcal{S}_{\nu,T}$. □

Proof of Theorem 3.1. In light of Lemma 3.2, there is a sequence $\{j_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $\underline{\tau}(\nu) = \lim_{n\to\infty}\downarrow\tau^{j_n}(\nu)$, a.s. Since the family of processes $\{Y^i\}_{i\in\mathcal{I}}$ is "$\mathscr{E}$-uniformly-right-continuous", we can find a subsequence of $\{j_n\}_{n\in\mathbb{N}}$ (we still denote it by $\{j_n\}_{n\in\mathbb{N}}$) such that

$$\lim_{n\to\infty} \operatorname*{esssup}_{i\in\mathcal{I}} \Big|\widetilde{\mathcal{E}}_i\big[Y^i_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - Y^i_{\underline{\tau}(\nu)}\Big| = 0, \quad \text{a.s.} \qquad (6.36)$$

Fix $i\in\mathcal{I}$ and $n\in\mathbb{N}$; we know from Definition 3.2 of [1] that there exists a $k_n = k(i, j_n, \underline{\tau}(\nu))\in\mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n} = \mathcal{E}^{\underline{\tau}(\nu)}_{i,j_n}$. For any $t\in[0,T]$, Lemma 3.3 implies that $R^{k_n,0}_{\underline{\tau}(\nu)\vee t} = R^{j_n,0}_{\underline{\tau}(\nu)\vee t}$, a.s. Since $R^{k_n,0}$ and $R^{j_n,0}$ are both RCLL processes, it holds a.s. that $R^{k_n,0}_{\underline{\tau}(\nu)\vee t} = R^{j_n,0}_{\underline{\tau}(\nu)\vee t}$ for any $t\in[0,T]$, which together with the fact that $\underline{\tau}(\nu)\le\tau^{k_n}(\nu)\wedge\tau^{j_n}(\nu)$, a.s. implies that

$$\begin{aligned}\tau^{k_n}(\nu) &= \inf\Big\{t\in\big[\nu,T\big] : R^{k_n,0}_t = Y_t\Big\} = \inf\Big\{t\in\big[\underline{\tau}(\nu),T\big] : R^{k_n,0}_t = Y_t\Big\}\\ &= \inf\Big\{t\in\big[\underline{\tau}(\nu),T\big] : R^{j_n,0}_t = Y_t\Big\} = \inf\Big\{t\in\big[\nu,T\big] : R^{j_n,0}_t = Y_t\Big\}\\ &= \tau^{j_n}(\nu), \quad \text{a.s.} \qquad (6.37)\end{aligned}$$

Then (2.5), (6.37) and (3.3) of [1] show that a.s.

$$\begin{aligned} R^{k_n}(\nu) + H^i_\nu &= \widetilde{\mathcal{E}}_{k_n}\big[Y_{\tau^{k_n}(\nu)} + H^{k_n}_{\nu,\tau^{k_n}(\nu)}|\mathcal{F}_\nu\big] + H^{k_n}_\nu \\ &= \widetilde{\mathcal{E}}_{k_n}\big[Y^{k_n}_{\tau^{k_n}(\nu)}|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\ \Big[\widetilde{\mathcal{E}}_{j_n}\big[Y^{k_n}_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big]\Big|\,\mathcal{F}_\nu\Big] \\ &= \widetilde{\mathcal{E}}_i\ \Big[\widetilde{\mathcal{E}}_{j_n}\big[Y_{\tau^{j_n}(\nu)} + H^{j_n}_{\underline{\tau}(\nu),\tau^{j_n}(\nu)} + H^i_{\underline{\tau}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big]\Big|\,\mathcal{F}_\nu\Big] \\ &= \widetilde{\mathcal{E}}_i\Big[\widetilde{\mathcal{E}}_{j_n}\big[Y^{j_n}_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - H^{j_n}_{\underline{\tau}(\nu)} + H^i_{\underline{\tau}(\nu)}|\mathcal{F}_\nu\Big] \\ &\le \widetilde{\mathcal{E}}_i\Big[\Big|\widetilde{\mathcal{E}}_{j_n}\big[Y^{j_n}_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - Y^{j_n}_{\underline{\tau}(\nu)}\Big| + Y^i_{\underline{\tau}(\nu)}|\mathcal{F}_\nu\Big] \\ &\le \widetilde{\mathcal{E}}_i\Big[\operatorname*{esssup}_{l\in\mathcal{I}}\Big|\widetilde{\mathcal{E}}_l\big[Y^l_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - Y^l_{\underline{\tau}(\nu)}\Big| + Y^i_{\underline{\tau}(\nu)}|\mathcal{F}_\nu\Big]. \end{aligned} \tag{6.38}$$

For any $l \in \mathcal{I}$, Proposition 2.7(3) of [1], and (2.7), (2.4) and (3.3) imply that

$$\begin{aligned} &\Big|\widetilde{\mathcal{E}}_l\big[Y^l_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - Y^l_{\underline{\tau}(\nu)}\Big| \\ &= \Big|\widetilde{\mathcal{E}}_l\big[Y_{\tau^{j_n}(\nu)} + H^l_{\underline{\tau}(\nu),\tau^{j_n}(\nu)} - C_*|\mathcal{F}_{\underline{\tau}(\nu)}\big] - (Y_{\underline{\tau}(\nu)} - C_Y) + C_H\Big| \\ &\le \widetilde{\mathcal{E}}_l\big[Y_{\tau^{j_n}(\nu)} + H^l_{\underline{\tau}(\nu),\tau^{j_n}(\nu)} - C_*|\mathcal{F}_{\underline{\tau}(\nu)}\big] + (Y_{\underline{\tau}(\nu)} - C_Y) - C_H \\ &= \widetilde{\mathcal{E}}_l\big[Y^l_{\tau^{j_n}(\nu)} + H^l_{\underline{\tau}(\nu),\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] + Y_{\underline{\tau}(\nu)} - 2C_* \le 2R^l\big(\underline{\tau}(\nu)\big) - 2C_*, \quad \text{a.s.} \end{aligned}$$

Taking the essential supremum over $l \in \mathcal{I}$, we can deduce from (2.8) and (3.3) that

$$C_* \le \operatorname*{esssup}_{l\in\mathcal{I}}\Big|\widetilde{\mathcal{E}}_l\big[Y^l_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - Y^l_{\underline{\tau}(\nu)}\Big| + Y^i_{\underline{\tau}(\nu)} \le 3R^l\big(\underline{\tau}(\nu)\big) - 2C_* + H^i_{\underline{\tau}(\nu)}, \quad \text{a.s.},$$

where $3R^l\big(\underline{\tau}(\nu)\big) - 2C_* + H^i_{\underline{\tau}(\nu)} \in \mathrm{Dom}(\mathscr{E})$ thanks to Proposition 3.1(1), (S1′) and (D2). Applying Proposition 2.9 of [1] and Proposition 2.7(3) of [1], we can deduce from (6.38) and (6.36) that

$$\begin{aligned} \overline{V}(\nu) &= \operatorname*{essinf}_{j\in\mathcal{I}} R^j(\nu) \le \varliminf_{n\to\infty} R^{k_n}(\nu) \\ &\le \lim_{n\to\infty}\widetilde{\mathcal{E}}_i\Big[\operatorname*{esssup}_{l\in\mathcal{I}}\Big|\widetilde{\mathcal{E}}_l\big[Y^l_{\tau^{j_n}(\nu)}|\mathcal{F}_{\underline{\tau}(\nu)}\big] - Y^l_{\underline{\tau}(\nu)}\Big| + Y^i_{\underline{\tau}(\nu)}|\mathcal{F}_\nu\Big] - H^i_\nu \\ &= \widetilde{\mathcal{E}}_i\big[Y^i_{\underline{\tau}(\nu)}|\mathcal{F}_\nu\big] - H^i_\nu = \widetilde{\mathcal{E}}_i\big[Y_{\underline{\tau}(\nu)} + H^i_{\nu,\underline{\tau}(\nu)}|\mathcal{F}_\nu\big], \quad \text{a.s.} \end{aligned}$$

Taking the essential infimum of the right-hand side over $i \in \mathcal{I}$ yields that

$$\begin{aligned} \overline{V}(\nu) &\le \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_{\underline{\tau}(\nu)} + H^i_{\nu,\underline{\tau}(\nu)}|\mathcal{F}_\nu\big] \\ &\le \operatorname*{esssup}_{\rho\in\mathcal{S}_{\nu,T}}\Big(\operatorname*{essinf}_{i\in\mathcal{I}}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\Big) = \underline{V}(\nu) \le \overline{V}(\nu), \quad \text{a.s.} \end{aligned}$$

Hence, it follows from (3.3) that

$$\underline{V}(\nu) = \operatorname*{essinf}_{i\in\mathcal{I}}\widetilde{\mathcal{E}}_i\big[Y_{\underline{\tau}(\nu)} + H^i_{\nu,\underline{\tau}(\nu)}|\mathcal{F}_\nu\big] = \overline{V}(\nu) = \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\nu) \ge Y_\nu, \quad \text{a.s.} \quad \square$$

Proof of Proposition 3.2. By Lemma 3.2, there is a sequence $\{i_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $\sigma \triangleq \underline{\tau}(\nu) = \lim_{n\to\infty}\downarrow \tau^{i_n}(\nu)$, a.s. For any $n \in \mathbb{N}$, since $\sigma \le \tau^{i_n}(\nu)$, a.s., we have

$$\tau^{i_n}(\nu) = \inf\{t \in [\nu, T] : R^{i_n,0}_t = Y_t\} = \inf\{t \in [\sigma, T] : R^{i_n,0}_t = Y_t\} = \tau^{i_n}(\sigma), \quad \text{a.s.}$$

Then (3.9) and (3.7) imply that

$$\begin{aligned} V(\sigma) &= \overline{V}(\sigma) \le R^{i_n}(\sigma) = \widetilde{\mathcal{E}}_{i_n}[Y_{\tau^{i_n}(\sigma)} + H^{i_n}_{\sigma,\tau^{i_n}(\sigma)}|\mathcal{F}_\sigma] = \widetilde{\mathcal{E}}_{i_n}[Y_{\tau^{i_n}(\nu)} + H^{i_n}_{\sigma,\tau^{i_n}(\nu)}|\mathcal{F}_\sigma] \\ &= \widetilde{\mathcal{E}}_{i_n}[Y^{i_n}_{\tau^{i_n}(\nu)}|\mathcal{F}_\sigma] - H^{i_n}_\sigma = \widetilde{\mathcal{E}}_{i_n}[Y^{i_n}_{\tau^{i_n}(\nu)}|\mathcal{F}_\sigma] - Y^{i_n}_\sigma + Y_\sigma \\ &\le \operatorname*{esssup}_{i\in\mathcal{I}} |\widetilde{\mathcal{E}}_i[Y^i_{\tau^{i_n}(\nu)}|\mathcal{F}_\sigma] - Y^i_\sigma| + Y_\sigma, \quad \text{a.s.} \end{aligned}$$

As $n \to \infty$, the "$\mathscr{E}$-uniform-right-continuity" of $\{Y^i\}_{i\in\mathcal{I}}$ implies that $V(\sigma) \le Y_\sigma$, a.s., while the reverse inequality is obvious from (3.9). $\square$

Proof of Proposition 3.3. In light of Lemma 3.1 and (3.9), there exists a sequence $\{j_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$V(\nu) = \overline{V}(\nu) = \lim_{n\to\infty} \downarrow R^{j_n}(\nu), \quad \text{a.s.}$$

For any $n \in \mathbb{N}$, Definition 3.2 of [1] assures a $k_n = k(i, j_n, \nu) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n} = \mathcal{E}^\nu_{i,j_n}$. Applying Proposition 2.7(5) of [1] to $\widetilde{\mathcal{E}}_i$, we can deduce from (3.3) of [1] and (3.5) that

$$\begin{aligned} \widetilde{\mathcal{E}}_{k_n}\big[V(\rho) + H^{j_n}_{\nu,\rho}|\mathcal{F}_\nu\big] &\le \widetilde{\mathcal{E}}_{k_n}\big[R^{j_n}(\rho) + H^{j_n}_{\nu,\rho}|\mathcal{F}_\nu\big] = \mathcal{E}^\nu_{i,j_n}\big[R^{j_n}(\rho) + H^{j_n}_{\nu,\rho}|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_{j_n}\big[R^{j_n}(\rho) + H^{j_n}_{\nu,\rho}|\mathcal{F}_\nu\big]|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_{j_n}\big[R^{j_n}(\rho) + H^{j_n}_{\nu,\rho}|\mathcal{F}_\nu\big] \le R^{j_n}(\nu), \quad \text{a.s.} \end{aligned}$$

Then Proposition 2.7(3) of [1] and (2.5) imply that

$$\begin{aligned} \operatorname*{essinf}_{k\in\mathcal{I}} \widetilde{\mathcal{E}}_k\big[V^k(\rho)|\mathcal{F}_\nu\big] &\le \widetilde{\mathcal{E}}_{k_n}\big[V^{k_n}(\rho)|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_{k_n}\big[V(\rho) + H^{j_n}_{\nu,\rho}|\mathcal{F}_\nu\big] + H^i_\nu \le R^{j_n}(\nu) + H^i_\nu, \quad \text{a.s.} \end{aligned}$$

Letting $n \to \infty$ gives (3.10).

