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Optimal stopping for non-linear expectations—Part I☆

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Abstract

We develop a theory for solving continuous time optimal stopping problems for non-linear expectations. Our motivation is to consider problems in which the stopper uses risk measures to evaluate future rewards. Our development is presented in two parts. In the first part, we will develop the stochastic analysis tools that will be essential in solving the optimal stopping problems, which will be presented in Bayraktar and Yao (2011) [1].
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1. Introduction

We solve continuous time optimal stopping problems in which the reward is evaluated using *non-linear* expectations. Our purpose is to use criteria other than the expected value to evaluate the present value of future rewards. Such criteria include *risk measures*, which are not necessarily linear. Given a filtered probability space $(\Omega, \mathcal{F}, P, \mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]})$ satisfying the *usual assumptions*, we define a filtration-consistent non-linear expectation ($\mathbf{F}$-expectation for short) with domain Λ as a collection of operators $\big\{\mathcal{E}[\cdot|\mathcal{F}_t] : \Lambda \mapsto \Lambda_t \triangleq \Lambda \cap L^0(\mathcal{F}_t)\big\}_{t\in[0,T]}$ satisfying "Monotonicity", "Time-Consistency", "Zero–one Law" and "Translation-Invariance". This definition is similar to the one proposed in [15]. A notable example of an $\mathbf{F}$-expectation is the so-called g-*expectation*, introduced by [14]. A fairly large class of *convex risk measures* (see e.g. [6] for the definition of risk measures) are g-expectations (see [4,15,12,7]).

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We consider two optimal stopping problems. In the first one, the stopper aims to find an optimal stopping time when there are multiple priors and the *Nature* is in cooperation with the stopper; i.e., the stopper finds an optimal stopping time that attains

$$Z(0) \triangleq \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho + H^i_\rho|\mathcal{F}_0], \tag{1.1}$$

in which $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ is a *stable* class of $\mathbf{F}$-expectations, $\mathcal{S}_{0,T}$ is the set of stopping times that take value in $[0, T]$. The reward process Y is a right-continuous $\mathbf{F}$-adapted process and for any $\nu \in \mathcal{S}_{0,T}$, Y_ν belongs to $\Lambda^\# \triangleq \{\xi \in \Lambda \mid \xi \geq c, \text{ a.s. for some } c \in \mathbb{R}\}$, where Λ is the common domain of the elements in $\mathscr{E}$. On the other hand, the *model-dependent* reward processes $\{H^i\}_{i\in\mathbb{N}}$ is a family of right-continuous adapted processes with $H^i_0 = 0$ that is *consistent* with $\mathscr{E}$. We will express the solution of this problem in terms of the *$\mathscr{E}$-upper Snell envelope* Z^0 of Y_t, the smallest RCLL $\mathbf{F}$-adapted process dominating Y such that $Z^{i,0} \triangleq \{Z^0_t + H^i_t\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}_i$-supermartingale for each $i \in \mathcal{I}$.

The construction of the Snell envelope is not straightforward. First, for any $i \in \mathcal{I}$, the conditional expectation $\mathcal{E}_i[\xi|\mathcal{F}_\nu]$, $\xi \in \Lambda$ and $\nu \in \mathcal{S}_{0,T}$ may not be well defined. However, we show that $t \to \mathcal{E}_i[\xi|\mathcal{F}_t]$ admits a right-continuous modification $t \to \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_\cdot]$ for any $\xi \in \Lambda$ and that $\widetilde{\mathcal{E}}_i$ is itself an $\mathbf{F}$-expectation on $\Lambda^\#$ such that $\widetilde{\mathcal{E}}_i[\cdot|\mathcal{F}_\nu]$ is well defined on $\Lambda^\#$ for any $\nu \in \mathcal{S}_{0,T}$. In terms of $\widetilde{\mathcal{E}}_i$ we have that

$$Z(0) = \sup_{(i,\rho)\in\mathcal{I}\times\mathcal{S}_{0,T}} \widetilde{\mathcal{E}}_i[Y_\rho + H^i_\rho|\mathcal{F}_0]. \tag{1.2}$$

Finding a RCLL modification requires the development of an upcrossing theorem. This theorem relies on the strict monotonicity of $\mathcal{E}_i$ and other mild hypotheses, one of which is equivalent to having lower semi-continuity (i.e. Fatou's lemma). Thanks to the right continuity of $t \to \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_t]$, we also have an optional sampling theorem for right-continuous $\widetilde{\mathcal{E}}_i$-supermartingales. Another important tool in finding an optimal stopping time, the dominated convergence theorem is also developed under another mild assumption. These developments are presented in Section 2.

The stability assumption we make on the family $\mathscr{E}$ is another essential ingredient in the construction of the Snell envelope. It guarantees that the class $\mathscr{E}$ is closed under *pasting*: for any $i, j \in \mathcal{I}$ and $\nu \in \mathcal{S}_{0,T}$ there exists a $k \in \mathcal{I}$ such that $\widetilde{\mathcal{E}}_k[\xi|\mathcal{F}_\sigma] = \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee\sigma}]|\mathcal{F}_\sigma\big]$, for any $\sigma \in \mathcal{S}_{0,T}$. Under this assumption it can then be seen, for example, that the collection of random variables $\big\{\widetilde{\mathcal{E}}_i\big[X(\rho) + H^i_\rho - H^i_\nu|\mathcal{F}_\nu\big],\ (i,\rho) \in \mathcal{I}\times\mathcal{S}_{\nu,T}\big\}$ is directed upwards. When the constituents of $\mathscr{E}$ are linear expectations, the notion of stability of this collection is given by [6, Definition 6.44], who showed that pasting two probability measures equivalent to P at a stopping time one will result in another probability measure equivalent to P. Our result in Proposition 3.1 shows that we have the same pasting property for $\mathbf{F}$-expectations. As we shall see, the stability assumption is crucial in showing that the Snell envelope is a supermartingale. This property of the Snell envelope is a generalization of *time consistency*, i.e.,

$$\begin{aligned}&\operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_\nu] = \operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i\left[\operatorname*{esssup}_{i\in\mathcal{I}} \widetilde{\mathcal{E}}_i[\xi|\mathcal{F}_\sigma]\middle|\mathcal{F}_\nu\right],\\ &\quad \text{a.s.}, \forall\, \nu,\sigma \in \mathcal{S}_{0,T} \text{ with } \nu \leq \sigma, \text{ a.s.}\end{aligned} \tag{1.3}$$

[5, Theorem 12] showed in the linear expectations case that the time consistency (1.3) is equivalent to the stability.

When the reward $t \to Y_t + H^i_t$ is "$\mathscr{E}$-uniformly-left-continuous" and each non-linear expectation in $\mathscr{E}$ is convex, we can find an optimal stopping time $\overline{\tau}(0)$ for (1.1) in terms of the Snell envelope. Then we can solve the problem

$$\sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}_i[Y_\rho + H^i_\rho|\mathcal{F}_0], \tag{1.4}$$

when $\mathcal{E}_i[\cdot|\mathcal{F}_t]$ has among other properties strict monotonicity, lower semi-continuity, dominated convergence theorem and the upcrossing lemma. Note that although, $\text{esssup}_{i\in\mathcal{I}}\widetilde{\mathcal{E}}_i[\cdot|\mathcal{F}_t]$ has similar properties to $\widetilde{\mathcal{E}}_i[\cdot|\mathcal{F}_t]$ (and that might lead one to think that (1.1) can actually be considered as a special case of (1.4)), the former does not satisfy strict monotonicity, the upcrossing lemma, and the dominated convergence theorem. One motivation for considering optimal stopping with multiple priors is to solve optimal stopping problems for "non-linear expectations" which do not satisfy these properties.

We show that the collection of g-expectations with uniformly Lipschitz generators satisfy the uniform left continuity assumption. Moreover, a g-expectation satisfies all the assumptions we ask of each $\mathcal{E}_i$ for the upcrossing theorem, Fatou's lemma and the dominated convergence theorem to hold; and pasting of g-expectations results in another g-expectation. As a result the case of g-expectations presents a non-conventional example in which we can determine an optimal stopping time for (1.1). In fact, in the g-expectation example we can even find an optimal *prior* $i^* \in \mathcal{I}$, i.e.,

$$Z(0) = \mathcal{E}_{i^*}[Y_{\overline{\tau}(0)} + H^{i^*}_{\overline{\tau}(0)}|\mathcal{F}_0]. \tag{1.5}$$

In the second problem, the *stopper* tries to find a robust optimal stopping time that attains

$$V(0) \triangleq \sup_{\rho\in\mathcal{S}_{0,T}} \inf_{i\in\mathcal{I}} \mathcal{E}_i\big[Y_\rho + H^i_\rho|\mathcal{F}_0\big]. \tag{1.6}$$

Under the "$\mathscr{E}$-uniform-right-continuity" assumption, we find an optimal stopping time in terms of the *$\mathscr{E}$-lower Snell envelope*. An immediate by-product is the following minimax theorem

$$V(0) = \inf_{i\in\mathcal{I}} \sup_{\rho\in\mathcal{S}_{0,T}} \mathcal{E}_i\big[Y_\rho + H^i_\rho|\mathcal{F}_0\big]. \tag{1.7}$$

Our work was inspired by [10,11], who developed a martingale approach to solving (1.1) and (1.6), when $\mathscr{E}$ is a class of linear expectations. In particular, [10] considered the *controller-stopper* problem

$$\sup_{\rho\in\mathcal{S}_{0,T}} \sup_{U\in\mathfrak{U}} \mathbf{E}^u\left[g\big(X(\rho)\big) + \int_0^\rho h(s,X,U_s)\mathrm{d}s\right], \tag{1.8}$$

where $X(t) = x + \int_0^t f(s,X,U_s)\mathrm{d}s + \int_0^t \sigma(s,X)\mathrm{d}W^U_s$. In this problem, the stability condition is automatically satisfied. Here, g and h are assumed to be bounded measurable functions. Our results on g-expectations extend the results of [10] from bounded rewards to rewards satisfying linear growth. [5,9] also considered (1.1) when the $\mathcal{E}_i$'s are linear expectations. The latter paper made a *convexity* assumption on the collection of equivalent probability measures instead of a stability assumption. On the other hand, the discrete time version of the robust optimization problem was analyzed by [6]. Also see [3, Sections 5.2 and 5.3].

The rest of Part I is organized as follows: In Section 1.1 we will introduce some notations that will be used in both Parts I and II. In Section 2, we define what we mean by an $\mathbf{F}$-expectation

$\mathcal{E}$, propose some basic hypotheses on $\mathcal{E}$ and discuss their implications such as Fatou's lemma, dominated convergence theorem and upcrossing lemma. We show that $t \to \mathcal{E}[\cdot|\mathcal{F}_t]$ admits a right-continuous modification which is also an **F**-expectation and satisfies Fatou's lemma and the dominated convergence theorem. This step is essential since $\mathcal{E}[\cdot|\mathcal{F}_\nu],\ \nu \in \mathcal{S}_{0,T}$ may not be well defined. We also show that the optional sampling theorem holds. The results in Section 2 will be the backbone of our analysis in Part II. In Section 3 we introduce the stable class of **F**-expectations and review the properties of essential extremum. The proofs of our results are presented in Section 4.

The optimal stopping problems (1.2) and (1.6) and their applications will be deferred to Part II.

1.1. Notation

Throughout this paper, we fix a finite time horizon $T > 0$ and consider a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a right continuous filtration $\mathbf{F} \triangleq \{\mathcal{F}_t\}_{t\in[0,T]}$, not necessarily a Brownian one, such that $\mathcal{F}_0$ is generated by all P-null sets in $\mathcal{F}$ (in fact, $\mathcal{F}_0$ collects all measurable sets with probability 0 or 1). Let $\mathcal{S}_{0,T}$ be the collection of all **F**-stopping times ν such that $0 \le \nu \le T$, a.s. For any $\nu, \sigma \in \mathcal{S}_{0,T}$ with $\nu \le \sigma$, a.s., we define $\mathcal{S}_{\nu,\sigma} \triangleq \{\rho \in \mathcal{S}_{0,T} \mid \nu \le \rho \le \sigma,\ \text{a.s.}\}$ and let $\mathcal{S}^F_{\nu,\sigma}$ denote the set of all finite-valued stopping times in $\mathcal{S}_{\nu,\sigma}$. We let $\mathcal{D} = \{k2^{-n} \mid k \in \mathbb{Z},\ n \in \mathbb{N}\}$ denote the set of all dyadic rational numbers and set $\mathcal{D}_T \triangleq \big([0,T) \cap \mathcal{D}\big) \cup \{T\}$. For any $t \in [0,T]$ and $n \in \mathbb{N}$, we also define

$$q_n^-(t) \triangleq \left(\frac{\lfloor 2^n t\rfloor - 1}{2^n}\right)^+ \quad \text{and} \quad q_n^+(t) \triangleq \frac{\lceil 2^n t\rceil}{2^n} \wedge T. \tag{1.9}$$

It is clear that $q_n^-(t), q_n^+(t) \in \mathcal{D}_T$.

In what follows we let $\mathcal{F}'$ be a generic sub-σ-field of $\mathcal{F}$ and let $\mathbb{B}$ be a generic Banach space with norm $|\cdot|_{\mathbb{B}}$. The following spaces of functions will be used in the sequel.

(1) For $0 \le p \le \infty$, we define
- $L^p(\mathcal{F}';\mathbb{B})$ to be the space of all $\mathbb{B}$-valued, $\mathcal{F}'$-measurable random variables ξ such that $E(|\xi|^p_{\mathbb{B}}) < \infty$. In particular, if $p = 0$, $L^0(\mathcal{F}';\mathbb{B})$ stands for the space of all $\mathbb{B}$-valued, $\mathcal{F}'$-measurable random variables; and if $p = \infty$, $L^\infty(\mathcal{F}';\mathbb{B})$ denotes the space of all $\mathbb{B}$-valued, $\mathcal{F}'$-measurable random variables ξ with $\|\xi\|_\infty \triangleq \operatorname{esssup}_{\omega\in\Omega}|\xi(\omega)|_{\mathbb{B}} < \infty$.
- $L^p_{\mathbf{F}}([0,T];\mathbb{B})$ to be the space of all $\mathbb{B}$-valued, **F**-adapted processes X such that $E\int_0^T |X_t|^p_{\mathbb{B}}\mathrm{d}t < \infty$. In particular, if $p = 0$, $L^0_{\mathbf{F}}([0,T];\mathbb{B})$ stands for the space of all $\mathbb{B}$-valued, **F**-adapted processes; and if $p = \infty$, $L^\infty_{\mathbf{F}}([0,T];\mathbb{B})$ denotes the space of all $\mathbb{B}$-valued, **F**-adapted processes X with $\|X\|_\infty \triangleq \operatorname{esssup}_{(t,\omega)\in[0,T]\times\Omega}|X_t(\omega)|_{\mathbb{B}} < \infty$.
- $\mathbb{C}^p_{\mathbf{F}}([0,T];\mathbb{B}) \triangleq \{X \in L^p_{\mathbf{F}}([0,T];\mathbb{B}) :\ X \text{ has continuous paths}\}$.
- $\mathcal{H}^p_{\mathbf{F}}([0,T];\mathbb{B}) \triangleq \{X \in L^p_{\mathbf{F}}([0,T];\mathbb{B}) : X \text{ is predictably measurable}\}$.

(2) For $p \ge 1$, we define a Banach space

$$M^p_{\mathbf{F}}([0,T];\mathbb{B}) = \left\{X \in \mathcal{H}^0_{\mathbf{F}}([0,T];\mathbb{B}) :\ \|X\|_{M^p} \triangleq \left\{E\left[\left(\int_0^T |X_s|^2_{\mathbb{B}}\mathrm{d}s\right)^{p/2}\right]\right\}^{1/p} < \infty\right\},$$

and denote $M_{\mathbf{F}}([0,T];\mathbb{B}) \triangleq \cap_{p\ge1} M^p_{\mathbf{F}}([0,T];\mathbb{B})$.

(3) We further define

$$L^e(\mathcal{F}';\mathbb{B}) \triangleq \Big\{\xi \in L^0(\mathcal{F}';\mathbb{B}) : E\big[e^{\lambda|\xi|_{\mathbb{B}}}\big] < \infty \text{ for all } \lambda > 0\Big\},$$

$$\mathbb{C}^e_{\mathbf{F}}([0,T];\mathbb{B}) \triangleq \left\{X \in \mathbb{C}^0_{\mathbf{F}}([0,T];\mathbb{B}) : E\left[\exp\left\{\lambda \sup_{t\in[0,T]} |X_t|_{\mathbb{B}}\right\}\right] < \infty \right.$$
$$\left. \text{for all } \lambda > 0\right\}.$$

If $d = 1$, we shall drop $\mathbb{B} = \mathbb{R}$ from the above notations (e.g., $L^p_{\mathbf{F}}([0,T]) = L^p_{\mathbf{F}}([0,T];\mathbb{R})$, $L^p(\mathcal{F}_T) = L^p(\mathcal{F}_T;\mathbb{R})$). In this paper, all $\mathbf{F}$-adapted processes are supposed to be real-valued unless specified otherwise.

2. F-expectations and their properties

We will define non-linear expectations on subspaces of $L^0(\mathcal{F}_T)$ satisfying certain algebraic properties, which are listed in the definition below.

Definition 2.1. Let $\mathscr{D}_T$ denote the collection of all non-empty subsets Λ of $L^0(\mathcal{F}_T)$ satisfying:

(D1) $0, 1 \in \Lambda$;
(D2) Λ is closed under addition and under multiplication with indicator random variables. Namely, for any $\xi, \eta \in \Lambda$ and $A \in \mathcal{F}_T$, both $\xi + \eta$ and $\mathbf{1}_A\xi$ belong to Λ;
(D3) Λ is positively solid: For any $\xi, \eta \in L^0(\mathcal{F}_T)$ with $0 \le \xi \le \eta$, a.s., if $\eta \in \Lambda$, then $\xi \in \Lambda$ as well.

