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Dynamic Programming Principles for Optimal Stopping with Expectation Constraint

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Abstract

We analyze an optimal stopping problem with a constraint on the expected cost. When the reward function and cost function are Lipschitz continuous in state variable, we show that the value of such an optimal stopping problem is a continuous function in current state and in budget level. Then we derive a dynamic programming principle (DPP) for the value function in which the conditional expected cost acts as an additional state process. As the optimal stopping problem with expectation constraint can be transformed to a stochastic optimization problem with supermartingale controls, we explore a *second* DPP of the value function and thus resolve an open question recently raised in [S. Ankirchner, M. Klein, and T. Kruse, A verification theorem for optimal stopping problems with expectation constraints, Appl. Math. Optim., 2017, pp. 1-33]. Based on these two DPPs, we characterize the value function as a viscosity solution to the related fully non-linear parabolic Hamilton-Jacobi-Bellman equation.

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1 Introduction

In this article, we analyze a continuous-time optimal stopping problem with expectation constraint on the accumulated cost. Suppose that the game begins at time t over the canonical space Ω^t of continuous paths. Under the Wiener measure P_t, the coordinator process $W^t = \{W^t_s\}_{s\in[t,\infty)}$ of Ω^t is a Brownian motion. Let $\overline{\mathbf{F}}^t = \big\{\overline{\mathcal{F}}^t_s\big\}_{s\in[t,\infty)}$ be the P_t-augmentation of the filtration generated by W^t, and let the $\mathbb{R}^l-$valued state flow $\mathcal{X}^{t,x}$ evolve from position $x\in\mathbb{R}^l$ according to a stochastic differential equation

$$\mathcal{X}_s = x + \int_t^s b(r,\mathcal{X}_r)dr + \int_t^s \sigma(r,\mathcal{X}_r)\, dW^t_r, \quad s\in[t,\infty). \tag{1.1}$$

We aim to maximize the sum $\mathcal{R}(t,x,\tau)$ of a running reward $\int_t^\tau f(r,\mathcal{X}^{t,x}_r)dr$ and a terminal reward $\pi(\tau,\mathcal{X}^{t,x}_\tau)$ by choosing an $\overline{\mathbf{F}}^t-$stopping time τ, which, however, has to satisfy a budget constraint $E_t[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr]\le y$. So the value of such a optimal stopping problem with expectation constraint is in form of

$$\mathcal{V}(t,x,y) := \sup_{\tau\in\mathcal{T}^t_x(y)} E_t\big[\mathcal{R}(t,x,\tau)\big], \tag{1.2}$$

with $\mathcal{T}^t_x(y) := \big\{\tau : E_t[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr] \le y\big\}$ and $E_t[\cdot] = E_{P_t}[\cdot]$. In particular, when the cost rate $g(r,x)$ is a power function of r, the budget constraint specifies as a moment constraint on stopping times.

Kennedy [37] initiated the study of optimal stopping problem with expectation constraint. The author used a *Lagrange multiplier* method to reduce a discrete-time optimal stopping problem with first-moment constraint to an

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unconstrained optimal stopping problem and showed that the optimal value of the dual problem is equal to that of the primal problem. Since then, the Lagrangian technique has been prevailing in research of optimal stopping problems with expectation constraints.

In the present paper, we develop a new approach to analyze the optimal stopping problem with expectation constraint (1.2). Our main contributions are obtaining the continuity of the value function $\mathcal{V}$ and establishing two dynamic programming principles (DPPs) for $\mathcal{V}$.

When reward/cost functions f, π, g are Lipschitz continuous in state variable x and the cost function g is non-degenerate in sense of (g3), we first demonstrate over a general probability setting that the value function is continuous in (t,x,y) by utilizing a priori estimates of the state process $\mathcal{X}^{t,x}$ and delicately constructing approximate stopping strategies (see Theorem 2.1). This continuity result together with the properties of shifted processes then allow us to derive in Theorem 4.1 a DPP for the value function $\mathcal{V}$ over the canonical space:

$$\mathcal{V}(t,x,y) = \sup_{\tau\in\mathcal{T}_x^t(y)} E_t\bigg[\mathbf{1}_{\{\tau\le\zeta(\tau)\}}\mathcal{R}(t,x,\tau)+\mathbf{1}_{\{\tau>\zeta(\tau)\}}\Big(\mathcal{V}\big(\zeta(\tau),\mathcal{X}^{t,x}_{\zeta(\tau)},\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)}\big)+\int_t^{\zeta(\tau)} f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg]. \tag{1.3}$$