Now we assume that $\nu \le \rho \le \underline{\tau}(\nu)$, a.s. Applying Lemma 3.1 and (3.9) once again, we can find another sequence $\{j'_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $V(\rho) = \overline{V}(\rho) = \lim_{n\to\infty} \downarrow R^{j'_n}(\rho)$, a.s. For any $n \in \mathbb{N}$, Definition 3.2 of [1] assures a $k'_n = k(i, j'_n, \rho) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k'_n} = \mathcal{E}^\rho_{i,j'_n}$. Since $\rho \le \underline{\tau}(\nu) \le \tau^{k'_n}(\nu)$, a.s., using (3.7) with $i = k'_n$ and applying Proposition 2.7(5) of [1] to $\widetilde{\mathcal{E}}_{j'_n}$, we can deduce from (2.5), (3.3) of [1] as well as Lemma 3.3 that

$$\begin{aligned} V^i(\nu) &= V(\nu) + H^i_\nu = V(\nu) + H^{k'_n}_\nu \le R^{k'_n}(\nu) + H^{k'_n}_\nu = \widetilde{\mathcal{E}}_{k'_n}\big[R^{k'_n}(\rho) + H^{k'_n}_\rho|\mathcal{F}_\nu\big] \\ &= \mathcal{E}^\rho_{i,j'_n}\big[R^{k'_n}(\rho) + H^{k'_n}_\rho|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_{j'_n}\big[R^{k'_n}(\rho) + H^{k'_n}_\rho|\mathcal{F}_\rho\big]|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[R^{k'_n}(\rho) + H^{k'_n}_\rho|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_i\big[R^{j'_n}(\rho) + H^i_\rho|\mathcal{F}_\nu\big], \quad \text{a.s.} \end{aligned} \tag{6.39}$$

It follows from (3.3) that $C_* \le Y_\rho + H^i_\rho \le R^{j'_n}(\rho) + H^i_\rho \le R^{j'_1}(\rho) + H^i_\rho$, a.s., where $R^{j'_1}(\rho) + H^i_\rho \in \mathrm{Dom}(\mathscr{E})$ thanks to Proposition 3.1(1), (S1′) and (D2). As $n \to \infty$ in (6.39), Proposition 2.9 of [1] implies that

$$\begin{aligned} V^i(\nu) &\le \lim_{n\to\infty} \widetilde{\mathcal{E}}_i\big[R^{j'_n}(\rho) + H^i_\rho|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_i\big[V(\rho) + H^i_\rho|\mathcal{F}_\nu\big] \\ &= \widetilde{\mathcal{E}}_i\big[V^i(\rho)|\mathcal{F}_\nu\big], \quad \text{a.s., proving (3.11).} \end{aligned}$$

Finally, we show that $\big\{V^i\big(\underline{\tau}(0)\wedge t\big)\big\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-submartingale: Fix $0\le s<t\le T$; we set $\nu \triangleq \underline{\tau}(0)\wedge s$, $\rho \triangleq \underline{\tau}(0)\wedge t$. Since $\nu\le\rho\le\underline{\tau}(0)\le\underline{\tau}(\nu)$, a.s., (3.11), Corollary 2.3 of [1] and Proposition 2.7(5) of [1] show that a.s.

$$\begin{aligned}V^i\big(\underline{\tau}(0)\wedge s\big) &= V^i(\nu)\le \widetilde{\mathcal{E}}_i\big[V^i(\rho)|\mathcal{F}_\nu\big]=\widetilde{\mathcal{E}}_i\big[V^i\big(\underline{\tau}(0)\wedge t\big)|\mathcal{F}_{\underline{\tau}(0)\wedge s}\big]\\ &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i\big[V^i\big(\underline{\tau}(0)\wedge t\big)|\mathcal{F}_{\underline{\tau}(0)}\big]|\mathcal{F}_s\big]=\widetilde{\mathcal{E}}_i\big[V^i\big(\underline{\tau}(0)\wedge t\big)|\mathcal{F}_s\big]. \quad\square\end{aligned}$$

Proof of Theorem 3.2.

Proof of (1). *Step* 1: For any $\rho,\nu\in\mathcal{S}_{0,T}$, we define

$$\Psi^\rho(\nu)\triangleq \underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\rho\wedge\nu,\rho}|\mathcal{F}_{\rho\wedge\nu}\big]+H^{i'}_{\rho\wedge\nu}\in\mathcal{F}_{\rho\wedge\nu}.$$

It follows from (2.7), (2.4), and Proposition 2.7(5) of [1] that a.s.

$$\begin{aligned}C_Y+2C_H &= \underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i[C_Y+C_H|\mathcal{F}_{\rho\wedge\nu}]+C_H\le\Psi^\rho(\nu)\\ &\le \widetilde{\mathcal{E}}_{i'}\big[Y_\rho+H^{i'}_{\rho\wedge\nu,\rho}|\mathcal{F}_{\rho\wedge\nu}\big]+H^{i'}_{\rho\wedge\nu}\le R^{i'}(\rho\wedge\nu)+H^{i'}_{\rho\wedge\nu},\end{aligned}\tag{6.40}$$

where $R^{i'}(\rho\wedge\nu)+H^{i'}_{\rho\wedge\nu}\in\mathrm{Dom}(\mathscr{E})$ due to Proposition 3.1(1), (S1′) and (D2). Then Lemma 3.2 of [1] implies that $\Psi^\rho(\nu)\in\mathrm{Dom}(\mathscr{E})$. Applying Proposition 2.7(2)–(3) of [1] and Lemma 3.3 of [1], we can rewrite $\Psi^\rho(\nu)$ as follows:

$$\begin{aligned}\Psi^\rho(\nu)-H^{i'}_{\rho\wedge\nu} &= \underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\rho\le\nu\}}Y_{\rho\wedge\nu}+\mathbf{1}_{\{\rho>\nu\}}(Y_\rho+H^i_{\nu,\rho})|\mathcal{F}_{\rho\wedge\nu}\big]\\ &= \underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \Big(\mathbf{1}_{\{\rho\le\nu\}}Y_{\rho\wedge\nu}+\mathbf{1}_{\{\rho>\nu\}}\widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\nu,\rho}|\mathcal{F}_\nu\big]\Big)\\ &= \mathbf{1}_{\{\rho\le\nu\}}Y_\rho+\mathbf{1}_{\{\rho>\nu\}}\underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\nu,\rho}|\mathcal{F}_\nu\big],\quad\text{a.s.}\end{aligned}$$

Let $\sigma\in\mathcal{S}_{0,T}$. Lemma 3.3(2) of [1] and Proposition 2.7(2) of [1] once again imply that

$$\begin{aligned}\mathbf{1}_{\{\nu=\sigma\}}\Psi^\rho(\nu) &= \mathbf{1}_{\{\rho\le\nu=\sigma\}}Y_\rho+\mathbf{1}_{\{\rho>\nu=\sigma\}}\underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\nu,\rho}|\mathcal{F}_\nu\big]+\mathbf{1}_{\{\nu=\sigma\}}H^{i'}_{\rho\wedge\nu}\\ &= \mathbf{1}_{\{\rho\le\nu=\sigma\}}Y_\rho+\mathbf{1}_{\{\rho>\nu\}}\underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\nu=\sigma\}}\big(Y_\rho+H^i_{\sigma,\rho}\big)|\mathcal{F}_\nu\big]+\mathbf{1}_{\{\nu=\sigma\}}H^{i'}_{\rho\wedge\sigma}\\ &= \mathbf{1}_{\{\rho\le\nu=\sigma\}}Y_\rho+\mathbf{1}_{\{\rho>\nu\}}\underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \mathbf{1}_{\{\nu=\sigma\}}\widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\sigma,\rho}|\mathcal{F}_\sigma\big]+\mathbf{1}_{\{\nu=\sigma\}}H^{i'}_{\rho\wedge\sigma}\\ &= \mathbf{1}_{\{\rho\le\sigma=\nu\}}Y_\rho+\mathbf{1}_{\{\rho>\sigma=\nu\}}\underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\sigma,\rho}|\mathcal{F}_\sigma\big]+\mathbf{1}_{\{\nu=\sigma\}}H^{i'}_{\rho\wedge\sigma}\\ &= \mathbf{1}_{\{\nu=\sigma\}}\Psi^\rho(\sigma),\quad\text{a.s.}\end{aligned}\tag{6.41}$$

Step 2: Fix $\rho\in\mathcal{S}_{0,T}$. For any $\nu\in\mathcal{S}_{0,T}$ and $\sigma\in\mathcal{S}_{\nu,T}$, letting $(\nu,\mathcal{I}',\mathcal{U})=(\rho\wedge\sigma,\mathcal{I},\{\rho\})$ and $X(\rho)=Y_\rho$ in Lemma 2.2, we can find a sequence $\{j_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$\underset{i\in\mathcal{I}}{\operatorname{essinf}}\ \widetilde{\mathcal{E}}_i\big[Y_\rho+H^i_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big]=\lim_{n\to\infty}\downarrow\widetilde{\mathcal{E}}_{j_n}\big[Y_\rho+H^{j_n}_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big],\quad\text{a.s.}$$

Definition 3.2 of [1] assures the existence of a $k_n=k(i',j_n,\rho\wedge\sigma)\in\mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n}=\mathcal{E}^{\rho\wedge\sigma}_{i',j_n}$. Applying Proposition 2.7(3) of [1] to $\widetilde{\mathcal{E}}_{k_n}$, we can deduce from (2.5) and (3.3) of [1] that a.s.

$$\begin{aligned}\Psi^\rho(\nu) &\le \widetilde{\mathcal{E}}_{k_n}\big[Y_\rho+H^{k_n}_{\rho\wedge\nu,\rho}|\mathcal{F}_{\rho\wedge\nu}\big]+H^{i'}_{\rho\wedge\nu}=\widetilde{\mathcal{E}}_{k_n}\big[Y_\rho+H^{k_n}_{\rho\wedge\nu,\rho}|\mathcal{F}_{\rho\wedge\nu}\big]+H^{k_n}_{\rho\wedge\nu}\\ &= \widetilde{\mathcal{E}}_{k_n}\big[Y^{k_n}_\rho|\mathcal{F}_{\rho\wedge\nu}\big]=\widetilde{\mathcal{E}}_{i'}\big[\widetilde{\mathcal{E}}_{j_n}\big[Y^{k_n}_\rho|\mathcal{F}_{\rho\wedge\sigma}\big]|\mathcal{F}_{\rho\wedge\nu}\big].\end{aligned}$$

For any $n \in \mathbb{N}$, Proposition 2.7(3) of [1], (5), and (2.8) as well as (2.5) imply that a.s.

$$\begin{aligned}C_* &\le \widetilde{\mathcal{E}}_{j_n}\big[Y_\rho^{k_n}|\mathcal{F}_{\rho\wedge\sigma}\big] = \widetilde{\mathcal{E}}_{j_n}\big[Y_\rho + H^{j_n}_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big] + H^{i'}_{\rho\wedge\sigma}\\ &\le \widetilde{\mathcal{E}}_{j_1}\big[Y_\rho + H^{j_1}_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big] + H^{i'}_{\rho\wedge\sigma} \le R^{j_1}(\rho\wedge\sigma) + H^{i'}_{\rho\wedge\sigma},\end{aligned}$$

where $R^{j_1}(\rho\wedge\sigma) + H^{i'}_{\rho\wedge\sigma} \in \mathrm{Dom}(\mathscr{E})$ thanks to Proposition 3.1(1), (S1′) and (D2). Then Proposition 2.9 of [1], Corollary 2.3 of [1] and Proposition 2.7(5) of [1] show that

$$\begin{aligned}\Psi^\rho(\nu) &\le \lim_{n\to\infty}\downarrow \widetilde{\mathcal{E}}_{i'}\big[\widetilde{\mathcal{E}}_{j_n}\big[Y_\rho^{k_n}|\mathcal{F}_{\rho\wedge\sigma}\big]|\mathcal{F}_{\rho\wedge\nu}\big]\\ &= \lim_{n\to\infty}\downarrow \widetilde{\mathcal{E}}_{i'}\Big[\widetilde{\mathcal{E}}_{j_n}\big[Y_\rho + H^{j_n}_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big] + H^{i'}_{\rho\wedge\sigma}|\mathcal{F}_{\rho\wedge\nu}\Big]\\ &= \widetilde{\mathcal{E}}_{i'}\big[\lim_{n\to\infty}\downarrow \widetilde{\mathcal{E}}_{j_n}\big[Y_\rho + H^{j_n}_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big] + H^{i'}_{\rho\wedge\sigma}|\mathcal{F}_{\rho\wedge\nu}\big]\\ &= \widetilde{\mathcal{E}}_{i'}\big[\operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_{i}\big[Y_\rho + H^{i}_{\rho\wedge\sigma,\rho}|\mathcal{F}_{\rho\wedge\sigma}\big] + H^{i'}_{\rho\wedge\sigma}|\mathcal{F}_{\rho\wedge\nu}\big]\\ &= \widetilde{\mathcal{E}}_{i'}\big[\Psi^\rho(\sigma)|\mathcal{F}_{\rho\wedge\nu}\big] = \widetilde{\mathcal{E}}_{i'}\big[\widetilde{\mathcal{E}}_{i'}\big[\Psi^\rho(\sigma)|\mathcal{F}_\rho\big]|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_{i'}\big[\Psi^\rho(\sigma)|\mathcal{F}_\nu\big], \quad \text{a.s.,}\end{aligned} \tag{6.42}$$

which implies that $\{\Psi^\rho(t)\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_{i'}$-submartingale. Hence, $\{-\Psi^\rho(t)\}_{t\in[0,T]}$ is an $\mathcal{E}'$-supermartingale by assumption (3.12). Since $\mathcal{E}'$ satisfies (H0), (H1), (2.3) of [1] and since $\mathrm{Dom}(\mathcal{E}') \in \widetilde{\mathscr{D}}_T$ (which results from $\mathrm{Dom}(\mathscr{E}) \in \widetilde{\mathscr{D}}_T$ and (3.12)), Theorem 2.3 of [1] shows that $\Psi_t^{\rho,+} \triangleq \underline{\lim}_{n\to\infty}\Psi^\rho\big(q_n^+(t)\big)$, $t\in[0,T]$ is an RCLL process and that

$$P\Big(\Psi_t^{\rho,+} = \lim_{n\to\infty}\Psi^\rho\big(q_n^+(t)\big) \text{ for any } t\in[0,T]\Big) = 1. \tag{6.43}$$

Step 3: For any $\nu \in \mathcal{S}_{0,T}$ and $n \in \mathbb{N}$, $q_n^+(\nu)$ takes values in a finite set $\mathcal{D}_T^n \triangleq \big([0,T)\cap\{k2^{-n}\}_{k\in\mathbb{Z}}\big)\cup\{T\}$. Given an $\alpha\in\mathcal{D}_T^n$, it holds for any $m\ge n$ that $q_m^+(\alpha)=\alpha$ since $\mathcal{D}_T^n \subset \mathcal{D}_T^m$. It follows from (6.43) that

$$\Psi_\alpha^{\rho,+} = \lim_{m\to\infty}\Psi^\rho\big(q_m^+(\alpha)\big) = \Psi^\rho(\alpha), \quad \text{a.s.}$$