Remark 2.1. (1) Each $\Lambda \in \mathscr{D}_T$ is also closed under maximization "$\vee$" and under minimization "$\wedge$": In fact, for any $\xi, \eta \in \Lambda$, since the set $\{\xi > \eta\} \in \mathcal{F}_T$, (D2) implies that $\xi \vee \eta = \xi\mathbf{1}_{\{\xi>\eta\}} + \eta\mathbf{1}_{\{\xi\le\eta\}} \in \Lambda$. Similarly, $\xi \wedge \eta \in \Lambda$;
(2) For each $\Lambda \in \mathscr{D}_T$, (D1)–(D3) imply that $c \in \Lambda$ for any $c \ge 0$;
(3) $\mathscr{D}_T$ is closed under intersections: If $\{\Lambda_i\}_{i\in\mathcal{I}}$ is a subset of $\mathscr{D}_T$, then $\cap_{i\in I}\Lambda_i \in \mathscr{D}_T$; $\mathscr{D}_T$ is closed under unions of increasing sequences: If $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathscr{D}_T$ such that $\Lambda_n \subset \Lambda_{n+1}$ for any $n \in \mathbb{N}$, then $\cup_{n\in\mathbb{N}}\Lambda_n \in \mathscr{D}_T$;
(4) It is clear that $L^p(\mathcal{F}_T) \in \mathscr{D}_T$ for all $0 \le p \le \infty$.

Definition 2.2. An $\mathbf{F}$-consistent non-linear expectation ($\mathbf{F}$-expectation for short) is a pair $(\mathcal{E}, \Lambda)$ in which $\Lambda \in \mathscr{D}_T$ and $\mathcal{E}$ denotes a family of operators $\big\{\mathcal{E}[\cdot|\mathcal{F}_t] : \Lambda \mapsto \Lambda_t \triangleq \Lambda \cap L^0(\mathcal{F}_t)\big\}_{t\in[0,T]}$ satisfying the following hypothesis for any $\xi, \eta \in \Lambda$ and $t \in [0,T]$:

(A1) "*Monotonicity (positively strict)*": $\mathcal{E}[\xi|\mathcal{F}_t] \le \mathcal{E}[\eta|\mathcal{F}_t]$, a.s. if $\xi \le \eta$, a.s.; Moreover, if $0 \le \xi \le \eta$ a.s. and $\mathcal{E}[\xi|\mathcal{F}_0] = \mathcal{E}[\eta|\mathcal{F}_0]$, then $\xi = \eta$, a.s.;
(A2) "*Time Consistency*": $\mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] = \mathcal{E}[\xi|\mathcal{F}_s]$, a.s. for any $0 \le s \le t \le T$;
(A3) "*Zero–one Law*": $\mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_t]$, a.s. for any $A \in \mathcal{F}_t$;
(A4) "*Translation Invariance*": $\mathcal{E}[\xi + \eta|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t] + \eta$, a.s. if $\eta \in \Lambda_t$.

We denote the domain Λ by $\mathrm{Dom}(\mathcal{E})$ and define

$$\mathrm{Dom}_\nu(\mathcal{E}) \triangleq \mathrm{Dom}(\mathcal{E}) \cap L^0(\mathcal{F}_\nu), \quad \forall\, \nu \in \mathcal{S}_{0,T}.$$

For any $\xi, \eta \in \mathrm{Dom}(\mathcal{E})$ with $\xi = \eta$, a.s., (A1) implies that $\mathcal{E}[\xi|\mathcal{F}_t] = \mathcal{E}[\eta|\mathcal{F}_t]$, a.s. for any $t \in [0,T]$, which shows that the $\mathbf{F}$-expectation $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is well-defined. Moreover, since

$\mathrm{Dom}_0(\mathcal{E}) = \mathrm{Dom}(\mathcal{E}) \cap L^0(\mathcal{F}_0) \subset L^0(\mathcal{F}_0) = \mathbb{R}$, $\mathcal{E}[\cdot|\mathcal{F}_0]$ is a real-valued function on $\mathrm{Dom}(\mathcal{E})$. In the rest of the paper, we will substitute $\mathcal{E}[\cdot]$ for $\mathcal{E}[\cdot|\mathcal{F}_0]$.

Remark 2.2. Our definition of **F**-expectations is similar to that of $\mathcal{F}_t^X$-consistent non-linear expectations introduced in [15, page 4].

Example 2.1. The following pairs satisfy (A1)–(A4); thus they are **F**-expectations:

(1) $\big(\{E[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}, L^1(\mathcal{F}_T)\big)$: the linear expectation E is a special **F**-expectation with domain $L^1(\mathcal{F}_T)$;
(2) $\big(\{\mathcal{E}_g[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}, L^2(\mathcal{F}_T)\big)$: the g-expectation with generator $g(t,z)$ Lipschitz in z (see [14,4] or Section 5.1 of [1]);
(3) $\big(\{\mathcal{E}_g[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}, L^e(\mathcal{F}_T)\big)$: the g-expectation with generator $g(t,z)$ having quadratic growth in z (see Section 5.4 of [1]).

F-expectations can alternatively be introduced in a more classical way:

Proposition 2.1. *Let $\mathcal{E}^o : \Lambda \mapsto \mathbb{R}$ be a mapping on some $\Lambda \in \mathscr{D}_T$ satisfying:*

(a1) *For any $\xi, \eta \in \Lambda$ with $\xi \le \eta$, a.s., we have $\mathcal{E}^o[\xi] \le \mathcal{E}^o[\eta]$. Moreover, if $\mathcal{E}^o[\xi] = \mathcal{E}^o[\eta]$, then $\xi = \eta$, a.s.;*
(a2) *For any $\xi \in \Lambda$ and $t \in [0,T]$, there exists a unique random variable $\xi_t \in \Lambda_t$ such that $\mathcal{E}^o[\mathbf{1}_A\xi + \gamma] = \mathcal{E}^o\big[\mathbf{1}_A\xi_t + \gamma\big]$ holds for any $A \in \mathcal{F}_t$ and $\gamma \in \Lambda_t$.*

Then $\{\mathcal{E}^o[\xi|\mathcal{F}_t] \triangleq \xi_t,\ \xi \in \Lambda\}_{t\in[0,T]}$ defines an **F***-expectation with domain Λ.*

Remark 2.3. For a mapping $\mathcal{E}^o$ on some $\Lambda \in \mathscr{D}_T$ satisfying (a1) and (a2), the implied operator $\mathcal{E}^o[\cdot|\mathcal{F}_0]$ is also from Λ to $\mathbb{R}$, which, however, may not be equal to $\mathcal{E}^o$. In fact, one can only deduce that $\mathcal{E}^o[\xi] = \mathcal{E}^o\big[\mathcal{E}^o[\xi|\mathcal{F}_0]\big]$ for any $\xi \in \Lambda$.

From now on, when we say an **F**-expectation $\mathcal{E}$, we will refer to the pair $\big(\mathcal{E}, \mathrm{Dom}(\mathcal{E})\big)$. Besides (A1)–(A4), the **F**-expectation $\mathcal{E}$ has the following properties:

Proposition 2.2. *For any $\xi, \eta \in \mathrm{Dom}(\mathcal{E})$ and $t \in [0,T]$, we have*

(1) *"Local Property": $\mathcal{E}[\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_t] + \mathbf{1}_{A^c}\mathcal{E}[\eta|\mathcal{F}_t]$, a.s. for any $A \in \mathcal{F}_t$;*
(2) *"Constant-Preserving": $\mathcal{E}[\xi|\mathcal{F}_t] = \xi$, a.s. if $\xi \in \mathrm{Dom}_t(\mathcal{E})$;*
(3) *"Comparison": Let $\xi, \eta \in L^0(\mathcal{F}_\nu)$ for some $\nu \in \mathcal{S}_{0,T}$. If $\eta \ge c$, a.s. for some $c \in \mathbb{R}$, then $\xi \le (\text{or} =)\eta$, a.s. if and only if $\mathcal{E}[\mathbf{1}_A\xi] \le (\text{or} =)\mathcal{E}[\mathbf{1}_A\eta]$ for all $A \in \mathcal{F}_\nu$.*

The following two subsets of $\mathrm{Dom}(\mathcal{E})$ will be of interest:

$$\begin{aligned}
&\mathrm{Dom}^+(\mathcal{E}) \triangleq \{\xi \in \mathrm{Dom}(\mathcal{E}) : \xi \ge 0,\ \text{a.s.}\},\\
&\mathrm{Dom}^\#(\mathcal{E}) \triangleq \{\xi \in \mathrm{Dom}(\mathcal{E}) : \xi \ge c,\ \text{a.s. for some } c = c(\xi) \in \mathbb{R}\}.
\end{aligned} \tag{2.1}$$

Remark 2.4. The restrictions of $\mathcal{E}$ on $\mathrm{Dom}^+(\mathcal{E})$ and on $\mathrm{Dom}^\#(\mathcal{E})$, namely $\big(\mathcal{E}, \mathrm{Dom}^+(\mathcal{E})\big)$ and $\big(\mathcal{E}, \mathrm{Dom}^\#(\mathcal{E})\big)$ respectively, are both **F**-expectations: To see this, first note that $\mathrm{Dom}^+(\mathcal{E})$ and $\mathrm{Dom}^\#(\mathcal{E})$ both belong to $\mathscr{D}_T$. For any $t \in [0,T]$, (A1) and Proposition 2.2(2) imply that for any $\xi \in \mathrm{Dom}^\#(\mathcal{E})$

$$\mathcal{E}[\xi|\mathcal{F}_t] \ge \mathcal{E}\big[c(\xi)|\mathcal{F}_t\big] = c(\xi), \quad \text{a.s., thus } \mathcal{E}[\xi|\mathcal{F}_t] \in \mathrm{Dom}^\#(\mathcal{E}),$$

which shows that $\mathcal{E}[\cdot|\mathcal{F}_t]$ maps $\mathrm{Dom}^\#(\mathcal{E})$ into $\mathrm{Dom}^\#(\mathcal{E})\cap L^0(\mathcal{F}_t)$. Then it is easy to check that the restriction of $\mathcal{E} = \big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ on $\mathrm{Dom}^\#(\mathcal{E})$ satisfies (A1)–(A4), thus it is an **F**-expectation. Similarly, $\big(\mathcal{E}, \mathrm{Dom}^+(\mathcal{E})\big)$ is also an **F**-expectation.

We should remark that restricting $\mathcal{E}$ on any subset Λ' of $\mathrm{Dom}(\mathcal{E})$, with $\Lambda' \in \mathscr{D}_T$, may not result in an **F**-expectation, i.e. $(\mathcal{E}, \Lambda')$ may not be an **F**-expectation.

Definition 2.3. (1) An **F**-adapted process $X = \{X_t\}_{t\in[0,T]}$ is called an "$\mathcal{E}$-process" if $X_t \in \mathrm{Dom}(\mathcal{E}), \forall\, t \in [0, T]$;
(2) An $\mathcal{E}$-process X is said to be an $\mathcal{E}$-supermartingale (resp. $\mathcal{E}$-martingale, $\mathcal{E}$-submartingale) if for any $0 \le s < t \le T$, $\mathcal{E}[X_t|\mathcal{F}_s] \le$ (resp. $=, \ge$) X_s, a.s.

Given a $\nu \in \mathcal{S}^F_{0,T}$ taking values in a finite set $\{t_1 < \cdots < t_n\}$, if X is an $\mathcal{E}$-process, (D2) implies that $X_\nu = \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}} X_{t_i} \in \mathrm{Dom}(\mathcal{E})$, thus $X_\nu \in \mathrm{Dom}_\nu(\mathcal{E})$. Since $\big\{X^\xi_t \triangleq \mathcal{E}[\xi|\mathcal{F}_t]\big\}_{t\in[0,T]}$ is an $\mathcal{E}$-process for any $\xi \in \mathrm{Dom}(\mathcal{E})$, we can define an operator $\mathcal{E}[\cdot|\mathcal{F}_\nu]$ from $\mathrm{Dom}(\mathcal{E})$ to $\mathrm{Dom}_\nu(\mathcal{E})$ by

$$\mathcal{E}[\xi|\mathcal{F}_\nu] \triangleq X^\xi_\nu, \quad \text{for any } \xi \in \mathrm{Dom}(\mathcal{E}),$$

which allows us to state a basic Optional Sampling Theorem for $\mathcal{E}$.

Proposition 2.3 (*Optional Sampling Theorem*)**.** *Let X be an $\mathcal{E}$-supermartingale (resp. $\mathcal{E}$-martingale, $\mathcal{E}$-submartingale). Then for any $\nu, \sigma \in \mathcal{S}^F_{0,T}$, $\mathcal{E}[X_\nu|\mathcal{F}_\sigma] \le$ (resp. $=, \ge$)$X_{\nu\wedge\sigma}$, a.s.*

In particular, applying Proposition 2.3 to each $\mathcal{E}$-martingale $\{\mathcal{E}[\xi|\mathcal{F}_t]\}_{t\in[0,T]}$, in which $\xi \in \mathrm{Dom}(\mathcal{E})$, yields the following result.

Corollary 2.1. *For any $\xi \in \mathrm{Dom}(\mathcal{E})$ and $\nu, \sigma \in \mathcal{S}^F_{0,T}$, $\mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma\big] = \mathcal{E}[\xi|\mathcal{F}_{\nu\wedge\sigma}]$, a.s.*

Remark 2.5. Corollary 2.1 extends the "Time-Consistency" (A2) to the case of finite-valued stopping times.

$\mathcal{E}[\cdot|\mathcal{F}_\nu]$ inherits other properties of $\mathcal{E}[\cdot|\mathcal{F}_t]$ as well:

Proposition 2.4. *For any $\xi, \eta \in \mathrm{Dom}(\mathcal{E})$ and $\nu \in \mathcal{S}^F_{0,T}$, it holds that*
(1) *"Monotonicity (positively strict)": $\mathcal{E}[\xi|\mathcal{F}_\nu] \le \mathcal{E}[\eta|\mathcal{F}_\nu]$, a.s. if $\xi \le \eta$, a.s.; Moreover, if $0 \le \xi \le \eta$, a.s. and $\mathcal{E}[\xi|\mathcal{F}_\sigma] = \mathcal{E}[\eta|\mathcal{F}_\sigma]$, a.s. for some $\sigma \in \mathcal{S}^F_{0,T}$, then $\xi = \eta$, a.s.;*
(2) *"Zero–one Law": $\mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_\nu] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_\nu]$, a.s. for any $A \in \mathcal{F}_\nu$;*
(3) *"Translation Invariance": $\mathcal{E}[\xi + \eta|\mathcal{F}_\nu] = \mathcal{E}[\xi|\mathcal{F}_\nu] + \eta$, a.s. if $\eta \in \mathrm{Dom}_\nu(\mathcal{E})$;*
(4) *"Local Property": $\mathcal{E}[\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta|\mathcal{F}_\nu] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_\nu] + \mathbf{1}_{A^c}\mathcal{E}[\eta|\mathcal{F}_\nu]$, a.s. for any $A \in \mathcal{F}_\nu$;*
(5) *"Constant-Preserving": $\mathcal{E}[\xi|\mathcal{F}_\nu] = \xi$, a.s., if $\xi \in \mathrm{Dom}_\nu(\mathcal{E})$.*

We make the following basic hypotheses on the **F**-expectation $\mathcal{E}$. These hypotheses will be essential in developing Fatou's lemma, the Dominated Convergence Theorem and the Upcrossing Theorem.

Hypotheses.

(H0) For any $A \in \mathcal{F}_T$ with $P(A) > 0$, we have $\lim_{n\to\infty} \mathcal{E}[n\mathbf{1}_A] = \infty$;
(H1) For any $\xi \in \mathrm{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \uparrow \mathbf{1}_{A_n} = 1$, a.s., we have $\lim_{n\to\infty} \uparrow \mathcal{E}[\mathbf{1}_{A_n}\xi] = \mathcal{E}[\xi]$;

(H2) For any $\xi, \eta \in \mathrm{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \downarrow \mathbf{1}_{A_n} = 0$, a.s., we have $\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta] = \mathcal{E}[\xi]$.

Remark 2.6. The linear expectation E on $L^1(\mathcal{F}_T)$ clearly satisfies (H0)–(H2). We will show that Lipschitz and quadratic g-expectations also satisfy (H0)–(H2) in Propositions 4.1 and 5.5 of [1] respectively.

The **F**-expectation $\mathcal{E}$ satisfies the following Fatou's Lemma and the Dominated Convergence Theorem.

Theorem 2.1 (*Fatou's lemma*)**.** (H1) *is equivalent to the lower semi-continuity of $\mathcal{E}$: If a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}^+(\mathcal{E})$ converges a.s. to some $\xi \in \mathrm{Dom}^+(\mathcal{E})$, then for any $\nu \in \mathcal{S}^F_{0,T}$, we have*

$$\mathcal{E}[\xi|\mathcal{F}_\nu] \le \varliminf_{n\to\infty} \mathcal{E}[\xi_n|\mathcal{F}_\nu], \quad a.s., \tag{2.2}$$

where the right hand side of (2.2) could be equal to infinity with non-zero probability.

Remark 2.7. In the case of the linear expectation E, a converse to (2.2) holds: For any non-negative sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset L^1(\mathcal{F}_T)$ that converges a.s. to some $\xi \in L^0(\mathcal{F}_T)$, if $\varliminf_{n\to\infty} E[\xi_n] < \infty$, then $\xi \in L^1(\mathcal{F}_T)$. However, this statement may not be the case for an arbitrary **F**-expectation. That is, $\varliminf_{n\to\infty}\mathcal{E}[\xi_n] < \infty$ may not imply that $\xi \in \mathrm{Dom}^+(\mathcal{E})$ given that $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}^+(\mathcal{E})$ is a sequence convergent a.s. to some $\xi \in L^0(\mathcal{F}_T)$. (See Example 5.1 of [1] for a counterexample in the case of a Lipschitz g-expectation.)

Theorem 2.2 (*Dominated Convergence Theorem*)**.** *Assume* (H1) *and* (H2) *hold. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence in* $\mathrm{Dom}^+(\mathcal{E})$ *that converges a.s. If there is an $\eta \in \mathrm{Dom}^+(\mathcal{E})$ such that $\xi_n \le \eta$ a.s. for any $n \in \mathbb{N}$, then the limit ξ of $\{\xi_n\}_{n\in\mathbb{N}}$ belongs to* $\mathrm{Dom}^+(\mathcal{E})$*, and for any $\nu \in \mathcal{S}^F_{0,T}$, we have*

$$\lim_{n\to\infty} \mathcal{E}[\xi_n|\mathcal{F}_\nu] = \mathcal{E}[\xi|\mathcal{F}_\nu], \quad a.s.$$

Next, we will derive an Upcrossing Theorem for $\mathcal{E}$-supermartingales, which is crucial in obtaining an RCLL (right-continuous, with limits from the left) modification for the process $\{\mathcal{E}[\xi|\mathcal{F}_t]\}_{t\in[0,T]}$ as long as $\xi \in \mathrm{Dom}(\mathcal{E})$ is bounded from below. Obtaining a right continuous modification is crucial, since otherwise the conditional expectation $\mathcal{E}[\xi|\mathcal{F}_\nu]$ may not be well defined for any $\nu \in \mathcal{S}_{0,T}$.