Then one can deduce from (6.41) that

$$\begin{aligned}\Psi^{\rho,+}_{q_n^+(\nu)} &= \sum_{\alpha\in\mathcal{D}_T^n}\mathbf{1}_{\{q_n^+(\nu)=\alpha\}}\Psi_\alpha^{\rho,+} = \sum_{\alpha\in\mathcal{D}_T^n}\mathbf{1}_{\{q_n^+(\nu)=\alpha\}}\Psi^\rho(\alpha)\\ &= \sum_{\alpha\in\mathcal{D}_T^n}\mathbf{1}_{\{q_n^+(\nu)=\alpha\}}\Psi^\rho\big(q_n^+(\nu)\big) = \Psi^\rho\big(q_n^+(\nu)\big), \quad \text{a.s.}\end{aligned}$$

Thus the right-continuity of the process $\Psi^{\rho,+}$ implies that

$$\Psi_\nu^{\rho,+} = \lim_{n\to\infty}\Psi^{\rho,+}_{q_n^+(\nu)} = \lim_{n\to\infty}\Psi^\rho\big(q_n^+(\nu)\big), \quad \text{a.s.} \tag{6.44}$$

We have assumed that $\operatorname{esssup}_{t\in\mathcal{D}_T}\widetilde{\mathcal{E}}_j[Y_\rho^j|\mathcal{F}_t] \in \mathrm{Dom}(\mathscr{E})$ for some $j = j(\rho)\in\mathcal{I}$. It holds a.s. that

$$\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_t\big] \le \operatorname*{esssup}_{s\in\mathcal{D}_T}\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_s\big], \quad \forall t\in\mathcal{D}_T.$$

Since $\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_\cdot\big]$ is an RCLL process, it holds except on a null set N that

$$\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_t\big] \le \operatorname*{esssup}_{s\in\mathcal{D}_T}\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_s\big], \quad \forall t\in[0,T], \quad \text{thus}$$

$$\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_{\rho\wedge q_n^+(\nu)}\big] \le \operatorname*{esssup}_{s\in\mathcal{D}_T}\widetilde{\mathcal{E}}_j\big[Y_\rho^j|\mathcal{F}_s\big], \quad \forall n\in\mathbb{N}.$$

Then one can deduce from (6.40), (2.4) and Proposition 2.7(3) of [1] that

$$\begin{aligned} C_Y + 2C_H &\le \Psi^\rho\big(q_n^+(\nu)\big) \le \widetilde{\mathcal{E}}_j\big[Y_\rho + H^j_{\rho\wedge q_n^+(\nu),\rho}|\mathcal{F}_{\rho\wedge q_n^+(\nu)}\big] + \zeta^{i'} \\ &= \widetilde{\mathcal{E}}_j\big[Y^j_\rho - H^j_{\rho\wedge q_n^+(\nu)}|\mathcal{F}_{\rho\wedge q_n^+(\nu)}\big] + \zeta^{i'} \\ &\le \widetilde{\mathcal{E}}_j\big[Y^j_\rho - C_H|\mathcal{F}_{\rho\wedge q_n^+(\nu)}\big] + \zeta^{i'} = \widetilde{\mathcal{E}}_j\big[Y^j_\rho|\mathcal{F}_{\rho\wedge q_n^+(\nu)}\big] - C_H + \zeta^{i'} \\ &\le \operatorname*{esssup}_{s\in\mathcal{D}_T} \widetilde{\mathcal{E}}_j\big[Y^j_\rho|\mathcal{F}_s\big] - C_H + \zeta^{i'}, \quad \text{a.s.,} \end{aligned}$$

where the right-hand side belongs to Dom($\mathscr{E}$) due to (D2) and the assumption that $\zeta^{i'} \in$ Dom($\mathscr{E}$). Proposition 2.9 of [1], (6.44), (6.42) and Proposition 2.7(5) of [1] then imply that $\Psi^{\rho,+}_\nu = \lim_{n\to\infty} \Psi^\rho\big(q_n^+(\nu)\big) \in \mathrm{Dom}(\mathscr{E})$ and that

$$\Psi^\rho(\nu) \le \lim_{n\to\infty} \widetilde{\mathcal{E}}_{i'}\big[\Psi^\rho(q_n^+(\nu))|\mathcal{F}_\nu\big] = \widetilde{\mathcal{E}}_{i'}\big[\Psi^{\rho,+}_\nu|\mathcal{F}_\nu\big] = \Psi^{\rho,+}_\nu, \quad \text{a.s.} \tag{6.45}$$

In the last equality, we used the fact that $\Psi^{\rho,+}_\nu = \lim_{n\to\infty} \Psi^\rho\big(q_n^+(\nu)\big) \in \mathcal{F}_\nu$ by the right-continuity of the filtration **F**.

Step 4: Given $\nu \in \mathcal{S}_{0,T}$, we set $\gamma \triangleq \underline{\tau}(0)\wedge\nu$, $\gamma_n \triangleq \underline{\tau}(0)\wedge q_n^+(\nu)$, $\forall n\in\mathbb{N}$ and let $\rho\in\mathcal{S}_{\gamma,T}$. Since $\lim_{n\to\infty}\uparrow \mathbf{1}_{\{\underline{\tau}(0)>q_n^+(\nu)\}} = \mathbf{1}_{\{\underline{\tau}(0)>\nu\}}$ and since

$$\begin{aligned} &\{\underline{\tau}(0)>\nu\} \subset \big\{q_n^+(\nu) = q_n^+\big(\underline{\tau}(0)\wedge\nu\big)\big\}, \\ &\{\underline{\tau}(0)>q_n^+(\nu)\} \subset \big\{q_n^+(\nu) = \underline{\tau}(0)\wedge q_n^+(\nu)\big\}, \quad \forall n\in\mathbb{N}, \end{aligned}$$

one can deduce from (6.45), (6.44) and (6.41) that

$$\begin{aligned} &\mathbf{1}_{\{\underline{\tau}(0)>\nu\}}\Psi^\rho(\gamma) \le \mathbf{1}_{\{\underline{\tau}(0)>\nu\}}\Psi^{\rho,+}_\gamma = \mathbf{1}_{\{\underline{\tau}(0)>\nu\}}\lim_{n\to\infty}\Psi^\rho\big(q_n^+(\gamma)\big) \\ &\quad = \lim_{n\to\infty}\mathbf{1}_{\{\underline{\tau}(0)>\nu\}}\Psi^\rho\big(q_n^+\big(\underline{\tau}(0)\wedge\nu\big)\big) \\ &\quad = \lim_{n\to\infty}\mathbf{1}_{\{\underline{\tau}(0)>\nu\}}\Psi^\rho\big(q_n^+(\nu)\big) = \lim_{n\to\infty}\mathbf{1}_{\{\underline{\tau}(0)>q_n^+(\nu)\}}\Psi^\rho\big(q_n^+(\nu)\big) \\ &\quad = \lim_{n\to\infty}\mathbf{1}_{\{\underline{\tau}(0)>q_n^+(\nu)\}}\Psi^\rho\big(\underline{\tau}(0)\wedge q_n^+(\nu)\big) = \mathbf{1}_{\{\underline{\tau}(0)>\nu\}}\lim_{n\to\infty}\Psi^\rho\big(\gamma_n\big), \quad \text{a.s.} \end{aligned} \tag{6.46}$$

For any $n\in\mathbb{N}$, we see from (3.9) that

$$\begin{aligned} V(\gamma_n) = \underline{V}(\gamma_n) &= \operatorname*{esssup}_{\sigma\in\mathcal{S}_{\gamma_n,T}}\Big(\operatorname*{essinf}_{i\in\mathcal{I}}\widetilde{\mathcal{E}}_i\big[Y_\sigma + H^i_{\gamma_n,\sigma}|\mathcal{F}_{\gamma_n}\big]\Big) \\ &\ge \operatorname*{essinf}_{i\in\mathcal{I}}\widetilde{\mathcal{E}}_i\big[Y_{\rho\vee\gamma_n} + H^i_{\gamma_n,\rho\vee\gamma_n}|\mathcal{F}_{\gamma_n}\big], \quad \text{a.s.} \end{aligned} \tag{6.47}$$

Since $\{\underline{\tau}(0)\le\nu\}\subset\{\gamma_n=\gamma=\underline{\tau}(0)\}$, Proposition 2.7(2)–(3) of [1] imply that for any $i\in\mathcal{I}$

$$\begin{aligned} \mathbf{1}_{\{\underline{\tau}(0)\le\nu\}}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma_n,\,\rho}|\mathcal{F}_{\gamma_n}\big] &= \widetilde{\mathcal{E}}_i\Big[\mathbf{1}_{\{\underline{\tau}(0)\le\nu\}}\big(Y_\rho + H^i_{\rho\wedge\gamma,\,\rho}\big)|\mathcal{F}_{\gamma_n}\Big] \\ &= \mathbf{1}_{\{\underline{\tau}(0)\le\nu\}}\widetilde{\mathcal{E}}_i\Big[Y_\rho + H^i_{\rho\wedge\gamma,\,\rho}|\mathcal{F}_\gamma\Big], \quad \text{a.s.,} \end{aligned}$$

and that

$$\begin{aligned} \widetilde{\mathcal{E}}_i\big[Y_{\rho\vee\gamma_n} + H^i_{\gamma_n,\,\rho\vee\gamma_n}|\mathcal{F}_{\gamma_n}\big] &= \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\rho\le\gamma_n\}}Y_{\gamma_n} + \mathbf{1}_{\{\rho>\gamma_n\}}\big(Y_\rho + H^i_{\rho\wedge\gamma_n,\,\rho}\big)|\mathcal{F}_{\gamma_n}\big] \\ &= \mathbf{1}_{\{\rho\le\gamma_n\}}Y_{\gamma_n} + \mathbf{1}_{\{\rho>\gamma_n\}}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma_n,\,\rho}|\mathcal{F}_{\gamma_n}\big] \\ &= \mathbf{1}_{\{\rho\le\gamma_n\}}Y_{\gamma_n} + \mathbf{1}_{\{\rho>\gamma_n,\underline{\tau}(0)>\nu\}}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma_n,\,\rho}|\mathcal{F}_{\rho\wedge\gamma_n}\big] \\ &\quad + \mathbf{1}_{\{\rho>\gamma_n,\underline{\tau}(0)\le\nu\}}\widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma,\,\rho}|\mathcal{F}_{\rho\wedge\gamma}\big], \quad \text{a.s.} \end{aligned}$$

Then it follows from (6.47) and Lemma 3.3 of [1] that

$$\begin{aligned} V(\gamma_n) \ge\ & \mathbf{1}_{\{\rho\le\gamma_n\}} Y_{\gamma_n} + \mathbf{1}_{\{\rho>\gamma_n,\underline{\tau}(0)>\nu\}} \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma_n,\,\rho}|\mathcal{F}_{\rho\wedge\gamma_n}\big] \\ & + \mathbf{1}_{\{\rho>\gamma_n,\underline{\tau}(0)\le\nu\}} \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma,\,\rho}|\mathcal{F}_{\rho\wedge\gamma}\big] \\ =\ & \mathbf{1}_{\{\rho\le\gamma_n\}} Y_{\gamma_n} + \mathbf{1}_{\{\rho>\gamma_n,\underline{\tau}(0)>\nu\}} \left(\Psi^\rho(\gamma_n) - H^{i'}_{\rho\wedge\gamma_n}\right) \\ & + \mathbf{1}_{\{\rho>\gamma_n,\underline{\tau}(0)\le\nu\}} \left(\Psi^\rho(\gamma) - H^{i'}_{\rho\wedge\gamma}\right), \quad \text{a.s.} \end{aligned}$$

As $n\to\infty$, the right-continuity of processes Y and $H^{i'}$, (6.46), and Lemma 3.3 of [1] as well as Proposition 2.7(2)–(3) of [1] show that

$$\begin{aligned} \varliminf_{n\to\infty} V(\gamma_n) \ge\ & \mathbf{1}_{\{\rho=\gamma\}} Y_\gamma + \mathbf{1}_{\{\rho>\gamma,\underline{\tau}(0)>\nu\}} \left(\varliminf_{n\to\infty} \Psi^\rho(\gamma_n) - H^{i'}_{\rho\wedge\gamma}\right) \\ & + \mathbf{1}_{\{\rho>\gamma,\underline{\tau}(0)\le\nu\}} \left(\Psi^\rho(\gamma) - H^{i'}_{\rho\wedge\gamma}\right) \\ \ge\ & \mathbf{1}_{\{\rho=\gamma\}} Y_\gamma + \mathbf{1}_{\{\rho>\gamma\}} \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\rho\wedge\gamma,\rho}|\mathcal{F}_{\rho\wedge\gamma}\big] \\ =\ & \operatorname*{essinf}_{i\in\mathcal{I}} \left(\mathbf{1}_{\{\rho=\gamma\}} Y_\gamma + \mathbf{1}_{\{\rho>\gamma\}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\gamma,\rho}|\mathcal{F}_\gamma\big]\right) \\ =\ & \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\rho=\gamma\}} Y_\gamma + \mathbf{1}_{\{\rho>\gamma\}}\big(Y_\rho + H^i_{\gamma,\rho}\big)|\mathcal{F}_\gamma\big] \\ =\ & \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\gamma,\rho}|\mathcal{F}_\gamma\big], \quad \text{a.s.} \end{aligned}$$

Taking the essential supremum of the right-hand side over $\rho\in\mathcal{S}_{\gamma,T}$, we obtain

$$\varliminf_{n\to\infty} V(\gamma_n) \ge \operatorname*{esssup}_{\rho\in\mathcal{S}_{\gamma,T}} \left(\operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\big[Y_\rho + H^i_{\gamma,\rho}|\mathcal{F}_\gamma\big]\right) = \underline{V}(\gamma) = V(\gamma), \quad \text{a.s.} \tag{6.48}$$

On the other hand, for any $i\in\mathcal{I}$ and $n\in\mathbb{N}$ we have that $V(\gamma_n) = \overline{V}(\gamma_n) = \operatorname{essinf}_{l\in\mathcal{I}} R^l(\gamma_n) \le R^i(\gamma_n)$, a.s. Then (3.6) and the right-continuity of the process $R^{i,0}$ imply that

$$\varlimsup_{n\to\infty} V(\gamma_n) \le \lim_{n\to\infty} R^i(\gamma_n) = \lim_{n\to\infty} R^{i,0}_{\gamma_n} = R^{i,0}_\gamma = R^i(\gamma), \quad \text{a.s.}$$

Taking the essential infimum of $R^i(\gamma)$ over $i\in\mathcal{I}$ yields that

$$\varlimsup_{n\to\infty} V(\gamma_n) \le \operatorname*{essinf}_{i\in\mathcal{I}} R^i(\gamma) = \overline{V}(\gamma) = V(\gamma), \quad \text{a.s.}$$

This together with (6.48) shows that $\lim_{n\to\infty} V(\gamma_n) = V(\gamma)$, a.s., which implies that for any $\nu\in\mathcal{S}_{0,T}$ and $i\in\mathcal{I}$

$$\begin{aligned} \lim_{n\to\infty} V^i\left(\underline{\tau}(0)\wedge q_n^+(\nu)\right) &= \lim_{n\to\infty} \left(V\big(\underline{\tau}(0)\wedge q_n^+(\nu)\big) + H^i_{\underline{\tau}(0)\wedge q_n^+(\nu)}\right) \\ &= V\big(\underline{\tau}(0)\wedge\nu\big) + H^i_{\underline{\tau}(0)\wedge\nu} = V^i\big(\underline{\tau}(0)\wedge\nu\big), \quad \text{a.s.} \end{aligned} \tag{6.49}$$

Step 5: Proposition 3.3 shows that the stopped process $\left\{V^{i'}\big(\underline{\tau}(0)\wedge t\big)\right\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_{i'}$-submartingale; thus $\left\{-V^{i'}\big(\underline{\tau}(0)\wedge t\big)\right\}_{t\in[0,T]}$ is an $\mathcal{E}'$-supermartingale by (3.12). Then Theorem 2.3 of [1] implies that $V^{i',+}_t \triangleq \varliminf_{n\to\infty} V^{i'}\big(\underline{\tau}(0)\wedge q_n^+(t)\big), t\in[0,T]$ is an RCLL process and that $P\Big(V_t^{i',+} = \lim_{n\to\infty} V^{i'}\big(\underline{\tau}(0) \wedge q_n^+(t)\big) \text{ for any } t \in [0,T]\Big) = 1$. For any $\sigma, \zeta \in \mathcal{S}_{0,T}$, Lemma 3.3 of [1] and (3.4) show that

$$\begin{aligned}\mathbf{1}_{\{\sigma=\zeta\}} V(\sigma) &= \mathbf{1}_{\{\sigma=\zeta\}} \overline{V}(\sigma) = \underset{j\in\mathcal{I}}{\operatorname{essinf}} \Big(\mathbf{1}_{\{\sigma=\zeta\}} R^j(\sigma)\Big) = \underset{j\in\mathcal{I}}{\operatorname{essinf}} \Big(\mathbf{1}_{\{\sigma=\zeta\}} R^j(\zeta)\Big) \\ &= \mathbf{1}_{\{\sigma=\zeta\}} \overline{V}(\zeta) = \mathbf{1}_{\{\sigma=\zeta\}} V(\zeta), \quad \text{a.s.,}\end{aligned}$$

which implies that

$$\begin{aligned}\mathbf{1}_{\{\sigma=\zeta\}} V^{i'}(\sigma) &= \mathbf{1}_{\{\sigma=\zeta\}} V(\sigma) + \mathbf{1}_{\{\sigma=\zeta\}} H_\sigma^{i'} = \mathbf{1}_{\{\sigma=\zeta\}} V(\zeta) + \mathbf{1}_{\{\sigma=\zeta\}} H_\zeta^{i'} \\ &= \mathbf{1}_{\{\sigma=\zeta\}} V^{i'}(\zeta), \quad \text{a.s.} \end{aligned} \tag{6.50}$$

Let $\sigma \in \mathcal{S}_{0,T}^F$ take values in a finite set $\{t_1 < \cdots < t_m\}$. For any $\alpha \in \{1 \cdots m\}$ and $n \in \mathbb{N}$, since $\{\sigma = t_\alpha\} \subset \{\underline{\tau}(0) \wedge q_n^+(\sigma) = \underline{\tau}(0) \wedge q_n^+(t_\alpha)\}$, (6.50) implies that $\mathbf{1}_{\{\sigma=t_\alpha\}} V^{i'}\big(\underline{\tau}(0) \wedge q_n^+(\sigma)\big) = \mathbf{1}_{\{\sigma=t_\alpha\}} V^{i'}\big(\underline{\tau}(0) \wedge q_n^+(t_\alpha)\big)$, a.s. As $n \to \infty$, (6.49) shows that a.s.

$$\begin{aligned}\mathbf{1}_{\{\sigma=t_\alpha\}} V_\sigma^{i',+} &= \mathbf{1}_{\{\sigma=t_\alpha\}} V_{t_\alpha}^{i',+} = \lim_{n\to\infty} \mathbf{1}_{\{\sigma=t_\alpha\}} V^{i'}\big(\underline{\tau}(0) \wedge q_n^+(t_\alpha)\big) \\ &= \lim_{n\to\infty} \mathbf{1}_{\{\sigma=t_\alpha\}} V^{i'}\big(\underline{\tau}(0) \wedge q_n^+(\sigma)\big) = \mathbf{1}_{\{\sigma=t_\alpha\}} V^{i'}\big(\underline{\tau}(0) \wedge \sigma\big).\end{aligned}$$

Summing up the above expression over $\alpha \in \{1 \cdots m\}$ yields that $V_\sigma^{i',+} = V^{i'}\big(\underline{\tau}(0) \wedge \sigma\big)$, a.s. Then the right-continuity of the process $V^{i',+}$ and (6.49) imply that

$$V_\nu^{i',+} = \lim_{n\to\infty} V_{q_n^+(\nu)}^{i',+} = \lim_{n\to\infty} V^{i'}\big(\underline{\tau}(0) \wedge q_n^+(\nu)\big) = V^{i'}\big(\underline{\tau}(0) \wedge \nu\big), \quad \text{a.s.} \tag{6.51}$$

In particular, $V^{i',+}$ is an RCLL modification of the stopped process $\Big\{V^{i'}\big(\underline{\tau}(0) \wedge t\big)\Big\}_{t\in[0,T]}$. Therefore, $V^0 \triangleq \Big\{V_t^{i',+} - H_{\underline{\tau}(0)\wedge t}^{i'}\Big\}_{t\in[0,T]}$ is an RCLL modification of the stopped value process $\big\{V\big(\underline{\tau}(0) \wedge t\big)\big\}_{t\in[0,T]}$. For any $\nu \in \mathcal{S}_{0,T}$, (6.51) implies that

$$\begin{aligned}V_\nu^0 &= V_\nu^{i',+} - H_{\underline{\tau}(0)\wedge\nu}^{i'} = V^{i'}\big(\underline{\tau}(0) \wedge \nu\big) - H_{\underline{\tau}(0)\wedge\nu}^{i'} \\ &= V\big(\underline{\tau}(0) \wedge \nu\big), \quad \text{a.s., proving (3.13).} \quad \square\end{aligned}$$

Proof of (2). (3.13) and Proposition 3.2 imply that $V_{\underline{\tau}(0)}^0 = V\big(\underline{\tau}(0)\big) = Y_{\underline{\tau}(0)}$, a.s. Hence, we can deduce from the right-continuity of processes V^0 and Y that τ_V in (3.14) is a stopping time in $\mathcal{S}_{0,\underline{\tau}(0)}$ and that $Y_{\tau_V} = V_{\tau_V}^0 = V(\tau_V)$, a.s., where the second equality is due to (3.13). Then it follows from (3.11) that for any $i \in \mathcal{I}$

$$V(0) = V^i(0) \le \widetilde{\mathcal{E}}_i\big[V^i(\tau_V)\big] = \widetilde{\mathcal{E}}_i\big[Y_{\tau_V}^i\big] = \mathcal{E}_i\big[Y_{\tau_V}^i\big].$$

Taking the infimum of the right-hand side over $i \in \mathcal{I}$ yields that

$$V(0) \le \inf_{i\in\mathcal{I}} \mathcal{E}_i\big[Y_{\tau_V}^i\big] \le \sup_{\rho\in\mathcal{S}_{0,T}} \Big(\inf_{i\in\mathcal{I}} \mathcal{E}_i\big[Y_\rho^i\big]\Big) = \underline{V}(0) = V(0),$$

which implies that $\inf_{i\in\mathcal{I}} \mathcal{E}_i\big[Y_{\tau_V}^i\big] = \sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \mathcal{E}_i\big[Y_\rho^i\big]$. $\square$

Proof of Proposition 4.1. Fix $t \in [0, T]$. For any $\xi \in \mathrm{Dom}(\mathscr{E})$ and $i \in \mathcal{I}$, the definition of $\mathrm{Dom}(\mathscr{E})$ assures that there exists a $c(\xi) \in \mathbb{R}$ such that $c(\xi) \le \xi$, a.s. Then Proposition 2.7(5) of [1] shows that

$$c(\xi) = \widetilde{\mathcal{E}}_i[c(\xi)|\mathcal{F}_t] \le \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{6.52}$$

Taking the essential infimum of the right-hand side over $i \in \mathcal{I}$, we obtain for an arbitrary $i' \in \mathcal{I}$ that

$$c(\xi) \le \underline{\mathscr{E}}[\xi|\mathcal{F}_t] \le \widetilde{\mathcal{E}}_{i'}[\xi|\mathcal{F}_t], \quad \text{a.s.}$$

Since $\widetilde{\mathcal{E}}_{i'}[\xi|\mathcal{F}_t] \in \mathrm{Dom}^{\#}(\mathcal{E}_{i'}) = \mathrm{Dom}(\mathscr{E})$, Lemma 3.2 of [1] implies that $\underline{\mathscr{E}}[\xi|\mathcal{F}_t] \in \mathrm{Dom}(\mathscr{E})$; thus $\underline{\mathscr{E}}[\cdot|\mathcal{F}_t]$ is a mapping from $\mathrm{Dom}(\mathscr{E})$ to $\mathrm{Dom}_t(\mathscr{E}) = \mathrm{Dom}(\mathscr{E}) \cap L^0(\mathcal{F}_t)$.

A simple application of Lemma 3.3 of [1] shows that $\underline{\mathscr{E}}$ satisfies (A3), (A4) and (4.1). Hence, it only remains to show (A2) for $\underline{\mathscr{E}}$. Fix $0 \le s < t \le T$. Letting $(\nu, \mathcal{I}', \mathcal{U}) = (t, \mathcal{I}, \{T\})$ and taking $X(T) = \xi$ in Lemma 2.2, we can find a sequence $\{i_n\}_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$\underline{\mathscr{E}}[\xi|\mathcal{F}_t] = \operatorname*{essinf}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_t] = \lim_{n\to\infty} \downarrow \widetilde{\mathcal{E}}_{i_n}[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{6.53}$$

Now fix $j \in \mathcal{I}$. For any $n \in \mathbb{N}$, it follows from Definition 3.2 of [1] that there exists a $k_n = k(j, i_n, t) \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_{k_n} = \mathcal{E}^t_{j,i_n}$. Applying (3.3) of [1] yields that

$$\underline{\mathscr{E}}[\xi|\mathcal{F}_s] \le \widetilde{\mathcal{E}}_{k_n}[\xi|\mathcal{F}_s] = \mathcal{E}^t_{j,i_n}[\xi|\mathcal{F}_s] = \widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_{i_n}\big[\xi|\mathcal{F}_t\big]|\mathcal{F}_s\big], \quad \text{a.s.} \tag{6.54}$$

For any $n \in \mathbb{N}$, (6.52) and (6.53) show that $c(\xi) \le \widetilde{\mathcal{E}}_{i_n}\big[\xi|\mathcal{F}_t\big] \le \widetilde{\mathcal{E}}_{i_1}\big[\xi|\mathcal{F}_t\big]$, a.s., where $\widetilde{\mathcal{E}}_{i_1}\big[\xi|\mathcal{F}_t\big] \in \mathrm{Dom}^{\#}(\mathcal{E}_{i_1}) = \mathrm{Dom}(\mathscr{E})$. Proposition 2.9 of [1] and (6.54) then imply that

$$\widetilde{\mathcal{E}}_j\big[\underline{\mathscr{E}}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] = \lim_{n\to\infty} \widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_{i_n}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] \ge \underline{\mathscr{E}}[\xi|\mathcal{F}_s], \quad \text{a.s.}$$

Taking the essential infimum of the left-hand side over $j \in \mathcal{I}$, we obtain

$$\underline{\mathscr{E}}\big[\underline{\mathscr{E}}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] \ge \underline{\mathscr{E}}[\xi|\mathcal{F}_s], \quad \text{a.s.} \tag{6.55}$$

On the other hand, for any $i \in \mathcal{I}$ and $\rho \in \mathcal{S}_{t,T}$, applying Corollary 2.3 of [1], we obtain

$$\widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_s] = \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] \ge \widetilde{\mathcal{E}}_i[\underline{\mathscr{E}}[\xi|\mathcal{F}_t]|\mathcal{F}_s] \ge \underline{\mathscr{E}}\big[\underline{\mathscr{E}}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big], \quad \text{a.s.}$$

Taking the essential infimum of the left-hand side over $i \in \mathcal{I}$ yields that $\underline{\mathscr{E}}[\xi|\mathcal{F}_s] \ge \underline{\mathscr{E}}\big[\underline{\mathscr{E}}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big]$, a.s., which together with (6.55) proves (A2) for $\underline{\mathscr{E}}$. □

Proof of Proposition 5.1. By (5.2), it holds $dt \times dP$-a.s. that for any $z \in \mathbb{R}^d$

$$|g(t,z)| = |g(t,z) - g(t,0)| \le K_g|z|, \quad \text{thus} \quad \tilde{g}(t,z) \triangleq -K_g|z| \le g(t,z).$$

Clearly, $\tilde{g}$ is a generator satisfying (5.2). It is also positively homogeneous in z, i.e.

$$\tilde{g}(t,\alpha z) = -K_g|\alpha z| = -\alpha K_g|z| = \alpha\tilde{g}(t,z), \quad \forall \alpha \ge 0, \forall z \in \mathbb{R}^d.$$

Then [13, Example 10] (or [16, Proposition 8]) and (5.6) imply that for any $n \in \mathbb{N}$ and any $A \in \mathcal{F}_T$ with $P(A) > 0$

$$n\mathcal{E}_{\tilde{g}}[\mathbf{1}_A] = \mathcal{E}_{\tilde{g}}[n\mathbf{1}_A] \le \mathcal{E}_g[n\mathbf{1}_A]. \tag{6.56}$$

Since $\mathcal{E}_{\tilde{g}}[\mathbf{1}_A] > 0$ (which follows from the second part of (A1)), letting $n \to \infty$ in (6.56) yields (H0).