Let us first recall what the "number of upcrossings" is: Given a real-valued process $\{X_t\}_{t\in[0,T]}$ and two real numbers $a < b$, for any finite subset F of $[0, T]$, we can define the "number of upcrossings" $U_F(a, b; X(\omega))$ of the interval $[a, b]$ by the sample path $\{X_t(\omega)\}_{t\in F}$ as follows: Set $\nu_0 = -1$, and for any $j = 1, 2, \ldots$ we recursively define

$$\nu_{2j-1}(\omega) \triangleq \min\{t \in F : t > \nu_{2j-2}(\omega), X_t(\omega) < a\} \wedge T \in \mathcal{S}^F_{0,T},$$
$$\nu_{2j}(\omega) \triangleq \min\{t \in F : t > \nu_{2j-1}(\omega), X_t(\omega) > b\} \wedge T \in \mathcal{S}^F_{0,T},$$

with the convention that $\min\emptyset = \infty$. Then $U_F(a, b; X(\omega))$ is defined to be the largest integer j for which $\nu_{2j}(\omega) < T$. If $I \subset [0, T]$ is not a finite set, we define

$$U_I(a, b; X(\omega)) \triangleq \sup\{U_F(a, b; X(\omega)) : F \text{ is a finite subset of } I\}.$$

It will be convenient to introduce a subcollection of $\mathscr{D}_T$

$$\widetilde{\mathscr{D}}_T \triangleq \{\Lambda \in \mathscr{D}_T : \mathbb{R} \subset \Lambda\}.$$

Clearly, $\widetilde{\mathscr{D}}_T$ contains all $L^p(\mathcal{F}_T), 0 \le p \le \infty$. In particular, $L^\infty(\mathcal{F}_T)$ is the smallest element of $\widetilde{\mathscr{D}}_T$ in the following sense:

Lemma 2.1. *For each $\Lambda \in \widetilde{\mathscr{D}}_T$, $L^\infty(\mathcal{F}_T) \subset \Lambda$.*

Proof. For any $\xi \in L^\infty(\mathcal{F}_T)$, we have $-\|\xi\|_\infty, 2\|\xi\|_\infty \in \mathbb{R} \subset \Lambda$. Since $0 \le \xi + \|\xi\|_\infty \le 2\|\xi\|_\infty$, a.s., (D3) implies that $\xi + \|\xi\|_\infty \in \Lambda$. Then we can deduce from (D2) that $\xi = (\xi + \|\xi\|_\infty) + (-\|\xi\|_\infty) \in \Lambda$. □

For any **F**-adapted process X, we define its left-limit and right-limit processes as follows:

$$X_t^- \triangleq \varliminf_{n\to\infty} X_{q_n^-(t)} \quad \text{and} \quad X_t^+ \triangleq \varliminf_{n\to\infty} X_{q_n^+(t)}, \quad \text{for any } t \in [0, T],$$

where $q_n^-(t)$ and $q_n^+(t)$ are defined in (1.9). Since the filtration **F** is right-continuous, we see that both X^- and X^+ are **F**-adapted processes.

It is now the time to present our Upcrossing Theorem for $\mathcal{E}$-supermartingales.

Theorem 2.3 (*Upcrossing Theorem*)**.** *Assume that* (H0), (H1) *hold and that* $\mathrm{Dom}(\mathcal{E}) \in \widetilde{\mathscr{D}}_T$. *For any $\mathcal{E}$-supermartingale X, we assume either that $X_T \ge c$, a.s. for some $c \in \mathbb{R}$ or that the operator $\mathcal{E}[\cdot]$ is concave: For any $\xi, \eta \in$* $\mathrm{Dom}(\mathcal{E})$

$$\mathcal{E}[\lambda\xi + (1-\lambda)\eta] \ge \lambda\mathcal{E}[\xi] + (1-\lambda)\mathcal{E}[\eta], \quad \forall\, \lambda \in (0,1). \tag{2.3}$$

Then for any two real numbers $a < b$, it holds that $P\big(U_{\mathcal{D}_T}(a, b; X) < \infty\big) = 1$. Thus we have

$$P\Big(X_t^- = \lim_{n\to\infty} X_{q_n^-(t)} \text{ and } X_t^+ = \lim_{n\to\infty} X_{q_n^+(t)} \text{ for any } t \in [0,T]\Big) = 1. \tag{2.4}$$

As a result, X^+ is an RCLL process.

In the rest of this section, we assume that the **F**-expectation $\mathcal{E}$ satisfies (H0)–(H2) and that $\mathrm{Dom}(\mathcal{E}) \in \widetilde{\mathscr{D}}_T$. The following proposition will play a fundamental role throughout this paper.

Proposition 2.5. *Let X be a non-negative $\mathcal{E}$-supermartingale.*

(1) *Assume either that* $\operatorname{esssup}_{t\in\mathcal{D}_T} X_t \in \mathrm{Dom}^+(\mathcal{E})$ *or that for any sequence* $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}^+(\mathcal{E})$ *convergent a.s. to some $\xi \in L^0(\mathcal{F}_T)$,*

$$\varliminf_{n\to\infty} \mathcal{E}[\xi_n] < \infty \quad \text{implies } \xi \in \mathrm{Dom}^+(\mathcal{E}). \tag{2.5}$$

Then for any $\nu \in \mathcal{S}_{0,T}$, X_ν^- and X_ν^+ both belong to $\mathrm{Dom}^+(\mathcal{E})$;

(2) *If $X_t^+ \in \mathrm{Dom}^+(\mathcal{E})$ for any $t \in [0,T]$, then X^+ is an RCLL $\mathcal{E}$-supermartingale such that for any $t \in [0,T]$, $X_t^+ \le X_t$, a.s.;*

(3) *Moreover, if the function $t \mapsto \mathcal{E}[X_t]$ from $[0,T]$ to $\mathbb{R}$ is right continuous, then X^+ is an RCLL modification of X. Conversely, if X has a right-continuous modification, then the function $t \mapsto \mathcal{E}[X_t]$ is right continuous.*

Now we add one more hypothesis to the **F**-expectation $\mathcal{E}$:

(H3) For any $\xi \in \mathrm{Dom}^+(\mathcal{E})$ and $\nu \in \mathcal{S}_{0,T}$, $X_\nu^{\xi,+} \in \mathrm{Dom}^+(\mathcal{E})$.

In light of Proposition 2.5(1), (H3) holds if $\operatorname{esssup}_{t\in\mathcal{D}_T} \mathcal{E}[\xi|\mathcal{F}_t] \in \mathrm{Dom}^+(\mathcal{E})$ or if $\mathcal{E}$ satisfies (2.5).

For each $\xi \in \mathrm{Dom}^{\#}(\mathcal{E})$, we define $\xi' \triangleq \xi - c(\xi) \in \mathrm{Dom}^{+}(\mathcal{E})$. Clearly $X^{\xi'} \triangleq \big\{\mathcal{E}[\xi'|\mathcal{F}_t]\big\}_{t\in[0,T]}$ is a non-negative $\mathcal{E}$-martingale. By (A2), $\mathcal{E}\big[X_t^{\xi'}\big] = \mathcal{E}\big[\mathcal{E}[\xi'|\mathcal{F}_t]\big] = \mathcal{E}[\xi']$ for any $t \in [0,T]$, which means that $t \mapsto \mathcal{E}\big[X_t^{\xi'}\big]$ is a continuous function on $[0,T]$. Thanks to Proposition 2.5(2) and (H3), the process $X_t^{\xi',+} \triangleq \underline{\lim}_{n\to\infty} X^{\xi'}_{q_n^+(t)}$, $t \in [0,T]$ is an RCLL modification of $X^{\xi'}$. Then for any $\nu \in \mathcal{S}_{0,T}$, we define

$$\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \triangleq X_\nu^{\xi',+} + c(\xi) \tag{2.6}$$

as the conditional $\mathbf{F}$-expectation of ξ at the stopping time $\nu \in \mathcal{S}_{0,T}$. Since we have assumed $\mathrm{Dom}(\mathcal{E}) \in \widetilde{\mathscr{D}}_T$, Lemma 2.1, (H3), (D2) as well as the non-negativity of $X_\nu^{\xi',+}$ imply that

$$\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \in \mathrm{Dom}^{\#}(\mathcal{E}), \tag{2.7}$$

which shows that $\widetilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu]$ is an operator from $\mathrm{Dom}^{\#}(\mathcal{E})$ to $\mathrm{Dom}^{\#}_\nu(\mathcal{E}) \triangleq \mathrm{Dom}^{\#}(\mathcal{E}) \cap L^0(\mathcal{F}_\nu)$. In fact, $\big\{\widetilde{\mathcal{E}}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ defines a $\mathbf{F}$-expectation on $\mathrm{Dom}^{\#}(\mathcal{E})$, as the next result shows.

Proposition 2.6. *For any $\xi \in \mathrm{Dom}^{\#}(\mathcal{E})$, $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\cdot]$ is an RCLL modification of $\mathcal{E}[\xi|\mathcal{F}_\cdot]$. $\big\{\widetilde{\mathcal{E}}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ is an $\mathbf{F}$-expectation with domain $\mathrm{Dom}(\widetilde{\mathcal{E}}) = \mathrm{Dom}^{\#}(\mathcal{E}) \in \widetilde{\mathscr{D}}_T$ and satisfying* (H0)–(H2)*; thus all preceding results are applicable to $\widetilde{\mathcal{E}}$.*

Proof. As $\mathrm{Dom}(\mathcal{E}) \in \widetilde{\mathscr{D}}_T$ is assumed, we see that $\mathrm{Dom}^{\#}(\mathcal{E})$ also belongs to $\widetilde{\mathscr{D}}_T$. Fix $\xi \in \mathrm{Dom}^{\#}(\mathcal{E})$. Since $X^{\xi',+}$ is an RCLL modification of $X^{\xi'}$, (A4) implies that for any $t \in [0,T]$

$$\widetilde{\mathcal{E}}[\xi|\mathcal{F}_t] = X_t^{\xi',+} + c(\xi) = \mathcal{E}[\xi'|\mathcal{F}_t] + c(\xi) = \mathcal{E}[\xi' + c(\xi)|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{2.8}$$

Thus $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\cdot]$ is actually an RCLL modification of $\mathcal{E}[\xi|\mathcal{F}_\cdot]$. Then it is easy to show that the pair $\big(\widetilde{\mathcal{E}}, \mathrm{Dom}^{\#}(\mathcal{E})\big)$ satisfies (A1)–(A4) and (H0)–(H2); thus it is an $\mathbf{F}$-expectation. □

We restate Proposition 2.5 with respect to $\widetilde{\mathcal{E}}$ for future use.

Corollary 2.2. *Let X be an $\widetilde{\mathcal{E}}$-supermartingale such that $\mathrm{essinf}_{t\in[0,T]} X_t \ge c$, a.s. for some $c \in \mathbb{R}$.*

(1) *If $\mathrm{esssup}_{t\in\mathcal{D}_T} X_t \in \mathrm{Dom}^{\#}(\mathcal{E})$ or if (2.5) holds, then both X_ν^- and X_ν^+ belong to $\mathrm{Dom}^{\#}(\mathcal{E})$ for any $\nu \in \mathcal{S}_{0,T}$;*
(2) *If $X_t^+ \in \mathrm{Dom}^{\#}(\mathcal{E})$ for any $t \in [0,T]$, then X^+ is an RCLL $\widetilde{\mathcal{E}}$-supermartingale such that for any $t \in [0,T]$, $X_t^+ \le X_t$, a.s.*
(3) *Moreover, if the function $t \mapsto \widetilde{\mathcal{E}}[X_t]$ from $[0,T]$ to $\mathbb{R}$ is right continuous, then X^+ is an RCLL modification of X. Conversely, if X has a right-continuous modification, then the function $t \mapsto \widetilde{\mathcal{E}}[X_t]$ is right continuous.*

The next result is the Optional Sampling Theorem of $\widetilde{\mathcal{E}}$ for the stopping times in $\mathcal{S}_{0,T}$.

Theorem 2.4 (*Optional Sampling Theorem 2*)**.** *Let X be a right-continuous $\widetilde{\mathcal{E}}$-supermartingale (resp. $\widetilde{\mathcal{E}}$-martingale, $\widetilde{\mathcal{E}}$-submartingale) such that $\mathrm{essinf}_{t\in\mathcal{D}_T} X_t \ge c$, a.s. for some $c \in \mathbb{R}$. If $X_\nu \in \mathrm{Dom}^{\#}(\mathcal{E})$ for any $\nu \in \mathcal{S}_{0,T}$, then for any $\nu, \sigma \in \mathcal{S}_{0,T}$, we have*

$$\widetilde{\mathcal{E}}[X_\nu|\mathcal{F}_\sigma] \le (\textit{resp.} =, \ge) X_{\nu\wedge\sigma}, \quad \textit{a.s.}$$

Using the Optional Sampling Theorem, we are able to extend Corollary 2.1 and Proposition 2.4 to the operators $\widetilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu]$, $\nu \in \mathcal{S}_{0,T}$.

Corollary 2.3. *For any* $\xi \in \mathrm{Dom}^\#(\mathcal{E})$ *and* $\nu, \sigma \in \mathcal{S}_{0,T}$, *we have*

$$\widetilde{\mathcal{E}}\big[\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma\big] = \widetilde{\mathcal{E}}[\xi|\mathcal{F}_{\nu\wedge\sigma}], \quad a.s. \tag{2.9}$$

Proof. Since $\big(\widetilde{\mathcal{E}}, \mathrm{Dom}^\#(\mathcal{E})\big)$ is an **F**-expectation by Proposition 2.6, for any $\xi \in \mathrm{Dom}^\#(\mathcal{E})$, (A2) implies that the RCLL process $\widetilde{X}^\xi \triangleq \big\{\widetilde{\mathcal{E}}[\xi|\mathcal{F}_t]\big\}_{t\in[0,T]}$ is an $\widetilde{\mathcal{E}}$-martingale. For any $t \in [0, T]$, (2.8) and Proposition 2.2(2) show that

$$\widetilde{X}^\xi_t = \widetilde{\mathcal{E}}[\xi|\mathcal{F}_t] \ge \widetilde{\mathcal{E}}[c(\xi)|\mathcal{F}_t] = \mathcal{E}[c(\xi)|\mathcal{F}_t] = c(\xi), \quad \text{a.s.},$$

which implies that $\operatorname{essinf}_{t\in[0,T]}\widetilde{X}^\xi_t \ge c(\xi)$, a.s. Then (2.7) and Theorem 2.4 give rise to (2.9). □

Proposition 2.7. *For any* $\xi, \eta \in \mathrm{Dom}^\#(\mathcal{E})$ *and* $\nu \in \mathcal{S}_{0,T}$, *it holds that*

(1) *"Strict Monotonicity":* $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \le \widetilde{\mathcal{E}}[\eta|\mathcal{F}_\nu]$, *a.s. if* $\xi \le \eta$, *a.s.; Moreover, if* $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\sigma] = \widetilde{\mathcal{E}}[\eta|\mathcal{F}_\sigma]$, *a.s. for some* $\sigma \in \mathcal{S}_{0,T}$, *then* $\xi = \eta$, *a.s.;*
(2) *"Zero–one Law":* $\widetilde{\mathcal{E}}[\mathbf{1}_A\xi|\mathcal{F}_\nu] = \mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu]$, *a.s. for any* $A \in \mathcal{F}_\nu$;
(3) *"Translation Invariance":* $\widetilde{\mathcal{E}}[\xi + \eta|\mathcal{F}_\nu] = \widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] + \eta$, *a.s. if* $\eta \in \mathrm{Dom}^\#_\nu(\mathcal{E})$;
(4) *"Local Property":* $\widetilde{\mathcal{E}}[\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta|\mathcal{F}_\nu] = \mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] + \mathbf{1}_{A^c}\widetilde{\mathcal{E}}[\eta|\mathcal{F}_\nu]$, *a.s. for any* $A \in \mathcal{F}_\nu$;
(5) *"Constant-Preserving":* $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] = \xi$, *a.s., if* $\xi \in \mathrm{Dom}^\#_\nu(\mathcal{E})$.

Remark 2.8. Corollary 2.3, Proposition 2.7(2) and (2.8) imply that for any $\xi \in \mathrm{Dom}^\#(\mathcal{E})$ and $\nu \in \mathcal{S}_{0,T}$,

$$\begin{aligned}\mathcal{E}[\mathbf{1}_A\xi] &= \widetilde{\mathcal{E}}[\mathbf{1}_A\xi] = \widetilde{\mathcal{E}}\big[\widetilde{\mathcal{E}}[\mathbf{1}_A\xi|\mathcal{F}_\nu]\big] \\ &= \widetilde{\mathcal{E}}\big[\mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu]\big] = \mathcal{E}\big[\mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu]\big], \quad \forall A \in \mathcal{F}_\nu.\end{aligned} \tag{2.10}$$

In light of Proposition 2.2(3), $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu]$ is the unique element (up to a P-null set) in $\mathrm{Dom}^\#_\nu(\mathcal{E})$ that makes (2.10) hold. Therefore, we see that the random variable $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu]$ defined by (2.6) is exactly the conditional **F**-expectation of ξ at the stopping time ν in the classical sense.

In light of Corollary 2.3 and Proposition 2.7, we can generalize Fatou's Lemma (Theorem 2.1) and the Dominated Convergence Theorem (Theorem 2.2) to the conditional **F**-expectation $\widetilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu]$, $\nu \in \mathcal{S}_{0,T}$.

Proposition 2.8 (*Fatou's lemma 2*)**.** *Let* $\{\xi_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{Dom}^\#(\mathcal{E})$ *that converges a.s. to some* $\xi \in \mathrm{Dom}^\#(\mathcal{E})$ *and satisfies* $\operatorname{essinf}_{n\in\mathbb{N}}\xi_n \ge c$, *a.s. for some* $c \in \mathbb{R}$, *then for any* $\nu \in \mathcal{S}_{0,T}$, *we have*

$$\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \le \varliminf_{n\to\infty} \widetilde{\mathcal{E}}[\xi_n|\mathcal{F}_\nu], \quad a.s., \tag{2.11}$$

where the right hand side of (2.11) could be equal to infinity with non-zero probability.