Next, we consider a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset L^2(\mathcal{F}_T)$ with $\sup_{n\in\mathbb{N}} |\xi_n| \in L^2(\mathcal{F}_T)$. If ξ_n converges a.s., it is clear that $\xi \triangleq \lim_{n\to\infty} \xi_n \in L^2(\mathcal{F}_T)$. Applying Lemma 5.1 with $\mu = K_g$, we obtain

$$\begin{aligned}\big|\mathcal{E}_g[\xi_n] - \mathcal{E}_g[\xi]\big| &\le \mathcal{E}_{g_\mu}\big[|\xi_n - \xi|\big] \le \Bigg\| \sup_{t\in[0,T]} \mathcal{E}_{g_\mu}\big[|\xi_n - \xi|\,|\mathcal{F}_t\big] \Bigg\|_{L^2(\mathcal{F}_T)} \\ &\le C\mathrm{e}^{(K_g+K_g^2)T} \|\xi_n - \xi\|_{L^2(\mathcal{F}_T)},\end{aligned}$$

where we used the fact that $K_{g_\mu} = \mu$ in the last inequality. As $n \to \infty$, thanks to the Dominated Convergence Theorem of the linear expectation E, we have that $\|\xi_n - \xi\|^2_{L^2(\mathcal{F}_T)} = E|\xi_n - \xi|^2 \to 0$; thus $\lim_{n\to\infty} \mathcal{E}_g[\xi_n] = \mathcal{E}_g[\xi]$. Then (H1) and (H2) follow.

For any $\nu \in \mathcal{S}_{0,T}$ and $\xi \in L^{2,+}(\mathcal{F}_T) \triangleq \{\xi \in L^2(\mathcal{F}_T) : \xi \ge 0,\ \text{a.s.}\}$, Lemma 5.1(1) shows that $\sup_{t\in[0,T]} \big|\mathcal{E}_g[\xi|\mathcal{F}_t]\big| \in L^{2,+}(\mathcal{F}_T)$; consequently $\mathcal{E}_g[\xi|\mathcal{F}_\nu] \in L^{2,+}(\mathcal{F}_T)$. Since $X^\xi \triangleq \mathcal{E}_g[\xi|\mathcal{F}_\cdot]$ is a continuous process, $X^{\xi,+}_\nu = X^\xi_\nu = \mathcal{E}_g[\xi|\mathcal{F}_\nu] \in L^{2,+}(\mathcal{F}_T)$, which proves (H3). □

Proof of Proposition 5.2. Fix $\nu \in \mathcal{S}_{0,T}$. It is easy to check that the generator g^ν satisfies (5.2) with Lipschitz coefficient $K_1 \vee K_2$. For any $\xi \in L^2(\mathcal{F}_T)$, we set $\eta \triangleq \Gamma^{\xi,g_2}_\nu \in \mathcal{F}_\nu$ and define

$$\widetilde{\Theta}_t \triangleq \mathbf{1}_{\{\nu\le t\}} \Theta^{\xi,g_2}_t + \mathbf{1}_{\{\nu>t\}} \Theta^{\eta,g_1}_t, \quad \forall t \in [0,T].$$

It follows that

$$\begin{aligned} g^\nu(t,\widetilde{\Theta}_t) &= \mathbf{1}_{\{\nu\le t\}}\, g_2(t,\widetilde{\Theta}_t) + \mathbf{1}_{\{\nu>t\}} g_1(t,\widetilde{\Theta}_t) \\ &= \mathbf{1}_{\{\nu\le t\}}\, g_2(t,\Theta^{\xi,g_2}_t) + \mathbf{1}_{\{\nu>t\}} g_1(t,\Theta^{\eta,g_1}_t), \quad \forall t \in [0,T]. \end{aligned}$$

For any $t \in [0,T]$, since $\{\nu \le t\} \in \mathcal{F}_t$, one can deduce that

$$\begin{aligned} &\mathbf{1}_{\{\nu\le t\}}\left(\xi + \int_t^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s - \int_t^T \widetilde{\Theta}_s \mathrm{d}B_s\right) \\ &= \mathbf{1}_{\{\nu\le t\}}\xi + \int_t^T \mathbf{1}_{\{\nu\le t\}} g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s - \int_t^T \mathbf{1}_{\{\nu\le t\}} \widetilde{\Theta}_s \mathrm{d}B_s \\ &= \mathbf{1}_{\{\nu\le t\}}\xi + \int_t^T \mathbf{1}_{\{\nu\le t\}} g_2(s,\Theta^{\xi,g_2}_s)\,\mathrm{d}s - \int_t^T \mathbf{1}_{\{\nu\le t\}} \Theta^{\xi,g_2}_s \mathrm{d}B_s \\ &= \mathbf{1}_{\{\nu\le t\}}\left(\xi + \int_t^T g_2(s,\Theta^{\xi,g_2}_s)\,\mathrm{d}s - \int_t^T \Theta^{\xi,g_2}_s \mathrm{d}B_s\right) = \mathbf{1}_{\{\nu\le t\}}\Gamma^{\xi,g_2}_t, \quad \text{a.s.} \end{aligned} \tag{6.57}$$

The continuity of processes $\int_\cdot^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s$, $\int_\cdot^T \widetilde{\Theta}_s \mathrm{d}B_s$ and $\Gamma^{\xi,g_2}_\cdot$ then implies that except on a null set N

$$\mathbf{1}_{\{\nu\le t\}}\left(\xi + \int_t^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s - \int_t^T \widetilde{\Theta}_s \mathrm{d}B_s\right) = \mathbf{1}_{\{\nu\le t\}}\, \Gamma^{\xi,g_2}_t, \quad \forall t \in [0,T].$$

Taking $t = \nu(\omega)$ for any $\omega \in N^c$ yields that

$$\xi + \int_\nu^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s - \int_\nu^T \widetilde{\Theta}_s \mathrm{d}B_s = \Gamma^{\xi,g_2}_\nu = \eta, \quad \text{a.s.} \tag{6.58}$$

Now fix $t \in [0, T]$. We can deduce from (6.58) that

$$
\begin{aligned}
&\mathbf{1}_{\{\nu>t\}}\left(\xi+\int_t^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s-\int_t^T \widetilde{\Theta}_s \mathrm{d}B_s\right)\\
&\quad=\mathbf{1}_{\{\nu>t\}}\left(\eta+\int_t^\nu g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s-\int_t^\nu \widetilde{\Theta}_s \mathrm{d}B_s\right)\\
&\quad=\mathbf{1}_{\{\nu>t\}}\left(\eta+\int_t^\nu g_1(s,\Theta_s^{\eta,g_1})\mathrm{d}s-\int_t^\nu \Theta_s^{\eta,g_1} \mathrm{d}B_s\right),\quad \text{a.s.}
\end{aligned}
\tag{6.59}
$$

Moreover, Proposition 2.7(5) of [1] implies that

$$
\begin{aligned}
\mathcal{E}_{g_1}[\eta|\mathcal{F}_{t\wedge\nu}] &= \eta+\int_{t\wedge\nu}^T g_1(s,\Theta_s^{\eta,g_1})\mathrm{d}s-\int_{t\wedge\nu}^T \Theta_s^{\eta,g_1}\mathrm{d}B_s\\
&= \mathcal{E}_{g_1}[\eta|\mathcal{F}_\nu]+\int_{t\wedge\nu}^\nu g_1(s,\Theta_s^{\eta,g_1})\mathrm{d}s-\int_{t\wedge\nu}^\nu \Theta_s^{\eta,g_1}\mathrm{d}B_s\\
&= \eta+\int_{t\wedge\nu}^\nu g_1(s,\Theta_s^{\eta,g_1})\mathrm{d}s-\int_{t\wedge\nu}^\nu \Theta_s^{\eta,g_1}\mathrm{d}B_s,\quad \text{a.s.}
\end{aligned}
$$

Multiplying both sides by $\mathbf{1}_{\{\nu>t\}}$ and using (6.59), we obtain

$$
\begin{aligned}
&\mathbf{1}_{\{\nu>t\}}\left(\xi+\int_t^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s-\int_t^T \widetilde{\Theta}_s \mathrm{d}B_s\right)\\
&\quad=\mathbf{1}_{\{\nu>t\}}\mathcal{E}_{g_1}[\eta|\mathcal{F}_t]=\mathbf{1}_{\{\nu>t\}}\mathcal{E}_{g_1}[\Gamma_\nu^{\xi,g_2}|\mathcal{F}_t],\quad \text{a.s.,}
\end{aligned}
$$

which in conjunction with (6.57) shows that for any $t \in [0, T]$

$$
\begin{aligned}
\xi+\int_t^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s-\int_t^T \widetilde{\Theta}_s \mathrm{d}B_s &= \mathbf{1}_{\{\nu\le t\}}\Gamma_t^{\xi,g_2}+\mathbf{1}_{\{\nu>t\}}\mathcal{E}_{g_1}[\Gamma_\nu^{\xi,g_2}|\mathcal{F}_t]\\
&= \mathbf{1}_{\{\nu\le t\}}\mathcal{E}_{g_2}[\xi|\mathcal{F}_t]+\mathbf{1}_{\{\nu>t\}}\mathcal{E}_{g_1}\big[\mathcal{E}_{g_2}[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big]\\
&= \mathcal{E}^\nu_{g_1,g_2}[\xi|\mathcal{F}_t],\quad \text{a.s.}
\end{aligned}
$$

Since $\int_\cdot^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s$, $\int_\cdot^T \widetilde{\Theta}_s\mathrm{d}B_s$ and $\mathcal{E}^\nu_{g_1,g_2}[\xi|\mathcal{F}_\cdot]$ are all continuous processes, it holds except for a null set N' that

$$
\mathcal{E}^\nu_{g_1,g_2}[\xi|\mathcal{F}_t]=\xi+\int_t^T g^\nu(s,\widetilde{\Theta}_s)\mathrm{d}s-\int_t^T \widetilde{\Theta}_s\mathrm{d}B_s,\quad \forall t\in[0,T].
$$

One can easily show that $\left(\mathcal{E}^\nu_{g_1,g_2}[\xi|\mathcal{F}_\cdot],\widetilde{\Theta}\right)\in\mathbb{C}^2_{\mathbf{F}}([0,T])\times\mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Thus the pair is the unique solution to the BSDE(ξ, g^ν), namely $\mathcal{E}_{g^\nu}[\xi|\mathcal{F}_t]=\mathcal{E}^\nu_{g_1,g_2}[\xi|\mathcal{F}_t]$ for any $t\in[0,T]$. □

Proof of Theorem 5.1. We first note that for any $g\in\mathscr{G}'$, (5.7) implies that for every $\mathcal{E}_g$-submartingale X, $-X$ is an $\mathcal{E}_{g^-}$-supermartingale although g^- is concave (which means that $\mathcal{E}_{g^-}$ may not belong to $\mathscr{E}'$). Hence, condition (3.12) is satisfied.

Fix $g\in\mathscr{G}'$. Clearly $H_0^g=0$. For any $s,t\in\mathcal{D}_T$ with $s<t$, we can deduce from ($\tilde{\mathrm{h}}$1) and ($\tilde{\mathrm{h}}$2) that

$$
C_{\mathscr{H}'}\triangleq c'T\le\int_s^t c'\mathrm{d}s\le\int_s^t h_r^g\mathrm{d}r=H_{s,t}^g\le\int_s^t h'(r)\mathrm{d}r\le\int_0^T h'(r)\mathrm{d}r,\quad\text{a.s.,}\tag{6.60}
$$

which implies that $C_{\mathscr{H}'}\le\operatorname{essinf}_{s,t\in\mathcal{D}_T;s<t}H_{s,t}^g\le\operatorname{esssup}_{s,t\in\mathcal{D}_T;s<t}H_{s,t}^g\le\int_0^T h'(r)\mathrm{d}r$, a.s.; thus (S2) holds. Since $\int_0^T h'(r)\mathrm{d}r\in L^2(\mathcal{F}_T)$, it follows that $\operatorname{esssup}_{s,t\in\mathcal{D}_T;s<t}H_{s,t}^g\in L^{2,\#}(\mathcal{F}_T)\triangleq$

$\{\xi \in L^2(\mathcal{F}_T) : \xi \ge c,\ a.s. \text{ for some } c \in \mathbb{R}\} = \mathrm{Dom}(\mathscr{E}')$. We can also deduce from (6.60) that except on a null set N

$$C_{\mathscr{H}'} \le H^g_{s,t} \le \int_0^T h'(r)\mathrm{d}r, \quad \forall 0 \le s < t \le T.$$

Hence, for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s., we have $C_{\mathscr{H}'} \le H^g_{\nu,\rho} \le \int_0^T h'(r)\mathrm{d}r$, a.s., which implies that $H^g_{\nu,\rho} \in L^{2,\#}(\mathcal{F}_T) = \mathrm{Dom}(\mathscr{E}')$; so we get (S1′). Moreover, (S3) directly follows from ($\tilde{\text{h}}$3).

Next, we check that the process Y satisfies (Y1) and (2.6). By (5.11) and (Y3), it holds a.s. that $C_Y \le Y_t \le \zeta'_Y$ for any $t \in \mathcal{D}_T$. The right-continuity of the process Y then implies that except on a null set $\tilde{N}$

$$C_Y \le Y_t \le \zeta'_Y, \quad \forall t \in [0,T], \quad \text{thus} \quad C_Y \le Y_\rho \le \zeta'_Y, \quad \forall \rho \in \mathcal{S}_{0,T}. \tag{6.61}$$

Since $\zeta'_Y \in L^2(\mathcal{F}_T)$, it follows that $Y_\rho \in L^{2,\#}(\mathcal{F}_T) = \mathrm{Dom}(\mathscr{E}')$ for any $\rho \in \mathcal{S}_{0,T}$; thus (Y1) holds. Moreover, for any $g \in \mathscr{G}'$, $\rho \in \mathcal{S}_{0,T}$ and $t \in \mathcal{D}_T$, Proposition 2.2(2) of [1], (6.61) and Lemma 5.1(2) show that a.s.

$$\begin{aligned} C_Y + c'T &= \mathcal{E}_g[C_Y + c'T|\mathcal{F}_t] \le \mathcal{E}_g\left[Y_\rho + \int_0^\rho c'\mathrm{d}s|\mathcal{F}_t\right] \\ &\le \mathcal{E}_g[Y^g_\rho|\mathcal{F}_t] \le |\mathcal{E}_g[Y^g_\rho|\mathcal{F}_t]| = |\mathcal{E}_g[Y^g_\rho|\mathcal{F}_t] - \mathcal{E}_g[0|\mathcal{F}_t]| \\ &\le \mathcal{E}_{g_M}\big[|Y^g_\rho|\,|\mathcal{F}_t\big] \le \mathcal{E}_{g_M}\left[\big|Y_\rho\big| + \int_0^\rho |h^g_s|\mathrm{d}s|\mathcal{F}_t\right] \\ &\le \sup_{t\in[0,T]} \mathcal{E}_{g_M}\left[\zeta'_Y \vee (-C_Y) + \int_0^T h'(s) \vee (-c')\,\mathrm{d}s|\mathcal{F}_t\right]. \end{aligned}$$

Taking the essential supremum of $\mathcal{E}_g[Y^g_\rho|\mathcal{F}_t]$ over $(g,\rho,t) \in \mathscr{G}' \times \mathcal{S}_{0,T} \times \mathcal{D}_T$, we can deduce from (A4) that

$$\begin{aligned} C_Y + c'T &\le \operatorname*{esssup}_{(g,\rho,t)\in\mathscr{G}'\times\mathcal{S}_{0,T}\times\mathcal{D}_T} \mathcal{E}_g[Y^g_\rho|\mathcal{F}_t] \\ &\le \sup_{t\in[0,T]} \mathcal{E}_{g_M}\left[\zeta'_Y + \int_0^T h'(s)\mathrm{d}s|\mathcal{F}_t\right] - C_Y - c'T, \quad \text{a.s.} \end{aligned} \tag{6.62}$$

Lemma 5.1(1) implies that

$$\begin{aligned} &\left\| \sup_{t\in[0,T]} \mathcal{E}_{g_M}\left[\zeta'_Y + \int_0^T h'(s)\mathrm{d}s|\mathcal{F}_t\right]\right\|_{L^2(\mathcal{F}_T)} \\ &\quad \le C\mathrm{e}^{(M+M^2)T}\left\|\zeta'_Y + \int_0^T h'(s)\mathrm{d}s\right\|_{L^2(\mathcal{F}_T)} < \infty. \end{aligned}$$

Hence, we see from (6.62) that $\operatorname{esssup}_{(g,\rho,t)\in\mathscr{G}'\times\mathcal{S}_{0,T}\times\mathcal{D}_T} \mathcal{E}_g[Y^g_\rho|\mathcal{F}_t] \in L^{2,\#}(\mathcal{F}_T) = \mathrm{Dom}(\mathscr{E}')$, which is exactly (2.6).

Now we show that the family of processes $\{Y^g_t,\ t \in [0,T]\}_{g\in\mathscr{G}'}$ is both "$\mathscr{E}'$-uniformly-left-continuous" and "$\mathscr{E}'$-uniformly-right-continuous". For any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s., let $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\nu,T}$ be a sequence increasing a.s. to ρ. For any $g \in \mathscr{G}'$, Lemma 5.1(2) implies that a.s.

$$\left|\mathcal{E}_g\left[\frac{n}{n-1}Y_{\rho_n} + H^g_{\rho_n}|\mathcal{F}_\nu\right] - \mathcal{E}_g\left[Y^g_\rho|\mathcal{F}_\nu\right]\right| \le \mathcal{E}_{g_M}\left[\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho - \int_{\rho_n}^{\rho} h^g(s)\mathrm{d}s\right| \,|\,\mathcal{F}_\nu\right]$$
$$\le \mathcal{E}_{g_M}\left[\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho} \widetilde{h}'(s)\mathrm{d}s|\,\mathcal{F}_\nu\right],$$

where $g_M(z) \triangleq M|z|$, $z \in \mathbb{R}^d$ and $\widetilde{h}'(t) \triangleq h'(t) - c'$, $t \in [0,T]$. Taking the essential supremum of the left-hand side over $g \in \mathscr{G}'$ yields that

$$\underset{g\in\mathscr{G}'}{\operatorname{esssup}}\left|\mathcal{E}_g\left[\frac{n}{n-1}Y_{\rho_n} + H^g_{\rho_n}|\mathcal{F}_\nu\right] - \mathcal{E}_g\left[Y^g_\rho|\mathcal{F}_\nu\right]\right|$$
$$\le \mathcal{E}_{g_M}\left[\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho} \widetilde{h}'(s)\,\mathrm{d}s|\,\mathcal{F}_\nu\right], \quad \text{a.s.} \tag{6.63}$$

Moreover, Lemma 5.1(1) implies that

$$\left\|\mathcal{E}_{g_M}\left[\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho} \widetilde{h}'(s)\mathrm{d}s|\,\mathcal{F}_\nu\right]\right\|_{L^2(\mathcal{F}_T)}$$
$$\le \left\|\sup_{t\in[0,T]} \mathcal{E}_{g_M}\left[\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho} \widetilde{h}'(s)\mathrm{d}s|\,\mathcal{F}_t\right]\right\|_{L^2(\mathcal{F}_T)}$$
$$\le C\mathrm{e}^{(M+M^2)T}\left\|\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho} \widetilde{h}'(s)\mathrm{d}s\right\|_{L^2(\mathcal{F}_T)}. \tag{6.64}$$

Since $\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| \le \frac{n}{n-1}\left|Y_{\rho_n} - Y_\rho\right| + \frac{1}{n-1}\left|Y_\rho\right| \le 2\left|Y_{\rho_n} - Y_\rho\right| + \frac{1}{n-1}\left|Y_\rho\right|$ for any $n \ge 2$, the continuity of Y implies that $\lim_{n\to\infty}\Big(\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho}\widetilde{h}'(s)\mathrm{d}s\Big) = 0$, a.s. It also holds for any $n \ge 2$ that

$$\left|\frac{n}{n-1}Y_{\rho_n} - Y_\rho\right| + \int_{\rho_n}^{\rho} \widetilde{h}'(s)\mathrm{d}s \le 3\big(\zeta'_Y - C_Y\big) + \int_0^T h'(s)\mathrm{d}s - c'T, \quad \text{a.s.},$$

where the right-hand side belongs to $L^2(\mathcal{F}_T)$. Thus the Dominated Convergence Theorem implies that the sequence $\{|\frac{n}{n-1}Y_{\rho_n} - Y_\rho| + \int_{\rho_n}^{\rho}\widetilde{h}'(s)\mathrm{d}s\}_{n\in\mathbb{N}}$ converges to 0 in $L^2(\mathcal{F}_T)$, which together with (6.63) and (6.64) implies that the sequence $\{\operatorname{esssup}_{g\in\mathscr{G}'}|\mathcal{E}_g[\frac{n}{n-1}Y_{\rho_n} + H^g_{\rho_n}|\mathcal{F}_\nu] - \mathcal{E}_g[Y^g_\rho|\mathcal{F}_\nu]|\}_{n\in\mathbb{N}}$ also converges to 0 in $L^2(\mathcal{F}_T)$. Then we can find a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{n\to\infty}\underset{g\in\mathscr{G}'}{\operatorname{esssup}}\left|\mathcal{E}_g\left[\frac{n_k}{n_k-1}Y_{\rho_{n_k}} + H^g_{\rho_{n_k}}|\mathcal{F}_\nu\right] - \mathcal{E}_g[Y^g_\rho|\mathcal{F}_\nu]\right| = 0, \quad \text{a.s.}$$

Therefore, the family of processes $\{Y^g\}_{g\in\mathscr{G}'}$ is "$\mathscr{E}'$-uniformly-left-continuous". The "$\mathscr{E}'$-uniform-right-continuity" of $\{Y^g\}_{g\in\mathscr{G}'}$ can be shown similarly. □

Proof of Theorem 5.2. For any $U \in \mathfrak{U}$, Theorem 5.1 and Proposition 2.2 imply that $Z^{U,0} = \left\{Z^0_t + \int_0^t h^U_s\mathrm{d}s\right\}_{t\in[0,T]}$ is an $\mathcal{E}_{g_U}$-supermartingale. In light of the Doob–Meyer Decomposition of g-expectation (see e.g. [15, Theorem 3.3]), there exists an RCLL increasing process Δ^U null at 0 and a process $\Theta^U \in \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$ such that

$$Z^{U,0}_t = Z^{U,0}_T + \int_t^T g_U(s,\Theta^U_s)\mathrm{d}s + \Delta^U_T - \Delta^U_t - \int_t^T \Theta^U_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{6.65}$$

In what follows we will show that $U^*(t,\omega) \triangleq u^*\big(t,\omega,\Theta^{U^0}_t(\omega)\big)$, $(t,\omega) \in [0,T]\times\Omega$ is the optimal control desired, where $U^0 \equiv 0$ denotes the null control. Recall that $\overline{\tau}(0) = \inf\big\{t \in [0,T] \mid Z^0_t = Y_t\big\}$. Taking $t = \overline{\tau}(0)$ and $t = \overline{\tau}(0)\wedge t$ respectively in (6.65) and subtracting the former from the latter yields that

$$Z^{U,0}_{\overline{\tau}(0)\wedge t} = Z^{U,0}_{\overline{\tau}(0)} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} g_U(s,\Theta^U_s)\mathrm{d}s + \Delta^U_{\overline{\tau}(0)} - \Delta^U_{\overline{\tau}(0)\wedge t} - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta^U_s \mathrm{d}B_s, \quad t \in [0,T], \tag{6.66}$$

which is equivalent to

$$Z^{0}_{\overline{\tau}(0)\wedge t} = Z^{0}_{\overline{\tau}(0)} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} H(s,\Theta^U_s,U_s)\mathrm{d}s + \Delta^U_{\overline{\tau}(0)} - \Delta^U_{\overline{\tau}(0)\wedge t} - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta^U_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{6.67}$$

In particular, taking $U = U^0$, we obtain

$$Z^{0}_{\overline{\tau}(0)\wedge t} = Z^{0}_{\overline{\tau}(0)} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} H(s,\Theta^{U^0}_s,U^0_s)\mathrm{d}s + \Delta^{U^0}_{\overline{\tau}(0)} - \Delta^{U^0}_{\overline{\tau}(0)\wedge t} - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta^{U^0}_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{6.68}$$

Comparing the martingale parts of (6.67) and (6.68), we see that for any $U \in \mathfrak{U}$,

$$\Theta^U_t = \Theta^{U^0}_t, \quad \mathrm{d}t\times\mathrm{d}P\text{-a.s.} \tag{6.69}$$

on the stochastic interval $[\![0,\overline{\tau}(0)]\!] \triangleq \{(t,\omega)\in[0,T]\times\Omega : 0 \le t \le \overline{\tau}(0)\}$. Plugging this back into (6.67) yields that

$$Z^{0}_{\overline{\tau}(0)\wedge t} = Z^{0}_{\overline{\tau}(0)} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} H(s,\Theta^{U^0}_s,U_s)\mathrm{d}s + \Delta^U_{\overline{\tau}(0)} - \Delta^U_{\overline{\tau}(0)\wedge t} - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta^{U^0}_s \mathrm{d}B_s, \quad t \in [0,T]. \tag{6.70}$$

Let us define $g_{K_o}(z) \triangleq K_o|z|$, $z \in \mathbb{R}^d$. Note that it is not necessary that $g_{K_o} = g_U$ for some $U \in \mathfrak{U}$. For any $U \in \mathfrak{U}$, we set $\Gamma_t \triangleq \mathcal{E}_{g_U}\left[Z^{U,0}_{\overline{\tau}(0)}|\mathcal{F}_t\right]$ and $\hat{\Gamma}_t \triangleq \mathcal{E}_{g_{K_o}}\left[-\Delta^{U^*}_{\overline{\tau}(0)}|\mathcal{F}_t\right]$, $t \in [0,T]$, which are the solutions to the BSDE$\big(Z^{U,0}_{\overline{\tau}(0)}, g_U\big)$ and the BSDE$\big(-\Delta^{U^*}_{\overline{\tau}(0)}, g_{K_o}\big)$ respectively, i.e.,

$$\Gamma_t = Z^{U,0}_{\overline{\tau}(0)} + \int_t^T g_U(s,\Theta_s)\mathrm{d}s - \int_t^T \Theta_s \mathrm{d}B_s \quad \text{and}$$
$$\hat{\Gamma}_t = -\Delta^{U^*}_{\overline{\tau}(0)} + \int_t^T K_o\left|\hat{\Theta}_s\right|\mathrm{d}s - \int_t^T \hat{\Theta}_s \mathrm{d}B_s, \quad t \in [0,T],$$

where $\Theta, \hat{\Theta} \in \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$. Proposition 2.7(5) of [1] and Corollary 2.3 of [1] imply that for any $t \in [0,T]$

$$\begin{aligned}\Gamma_{\overline{\tau}(0)} - \Gamma_{\overline{\tau}(0)\wedge t} &= \mathcal{E}_{g_U}\left[Z^{U,0}_{\overline{\tau}(0)}|\mathcal{F}_{\overline{\tau}(0)}\right] - \mathcal{E}_{g_U}\left[Z^{U,0}_{\overline{\tau}(0)}|\mathcal{F}_{\overline{\tau}(0)\wedge t}\right] \\ &= Z^{U,0}_{\overline{\tau}(0)} - \mathcal{E}_{g_U}\left[\mathcal{E}_{g_U}\left[Z^{U,0}_{\overline{\tau}(0)}|\mathcal{F}_{\overline{\tau}(0)}\right]\Big|\mathcal{F}_t\right] \\ &= Z^{U,0}_{\overline{\tau}(0)} - \mathcal{E}_{g_U}\left[Z^{U,0}_{\overline{\tau}(0)}|\mathcal{F}_t\right] = Z^{U,0}_{\overline{\tau}(0)} - \Gamma_t, \quad \text{a.s.}\end{aligned}$$