Proposition 2.9 (*Dominated Convergence Theorem 2*)**.** *Let* $\{\xi_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{Dom}^\#(\mathcal{E})$ *that converges a.s. and that satisfies* $\operatorname{essinf}_{n\in\mathbb{N}}\xi_n \ge c$, *a.s. for some* $c \in \mathbb{R}$. *If there is an*

$\eta \in \mathrm{Dom}^{\#}(\mathcal{E})$ such that $\xi_n \le \eta$ a.s. for any $n \in \mathbb{N}$, then the limit ξ of $\{\xi_n\}_{n\in\mathbb{N}}$ belongs to $\mathrm{Dom}^{\#}(\mathcal{E})$ and for any $\nu \in \mathcal{S}_{0,T}$, we have

$$\lim_{n\to\infty} \widetilde{\mathcal{E}}[\xi_n|\mathcal{F}_\nu] = \widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu], \quad a.s. \tag{2.12}$$

Proof of Propositions 2.8 and 2.9. In the proofs of Theorems 2.1 and 2.2, we only need to replace $\{\xi_n\}_{n\in\mathbb{N}}$ and $\mathcal{E}[\cdot|\mathcal{F}_t]$ by $\{\xi_n - c\}_{n\in\mathbb{N}}$ and $\widetilde{\mathcal{E}}[\cdot|\mathcal{F}_\nu]$ respectively. Instead of (A1), (A3) and (A4), we apply Proposition 2.7(1)–(3). Moreover, since (A2) is only used on $\mathrm{Dom}^{+}(\mathcal{E})$ in the proofs of Theorems 2.1 and 2.2, we can substitute Corollary 2.3 for it. Eventually, a simple application of Proposition 2.7(3) yields (2.11) and (2.12). □

3. Collections of F-expectations

In this section, we will show that *pasting* of two **F**-expectations at a given stopping time is itself an **F**-expectation. Moreover, pasting preserves (H1) and (H2). We will then introduce the concept of a *stable* class of **F**-expectations, which are collections closed under pasting. We will solve the optimal stopping problems introduced in (1.1) and (1.6) over this class of **F**-expectations. Before we show the pasting property of **F**-expectations, we introduce the concept of convexity for an **F**-expectation and give one of the consequences of having convexity:

Definition 3.1. An **F**-expectation $\mathcal{E}$ is called "positively-convex" if for any $\xi, \eta \in \mathrm{Dom}^{+}(\mathcal{E})$, $\lambda \in (0, 1)$ and $t \in [0, T]$

$$\mathcal{E}[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_t] \le \lambda\mathcal{E}[\xi|\mathcal{F}_t] + (1-\lambda)\mathcal{E}[\eta|\mathcal{F}_t], \quad \text{a.s.}$$

Lemma 3.1. *Any positively-convex* **F***-expectation satisfies* (H0). *Moreover, an* **F***-expectation $\mathcal{E}$ is positively-convex if and only if the implied* **F***-expectation $\big(\widetilde{\mathcal{E}}, \mathrm{Dom}^{\#}(\mathcal{E})\big)$ is convex, i.e., for any $\xi, \eta \in \mathrm{Dom}^{\#}(\mathcal{E})$, $\lambda \in (0, 1)$ and $t \in [0, T]$*

$$\widetilde{\mathcal{E}}[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_t] \le \lambda\widetilde{\mathcal{E}}[\xi|\mathcal{F}_t] + (1-\lambda)\widetilde{\mathcal{E}}[\eta|\mathcal{F}_t], \quad a.s. \tag{3.1}$$

Proposition 3.1. *Let $\mathcal{E}_i, \mathcal{E}_j$ be two* **F***-expectations with the same domain $\Lambda \in \widetilde{\mathscr{D}}_T$ and satisfying* (H1)–(H3). *For any $\nu \in \mathcal{S}_{0,T}$, we define the pasting of $\mathcal{E}_i, \mathcal{E}_j$ at the stopping time ν to be the following RCLL* **F***-adapted process*

$$\mathcal{E}^{\nu}_{i,j}[\xi|\mathcal{F}_t] \triangleq \mathbf{1}_{\{\nu\le t\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_t] + \mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big], \quad \forall\, t \in [0, T] \tag{3.2}$$

for any $\xi \in \Lambda^{\#} = \{\xi \in \Lambda : \xi \ge c,$ a.s. for some $c = c(\xi) \in \mathbb{R}\}$. Then $\mathcal{E}^{\nu}_{i,j}$ is an **F***-expectation with domain $\Lambda^{\#} \in \widetilde{\mathscr{D}}_T$ and satisfying* (H1) *and* (H2). *Moreover, if $\mathcal{E}_i$ and $\mathcal{E}_j$ are both positively-convex, $\mathcal{E}^{\nu}_{i,j}$ is convex in the sense of* (3.1).

In particular, for any $\sigma \in \mathcal{S}_{0,T}$, applying Proposition 2.7(4) and (5), we obtain

$$\begin{aligned}
\mathcal{E}^{\nu}_{i,j}[\xi|\mathcal{F}_\sigma] &= \mathbf{1}_{\{\nu\le\sigma\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\sigma] + \mathbf{1}_{\{\nu>\sigma\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma\big] \\
&= \mathbf{1}_{\{\nu\le\sigma\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\sigma]|\mathcal{F}_\sigma\big] + \mathbf{1}_{\{\nu>\sigma\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma\big] \\
&= \widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\nu\le\sigma\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\sigma] + \mathbf{1}_{\{\nu>\sigma\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_\sigma\big] \\
&= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee\sigma}]|\mathcal{F}_\sigma\big], \quad \text{a.s.},
\end{aligned} \tag{3.3}$$

where we used the fact that $\{\nu > \sigma\} \in \mathcal{F}_{\nu\wedge\sigma}$ thanks to [8, Lemma 1.2.16].

Remark 3.1. Pasting may not preserve (H0). From now on, we will replace assumption (H0) by the positive convexity, which implies the former and is an invariant property under pasting thanks to the previous two results. Positive convexity is also important in constructing an optimal stopping time of (1.1) (see Theorem 2.1 of [1]).

All of the ingredients are in place to introduce what we mean by a stable class of **F**-expectations. As we will see in Lemma 2.2 of [1], stability assures that the essential supremum or infimum over the class can be approximated by an increasing or decreasing sequence in the class.

Definition 3.2. A class $\mathscr{E} = \{\mathcal{E}_i\}_{i\in\mathcal{I}}$ of **F**-expectations is said to be "stable" if

(1) All $\mathcal{E}_i,\ i \in \mathcal{I}$ are positively-convex **F**-expectations with the same domain $\Lambda \in \widetilde{\mathscr{D}}_T$ and they satisfy (H1)–(H3);
(2) $\mathscr{E}$ is closed under pasting: namely, for any $i, j \in \mathcal{I}$, $\nu \in \mathcal{S}_{0,T}$, there exists a $k = k(i, j, \nu) \in \mathcal{I}$ such that $\mathcal{E}^{\nu}_{i,j}$ coincides with $\widetilde{\mathcal{E}}_k$ on $\Lambda^{\#}$.

We shall denote $\text{Dom}(\mathscr{E}) \triangleq \Lambda^{\#}$, thus $\text{Dom}(\mathscr{E}) = \text{Dom}^{\#}(\mathcal{E}_i) \in \widetilde{\mathscr{D}}_T$ for any $i \in \mathcal{I}$. Moreover, if $\mathscr{E}' = \{\mathcal{E}_i\}_{i\in\mathcal{I}'}$ satisfies (2) for some non-empty subset $\mathcal{I}'$ of $\mathcal{I}$, then we call $\mathscr{E}'$ a stable subclass of $\mathscr{E}$, clearly $\text{Dom}(\mathscr{E}') = \text{Dom}(\mathscr{E})$.

Remark 3.2. The notion of "pasting" for linear expectations was given by [6, Definition 6.41]. The counterpart of Proposition 3.1 for the linear expectations, which states that pasting two probability measures equivalent to P results in another probability measure equivalent to P, is given by [6, Lemma 6.43]. Note that in the case of linear expectations, (H1), (H2) and the convexity are trivially preserved because pasting in that case gives us a linear expectation. On the other hand, the notion of stability for linear expectations was given by [6, Definition 6.44]. The stability is also referred to as "fork convexity" in stochastic control theory, "m-stability" in stochastic analysis or "rectangularity" in decision theory (see the introduction of [5,2] for details).

Example 3.1. (1) Let $\mathcal{P}$ be the set of all probability measures equivalent to P, then $\mathscr{E}_{\mathcal{P}} \triangleq \{E_Q\}_{Q\in\mathcal{P}}$ is a stable class of linear expectations; see [6, Proposition 6.45].
(2) Consider a collection $\mathfrak{U}$ of admissible control processes. For any $U \in \mathfrak{U}$, let P^U be the equivalent probability measure defined via [11, (5)] (or [10, (2.5)]), then $\mathscr{E}_{\mathfrak{U}} \triangleq \{E_{P^U}\}_{U\in\mathfrak{U}}$ is a stable class of linear expectations; see Section 5.3 of [1].
(3) For any $M > 0$, a family $\mathscr{E}_M$ of convex Lipschitz g-expectations with Lipschitz coefficient $K_g \le M$ is an example of stable class of non-linear expectations; see Section 5.1 of [1].

The following lemma gives us a tool for checking whether a random variable is inside the domain $\text{Dom}(\mathscr{E})$ of a stable class $\mathscr{E}$.

Lemma 3.2. *Given a stable class $\mathscr{E}$ of* **F***-expectations, a random variable ξ belongs to* $\text{Dom}(\mathscr{E})$ *if and only if $c \le \xi \le \eta$, a.s. for some $c \in \mathbb{R}$ and $\eta \in \text{Dom}(\mathscr{E})$.*

Proof. Consider a random variable ξ. If $\xi \in \text{Dom}(\mathscr{E})$, since $\text{Dom}(\mathscr{E}) = \text{Dom}^{\#}(\mathcal{E}_i)$ for any $i \in \mathcal{I}$, we know that there exists a $c = c(\xi) \in \mathbb{R}$ such that $\xi \ge c(\xi)$, a.s.

On the other hand, if $c \le \xi \le \eta$, a.s. for some $c \in \mathbb{R}$ and $\eta \in \text{Dom}(\mathscr{E})$, it follows that $0 \le \xi - c \le \eta - c$, a.s. Since $\text{Dom}(\mathscr{E}) \in \widetilde{\mathscr{D}}_T$, we see that $-c, c \in \mathbb{R} \subset \text{Dom}(\mathscr{E})$. Then (D2) shows that $\eta - c \in \text{Dom}(\mathscr{E})$ and thus (D3) implies that $\xi - c \in \text{Dom}(\mathscr{E})$, which further leads to $\xi = (\xi - c) + c \in \text{Dom}(\mathscr{E})$ thanks to (D2) again. $\square$

We end this section by reviewing some basic properties of the essential supremum and essential infimum (for their definitions, see e.g. [13, Proposition VI-**1**-1], or [6, Theorem A.32]).

Lemma 3.3. *Let $\{\xi_j\}_{j\in\mathcal{J}}$ and $\{\eta_j\}_{j\in\mathcal{J}}$ be two families of random variables of $L^0(\mathcal{F})$ with the same index set $\mathcal{J}$.*

(1) *If $\xi_j \le (=)\ \eta_j$, a.s. for any $j \in \mathcal{J}$, then $\operatorname{esssup}_{j\in\mathcal{J}}\xi_j \le (=) \operatorname{esssup}_{j\in\mathcal{J}}\eta_j$, a.s.*
(2) *For any $A \in \mathcal{F}$, it holds a.s. that $\operatorname{esssup}_{j\in\mathcal{J}}\big(\mathbf{1}_A\xi_j + \mathbf{1}_{A^c}\eta_j\big) = \mathbf{1}_A\operatorname{esssup}_{j\in\mathcal{J}}\xi_j + \mathbf{1}_{A^c}\operatorname{esssup}_{j\in\mathcal{J}}\eta_j$; In particular, $\operatorname{esssup}_{j\in\mathcal{J}}\big(\mathbf{1}_A\xi_j\big) = \mathbf{1}_A\operatorname{esssup}_{j\in\mathcal{J}}\xi_j$, a.s.*
(3) *For any random variable $\gamma \in L^0(\mathcal{F})$ and any $\alpha > 0$, we have $\operatorname{esssup}_{j\in\mathcal{J}}(\alpha\xi_j + \gamma) = \alpha\operatorname{esssup}_{j\in\mathcal{J}}\xi_j + \gamma$, a.s.*

Moreover, (1)–(3) *hold when we replace $\operatorname{esssup}_{j\in\mathcal{J}}$ by $\operatorname{essinf}_{j\in\mathcal{J}}$.*

4. Proofs

Proof of Proposition 2.1. For any $\xi \in \Lambda$ and $t \in [0,T]$, let us define $\mathcal{E}^o[\xi|\mathcal{F}_t] \triangleq \xi_t$. We will check that the system $\big\{\mathcal{E}^o[\xi|\mathcal{F}_t],\ \xi \in \Lambda\big\}_{t\in[0,T]}$ satisfies (A1)–(A4); thus it is an $\mathbf{F}$-expectation with domain Λ.
(1) For any $\eta \in \Lambda$ with $\xi \le \eta$, a.s., we set $A \triangleq \big\{\mathcal{E}^o[\xi|\mathcal{F}_t] > \mathcal{E}^o[\eta|\mathcal{F}_t]\big\} \in \mathcal{F}_t$, thus $\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t] \ge \mathbf{1}_A\mathcal{E}^o[\eta|\mathcal{F}_t]$. It follows from (a1) and (a2) that

$$\mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t]\big] \ge \mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\eta|\mathcal{F}_t]\big] = \mathcal{E}^o[\mathbf{1}_A\eta] \ge \mathcal{E}^o[\mathbf{1}_A\xi] = \mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t]\big],$$

which shows that $\mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t]\big] = \mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\eta|\mathcal{F}_t]\big]$. Then the "strict monotonicity" of (a1) further implies that $\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}^o[\eta|\mathcal{F}_t]$, a.s., thus $P(A) = 0$, i.e., $\mathcal{E}^o[\xi|\mathcal{F}_t] \le \mathcal{E}^o[\eta|\mathcal{F}_t]$, a.s.

Moreover, if $0 \le \xi \le \eta$, a.s. and $\mathcal{E}^o[\xi|\mathcal{F}_0] = \mathcal{E}^o[\eta|\mathcal{F}_0]$, applying (a2) with $A = \Omega$ and $\gamma = 0$, we obtain

$$\mathcal{E}^o[\xi] = \mathcal{E}^o\big[\mathcal{E}^o[\xi|\mathcal{F}_0]\big] = \mathcal{E}^o\big[\mathcal{E}^o[\eta|\mathcal{F}_0]\big] = \mathcal{E}^o[\eta].$$

Then the strict monotonicity of (a1) implies that $\xi = \eta$, a.s., proving (A1).
(2) Let $0 \le s \le t \le T$, for any $A \in \mathcal{F}_s \subset \mathcal{F}_t$ and $\gamma \in \Lambda_s \subset \Lambda_t$, one can deduce that

$$\mathcal{E}^o\Big[\mathbf{1}_A\mathcal{E}^o\big[\mathcal{E}^o[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] + \gamma\Big] = \mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t] + \gamma\big] = \mathcal{E}^o\big[\mathbf{1}_A\xi + \gamma\big].$$

Since $\mathcal{E}^o\big[\mathcal{E}^o[\xi|\mathcal{F}_t]|\mathcal{F}_s\big] \in \mathcal{F}_s$, (a2) implies that $\mathcal{E}^o[\xi|\mathcal{F}_s] = \xi_s = \mathcal{E}^o\big[\mathcal{E}^o[\xi|\mathcal{F}_t]|\mathcal{F}_s\big]$, proving (A2).
(3) Fix $A \in \mathcal{F}_t$, for any $\tilde{A} \in \mathcal{F}_t$ and $\gamma \in \Lambda_t$, we have

$$\begin{aligned}\mathcal{E}^o\big[\mathbf{1}_{\tilde{A}}\big(\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t]\big) + \gamma\big] &= \mathcal{E}^o\big[\mathbf{1}_{\tilde{A}\cap A}\mathcal{E}^o[\xi|\mathcal{F}_t] + \gamma\big]\\ &= \mathcal{E}^o\big[\mathbf{1}_{\tilde{A}\cap A}\xi + \gamma\big] = \mathcal{E}^o\big[\mathbf{1}_{\tilde{A}}(\mathbf{1}_A\xi) + \gamma\big].\end{aligned}$$

Since $\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t] \in \mathcal{F}_t$, (a2) implies that $\mathcal{E}^o[\mathbf{1}_A\xi|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t]$, proving (A3).
(4) For any $A \in \mathcal{F}_t$ and $\eta,\ \gamma \in \Lambda_t$, (D2) implies that $\mathbf{1}_A\eta + \gamma \in \Lambda_t$, thus we have

$$\begin{aligned}\mathcal{E}^o\big[\mathbf{1}_A\big(\mathcal{E}^o[\xi|\mathcal{F}_t] + \eta\big) + \gamma\big] &= \mathcal{E}^o\big[\mathbf{1}_A\mathcal{E}^o[\xi|\mathcal{F}_t] + (\mathbf{1}_A\eta + \gamma)\big] = \mathcal{E}^o\big[\mathbf{1}_A\xi + (\mathbf{1}_A\eta + \gamma)\big]\\ &= \mathcal{E}^o\big[\mathbf{1}_A(\xi + \eta) + \gamma\big].\end{aligned}$$

Then it follows from (a2) that $\mathcal{E}^o[\xi + \eta|\mathcal{F}_t] = \mathcal{E}^o[\xi|\mathcal{F}_t] + \eta$, proving (A4). □