Then the continuity of processes $\Gamma_\cdot$ and $Z^{U,0}_\cdot$ implies that

$$\begin{aligned}\Gamma_t - Z^{U,0}_{\overline{\tau}(0)\wedge t} &= Z^{U,0}_{\overline{\tau}(0)} - Z^{U,0}_{\overline{\tau}(0)\wedge t} + \Gamma_{\overline{\tau}(0)\wedge t} - \Gamma_{\overline{\tau}(0)} \\ &= Z^{U,0}_{\overline{\tau}(0)} - Z^{U,0}_{\overline{\tau}(0)\wedge t} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} g_U(s,\Theta_s)\mathrm{d}s - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta_s \mathrm{d}B_s \\ &= Z^{0}_{\overline{\tau}(0)} - Z^{0}_{\overline{\tau}(0)\wedge t} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} H(s,\Theta_s,U_s)\mathrm{d}s - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta_s \mathrm{d}B_s \\ &= -\Delta^{U^*}_{\overline{\tau}(0)} + \Delta^{U^*}_{\overline{\tau}(0)\wedge t} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \left[H(s,\Theta_s,U_s) - H(s,\Theta^{U^0}_s,U^*_s)\right]\mathrm{d}s \\ &\quad - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} (\Theta_s - \Theta^{U^0}_s)\mathrm{d}B_s, \quad t \in [0,T],\end{aligned}$$

where we used (6.70) with $U = U^*$ in the last inequality. Since it holds $\mathrm{d}t \times \mathrm{d}P$-a.s. that

$$\begin{aligned}H(t,\Theta_t,U_t) - H(t,\Theta^{U^0}_t,U^*_t) &= H(t,\Theta_t,U_t) - H\big(t,\Theta^{U^0}_t,u^*(t,\Theta^{U^0}_t)\big) \\ &\le H(t,\Theta_t,U_t) - H(t,\Theta^{U^0}_t,U_t) \\ &= g^o(t,\Theta_t,U_t) - g^o(t,\Theta^{U^0}_t,U_t) \\ &\le \left|g^o(t,\Theta_t,U_t) - g^o(t,\Theta^{U^0}_t,U_t)\right| \le K_o\left|\Theta_t - \Theta^{U^0}_t\right|,\end{aligned}$$

the Comparison Theorem for BSDEs (see e.g. [13, Theorem 35.3]) implies that

$$\hat{\Gamma}_t \ge \Gamma_t - Z^{U,0}_{\overline{\tau}(0)\wedge t} - \Delta^{U^*}_{\overline{\tau}(0)\wedge t}, \quad t \in [0,T].$$

In particular, when $t = 0$, we can deduce from (2.17) that $\mathcal{E}_{g_{K_o}}\left[-\Delta^{U^*}_{\overline{\tau}(0)}\right] \ge \mathcal{E}_{g_U}\left[Z^U\big(\overline{\tau}(0)\big)\right] - Z(0)$. Taking the supremum of the right-hand side over $U \in \mathfrak{U}$ and applying Theorem 2.1 with $\nu = 0$, we obtain

$$0 \ge \mathcal{E}_{g_{K_o}}\left[-\Delta^{U^*}_{\overline{\tau}(0)}\right] \ge \sup_{U\in\mathfrak{U}} \mathcal{E}_{g_U}\left[Z^U\big(\overline{\tau}(0)\big)\right] - Z(0) = 0,$$

and thus $\mathcal{E}_{g_{K_o}}\left[-\Delta^{U^*}_{\overline{\tau}(0)}\right] = 0$. The strict monotonicity of g-expectation (see e.g. [5, Proposition 2.2(iii)]) then implies that $\Delta^{U^*}_{\overline{\tau}(0)} = 0$, a.s. Plugging this back into (6.66) and using (6.69), we obtain

$$\begin{aligned}Z^{U^*,0}_{\overline{\tau}(0)\wedge t} &= Z^{U^*,0}_{\overline{\tau}(0)} + \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} g_{U^*}(s,\Theta^{U^0}_s)\mathrm{d}s - \int_{\overline{\tau}(0)\wedge t}^{\overline{\tau}(0)} \Theta^{U^0}_s \mathrm{d}B_s \\ &= Z^{U^*,0}_{\overline{\tau}(0)} + \int_t^T g_{U^*}(s,\mathbf{1}_{\{s\le\overline{\tau}(0)\}}\Theta^{U^0}_s)\mathrm{d}s - \int_t^T \mathbf{1}_{\{s\le\overline{\tau}(0)\}}\Theta^{U^0}_s \mathrm{d}B_s, \\ &\quad t \in [0,T], \qquad (6.71)\end{aligned}$$

which implies that $\mathcal{E}_{g_{U^*}}\left[Z^{U^*,0}_{\overline{\tau}(0)}|\mathcal{F}_t\right] = Z^{U^*,0}_{\overline{\tau}(0)\wedge t}$, $\forall t \in [0,T]$. Namely, $\left\{Z^{U^*,0}_{\overline{\tau}(0)\wedge t}\right\}_{t\in[0,T]}$ is a g_{U^*}-martingale. Eventually, letting $t = 0$ in (6.71), we can deduce from (2.17) and Theorem 2.1 that

$$Z(0) = Z^{U^*,0}_0 = \mathcal{E}_{g_{U^*}}\left[Z^{U^*,0}_{\overline{\tau}(0)}\right] = \mathcal{E}_{g_{U^*}}\left[Z\big(\overline{\tau}(0)\big) + \int_0^{\overline{\tau}(0)} h^{U^*}_s \mathrm{d}s\right]$$

$$= \mathcal{E}_{g_{U^*}}\left[Y_{\overline{\tau}(0)} + \int_0^{\overline{\tau}(0)} h^{U^*}_s \mathrm{d}s\right]. \quad \square$$

Proof of Proposition 5.3. Because of its linearity in z, the primary generator

$$g^o(t,\omega,z,u) \triangleq \left\langle \sigma^{-1}(t,X(\omega)) f\big(t,X(\omega),u\big), z\right\rangle$$
$$\forall (t,\omega,z,u) \in [0,T]\times\Omega\times\mathbb{R}^d\times S \tag{6.72}$$

satisfies ($\mathrm{g^o2}$) and ($\mathrm{g^o4}$). Then ($\mathrm{g^o1}$) follows from the continuity of the process $\{X(t)\}_{t\in[0,T]}$ as well as the measurability of the volatility σ and of the function f. Moreover, (5.15) and (5.20) imply that for a.e. $t \in [0,T]$

$$|g^o(t,\omega,z_1,u) - g^o(t,\omega,z_2,u)| = \left|\langle \sigma^{-1}(t,X(\omega)) f\big(t,X(\omega),u\big), z-z'\rangle\right|$$
$$\le \left\|\sigma^{-1}(t,X(\omega))\right\| \cdot \left|f\big(t,X(\omega),u\big)\right| \cdot |z-z'|$$
$$\le K^2|z-z'|, \quad \forall z_1,z_2 \in \mathbb{R}^d,\ \forall(\omega,u)\in\Omega\times S,$$

which shows that g^o satisfies ($\mathrm{g^o4}$) with $K_o = K^2$. Clearly, $\widetilde{\mathfrak{U}} = \mathcal{H}^0_{\mathbf{F}}([0,T];S)$ is closed under the pasting in the sense of (5.13). Hence, we know from the last section that $\{\mathcal{E}_{g_U}\}_{U\in\widetilde{\mathfrak{U}}}$ is a stable class of g-expectations, where g_U is defined in (5.12).

Fix $U \in \widetilde{\mathfrak{U}}$. For any $\xi \in L^2(\mathcal{F})$, we see from (5.4) that

$$\mathcal{E}_{g_U}[\xi|\mathcal{F}_t] = \xi + \int_t^T g_U(s,\Theta_s)\mathrm{d}s - \int_t^T \Theta_s \mathrm{d}B_s$$
$$= \xi + \int_t^T \left\langle \sigma^{-1}(s,X) f\big(s,X,U_s\big), \Theta_s\right\rangle \mathrm{d}s - \int_t^T \Theta_s \mathrm{d}B_s$$
$$= \xi - \int_t^T \Theta_s \mathrm{d}B^U_s, \quad t\in[0,T],$$

where $B^U_t \triangleq B_t - \int_0^t \sigma^{-1}(s,X) f(s,X,U_s)\,\mathrm{d}s$, $t \in [0,T]$ is a Brownian motion with respect to P_U. For any $t \in [0,T]$, taking $E_U[\cdot|\mathcal{F}_t]$ on both sides above yields that

$$\mathcal{E}_{g_U}[\xi|\mathcal{F}_t] = E_U\big[\mathcal{E}_{g_U}[\xi|\mathcal{F}_t]|\mathcal{F}_t\big] = E_U[\xi|\mathcal{F}_t] - E_U\left[\int_t^T \Theta_s \mathrm{d}B^U_s|\mathcal{F}_t\right]$$
$$= E_U[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{6.73}$$

Hence the g-expectation $\mathcal{E}_{g_U}$ coincides with the linear expectation E_U on $L^2(\mathcal{F}_T)$.

Clearly, the process $Y \triangleq \{\varphi(X(t))\}_{t\in[0,T]}$ satisfies (Y3) since φ is bounded from below by $-K$. We see from (5.19) that $Y_t = \varphi(X(t)) \le K|X(t)| \le K\|X\|^*_T$, $\forall t\in[0,T]$, which implies that

$$\zeta'_Y \triangleq \Big(\operatorname*{esssup}_{t\in\mathcal{D}_T} Y_t\Big)^+ \le K\|X\|^*_T, \quad \text{a.s.} \tag{6.74}$$

For any $t \in [0, T]$, the Burkholder–Davis–Gundy inequality, $(\sigma 1)$, and (5.15) as well as Fubini Theorem imply that

$$E\Big[\big(\|X\|_t^*\big)^2\Big] = E\Big[\sup_{s\in[0,t]} |X(s)|^2\Big] \le 2x^2 + 2E\left\{\sup_{s\in[0,t]}\left|\int_0^s \sigma(r, X)\mathrm{d}B_r\right|^2\right\}$$
$$\le 2x^2 + 2CE\int_0^t |\sigma(s, X)|^2\mathrm{d}s$$
$$\le 2x^2 + 4C\int_0^t |\sigma(s, \vec{0})|^2\mathrm{d}s + 4CE\int_0^t |\sigma(s, X) - \sigma(s, \vec{0})|^2\mathrm{d}s$$
$$\le 2x^2 + 4C\int_0^T |\sigma(s, \vec{0})|^2\mathrm{d}s + 4Cn^2K^2\int_0^t E\Big[\big(\|X\|_s^*\big)^2\Big]\mathrm{d}s.$$

Then applying Gronwall's inequality yields that

$$E\Big[\big(\|X\|_T^*\big)^2\Big] \le \left(2x^2 + 4C\int_0^T |\sigma(s, \vec{0})|^2\mathrm{d}s\right)\mathrm{e}^{4Cn^2K^2T} < \infty, \tag{6.75}$$

which together with (6.74) shows that $\zeta'_Y \in L^2(\mathcal{F}_T)$, proving (5.11).

Next, we define a function $h^o(t, \omega, u) \triangleq h(t, X(\omega), u), \forall (t, \omega, u) \in [0, T] \times \Omega \times S$. The continuity of the process $\{X(t)\}_{t\in[0,T]}$ and the measurability of the function h imply that h^o is $\mathscr{P} \otimes \mathfrak{S}/\mathscr{B}(\mathbb{R})$-measurable. We see from (5.20) that h^o satisfies $(\hat{\text{h}}1)$. It also follows from (5.20) that for a.e. $t \in [0, T]$ and for any $\omega \in \Omega$,

$$h_t^U(\omega) \triangleq h^o(t, \omega, U_t(\omega)) = h(t, X(\omega), U_t(\omega)) \le K\|X(\omega)\|_T^*, \quad \forall U \in \widetilde{\mathfrak{U}}.$$

Taking the essential supremum of $h_t^U(\omega)$ over $U \in \widetilde{\mathfrak{U}}$ with respect to the product measure space $([0, T] \times \Omega, \mathscr{P}, \lambda \times P)$ yields that $\hat{h}(t, \omega) \triangleq \left(\operatorname{esssup}_{U\in\widetilde{\mathfrak{U}}} h_t^U(\omega)\right)^+ \le K\|X(\omega)\|_T^*$, $\mathrm{d}t \times \mathrm{d}P$-a.s., which leads to the relation $\int_0^T \hat{h}(t, \omega)\mathrm{d}t \le KT\|X(\omega)\|_T^*$, a.s. Hence, (6.75) implies that $\int_0^T \hat{h}(t, \omega)\mathrm{d}t \in L^2(\mathcal{F}_T)$, proving $(\hat{\text{h}}2)$ for h^o.