Proof of Proposition 2.2. (1) For any $A \in \mathcal{F}_t$, using (A3) twice, we obtain

$$\begin{aligned}\mathcal{E}[\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta|\mathcal{F}_t] &= \mathbf{1}_A\mathcal{E}[\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta|\mathcal{F}_t] + \mathbf{1}_{A^c}\mathcal{E}[\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta|\mathcal{F}_t]\\ &= \mathcal{E}[\mathbf{1}_A(\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta)|\mathcal{F}_t] + \mathcal{E}[\mathbf{1}_{A^c}(\mathbf{1}_A\xi + \mathbf{1}_{A^c}\eta)|\mathcal{F}_t]\\ &= \mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_t] + \mathcal{E}[\mathbf{1}_{A^c}\eta|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_t] + \mathbf{1}_{A^c}\mathcal{E}[\eta|\mathcal{F}_t], \quad \text{a.s.}\end{aligned}$$

(2) Applying (A3) with a null set A and $\xi = 0$, we obtain $\mathcal{E}[0|\mathcal{F}_t] = \mathcal{E}[\mathbf{1}_A 0|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}[0|\mathcal{F}_t] = 0$, a.s. If $\xi \in \mathrm{Dom}_t(\mathcal{E})$, (A4) implies that $\mathcal{E}[\xi|\mathcal{F}_t] = \mathcal{E}[0+\xi|\mathcal{F}_t] = \mathcal{E}[0|\mathcal{F}_t] + \xi = \xi$, a.s.
(3) If $\xi \le \eta$, a.s., (A1) directly implies that for any $A \in \mathcal{F}_\nu$, $\mathcal{E}[\mathbf{1}_A\xi] \le \mathcal{E}[\mathbf{1}_A\eta]$. On the other hand, suppose that $\mathcal{E}[\mathbf{1}_A\xi] \le \mathcal{E}[\mathbf{1}_A\eta]$ for any $A \in \mathcal{F}_\nu$. We set $\tilde{A} \triangleq \{\xi > \eta\} \in \mathcal{F}_\nu$, thus $\mathbf{1}_{\tilde{A}}\xi \ge \mathbf{1}_{\tilde{A}}\eta \ge c \wedge 0$, a.s. Using (A1) we see that $\mathcal{E}[\mathbf{1}_{\tilde{A}}\xi] \ge \mathcal{E}[\mathbf{1}_{\tilde{A}}\eta]$; hence $\mathcal{E}[\mathbf{1}_{\tilde{A}}\xi] = \mathcal{E}[\mathbf{1}_{\tilde{A}}\eta]$. Then (A4) implies that

$$\mathcal{E}[\mathbf{1}_{\tilde{A}}\xi - c\wedge 0] = \mathcal{E}[\mathbf{1}_{\tilde{A}}\xi] - c\wedge 0 = \mathcal{E}[\mathbf{1}_{\tilde{A}}\eta] - c\wedge 0 = \mathcal{E}[\mathbf{1}_{\tilde{A}}\eta - c\wedge 0].$$

Applying the second part of (A1), we obtain that $\mathbf{1}_{\tilde{A}}\xi - c\wedge 0 = \mathbf{1}_{\tilde{A}}\eta - c\wedge 0$, a.s., which implies that $P(\tilde{A}) = 0$, i.e. $\xi \le \eta$, a.s. □

Proof of Proposition 2.3. We shall only consider the $\mathcal{E}$-supermartingale case, as the other cases can be deduced similarly. We first show that for any $s \in [0, T]$ and $\nu \in \mathcal{S}^F_{0,T}$

$$\mathcal{E}[X_\nu|\mathcal{F}_s] \le X_{\nu\wedge s}, \quad \text{a.s.} \tag{4.1}$$

To see this, we note that since $\{\nu \le s\} \in \mathcal{F}_s$, (A3) and Proposition 2.2(2) imply that

$$\begin{aligned}\mathcal{E}[X_\nu|\mathcal{F}_s] &= \mathbf{1}_{\{\nu>s\}}\mathcal{E}[X_\nu|\mathcal{F}_s] + \mathbf{1}_{\{\nu\le s\}}\mathcal{E}[X_\nu|\mathcal{F}_s]\\ &= \mathcal{E}\big[\mathbf{1}_{\{\nu>s\}}X_{\nu\vee s}|\mathcal{F}_s\big] + \mathcal{E}\big[\mathbf{1}_{\{\nu\le s\}}X_{\nu\wedge s}|\mathcal{F}_s\big]\\ &= \mathbf{1}_{\{\nu>s\}}\mathcal{E}\big[X_{\nu\vee s}|\mathcal{F}_s\big] + \mathbf{1}_{\{\nu\le s\}}\mathcal{E}\big[X_{\nu\wedge s}|\mathcal{F}_s\big]\\ &= \mathbf{1}_{\{\nu>s\}}\mathcal{E}\big[X_{\nu\vee s}|\mathcal{F}_s\big] + \mathbf{1}_{\{\nu\le s\}}X_{\nu\wedge s}, \quad \text{a.s.}\end{aligned} \tag{4.2}$$

Suppose that $\nu_s \triangleq \nu \vee s$ takes values in a finite subset $\{t_1 < \cdots < t_n\}$ of $[s, T]$. Then (A4) implies that

$$\mathcal{E}[X_{\nu_s}|\mathcal{F}_{t_{n-1}}] = \mathcal{E}\big[\mathbf{1}_{\{\nu_s=t_n\}}X_{t_n}|\mathcal{F}_{t_{n-1}}\big] + \sum_{i=1}^{n-1}\mathbf{1}_{\{\nu_s=t_i\}}X_{t_i}, \quad \text{a.s.}$$

Since $\{\nu_s = t_n\} = \{\nu_s > t_{n-1}\} \in \mathcal{F}_{t_{n-1}}$, (A3) shows that

$$\mathcal{E}\big[\mathbf{1}_{\{\nu_s=t_n\}}X_{t_n}|\mathcal{F}_{t_{n-1}}\big] = \mathbf{1}_{\{\nu_s=t_n\}}\mathcal{E}[X_{t_n}|\mathcal{F}_{t_{n-1}}] \le \mathbf{1}_{\{\nu_s=t_n\}}X_{t_{n-1}}, \quad \text{a.s.}$$

Thus it holds a.s. that $\mathcal{E}[X_{\nu_s}|\mathcal{F}_{t_{n-1}}] \le \mathbf{1}_{\{\nu_s>t_{n-2}\}}X_{t_{n-1}} + \sum_{i=1}^{n-2}\mathbf{1}_{\{\nu_s=t_i\}}X_{t_i}$. Applying $\mathcal{E}[\cdot|\mathcal{F}_{t_{n-2}}]$ on both sides, we can further deduce from (A2)–(A4) that

$$\begin{aligned}\mathcal{E}\big[X_{\nu_s}|\mathcal{F}_{t_{n-2}}\big] &= \mathcal{E}\Big[\mathcal{E}\big[X_{\nu_s}|\mathcal{F}_{t_{n-1}}\big]|\mathcal{F}_{t_{n-2}}\Big] \le \mathbf{1}_{\{\nu_s>t_{n-2}\}}\mathcal{E}[X_{t_{n-1}}|\mathcal{F}_{t_{n-2}}] + \sum_{i=1}^{n-2}\mathbf{1}_{\{\nu_s=t_i\}}X_{t_i}\\ &\le \mathbf{1}_{\{\nu_s>t_{n-2}\}}X_{t_{n-2}} + \sum_{i=1}^{n-2}\mathbf{1}_{\{\nu_s=t_i\}}X_{t_i} = \mathbf{1}_{\{\nu_s>t_{n-3}\}}X_{t_{n-2}} + \sum_{i=1}^{n-3}\mathbf{1}_{\{\nu_s=t_i\}}X_{t_i}, \quad \text{a.s.}\end{aligned}$$

Inductively, it follows that $\mathcal{E}[X_{\nu_s}|\mathcal{F}_{t_1}] \le X_{t_1}$, a.s. Applying (A2) once again, we obtain

$$\mathcal{E}\big[X_{\nu_s}|\mathcal{F}_s\big] = \mathcal{E}\Big[\mathcal{E}\big[X_{\nu_s}|\mathcal{F}_{t_1}\big]|\mathcal{F}_s\Big] \le \mathcal{E}[X_{t_1}|\mathcal{F}_s] \le X_s, \quad \text{a.s.},$$

which together with (4.2) implies that $\mathcal{E}[X_\nu|\mathcal{F}_s] \le \mathbf{1}_{\{\nu>s\}}X_s + \mathbf{1}_{\{\nu\le s\}}X_{\nu\wedge s} = X_{\nu\wedge s}$, a.s., proving (4.1).

Let $\sigma \in \mathcal{S}^F_{0,T}$ taking values in a finite set $\{s_1 < \cdots < s_m\}$, then

$$\mathcal{E}[X_\nu|\mathcal{F}_\sigma] = \sum_{j=1}^m \mathbf{1}_{\{\sigma=s_j\}}\mathcal{E}[X_\nu|\mathcal{F}_{s_j}] \le \sum_{j=1}^m \mathbf{1}_{\{\sigma=s_j\}} X_{\nu\wedge s_j} = X_{\nu\wedge\sigma}, \quad \text{a.s.} \quad \square$$

Proof of Proposition 2.4. Given $\xi \in \mathrm{Dom}(\mathcal{E})$, we let $\nu \in \mathcal{S}^F_{0,T}$ take values in a finite set $\{t_1 < \cdots < t_n\}$.
(1) For any $\eta \in \mathrm{Dom}(\mathcal{E})$ with $\xi \le \eta$, $a.s.$, (A1) implies that

$$\mathcal{E}[\xi|\mathcal{F}_\nu] = \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\xi|\mathcal{F}_{t_i}] \le \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\eta|\mathcal{F}_{t_i}] = \mathcal{E}[\eta|\mathcal{F}_\nu], \quad \text{a.s.}$$

Moreover, if $0 \le \xi \le \eta$, a.s. and $\mathcal{E}[\xi|\mathcal{F}_\sigma] = \mathcal{E}[\eta|\mathcal{F}_\sigma]$, a.s. for some $\sigma \in \mathcal{S}^F_{0,T}$, we can apply Corollary 2.1 to obtain

$$\mathcal{E}[\xi] = \mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_\sigma]\big] = \mathcal{E}\big[\mathcal{E}[\eta|\mathcal{F}_\sigma]\big] = \mathcal{E}[\eta].$$

The second part of (A1) then implies that $\xi = \eta$, a.s., proving (1).
(2) For any $A \in \mathcal{F}_\nu$, it is clear that $A \cap \{\nu = t_i\} \in \mathcal{F}_{t_i}$ for each $i \in \{1, \ldots, n\}$. Hence we can deduce from (A3) that

$$\begin{aligned}\mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_\nu] &= \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_{t_i}] = \sum_{i=1}^n \mathcal{E}[\mathbf{1}_{\{\nu=t_i\}\cap A}\xi|\mathcal{F}_{t_i}] = \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}\cap A}\mathcal{E}[\xi|\mathcal{F}_{t_i}] \\ &= \mathbf{1}_A \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\xi|\mathcal{F}_{t_i}] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_\nu], \quad \text{a.s., proving (2).}\end{aligned}$$

(3) For any $\eta \in \mathrm{Dom}_\nu(\mathcal{E})$, since $\mathbf{1}_{\{\nu=t_i\}}\eta \in \mathrm{Dom}_{t_i}(\mathcal{E})$ for each $i \in \{1, \ldots, n\}$, (A3) and (A4) imply that

$$\begin{aligned}\mathcal{E}[\xi+\eta|\mathcal{F}_\nu] &= \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\xi+\eta|\mathcal{F}_{t_i}] = \sum_{i=1}^n \mathcal{E}[\mathbf{1}_{\{\nu=t_i\}}\xi + \mathbf{1}_{\{\nu=t_i\}}\eta|\mathcal{F}_{t_i}] \\ &= \sum_{i=1}^n \Big(\mathcal{E}[\mathbf{1}_{\{\nu=t_i\}}\xi|\mathcal{F}_{t_i}] + \mathbf{1}_{\{\nu=t_i\}}\eta\Big) \\ &= \sum_{i=1}^n \mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\xi|\mathcal{F}_{t_i}] + \eta = \mathcal{E}[\xi|\mathcal{F}_\nu] + \eta, \quad \text{a.s., proving (3).}\end{aligned}$$

The proof of (4) and (5) is similar to that of Proposition 2.2(1) and (2) by applying the just obtained "Zero–one Law" and "Translation Invariance". $\square$

Proof of Theorem 2.1. (H1) is an easy consequence of the lower semi-continuity (2.2). In fact, for any $\xi \in \mathrm{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \uparrow \mathbf{1}_{A_n} = 1$ a.s., $\{\mathbf{1}_{A_n}\xi\}_{n\in\mathbb{N}}$ is an increasing sequence converging to ξ. Then applying the lower semi-continuity with $\nu = 0$ and using (A1), we obtain $\mathcal{E}[\xi] \le \lim_{n\to\infty} \uparrow \mathcal{E}[\mathbf{1}_{A_n}\xi] \le \mathcal{E}[\xi]$; so (H1) follows.

On the other hand, to show that (H1) implies the lower semi-continuity, we first extend (H1) as follows: For any $\xi \in \mathrm{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \uparrow \mathbf{1}_{A_n} = 1$, a.s., it holds for any $t \in [0, T]$ that

$$\lim_{n\to\infty} \uparrow \mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{4.3}$$

In fact, by (A1), it holds a.s. that $\big\{\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]\big\}_{n\in\mathbb{N}}$ is an increasing sequence bounded from above by $\mathcal{E}[\xi|\mathcal{F}_t]$. Hence, $\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]\le\mathcal{E}[\xi|\mathcal{F}_t]$, a.s. Assuming that $\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]<\mathcal{E}[\xi|\mathcal{F}_t]$ with a positive probability, we can find an $\varepsilon>0$ such that the set $A_\varepsilon=\big\{\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]\le\mathcal{E}[\xi|\mathcal{F}_t]-\varepsilon\big\}\in\mathcal{F}_t$ still has positive probability. Hence for any $n\in\mathbb{N}$, we have

$$\mathbf{1}_{A_\varepsilon}\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]\le\mathbf{1}_{A_\varepsilon}\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]\le\mathbf{1}_{A_\varepsilon}\big(\mathcal{E}[\xi|\mathcal{F}_t]-\varepsilon\big),\quad\text{a.s.}$$

Then (A1)–(A4) imply that

$$\begin{aligned}\mathcal{E}[\mathbf{1}_{A_\varepsilon}\mathbf{1}_{A_n}\xi]+\varepsilon&=\mathcal{E}[\mathbf{1}_{A_\varepsilon}\mathbf{1}_{A_n}\xi+\varepsilon]=\mathcal{E}\big[\mathcal{E}[\mathbf{1}_{A_\varepsilon}\mathbf{1}_{A_n}\xi+\varepsilon|\mathcal{F}_t]\big]=\mathcal{E}\big[\mathbf{1}_{A_\varepsilon}\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]+\varepsilon\big]\\&\le\mathcal{E}\big[\mathbf{1}_{A_\varepsilon}\mathcal{E}[\xi|\mathcal{F}_t]+\varepsilon\mathbf{1}_{A_\varepsilon^c}\big]=\mathcal{E}\big[\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon\mathbf{1}_{A_\varepsilon^c}|\mathcal{F}_t]\big]=\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon\mathbf{1}_{A_\varepsilon^c}].\end{aligned}$$

Using (A4), (H1) and (A1), we obtain

$$\begin{aligned}\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon]&=\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi]+\varepsilon=\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n}\mathbf{1}_{A_\varepsilon}\xi]+\varepsilon\\&\le\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon\mathbf{1}_{A_\varepsilon^c}]\le\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon],\end{aligned}$$

thus $\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon]=\mathcal{E}[\mathbf{1}_{A_\varepsilon}\xi+\varepsilon\mathbf{1}_{A_\varepsilon^c}]$. Then the second part of (A1) implies that $\mathbf{1}_{A_\varepsilon}\xi+\varepsilon=\mathbf{1}_{A_\varepsilon}\xi+\varepsilon\mathbf{1}_{A_\varepsilon^c}$, a.s., which can hold only if $P(A_\varepsilon)=0$. This results in a contradiction. Thus $\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n}\xi|\mathcal{F}_t]=\mathcal{E}[\xi|\mathcal{F}_t]$, a.s., proving (4.3).

Next, we show that (2.2) holds for each deterministic stopping time $\nu=t\in[0,T]$. For any $j,n\in\mathbb{N}$, we define $A_n^j\triangleq\cap_{k=n}^\infty\{|\xi-\xi_k|<1/j\}\in\mathcal{F}_T$. (A1) and (A4) imply that for any $k\ge n$

$$\mathcal{E}[\mathbf{1}_{A_n^j}\xi|\mathcal{F}_t]\le\mathcal{E}[\mathbf{1}_{\{|\xi-\xi_k|<1/j\}}\xi|\mathcal{F}_t]\le\mathcal{E}[\xi_k+1/j|\mathcal{F}_t]=\mathcal{E}[\xi_k|\mathcal{F}_t]+1/j,\quad\text{a.s.}$$

Hence, except on a null set N_n^j, the above inequality holds for any $k\ge n$. As $k\to\infty$, it holds on $\big(N_n^j\big)^c$ that

$$\mathcal{E}[\mathbf{1}_{A_n^j}\xi|\mathcal{F}_t]\le\underline{\lim}_{k\to\infty}\mathcal{E}[\xi_k|\mathcal{F}_t]+1/j.$$

(Here it is not necessary that $\underline{\lim}_{k\to\infty}\mathcal{E}[\xi_k|\mathcal{F}_t]<\infty$, a.s.) Since $\xi_n\to\xi$, a.s. as $n\to\infty$, it is clear that $\lim_{n\to\infty}\uparrow\mathbf{1}_{A_n^j}=1$, a.s. Then (4.3) implies that $\mathcal{E}[\xi|\mathcal{F}_t]=\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n^j}\xi|\mathcal{F}_t]$ holds except on a null set N_0^j. Let $N^j=\cup_{n=0}^\infty N_n^j$. It then holds on $\big(N^j\big)^c$ that

$$\mathcal{E}[\xi|\mathcal{F}_t]=\lim_{n\to\infty}\uparrow\mathcal{E}[\mathbf{1}_{A_n^j}\xi|\mathcal{F}_t]\le\underline{\lim}_{k\to\infty}\mathcal{E}[\xi_k|\mathcal{F}_t]+1/j.$$

As $j\to\infty$, it holds except on the null set $\cup_{j=1}^\infty N^j$ that

$$\mathcal{E}[\xi|\mathcal{F}_t]\le\underline{\lim}_{n\to\infty}\mathcal{E}[\xi_n|\mathcal{F}_t].\tag{4.4}$$

Let $\nu\in\mathcal{S}_{0,T}^F$ taking values in a finite set $\{t_1<\cdots<t_n\}$. Then we can deduce from (4.4) that

$$\begin{aligned}\mathcal{E}[\xi|\mathcal{F}_\nu]&=\sum_{i=1}^n\mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\xi|\mathcal{F}_{t_i}]\le\sum_{i=1}^n\mathbf{1}_{\{\nu=t_i\}}\underline{\lim}_{n\to\infty}\mathcal{E}[\xi_n|\mathcal{F}_{t_i}]=\underline{\lim}_{n\to\infty}\sum_{i=1}^n\mathbf{1}_{\{\nu=t_i\}}\mathcal{E}[\xi_n|\mathcal{F}_{t_i}]\\&=\underline{\lim}_{n\to\infty}\mathcal{E}[\xi_n|\mathcal{F}_\nu],\quad\text{a.s.,}\end{aligned}\tag{4.5}$$

which completes the proof. □

Proof of Theorem 2.2. We first show an extension of (H2): For any $\xi, \eta \in \mathrm{Dom}^+(\mathcal{E})$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \downarrow \mathbf{1}_{A_n} = 0$, a.s., it holds a.s. that

$$\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t], \quad \text{a.s.} \tag{4.6}$$

In fact, by (A1), it holds a.s. that $\{\mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t]\}_{n\in\mathbb{N}}$ is a decreasing sequence bounded from below by $\mathcal{E}[\xi|\mathcal{F}_t]$. Hence, $\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] \geq \mathcal{E}[\xi|\mathcal{F}_t]$, a.s. Assume that $\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] > \mathcal{E}[\xi|\mathcal{F}_t]$ with a positive probability, then we can find an $\varepsilon > 0$ such that the set $A'_\varepsilon = \{\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] \geq \mathcal{E}[\xi|\mathcal{F}_t] + \varepsilon\} \in \mathcal{F}_t$ still has positive probability. For any $n \in \mathbb{N}$, (A4) implies that

$$\begin{aligned}\mathbf{1}_{A'_\varepsilon}\mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] &\geq \mathbf{1}_{A'_\varepsilon} \lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] \\ &\geq \mathbf{1}_{A'_\varepsilon}\big(\mathcal{E}[\xi|\mathcal{F}_t] + \varepsilon\big) = \mathbf{1}_{A'_\varepsilon}\mathcal{E}[\xi + \varepsilon|\mathcal{F}_t], \quad \text{a.s.}\end{aligned}$$

Applying (A1)–(A3), we obtain

$$\begin{aligned}\mathcal{E}[\mathbf{1}_{A'_\varepsilon}\xi + \mathbf{1}_{A_n}\mathbf{1}_{A'_\varepsilon}\eta] &= \mathcal{E}\left[\mathcal{E}[\mathbf{1}_{A'_\varepsilon}\xi + \mathbf{1}_{A_n}\mathbf{1}_{A'_\varepsilon}\eta|\mathcal{F}_t]\right] \\ &= \mathcal{E}\big[\mathbf{1}_{A'_\varepsilon}\mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t]\big] \geq \mathcal{E}\big[\mathbf{1}_{A'_\varepsilon}\mathcal{E}[\xi + \varepsilon|\mathcal{F}_t]\big] \\ &= \mathcal{E}\big[\mathcal{E}[\mathbf{1}_{A'_\varepsilon}(\xi + \varepsilon)|\mathcal{F}_t]\big] = \mathcal{E}\big[\mathbf{1}_{A'_\varepsilon}(\xi + \varepsilon)\big].\end{aligned}$$

Thanks to (H2) we further have

$$\mathcal{E}[\mathbf{1}_{A'_\varepsilon}\xi] = \lim_{n\to\infty} \downarrow \mathcal{E}[\mathbf{1}_{A'_\varepsilon}\xi + \mathbf{1}_{A_n}\mathbf{1}_{A'_\varepsilon}\eta] \geq \mathcal{E}\big[\mathbf{1}_{A'_\varepsilon}(\xi + \varepsilon)\big] \geq \mathcal{E}\big[\mathbf{1}_{A'_\varepsilon}\xi\big],$$

thus $\mathcal{E}[\mathbf{1}_{A'_\varepsilon}\xi] = \mathcal{E}[\mathbf{1}_{A'_\varepsilon}(\xi + \varepsilon)\big]$. Then the second part of (A1) implies that $P(A'_\varepsilon) = 0$, which yields a contradiction. Therefore, $\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + \mathbf{1}_{A_n}\eta|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t]$, a.s., proving (4.6).

Since the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded above by η, it holds a.s. that $\xi = \lim_{n\to\infty} \xi_n \leq \eta$, thus (D3) implies that $\xi \in \mathrm{Dom}(\mathcal{E})$. Then Fatou's Lemma (Theorem 2.1) implies that for any $\nu \in \mathcal{S}^F_{0,T}$,

$$\mathcal{E}[\xi|\mathcal{F}_\nu] \leq \varliminf_{n\to\infty} \mathcal{E}[\xi_n|\mathcal{F}_\nu], \quad \text{a.s.} \tag{4.7}$$

On the other hand, we first fix $t \in [0, T]$. For any $j, n \in \mathbb{N}$, define $A^j_n \triangleq \cap_{k=n}^\infty \{|\xi - \xi_k| < 1/j\} \in \mathcal{F}_T$. Then one can deduce that for any $k \geq n$

$$\mathcal{E}[\xi_k|\mathcal{F}_t] \leq \mathcal{E}[\mathbf{1}_{A^j_n}(\xi + 1/j) + \mathbf{1}_{(A^j_n)^c}\eta|\mathcal{F}_t] \leq \mathcal{E}[\xi + 1/j + \mathbf{1}_{(A^j_n)^c}(\eta - \xi)|\mathcal{F}_t], \quad \text{a.s.}$$

Hence, except on a null set N^j_n, the above inequality holds for any $k \geq n$. As $k \to \infty$, it holds on $\big(N^j_n\big)^c$ that

$$\varlimsup_{k\to\infty} \mathcal{E}[\xi_k|\mathcal{F}_t] \leq \mathcal{E}[\xi + 1/j + \mathbf{1}_{(A^j_n)^c}(\eta - \xi)|\mathcal{F}_t].$$

Since $\xi \in L^0(\mathcal{F}_T)$ and $\xi_n \to \xi$, a.s. as $n \to \infty$, it is clear that $\lim_{n\to\infty} \uparrow \mathbf{1}_{A^j_n} = 1$, a.s. Then (4.6) and (A4) imply that except on a null set N^j_0, we have

$$\lim_{n\to\infty} \downarrow \mathcal{E}[\xi + 1/j + \mathbf{1}_{(A^j_n)^c}(\eta - \xi)|\mathcal{F}_t] = \mathcal{E}[\xi + 1/j|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t] + 1/j.$$

Let $N^j = \cup_{n=0}^\infty N^j_n$, thus it holds on $\big(N^j\big)^c$ that

$$\varlimsup_{k\to\infty} \mathcal{E}[\xi_k|\mathcal{F}_t] \leq \mathcal{E}[\xi|\mathcal{F}_t] + 1/j.$$

As $j \to \infty$, it holds except on the null set $\cup_{j=1}^{\infty} N^j$ that $\overline{\lim}_{n\to\infty}\mathcal{E}[\xi_n|\mathcal{F}_t] \le \mathcal{E}[\xi|\mathcal{F}_t]$. Then for any $\nu \in \mathcal{S}_{0,T}^F$, using an argument similar to (4.5) yields that

$$\overline{\lim_{n\to\infty}}\, \mathcal{E}[\xi_n|\mathcal{F}_\nu] \le \mathcal{E}[\xi|\mathcal{F}_\nu], \quad \text{a.s.,}$$

which together with (4.7) proves the theorem. □

Proof of Theorem 2.3. Let $F = \{t_1 < t_2 < \cdots < t_d\}$ be any finite subset of $\mathcal{D}_T$. For $j = 1, \ldots, d$, we define $A_j = \{\nu_j < T\} \in \mathcal{F}_{\nu_j}$, clearly, $A_j \supset A_{j+1}$. Let $d' = \left\lfloor \frac{d}{2} \right\rfloor$, one can deduce that $U_F\big(a, b; X\big) = \sum_{j=1}^{d'} \mathbf{1}_{A_{2j}}$ and that

$$\begin{aligned}\mathbf{1}_{\cup_{j=1}^{d'}(A_{2j-1}\backslash A_{2j})}(X_T - a) &\ge \mathbf{1}_{\cup_{j=1}^{d'}(A_{2j-1}\backslash A_{2j})}\mathbf{1}_{\{X_T<a\}}(X_T - a)\\ &\ge \mathbf{1}_{\{X_T<a\}}(X_T - a) = -(a - X_T)^+.\end{aligned}$$

Since $X_T \in \mathrm{Dom}(\mathcal{E})$ and $L^\infty(\mathcal{F}_T) \subset \mathrm{Dom}(\mathcal{E})$ (by Lemma 2.1), we can deduce from (D2) that

$$(b-a)U_F\big(a, b; X\big) - (a - X_T)^+ = \sum_{j=1}^{d'} \mathbf{1}_{A_{2j}}(b-a) + \mathbf{1}_{\{X_T<a\}}(X_T - a) \in \mathrm{Dom}(\mathcal{E}).$$

Then Proposition 2.4(1)–(3) and Proposition 2.3 imply that

$$\begin{aligned}&\mathcal{E}\Big[(b-a)U_F\big(a, b; X\big) - (a - X_T)^+|\mathcal{F}_{\nu_{2d'}}\Big]\\ &\quad\le (b-a)\sum_{j=1}^{d'} \mathbf{1}_{A_{2j}} + \mathcal{E}\Big[\mathbf{1}_{\cup_{j=1}^{d'}(A_{2j-1}\backslash A_{2j})}(X_T - a)|\mathcal{F}_{\nu_{2d'}}\Big]\\ &\quad= (b-a)\sum_{j=1}^{d'} \mathbf{1}_{A_{2j}} + \mathbf{1}_{\cup_{j=1}^{d'}(A_{2j-1}\backslash A_{2j})}\big(\mathcal{E}[X_T|\mathcal{F}_{\nu_{2d'}}] - a\big)\\ &\quad\le (b-a)\sum_{j=1}^{d'} \mathbf{1}_{A_{2j}} + \mathbf{1}_{\cup_{j=1}^{d'}(A_{2j-1}\backslash A_{2j})}\big(X_{\nu_{2d'}} - a\big), \quad \text{a.s.}\end{aligned}$$

Applying $\mathcal{E}[\cdot|\mathcal{F}_{\nu_{2d'-1}}]$ to the above inequality, using Proposition 2.4(1)–(3) and Proposition 2.3 again, we obtain

$$\begin{aligned}&\mathcal{E}\Big[(b-a)U_F\big(a, b; X\big) - (a - X_T)^+\Big|\,\mathcal{F}_{\nu_{2d'-1}}\Big]\\ &\quad\le \mathcal{E}\left[(b-a)\sum_{j=1}^{d'-1} \mathbf{1}_{A_{2j}} + \big(\mathbf{1}_{A_{2d'-1}} + \mathbf{1}_{\cup_{j=1}^{d'-1}(A_{2j-1}\backslash A_{2j})}\big)\big(X_{\nu_{2d'}} - a\big)\,\Big|\,\mathcal{F}_{\nu_{2d'-1}}\right]\\ &\quad= (b-a)\sum_{j=1}^{d'-1} \mathbf{1}_{A_{2j}} + \mathcal{E}\Big[\big(\mathbf{1}_{A_{2d'-1}} + \mathbf{1}_{\cup_{j=1}^{d'-1}(A_{2j-1}\backslash A_{2j})}\big)\big(X_{\nu_{2d'}} - a\big)\,\Big|\,\mathcal{F}_{\nu_{2d'-1}}\Big]\\ &\quad= (b-a)\sum_{j=1}^{d'-1} \mathbf{1}_{A_{2j}} + \big(\mathbf{1}_{A_{2d'-1}} + \mathbf{1}_{\cup_{j=1}^{d'-1}(A_{2j-1}\backslash A_{2j})}\big)\big(\mathcal{E}[X_{\nu_{2d'}}|\mathcal{F}_{\nu_{2d'-1}}] - a\big)\\ &\quad\le (b-a)\sum_{j=1}^{d'-1} \mathbf{1}_{A_{2j}} + \big(\mathbf{1}_{A_{2d'-1}} + \mathbf{1}_{\cup_{j=1}^{d'-1}(A_{2j-1}\backslash A_{2j})}\big)(X_{\nu_{2d'-1}} - a)\\ &\quad\le (b-a)\sum_{j=1}^{d'-1} \mathbf{1}_{A_{2j}} + \mathbf{1}_{\cup_{j=1}^{d'-1}(A_{2j-1}\backslash A_{2j})}(X_{\nu_{2d'-1}} - a), \quad \text{a.s.,}\end{aligned}$$

where we used the fact that $X_{\nu_{2d'}} > b$ on $A_{2d'}$ in the first inequality and the fact that $X_{\nu_{2d'-1}} < a$ on $A_{2d'-1}$ in the last inequality. Similarly, applying $\mathcal{E}[\cdot|\mathcal{F}_{\nu_{2d'-2}}]$ to the above inequality yields that

$$\mathcal{E}\big[(b-a)U_F\big(a,b;X\big)-(a-X_T)^+|\mathcal{F}_{\nu_{2d'-2}}\big] \le (b-a)\sum_{j=1}^{d'-1}\mathbf{1}_{A_{2j}}+\mathbf{1}_{\cup_{j=1}^{d'-1}(A_{2j-1}\backslash A_{2j})}(X_{\nu_{2d'-2}}-a), \quad \text{a.s.}$$

Iteratively applying $\mathcal{E}[\cdot|\mathcal{F}_{\nu_{2d'-3}}]$, $\mathcal{E}[\cdot|\mathcal{F}_{\nu_{2d'-4}}]$ and so on, we eventually obtain that

$$\mathcal{E}\big[(b-a)U_F\big(a,b;X\big)-(a-X_T)^+\big]\le 0. \tag{4.8}$$

We assume first that $X_T \ge c$, a.s. for some $c \in \mathbb{R}$. Since $(a-X_T)^+ \le |a|+|c|$, it directly follows from (A4) that

$$0\ge \mathcal{E}\big[(b-a)U_F\big(a,b;X\big)-(a-X_T)^+\big]\ge \mathcal{E}\big[(b-a)U_F\big(a,b;X\big)\big]-(|a|+|c|). \tag{4.9}$$

Let $\{F_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathcal{D}_T$ with $\cup_{n\in\mathbb{N}}F_n = \mathcal{D}_T$, thus $\lim_{n\to\infty}\uparrow U_{F_n}(a,b;X)=U_{\mathcal{D}_T}(a,b;X)$. Fix $M\in\mathbb{N}$, we see that

$$\lim_{n\to\infty}\uparrow \mathbf{1}_{\{U_{F_n}(a,b;X)>M\}}=\mathbf{1}_{\cup_n\{U_{F_n}(a,b;X)>M\}}=\mathbf{1}_{\{U_{\mathcal{D}_T}(a,b;X)>M\}}. \tag{4.10}$$

For any $n\in\mathbb{N}$, we know from (4.9) that $\mathcal{E}[(b-a)M\mathbf{1}_{\{U_{F_n}(a,b;X)>M\}}]\le\mathcal{E}[(b-a)U_{F_n}(a,b;X)]\le |a|+|c|$, thus Fatou's Lemma (Theorem 2.1) implies that

$$\mathcal{E}\big[(b-a)M\mathbf{1}_{\{U_{\mathcal{D}_T}(a,b;X)=\infty\}}\big]\le\mathcal{E}\big[(b-a)M\mathbf{1}_{\{U_{\mathcal{D}_T}(a,b;X)>M\}}\big] \le \lim_{n\to\infty}\uparrow\mathcal{E}\big[(b-a)M\mathbf{1}_{\{U_{F_n}(a,b;X)>M\}}\big]\le|a|+|c|. \tag{4.11}$$

On the other hand, if $\mathcal{E}[\cdot]$ is concave, then we can deduce from (4.8) that

$$0\ge\mathcal{E}\big[(b-a)U_F\big(a,b;X\big)-(a-X_T)^+\big] \ge\frac{1}{2}\mathcal{E}\big[2(b-a)U_F\big(a,b;X\big)\big]+\frac{1}{2}\mathcal{E}\big[-2(a-X_T)^+\big].$$

Mimicking the arguments in (4.10) and (4.11), we obtain that

$$\mathcal{E}\big[(b-a)2M\mathbf{1}_{\{U_{\mathcal{D}_T}(a,b;X)=\infty\}}\big]\le-\mathcal{E}\big[-2(a-X_T)^+\big]$$

where $-2(a-X_T)^+=\mathbf{1}_{\{X_T<a\}}2(X_T-a)\in\mathrm{Dom}(\mathcal{E})$ thanks to (D2). Also note that (A1) and Proposition 2.4(5) imply that $\mathcal{E}\big[-2(a-X_T)^+\big]\le\mathcal{E}[0]=0$.