We can apply the optimal stopping theory developed in Section 2 to the triple $\big(\{\mathcal{E}_{g_U}\}_{U\in\widetilde{\mathfrak{U}}}, \{h^U\}_{U\in\widetilde{\mathfrak{U}}}, Y\big)$ and use (6.73) to obtain (5.16). In addition, if there exists a measurable mapping $u^* : [0, T] \times \Omega \times \mathbb{R}^d \mapsto S$ satisfying (5.17), then (6.72) indicates that for any $(t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d$

$$\sup_{u\in S}\Big(g^o(t, \omega, z, u) + h^o(t, \omega, u)\Big) = \sup_{u\in S} \widetilde{H}(t, X(\omega), z, u)$$
$$= \widetilde{H}\big(t, X(\omega), z, u^*(t, X(\omega), z)\big)$$
$$= g^o(t, \omega, z, u^*(t, X(\omega), z)) + h^o(t, \omega, u^*(t, X(\omega), z)),$$

which shows that (5.14) holds for the mapping $\widetilde{u}^*(t, \omega, z) = u^*(t, X(\omega), z)$, $(t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R}^d$. Therefore, an application of Theorem 5.2 yields (5.18) for some $U^* \in \widetilde{\mathfrak{U}}$. □

Proof of Proposition 5.4. (5.23) directly follows from [4, Theorem 5]. To see the second statement, we set $\Delta\Gamma \triangleq \Gamma^{\xi_1,\hat{g}} - \Gamma^{\xi_2,\hat{g}}$ and $\Delta\Theta \triangleq \Theta^{\xi_1,\hat{g}} - \Theta^{\xi_2,\hat{g}}$; then (5.21)(i) implies that

$$\mathrm{d}\Delta\Gamma_t = -\big(\hat{g}(t, \Theta_t^{\xi_1,\hat{g}}) - \hat{g}(t, \Theta_t^{\xi_2,\hat{g}})\big)\mathrm{d}t + \Delta\Theta_t\mathrm{d}B_t$$
$$= -\int_0^1 \frac{\partial\hat{g}}{\partial z}(t, \lambda\Delta\Theta_t + \Theta_t^{\xi_2,\hat{g}})\Delta\Theta_t\mathrm{d}\lambda\,\mathrm{d}t + \Delta\Theta_t\mathrm{d}B_t$$
$$= \Delta\Theta_t\big(-a_t\mathrm{d}t + \mathrm{d}B_t\big), \quad t \in [0, T],$$

where $a_t \triangleq \int_0^1 \frac{\partial \hat{g}}{\partial z}(\lambda \Delta \Theta_t + \Theta_t^{\xi_2,\hat{g}}), \mathrm{d}\lambda$, $t \in [0,T]$. Since $M_{\mathbf{F}}([0,T];\mathbb{R}^d) \subset M^2_{\mathbf{F}}([0,T];\mathbb{R}^d) = \mathcal{H}^2_{\mathbf{F}}([0,T];\mathbb{R}^d)$, one can deduce from (5.21)(ii) that

$$\begin{aligned} E\int_0^T |a_t|^2 \mathrm{d}t &\le E\int_0^T \int_0^1 \left|\frac{\partial \hat{g}}{\partial z}(\lambda \Delta \Theta_t + \Theta_t^{\xi_2,\hat{g}})\right|^2 \mathrm{d}\lambda\, \mathrm{d}t \\ &\le 2\kappa^2 T + 2\kappa^2 E\int_0^T \int_0^1 |\lambda \Theta_t^{\xi_1,\hat{g}} + (1-\lambda)\Theta_t^{\xi_2,\hat{g}}|^2 \mathrm{d}\lambda\, \mathrm{d}t \\ &\le 2\kappa^2 T + \frac{4}{3}\kappa^2 E\int_0^T \left(\left|\Theta_t^{\xi_1,\hat{g}}\right|^2 + \left|\Theta_t^{\xi_2,\hat{g}}\right|^2\right)\mathrm{d}t < \infty. \end{aligned}$$

Moreover, Doob's martingale inequality shows that

$$E\left[\sup_{t\in[0,T]}\left|\int_0^t a_s \mathrm{d}B_s\right|^2\right] \le 4E\left[\left|\int_0^T a_s \mathrm{d}B_s\right|^2\right] = 4E\int_0^T |a_t|^2 \mathrm{d}t < \infty. \tag{6.76}$$

Thus, we can define the process $Q_t \triangleq \exp\left\{-\frac{1}{2}\int_0^t |a_s|^2 \mathrm{d}s + \int_0^t a_s \mathrm{d}B_s\right\}$, $t \in [0,T]$ as well as the stopping times

$$\nu_n \triangleq \inf\left\{t \in [\nu, T] : Q_t \vee |\Delta\Gamma_t| > n\right\} \wedge T, \quad \forall n \in \mathbb{N}.$$

It is clear that $\lim_{n\to\infty} \uparrow \nu_n = T$, a.s., and (6.76) assures that there exists a null set N such that for any $\omega \in N^c$, $T = \nu_m(\omega)$ for some $m = m(\omega) \in \mathbb{N}$.

For any $n \in \mathbb{N}$, integrating by parts on $[\nu, \nu_n]$ yields that

$$\begin{aligned} Q_{\nu_n}\Delta\Gamma_{\nu_n} &= Q_\nu \Delta\Gamma_\nu - \int_\nu^{\nu_n} Q_t \Delta\Theta_t a_t \mathrm{d}t + \int_\nu^{\nu_n} Q_t \Delta\Theta_t \mathrm{d}B_t + \int_\nu^{\nu_n} \Delta\Gamma_t Q_t a_t \mathrm{d}B_t \\ &\quad + \int_\nu^{\nu_n} Q_t \Delta\Theta_t a_t \mathrm{d}t \\ &= \int_\nu^{\nu_n} \left(Q_t \Delta\Theta_t + \Delta\Gamma_t Q_t a_t\right)\mathrm{d}B_t \end{aligned}$$

which implies that $E\left[Q_{\nu_n}\Delta\Gamma_{\nu_n}\right] = 0$. Thus we can find a null set N_n such that $\Delta\Gamma_{\nu_n(\omega)}(\omega) = 0$, $\forall \omega \in N_n^c$. Eventually, for any $\omega \in \left\{N \cup \left(\cup_{n\in\mathbb{N}} N_n\right)\right\}^c$, we have

$$\xi^1(\omega) = \Gamma_T^{\xi_1,\hat{g}}(\omega) = \lim_{n\to\infty} \Gamma_{\nu_n(\omega)}^{\xi_1,\hat{g}}(\omega) = \lim_{n\to\infty} \Gamma_{\nu_n(\omega)}^{\xi_2,\hat{g}}(\omega) = \Gamma_T^{\xi_2,\hat{g}}(\omega) = \xi^2(\omega). \quad \square$$

Proof of Proposition 5.5. Let $\{A_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathcal{F}_T$ such that $\lim_{n\to\infty} \downarrow \mathbf{1}_{A_n} = 0$, a.s. For any $\xi, \eta \in L^{e,+}(\mathcal{F}_T) \triangleq \{\xi \in L^e(\mathcal{F}_T) : \xi \ge 0, \text{ a.s.}\}$, since $E\left[e^{\lambda|\xi|}\right] < \infty$ and since $\sup_{n\in\mathbb{N}} E\left[e^{\lambda|\xi + \mathbf{1}_{A_n}\eta|}\right] \le E\left[e^{\lambda|\xi|}e^{\lambda|\eta|}\right] \le \frac{1}{2}E\left[e^{2\lambda|\xi|}\right] + \frac{1}{2}E\left[e^{2\lambda|\eta|}\right] < \infty$ holds for each $\lambda > 0$, Lemma 5.2 implies that

$$0 = \lim_{n\to\infty} E\left[\sup_{t\in[0,T]}\left|\mathcal{E}_{\hat{g}}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] - \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_t]\right|\right] \ge \lim_{n\to\infty}\left|\mathcal{E}_{\hat{g}}[\xi + \mathbf{1}_{A_n}\eta] - \mathcal{E}_{\hat{g}}[\xi]\right| \ge 0,$$

and thus $\mathcal{E}_{\hat{g}}$ satisfies (H2). Similarly, we can show that (H1) also holds for $\mathcal{E}_{\hat{g}}$.

Moreover, for any $\nu \in \mathcal{S}_{0,T}$ and $\xi \in L^{e,+}(\mathcal{F}_T)$, since the process $\Gamma^{\xi,\hat{g}}$ belongs to $\mathbb{C}^e_{\mathbf{F}}([0,T])$, one can deduce that $\mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_\nu] = \Gamma_\nu^{\xi,\hat{g}} \in L^{e,+}(\mathcal{F}_T)$. Then the continuity of the process $X^\xi \triangleq \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_\cdot]$ implies that $X_\nu^{\xi,+} = X_\nu^\xi = \mathcal{E}_{\hat{g}}[\xi|\mathcal{F}_\nu] \in L^{e,+}(\mathcal{F}_T)$, which proves (H3). $\square$

Proof of Theorem 5.3. This proof is just an application of the optimal stopping theory developed in Section 2 to the singleton $\{\mathcal{E}_{\hat{g}}\}$. Hence, it suffices to check that Y satisfies (Y1), (Y2) and (2.21).

Like for (6.61), it holds except on a null set N that

$$C_Y \le Y_t \le \zeta'_Y, \quad \forall t \in [0,T], \quad \text{thus} \quad C_Y \le Y_\rho \le \zeta'_Y, \quad \forall \rho \in \mathcal{S}_{0,T}. \tag{6.77}$$

Since $\zeta'_Y \in L^e(\mathcal{F}_T)$, it holds for any $\rho \in \mathcal{S}_{0,T}$ that

$$E\Big[e^{\lambda|Y_\rho|}\Big] \le E\Big[e^{\lambda(\zeta'_Y - C_Y)}\Big] = e^{-\lambda C_Y} E\Big[e^{\lambda \zeta'_Y}\Big] < \infty, \quad \forall \lambda > 0, \tag{6.78}$$

which implies that $Y_\rho \in L^{e,\#}(\mathcal{F}_T) = \mathrm{Dom}\big(\{\mathcal{E}_{\hat{g}}\}\big)$. Hence (Y1) holds.

Next, for any $\rho \in \mathcal{S}_{0,T}$ and $t \in \mathcal{D}_T$, Proposition 2.2(2) of [1], (6.77) show that

$$C_Y = \mathcal{E}_{\hat{g}}[C_Y|\mathcal{F}_t] \le \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_t] \le \mathcal{E}_{\hat{g}}[\zeta'_Y|\mathcal{F}_t] = \Gamma_t^{\zeta'_Y,\hat{g}} \le \sup_{t\in[0,T]} \left|\Gamma_t^{\zeta'_Y,\hat{g}}\right|, \quad \text{a.s.}$$

Taking the essential supremum of $\mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_t]$ over $(\rho,t) \in \mathcal{S}_{0,T} \times \mathcal{D}_T$ yields that

$$C_Y \le \operatorname*{esssup}_{(\rho,t)\in\mathcal{S}_{0,T}\times\mathcal{D}_T} \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_t] \le \sup_{t\in[0,T]} \left|\Gamma_t^{\zeta'_Y,\hat{g}}\right|, \quad \text{a.s.}$$

Since $\Gamma^{\zeta'_Y,\hat{g}} \in \mathbb{C}^e_{\mathbf{F}}([0,T])$, or equivalently $\sup_{t\in[0,T]}\left|\Gamma_t^{\zeta'_Y,\hat{g}}\right| \in L^e(\mathcal{F}_T)$, we can deduce that $\operatorname{esssup}_{(\rho,t)\in\mathcal{S}_{0,T}\times\mathcal{D}_T} \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_t] \in L^{e,\#}(\mathcal{F}_T) = \mathrm{Dom}\big(\{\mathcal{E}_{\hat{g}}\}\big)$, which together with Remark 2.2(2) proves (Y2).

Moreover, for any $\nu, \rho \in \mathcal{S}_{0,T}$ with $\nu \le \rho$, a.s. and any sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_{\nu,T}$ increasing a.s. to ρ, the continuity of the process Y implies that $\frac{n}{n-1}Y_{\rho_n}$ converges to Y_ρ a.s. By (6.77), one can deduce that

$$\begin{aligned}\sup_{n\in\mathbb{N}} E\left[\exp\left\{\lambda\left|\frac{n}{n-1}Y_{\rho_n}\right|\right\}\right] &\le \sup_{n\in\mathbb{N}} E\Big[e^{2\lambda|Y_{\rho_n}|}\Big] \le E\Big[e^{2\lambda(\zeta'_Y - C_Y)}\Big] \\ &= e^{-2\lambda C_Y} E\Big[e^{2\lambda\zeta'_Y}\Big] < \infty, \quad \forall \lambda > 0,\end{aligned}$$

which together with (6.78) allows us to apply Lemma 5.2:

$$\begin{aligned}0 &= \lim_{n\to\infty} E\left[\sup_{t\in[0,T]}\left|\mathcal{E}_{\hat{g}}\left[\frac{n}{n-1}Y_{\rho_n}|\mathcal{F}_t\right] - \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_t]\right|\right] \\ &\ge \lim_{n\to\infty} E\left[\left|\mathcal{E}_{\hat{g}}\left[\frac{n}{n-1}Y_{\rho_n}|\mathcal{F}_\nu\right] - \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_\nu]\right|\right] \ge 0.\end{aligned}$$

Hence, $\lim_{n\to\infty} E\left[\left|\mathcal{E}_{\hat{g}}\left[\frac{n}{n-1}Y_{\rho_n}|\mathcal{F}_\nu\right] - \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_\nu]\right|\right] = 0$. Then we can find a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\lim_{n\to\infty}\left|\mathcal{E}_{\hat{g}}\left[\frac{n_k}{n_k-1}Y_{\rho_{n_k}}|\mathcal{F}_\nu\right] - \mathcal{E}_{\hat{g}}[Y_\rho|\mathcal{F}_\nu]\right| = 0, \quad \text{a.s., proving (2.21) for} Y. \quad \square$$

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