Using (H0) in both cases above yields that $P(U_{\mathcal{D}_T}(a,b;X)=\infty)=0$, i.e., $U_{\mathcal{D}_T}(a,b;X)<\infty$, a.s. Then a classical argument (see e.g. [8, Proposition 1.3.14]) shows that

$$P\Big(\text{both}\lim_{s\nearrow t,\,s\in\mathcal{D}_T}X_s\text{ and}\lim_{s\searrow t,\,s\in\mathcal{D}_T}X_s\text{ exist for any }t\in[0,T]\Big)=1.$$

This completes the proof. □

Proof of Proposition 2.5. We can deduce from (2.4) that except on a null set N

$$X_t^-=\lim_{n\to\infty}X_{q_n^-(t)}\le\operatorname*{esssup}_{s\in\mathcal{D}_T}X_s\quad\text{and}$$
$$X_t^+=\lim_{n\to\infty}X_{q_n^+(t)}\le\operatorname*{esssup}_{s\in\mathcal{D}_T}X_s\quad\text{for any }t\in[0,T], \tag{4.12}$$

thus $X^-_\nu = \lim_{n\to\infty} X_{q^-_n(\nu)} \le \operatorname*{esssup}_{s\in\mathcal{D}_T} X_s$ and

$$X^+_\nu = \lim_{n\to\infty} X_{q^+_n(\nu)} \le \operatorname*{esssup}_{s\in\mathcal{D}_T} X_s \quad \text{for any } \nu \in \mathcal{S}_{0,T}. \tag{4.13}$$

Proof of (1): Case I. For any $\nu \in \mathcal{S}_{0,T}$, if $\operatorname{esssup}_{s\in\mathcal{D}_T} X_s \in \mathrm{Dom}^+(\mathcal{E})$, (D3) and (4.13) directly imply that both X^-_ν and X^+_ν belong to $\mathrm{Dom}(\mathcal{E})$.
Case II. Assume that $\mathcal{E}$ satisfies (2.5). For any $n \in \mathbb{N}$, since X is an $\mathcal{E}$-supermartingale and since $q^-_n(\nu), q^+_n(\nu) \in \mathcal{S}^F_{0,T}$, Corollary 2.1 and Proposition 2.3 imply that

$$\mathcal{E}\big[X_{q^+_n(\nu)}\big] = \mathcal{E}\Big[\mathcal{E}\big[X_{q^+_n(\nu)}|\mathcal{F}_{q^+_{n+1}(\nu)}\big]\Big] \le \mathcal{E}\big[X_{q^+_{n+1}(\nu)}\big] \le X_0$$
$$\text{and} \quad \mathcal{E}\big[X_{q^-_{n+1}(\nu)}\big] = \mathcal{E}\Big[\mathcal{E}\big[X_{q^-_{n+1}(\nu)}|\mathcal{F}_{q^-_n(\nu)}\big]\Big] \le \mathcal{E}\big[X_{q^-_n(\nu)}\big] \le X_0.$$

Hence, $\big\{\mathcal{E}\big[X_{q^+_n(\nu)}\big]\big\}_{n\in\mathbb{N}}$ is an increasing non-negative sequence and $\big\{\mathcal{E}\big[X_{q^-_n(\nu)}\big]\big\}_{n\in\mathbb{N}}$ is a decreasing non-negative sequence, both of which are bounded from above by $X_0 \in [0,\infty)$. (2.5) and (4.13) then imply that both X^-_ν and X^+_ν belong to $\mathrm{Dom}(\mathcal{E})$, proving statement (1).
Proof of (2): Now suppose that $X^+_t \in \mathrm{Dom}^+(\mathcal{E})$ for any $t \in [0,T]$. First, we show that for $t \in [0,T]$ and $A \in \mathcal{F}_t$

$$\mathcal{E}\big[\mathbf{1}_A X^+_t\big] = \lim_{n\to\infty} \mathcal{E}\big[\mathbf{1}_A X_{q^+_n(t)}\big]. \tag{4.14}$$

Since the distribution function $x \mapsto P\{X^+_t \le x\}$ jumps up at most on a countable subset S of $[0,\infty)$, we can find a sequence $\{K_j\}_{j=1}^\infty \subset [0,\infty)\setminus S$ increasing to ∞. Fix $m, j \in \mathbb{N}$, (A1)–(A3) imply that for any $n \ge m$

$$\begin{aligned}
&\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}(X_{q^+_n(t)} \wedge K_j)\big]\\
&\quad = \mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}X_{q^+_n(t)}\big] \ge \mathcal{E}\Big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}\mathcal{E}\big[X_{q^+_m(t)}|\mathcal{F}_{q^+_n(t)}\big]\Big]\\
&\quad = \mathcal{E}\Big[\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}X_{q^+_m(t)}|\mathcal{F}_{q^+_n(t)}\big]\Big] = \mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}X_{q^+_m(t)}\big].
\end{aligned}$$

Since $K_j \notin S$, $P\{X^+_t = K_j\} = 0$, one can easily deduce from (4.12) that $\lim_{n\to\infty}\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}} = \mathbf{1}_{\{X^+_t<K_j\}}$, a.s. (In fact, for almost every $\omega \in \{X^+_t < K_j\}$ (resp. $\{X^+_t > K_j\}$), there exists an $N(\omega) \in \mathbb{N}$ such that $X_{q^+_n(t)} <$ (resp. $>$) K_j for any $n \ge N(\omega)$, which means $\lim_{n\to\infty}\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}(\omega) = 1(\text{resp. } 0) = \mathbf{1}_{\{X^+_t<K_j\}}(\omega)$). Applying the Dominated Convergence Theorem (Theorem 2.2) twice, we obtain

$$\begin{aligned}
&\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X^+_t<K_j\}}X^+_t\big]\\
&\quad = \mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X^+_t<K_j\}}(X^+_t \wedge K_j)\big] = \lim_{n\to\infty}\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}(X_{q^+_n(t)} \wedge K_j)\big]\\
&\quad \ge \lim_{n\to\infty}\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X_{q^+_n(t)}<K_j\}}X_{q^+_m(t)}\big] = \mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X^+_t<K_j\}}X_{q^+_m(t)}\big].
\end{aligned}$$

Since $\lim_{j\to\infty}\uparrow \mathbf{1}_{\{X^+_t<K_j\}} = 1$, a.s., the Dominated Convergence Theorem again implies that

$$\mathcal{E}\big[\mathbf{1}_A X^+_t\big] = \lim_{j\to\infty}\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X^+_t<K_j\}}X^+_t\big] \ge \lim_{j\to\infty}\mathcal{E}\big[\mathbf{1}_A\mathbf{1}_{\{X^+_t<K_j\}}X_{q^+_m(t)}\big] = \mathcal{E}\big[\mathbf{1}_A X_{q^+_m(t)}\big],$$

which leads to that $\mathcal{E}\big[\mathbf{1}_A X^+_t\big] \ge \overline{\lim}_{m\to\infty}\mathcal{E}\big[\mathbf{1}_A X_{q^+_m(t)}\big]$. Fatou's Lemma (Theorem 2.1) gives the reverse inequality, thus proving (4.14). Since X is an $\mathcal{E}$-supermartingale, using (4.14), (A2) and

(A3), we obtain

$$\mathcal{E}\big[\mathbf{1}_A X_t^+\big] = \lim_{n\to\infty} \mathcal{E}\big[\mathbf{1}_A X_{q_n^+(t)}\big] = \lim_{n\to\infty} \mathcal{E}\big[\mathcal{E}[\mathbf{1}_A X_{q_n^+(t)}|\mathcal{F}_t]\big]$$
$$= \lim_{n\to\infty} \mathcal{E}\Big[\mathbf{1}_A \mathcal{E}\big[X_{q_n^+(t)}|\mathcal{F}_t\big]\Big] \le \mathcal{E}\big[\mathbf{1}_A X_t\big]$$

for any $A \in \mathcal{F}_t$, which further implies that $X_t^+ \le X_t$, a.s. thanks to Proposition 2.2(3).

Next, we show that X^+ is an $\mathcal{E}$-supermartingale: For any $0 \le s < t \le T$, it is clear that $q_n^+(s) \le q_n^+(t)$ for any $n \in \mathbb{N}$. For any $A \in \mathcal{F}_s$, (A3) and Corollary 2.1 imply that for any $n \in \mathbb{N}$

$$\mathcal{E}\big[\mathbf{1}_A X_{q_n^+(s)}\big] \ge \mathcal{E}\Big[\mathbf{1}_A \mathcal{E}\big[X_{q_n^+(t)}|\mathcal{F}_{q_n^+(s)}\big]\Big] = \mathcal{E}\Big[\mathcal{E}\big[\mathbf{1}_A X_{q_n^+(t)}|\mathcal{F}_{q_n^+(s)}\big]\Big] = \mathcal{E}\big[\mathbf{1}_A X_{q_n^+(t)}\big].$$

As $n \to \infty$, (4.14), (A2) and (A3) imply that

$$\mathcal{E}\big[\mathbf{1}_A X_s^+\big] = \lim_{n\to\infty} \mathcal{E}\big[\mathbf{1}_A X_{q_n^+(s)}\big] \ge \lim_{n\to\infty} \mathcal{E}\big[\mathbf{1}_A X_{q_n^+(t)}\big] = \mathcal{E}\big[\mathbf{1}_A X_t^+\big] = \mathcal{E}\big[\mathcal{E}[\mathbf{1}_A X_t^+|\mathcal{F}_s]\big]$$
$$= \mathcal{E}\big[\mathbf{1}_A \mathcal{E}[X_t^+|\mathcal{F}_s]\big].$$

Then Proposition 2.2(3) implies that $X_s^+ \ge \mathcal{E}[X_t^+|\mathcal{F}_s]$, a.s., thus $\{X_t^+\}_{t\in[0,T]}$ is an RCLL $\mathcal{E}$-supermartingale.

Proof of (3): If $t \mapsto \mathcal{E}[X_t]$ is right continuous, for any $t \in [0, T]$, (4.14) implies that

$$\mathcal{E}[X_t^+] = \lim_{n\to\infty} \mathcal{E}\big[X_{q_n^+(t)}\big] = \mathcal{E}[X_t].$$

Then the second part of (A1) imply that $X_t^+ = X_t$, a.s., which means that X^+ is an RCLL modification of X. On the other hand, if $\widetilde{X}$ is a right-continuous modification of X, we see from (2.4) that except on a null set $\tilde{N}$

$$X_t^+ = \lim_{n\to\infty} X_{q_n^+(t)}, \qquad \widetilde{X}_t = \lim_{n\to\infty} \widetilde{X}_{q_n^+(t)}, \qquad \widetilde{X}_t = X_t, \quad \text{and}$$
$$\widetilde{X}_{q_n^+(t)} = X_{q_n^+(t)} \quad \text{for any } n \in \mathbb{N}.$$

Putting them together, it holds on $\tilde{N}^c$ that

$$X_t^+ = \lim_{n\to\infty} X_{q_n^+(t)} = \lim_{n\to\infty} \widetilde{X}_{q_n^+(t)} = \widetilde{X}_t = X_t. \tag{4.15}$$

Since X is an $\mathcal{E}$-supermartingale, (A2) implies that for any $0 \le t_1 < t_2 \le T, \mathcal{E}[X_{t_1}] \ge \mathcal{E}\big[\mathcal{E}[X_{t_2}|\mathcal{F}_{t_1}]\big] = \mathcal{E}[X_{t_2}]$, which shows that the function $t \mapsto \mathcal{E}[X_t]$ is decreasing. Then (4.14) and (4.15) imply that for any $t \in [0, T]$

$$\mathcal{E}[X_t] \ge \lim_{s\downarrow t} \mathcal{E}[X_s] = \lim_{n\to\infty} \mathcal{E}\big[X_{q_n^+(t)}\big] = \mathcal{E}[X_t^+] = \mathcal{E}[X_t],$$

thus $\lim_{s\downarrow t} \mathcal{E}[X_s] = \mathcal{E}[X_t]$, i.e., the function $t \mapsto \mathcal{E}[X_t]$ is right continuous. $\square$

Proof of Corollary 2.2. Since $\operatorname{essinf}_{t\in[0,T]} X_t \ge c$, a.s., we can deduce from (A4) that $X^c \triangleq \{X_t - c\}_{t\in[0,T]}$ is a non-negative $\mathcal{E}$-supermartingale. If $\operatorname{esssup}_{t\in\mathcal{D}_T} X_t \in \mathrm{Dom}^\#(\mathcal{E})$ ((D2) implies that $\operatorname{esssup}_{t\in\mathcal{D}_T} X_t \in \mathrm{Dom}^\#(\mathcal{E})$ is equivalent to $\operatorname{esssup}_{t\in\mathcal{D}_T} X_t^c \in \mathrm{Dom}^+(\mathcal{E})$) or if (2.5) holds, Proposition 2.5(1) shows that for any $\nu \in \mathcal{S}_{0,T}$, both $(X^c)_\nu^-$ and $(X^c)_\nu^+$ belong to $\mathrm{Dom}^+(\mathcal{E})$. Because

$$(X^c)_t^- = X_t^- - c \quad \text{and} \quad (X^c)_t^+ = X_t^+ - c, \quad \forall\, t \in [0, T], \tag{4.16}$$

(D2) and the non-negativity of $(X^c)^-$, $(X^c)^+$ imply that

$$X_\nu^- = (X^c)_\nu^- + c \in \mathrm{Dom}^\#(\mathcal{E}) \quad \text{and} \quad X_\nu^+ = (X^c)_\nu^+ + c \in \mathrm{Dom}^\#(\mathcal{E}).$$

On the other hand, if $X_t^+ \in \mathrm{Dom}^{\#}(\mathcal{E})$ for any $t \in [0,T]$, (D2) implies that the non-negative random variable $(X^c)_t^+ = X_t^+ - c$ belongs to $\mathrm{Dom}^+(\mathcal{E})$. Hence, Proposition 2.5(2) show that $(X^c)^+$ is an RCLL $\mathcal{E}$-supermartingale such that for any $t \in [0,T]$, $(X^c)_t^+ \le X_t^c$, a.s. Then (4.16), (2.8) and (A4) imply that X^+ is an RCLL $\widetilde{\mathcal{E}}$-supermartingale such that for any $t \in [0,T]$, $X_t^+ \le X_t$, a.s. Moreover, if $t \mapsto \widetilde{\mathcal{E}}[X_t]$ is a right-continuous function (which is equivalent to the right continuity of $t \mapsto \mathcal{E}[X_t^c]$), then we know from Proposition 2.5(2) that for any $t \in [0,T]$, $(X^c)_t^+ = X_t^c$, a.s., or equivalently, $X_t^+ = X_t$, a.s. Conversely, if X has a right-continuous modification, so does X^c, then Proposition 2.5(2) once again shows that $t \mapsto \mathcal{E}[X_t^c]$ is right continuous, which is equivalent to the right continuity of $t \mapsto \widetilde{\mathcal{E}}[X_t]$. This completes the proof. □

Proof of Theorem 2.4. We shall only consider the $\widetilde{\mathcal{E}}$-supermartingale case, as the other cases can be deduced easily by similar arguments. Fix $t \in [0,T]$, we let $\{\nu_n^t\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{S}_{t,T}^F$ such that $\lim_{n\to\infty} \nu_n^t = \nu \vee t$. Since $\mathrm{essinf}_{t\in\mathcal{D}_T} X_t \ge c$, a.s., it holds a.s. that $X_t \ge c$ for each $t \in \mathcal{D}_T$. The right-continuity of the process X then implies that except on a null set N, $X_t \ge c$ for any $t \in [0,T]$. Thus we see from (A4) that $X^c \triangleq \{X_t - c\}_{t\in[0,T]}$ is a non-negative $\mathcal{E}$-supermartingale. For any $n \in \mathbb{N}$ and $A \in \mathcal{F}_t \subset \mathcal{F}_{\nu\vee t}$, (A2), (A3) and Proposition 2.3 imply that

$$\mathcal{E}[\mathbf{1}_A X^c_{\nu_n^t}] = \mathcal{E}\big[\mathcal{E}[\mathbf{1}_A X^c_{\nu_n^t}|\mathcal{F}_t]\big] = \mathcal{E}\big[\mathbf{1}_A \mathcal{E}[X^c_{\nu_n^t}|\mathcal{F}_t]\big] \le \mathcal{E}[\mathbf{1}_A X_t^c]. \tag{4.17}$$

We also have that $\mathcal{E}\big[\mathbf{1}_A X^c_{\nu\vee t}\big] = \lim_{n\to\infty} \mathcal{E}\big[\mathbf{1}_A X^c_{\nu_n^t}\big]$. The proof is similar to that of (4.14). (We only need to replace X_t^+ by $X^c_{\nu\vee t}$ and $X_{q_n^+(t)}$ by $X^c_{\nu_n^t}$ in the proof of (4.14)). As $n \to \infty$ in (4.17), (A2) and (A3) imply that

$$\mathcal{E}[\mathbf{1}_A X_t^c] \ge \lim_{n\to\infty} \mathcal{E}\big[\mathbf{1}_A X^c_{\nu_n^t}\big] = \mathcal{E}[\mathbf{1}_A X^c_{\nu\vee t}] = \mathcal{E}\big[\mathcal{E}[\mathbf{1}_A X^c_{\nu\vee t}|\mathcal{F}_t]\big] = \mathcal{E}\big[\mathbf{1}_A \mathcal{E}[X^c_{\nu\vee t}|\mathcal{F}_t]\big].$$

Applying Proposition 2.2(3), we obtain that $\mathcal{E}[X^c_{\nu\vee t}|\mathcal{F}_t] \le X_t^c$, a.s. Then (A4) and (2.8) imply that

$$\widetilde{\mathcal{E}}[X_{\nu\vee t}|\mathcal{F}_t] = \mathcal{E}[X_{\nu\vee t}|\mathcal{F}_t] = \mathcal{E}[X^c_{\nu\vee t} + c|\mathcal{F}_t] = \mathcal{E}[X^c_{\nu\vee t}|\mathcal{F}_t] + c \le X_t^c + c = X_t, \quad \text{a.s.}$$

Since $\{\nu \le t\} \in \mathcal{F}_t$, we can deduce from (A3) and (A4) that

$$\begin{aligned}\widetilde{\mathcal{E}}[X_\nu|\mathcal{F}_t] &= \widetilde{\mathcal{E}}\big[\mathbf{1}_{\{\nu>t\}} X_{\nu\vee t} + \mathbf{1}_{\{\nu\le t\}} X_{\nu\wedge t}|\mathcal{F}_t\big] = \mathbf{1}_{\{\nu>t\}} \widetilde{\mathcal{E}}\big[X_{\nu\vee t}|\mathcal{F}_t\big] + \mathbf{1}_{\{\nu\le t\}} X_{\nu\wedge t} \\ &\le \mathbf{1}_{\{\nu>t\}} X_t + \mathbf{1}_{\{\nu\le t\}} X_{\nu\wedge t} = X_{\nu\wedge t} \quad \text{a.s.}\end{aligned}$$

Hence, we can find a null set $\widetilde{N}$ such that except on $\widetilde{N}^c$

$$\widetilde{\mathcal{E}}[X_\nu|\mathcal{F}_t] \le X_{\nu\wedge t}, \qquad \text{for any } t \in \mathcal{D}_T \text{ and the paths of } \widetilde{\mathcal{E}}[X_\nu|\mathcal{F}_\cdot] \text{ and } X_{\nu\wedge\cdot} \text{ are all RCLL.}$$

As a result, on $\widetilde{N}^c$

$$\widetilde{\mathcal{E}}[X_\nu|\mathcal{F}_t] \le X_{\nu\wedge t}, \quad \forall\, t \in [0,T], \qquad \text{thus } \widetilde{\mathcal{E}}[X_\nu|\mathcal{F}_\sigma] \le X_{\nu\wedge\sigma}, \quad \forall\, \sigma \in \mathcal{S}_{0,T}. \quad \square$$

Proof of Proposition 2.7. (1) If $\xi \le \eta$, a.s., by (A1), it holds except on a null set N that

$$\widetilde{\mathcal{E}}[\xi|\mathcal{F}_t] \le \widetilde{\mathcal{E}}[\eta|\mathcal{F}_t], \quad \text{for any } t \in \mathcal{D}_T \text{ and that the paths of } \widetilde{\mathcal{E}}[\xi|\mathcal{F}_\cdot] \text{ and } \widetilde{\mathcal{E}}[\eta|\mathcal{F}_\cdot] \text{ are all RCLL,}$$

which implies that on N^c

$$\widetilde{\mathcal{E}}[\xi|\mathcal{F}_t] \le \widetilde{\mathcal{E}}[\eta|\mathcal{F}_t], \quad \forall\, t \in [0,T], \qquad \text{thus } \widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] \le \widetilde{\mathcal{E}}[\eta|\mathcal{F}_\nu].$$

Moreover, if $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\sigma] = \widetilde{\mathcal{E}}[\eta|\mathcal{F}_\sigma]$, a.s. for some $\sigma \in \mathcal{S}_{0,T}$, we can apply (2.8) and Corollary 2.3 to get

$$\mathcal{E}[\xi] = \widetilde{\mathcal{E}}[\xi] = \widetilde{\mathcal{E}}\big[\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\sigma]\big] = \widetilde{\mathcal{E}}\big[\widetilde{\mathcal{E}}[\eta|\mathcal{F}_\sigma]\big] = \widetilde{\mathcal{E}}[\eta] = \mathcal{E}[\eta].$$

Then (A4) implies that $\mathcal{E}[\xi - c(\xi)] = \mathcal{E}[\xi] - c(\xi) = \mathcal{E}[\eta] - c(\xi) = \mathcal{E}[\eta - c(\xi)]$. Clearly, $0 \le \xi - c(\xi) \le \eta - c(\xi)$, a.s. The second part of (A1) then implies that $\xi - c(\xi) = \eta - c(\xi)$, a.s., i.e. $\xi = \eta$, a.s., proving (1).
(2) For any $A \in \mathcal{F}_\nu$ and $\eta \in \mathrm{Dom}^\#_\nu(\mathcal{E})$, we let $\{\nu_n\}_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{S}^F_{0,T}$ such that $\lim_{n\to\infty} \downarrow \nu_n = \nu$, a.s. For any $n \in \mathbb{N}$, since $A \in \mathcal{F}_{\nu_n}$ and $\eta \in \mathrm{Dom}^\#_{\nu_n}(\mathcal{E})$, Proposition 2.4(2) and (3) imply that

$$\widetilde{\mathcal{E}}[\mathbf{1}_A\xi|\mathcal{F}_{\nu_n}] = \mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_{\nu_n}], \quad \text{and} \quad \widetilde{\mathcal{E}}[\xi + \eta|\mathcal{F}_{\nu_n}] = \widetilde{\mathcal{E}}[\xi|\mathcal{F}_{\nu_n}] + \eta, \quad \text{a.s.} \tag{4.18}$$

Then we can find a null set N' such that except on N'

(4.18) holds for any $n \in \mathbb{N}$ and the paths of $\widetilde{\mathcal{E}}[\mathbf{1}_A\xi|\mathcal{F}_\cdot]$, $\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\cdot]$ and $\widetilde{\mathcal{E}}[\xi + \eta|\mathcal{F}_\cdot]$ are all RCLL.

As $n \to \infty$, it holds on $(N')^c$ that

$$\widetilde{\mathcal{E}}[\mathbf{1}_A\xi|\mathcal{F}_\nu] = \lim_{n\to\infty} \widetilde{\mathcal{E}}[\mathbf{1}_A\xi|\mathcal{F}_{\nu_n}] = \lim_{n\to\infty} \mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_{\nu_n}] = \mathbf{1}_A\widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu],$$

$$\text{and that} \quad \widetilde{\mathcal{E}}[\xi + \eta|\mathcal{F}_\nu] = \lim_{n\to\infty} \widetilde{\mathcal{E}}[\xi + \eta|\mathcal{F}_{\nu_n}] = \lim_{n\to\infty} \widetilde{\mathcal{E}}[\xi|\mathcal{F}_{\nu_n}] + \eta = \widetilde{\mathcal{E}}[\xi|\mathcal{F}_\nu] + \eta,$$

proving (2) and (3). Proofs of (4) and (5) are similar to those of Proposition 2.2(1) and (2). The proofs can be carried out by applying the just obtained "Zero–one Law" and "Translation Invariance". □

Proof of Lemma 3.1. (1) Let $\mathcal{E}$ be a positively-convex **F**-expectation. For any $A \in \mathcal{F}_T$ and $n \in \mathbb{N}$, (D1) and (D2) imply that $\mathbf{1}_A, n\mathbf{1}_A \in \mathrm{Dom}(\mathcal{E})$. Then the positive-convexity of $\mathcal{E}$ and Proposition 2.2(2) show that

$$\begin{aligned} \mathcal{E}[\mathbf{1}_A] &= \mathcal{E}\left[\frac{1}{n}\big(n\mathbf{1}_A\big)\right] \le \frac{1}{n}\mathcal{E}[n\mathbf{1}_A] + \left(1 - \frac{1}{n}\right)\mathcal{E}[0] \\ &= \frac{1}{n}\mathcal{E}[n\mathbf{1}_A] + \left(1 - \frac{1}{n}\right) \cdot 0 = \frac{1}{n}\mathcal{E}[n\mathbf{1}_A]. \end{aligned} \tag{4.19}$$

Since $P(A) > 0$, one can deduce from the second part of (A1) that $\mathcal{E}[\mathbf{1}_A] > 0$. Letting $n \to \infty$ in (4.19) yields that

$$\lim_{n\to\infty} \mathcal{E}[n\mathbf{1}_A] \ge \lim_{n\to\infty} n\mathcal{E}[\mathbf{1}_A] = \infty,$$

thus $\mathcal{E}$ satisfies (H0). Moreover, for any $\xi, \eta \in \mathrm{Dom}^\#(\mathcal{E})$, $\lambda \in (0, 1)$ and $t \in [0, T]$, we can deduce from (2.8), (A4) and the positive-convexity of $\mathcal{E}$ that

$$\begin{aligned} &\widetilde{\mathcal{E}}[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_t] = \mathcal{E}[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_t] \\ &\quad = \mathcal{E}[\lambda\big(\xi - c(\xi)\big) + (1-\lambda)\big(\eta - c(\eta)\big)|\mathcal{F}_t] + \lambda c(\xi) + (1-\lambda)c(\eta) \\ &\quad \le \lambda\mathcal{E}[\xi - c(\xi)|\mathcal{F}_t] + \lambda c(\xi) + (1-\lambda)\mathcal{E}[\eta - c(\eta)|\mathcal{F}_t] + (1-\lambda)c(\eta) \\ &\quad = \lambda\mathcal{E}[\xi|\mathcal{F}_t] + (1-\lambda)\mathcal{E}[\eta|\mathcal{F}_t] = \lambda\widetilde{\mathcal{E}}[\xi|\mathcal{F}_t] + (1-\lambda)\widetilde{\mathcal{E}}[\eta|\mathcal{F}_t], \quad \text{a.s.,} \end{aligned}$$

which shows that $\widetilde{\mathcal{E}}$ is convex in the sense of (3.1). On the other hand, if $\widetilde{\mathcal{E}}$ satisfies (3.1), since $\mathrm{Dom}^+(\mathcal{E}) \subset \mathrm{Dom}^\#(\mathcal{E})$, one can easily deduce from (2.8) that $\mathcal{E}$ is positively-convex. □

Proof of Proposition 3.1. We first check that $\mathcal{E}^{\nu}_{i,j}$ satisfies (A1)–(A4). Let $\xi,\eta\in\Lambda^{\#}$ and $t\in[0,T]$.
(1) If $\xi\le\eta$, a.s., applying Proposition 2.7(1) to $\widetilde{\mathcal{E}}_j$ yields that $\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}]\le\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_{\nu\vee t}]$, a.s. Then (A1) of $\widetilde{\mathcal{E}}_i$ and (3.3) imply that

$$\mathcal{E}^{\nu}_{i,j}\big[\xi|\mathcal{F}_t\big]=\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big]\le\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big]=\mathcal{E}^{\nu}_{i,j}\big[\eta|\mathcal{F}_t\big],\quad\text{a.s.}$$

Moreover, if $0\le\xi\le\eta$ a.s. and $\mathcal{E}^{\nu}_{i,j}[\xi]=\mathcal{E}^{\nu}_{i,j}[\eta]$ (i.e. $\widetilde{\mathcal{E}}_i[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]]=\widetilde{\mathcal{E}}_i[\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_\nu]]$ by (3.3)), the second part of (A1) implies that $\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]=\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_\nu]$, a.s. Further applying the second part of Proposition 2.7(1), we obtain $\xi=\eta$, a.s., proving (A1) for $\mathcal{E}^{\nu}_{i,j}$.
(2) Next, we let $0\le s\le t\le T$ and set $\Xi_t\triangleq\mathcal{E}^{\nu}_{i,j}\big[\xi|\mathcal{F}_t\big]$. Applying Proposition 2.7(2) to $\widetilde{\mathcal{E}}_i$ and $\widetilde{\mathcal{E}}_j$, we obtain

$$\begin{aligned}\mathcal{E}^{\nu}_{i,j}\big[\Xi_t|\mathcal{F}_s\big]&=\mathbf{1}_{\{\nu\le s\}}\widetilde{\mathcal{E}}_j\big[\Xi_t|\mathcal{F}_s\big]+\mathbf{1}_{\{\nu>s\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j\big[\Xi_t|\mathcal{F}_\nu\big]|\mathcal{F}_s\big]\\&=\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{\{\nu\le s\}}\Xi_t|\mathcal{F}_s\big]+\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{\{\nu>s\}}\Xi_t|\mathcal{F}_\nu\big]|\mathcal{F}_s\big],\quad\text{a.s.,}\end{aligned}$$

where we used the fact that $\{\nu>s\}\in\mathcal{F}_{\nu\wedge s}$ thanks to [8, Lemma 1.2.16]. Then (A3) and (A2) imply that

$$\begin{aligned}\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{\{\nu\le s\}}\Xi_t|\mathcal{F}_s\big]&=\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{\{\nu\le s\}}\widetilde{\mathcal{E}}_j\big[\xi|\mathcal{F}_t\big]|\mathcal{F}_s\big]\\&=\mathbf{1}_{\{\nu\le s\}}\widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_j\big[\xi|\mathcal{F}_t\big]|\mathcal{F}_s\big]=\mathbf{1}_{\{\nu\le s\}}\widetilde{\mathcal{E}}_j\big[\xi|\mathcal{F}_s\big],\quad\text{a.s.}\end{aligned}\tag{4.20}$$

On the other hand, we can deduce from (3.2) that

$$\begin{aligned}\mathbf{1}_{\{\nu>s\}}\Xi_t&=\mathbf{1}_{\{s<\nu\le t\}}\widetilde{\mathcal{E}}_j\big[\xi|\mathcal{F}_t\big]+\mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big]\\&=\mathbf{1}_{\{s<\nu\le t\}}\widetilde{\mathcal{E}}_j\big[\xi|\mathcal{F}_t\big]+\mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_{\nu\wedge t}\big],\quad\text{a.s.}\end{aligned}$$

Since both $\{s<\nu\le t\}=\{\nu>s\}\cap\{\nu>t\}^c$ and $\{\nu>t\}$ belong to $\mathcal{F}_{\nu\wedge t}$, Proposition 2.7(3) and (2) as well as Corollary 2.3 imply that

$$\begin{aligned}&\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{\{\nu>s\}}\Xi_t|\mathcal{F}_\nu\big]=\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{\{s<\nu\le t\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_t]|\mathcal{F}_\nu\big]+\mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_{\nu\wedge t}\big]\\&\quad=\mathbf{1}_{\{s<\nu\le t\}}\widetilde{\mathcal{E}}_j\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_t]|\mathcal{F}_\nu\big]+\mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big]\\&\quad=\mathbf{1}_{\{s<\nu\le t\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\wedge t}]+\widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big]\\&\quad=\widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{s<\nu\le t\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\wedge t}]+\mathbf{1}_{\{\nu>t\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big]=\widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{s<\nu\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big],\quad\text{a.s.}\end{aligned}$$

Taking $\widetilde{\mathcal{E}}_i\big[\cdot|\mathcal{F}_s\big]$ of both sides as well as using (A2) and (A3) of $\widetilde{\mathcal{E}}_i$, we obtain

$$\begin{aligned}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\mathbf{1}_{\{\nu>s\}}\Xi_t|\mathcal{F}_\nu]|\mathcal{F}_s\big]&=\widetilde{\mathcal{E}}_i\Big[\widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{s<\nu\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_t\big]|\mathcal{F}_s\Big]=\widetilde{\mathcal{E}}_i\big[\mathbf{1}_{\{s<\nu\}}\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_s\big]\\&=\mathbf{1}_{\{\nu>s\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_s\big],\quad\text{a.s.,}\end{aligned}$$

which together with (4.20) yields that

$$\mathcal{E}^{\nu}_{i,j}\big[\mathcal{E}^{\nu}_{i,j}[\xi|\mathcal{F}_t]|\mathcal{F}_s\big]=\mathbf{1}_{\{\nu\le s\}}\widetilde{\mathcal{E}}_j\big[\xi|\mathcal{F}_s\big]+\mathbf{1}_{\{\nu>s\}}\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]|\mathcal{F}_s\big]=\mathcal{E}^{\nu}_{i,j}\big[\xi|\mathcal{F}_s\big],$$

a.s., proving (A2) for $\mathcal{E}^{\nu}_{i,j}$.

(3) For any $A\in\mathcal{F}_t$, using (3.3), (A3) of $\widetilde{\mathcal{E}}_i$ as well as applying Proposition 2.7(2) to $\widetilde{\mathcal{E}}_j$, we obtain

$$\mathcal{E}^{\nu}_{i,j}\big[\mathbf{1}_A\xi|\mathcal{F}_t\big]=\widetilde{\mathcal{E}}_i\big[\mathbf{1}_A\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big]=\mathbf{1}_A\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big]=\mathbf{1}_A\mathcal{E}^{\nu}_{i,j}\big[\xi|\mathcal{F}_t\big],$$

a.s., proving (A3) for $\mathcal{E}^{\nu}_{i,j}$.

Similarly, we can show that (A4) holds for $\mathcal{E}^{\nu}_{i,j}$ as well. Therefore, $\mathcal{E}^{\nu}_{i,j}$ is an **F**-expectation with domain $\Lambda^{\#}$. Since $\Lambda \in \widetilde{\mathscr{D}}_T$, i.e. $\mathbb{R} \subset \Lambda$, it follows easily that $\mathbb{R} \subset \Lambda^{\#}$, which shows that $\Lambda^{\#} \in \widetilde{\mathscr{D}}_T$.
(4) Now we show that $\mathcal{E}^{\nu}_{i,j}$ satisfies (H1) and (H2): For any $\xi \in \Lambda^{+}$ and any $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_T$ with $\lim_{n\to\infty} \uparrow \mathbf{1}_{A_n} = 1$, a.s., the Dominated Convergence Theorem (Proposition 2.9) implies that $\lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_j[\mathbf{1}_{A_n}\xi|\mathcal{F}_\nu] = \widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]$, a.s. Furthermore, using (3.3) and applying the Dominated Convergence Theorem to $\widetilde{\mathcal{E}}_i$ yield that

$$\lim_{n\to\infty} \uparrow \mathcal{E}^{\nu}_{i,j}\big[\mathbf{1}_{A_n}\xi\big] = \lim_{n\to\infty} \uparrow \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j\big[\mathbf{1}_{A_n}\xi|\mathcal{F}_\nu\big]\big]$$
$$= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\nu]\big] = \mathcal{E}^{\nu}_{i,j}\big[\xi\big], \quad \text{proving (H1) for } \mathcal{E}^{\nu}_{i,j}.$$

With a similar argument, we can show that $\mathcal{E}^{\nu}_{i,j}$ also satisfies (H2).
(5) If both $\mathcal{E}_i$ and $\mathcal{E}_j$ are positively-convex, so are $\widetilde{\mathcal{E}}_i$ and $\widetilde{\mathcal{E}}_j$ thanks to (2.8). To see that $\mathcal{E}^{\nu}_{i,j}$ is convex in the sense of (3.1), we fix $\xi, \eta \in \Lambda^{\#}$, $\lambda \in (0,1)$ and $t \in [0,T]$. For any $s \in [0,T]$, we have

$$\widetilde{\mathcal{E}}_j[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_s] \le \lambda\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_s] + (1-\lambda)\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_s], \quad \text{a.s.}$$

Since $\widetilde{\mathcal{E}}_j[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_\cdot]$, $\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_\cdot]$ and $\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_\cdot]$ are all RCLL processes, it holds except on a null set N that

$$\widetilde{\mathcal{E}}_j[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_s] \le \lambda\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_s] + (1-\lambda)\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_s], \quad \forall\, s \in [0,T],$$
$$\text{thus } \widetilde{\mathcal{E}}_j[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_{\nu\vee t}] \le \lambda\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}] + (1-\lambda)\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_{\nu\vee t}].$$

Then (3.3) implies that

$$\begin{aligned}
\mathcal{E}^{\nu}_{i,j}[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_t] &= \widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\lambda\xi + (1-\lambda)\eta|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big] \\
&\le \widetilde{\mathcal{E}}_i\big[\lambda\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}] + (1-\lambda)\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big], \\
&\le \lambda\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\xi|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big] + (1-\lambda)\widetilde{\mathcal{E}}_i\big[\widetilde{\mathcal{E}}_j[\eta|\mathcal{F}_{\nu\vee t}]|\mathcal{F}_t\big] \\
&= \lambda\mathcal{E}^{\nu}_{i,j}[\xi|\mathcal{F}_t] + (1-\lambda)\mathcal{E}^{\nu}_{i,j}[\eta|\mathcal{F}_t], \quad \text{a.s.} \quad \square
\end{aligned}$$

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