Here the conditional expected cost $\mathcal{Y}^{t,x,\tau}_s := E_t\big[\int_{\tau\wedge s}^{\tau} g\big(r,\mathcal{X}^{t,x}_r\big)dr\big|\overline{\mathcal{F}}^t_s\big]$ acts as an additional state process and the intermediate horizon ζ can be a general $\overline{\mathbf{F}}^t-$stopping time depending on the stopping rule τ we select. For the "$\le$" part of (1.3), we exploit the flow property of shifted stochastic differential equations (Proposition 3.6) as well as the regular conditional probability distribution due to [57]; while in the "$\ge$" part, we carefully paste together local $\varepsilon-$optimal stopping strategies and utilize the continuity of value function $\mathcal{V}$.

Also, we can transform the optimal stopping problem with expectation constraint to an unconstrained stochastic optimization problem whose controls are supermartingales starting from budget level y: Let $\mathfrak{A}_t(y)$ denote all uniformly integrable continuous supermartingales $\alpha=\{\alpha_s\}_{s\in[t,\infty)}$ with $\alpha_t=y$. As shown in Proposition 4.2, for each nontrivial $\tau\in\mathcal{T}^t_x(y)$ there exists $\alpha\in\mathfrak{A}_t(y)$ such that τ coincides with the first hitting time $\tau(t,x,\alpha)$ of the process $Y^{t,x,\alpha}_s := \alpha_s-\int_t^s g(r,\mathcal{X}^{t,x}_r)dr$, $s\in[t,\infty)$ to 0 $\big($If $E_t[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr]=y$, α is indeed a true martingale$\big)$. So the value function $\mathcal{V}$ can be alternatively expressed as $\mathcal{V}(t,x,y)= \sup\limits_{\alpha\in\mathfrak{A}_t(y)} E_t\big[\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big]$. Correspondingly, we establish a second DPP for the value function $\mathcal{V}$ over the canonical space (Theorem 4.2)

$$\begin{aligned}\mathcal{V}(t,x,y) \;=\; \sup_{\alpha\in\mathfrak{A}_t(y)} E_t\bigg[&\mathbf{1}_{\{\tau(t,x,\alpha)\le\zeta(\alpha)\}}\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\\ &+\mathbf{1}_{\{\tau(t,x,\alpha)>\zeta(\alpha)\}}\Big(\mathcal{V}\big(\zeta(\alpha),\mathcal{X}^{t,x}_{\zeta(\alpha)},Y^{t,x,\alpha}_{\zeta(\alpha)}\big)+\int_t^{\zeta(\alpha)} f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg],\end{aligned} \tag{1.4}$$

and thus justify a postulate recently made by [2] (see Remark 3.3 therein). Although the "$\le$" part of (1.4) can be easily deduced from (1.3), the "$\ge$" part entails an intricate pasting of approximately optimal supermartingale controls.

In light of these two DPPs, we then show that the value function $\mathcal{V}$ of the optimal stopping problem with expectation constraint is a viscosity solution to a related fully non-linear parabolic Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases}-\partial_t u(t,x,y)-\frac{1}{2}trace\big(\sigma(t,x)\cdot\sigma^T(t,x)\cdot D_x^2 u(t,x,y)\big)-b^T(t,x)\cdot D_x u(t,x,y)\\ \qquad +g(t,x)\partial_y u(t,x,y)-\mathcal{H}u(t,x,y)=f(t,x), \quad \forall\,(t,x,y)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty),\\ u(t,x,0)=\pi(t,x), \quad \forall\,(t,x)\in[0,\infty)\times\mathbb{R}^l,\end{cases} \tag{1.5}$$

with the Hamiltonian $\mathcal{H}u(t,x,y):=\sup\limits_{a\in\mathbb{R}^d}\big\{\frac{1}{2}|a|^2\partial_y^2 u(t,x,y)+(D_x(\partial_y u(t,x,y)))^T\cdot\sigma(t,x)\cdot a\big\}$. As pointed out in [44], the non-linear HJB equation (1.5) is a Monge-Ampère type equation.

Relevant Literature. Since Arrow et al. [3] and Snell [56], the general theory of (unconstrained) optimal stopping has been plentifully developed over decades. Expositions of this theory are presented in the monographs [21, 46, 55, 27, 33, 51], which contain extensive bibliographies and references to the literature. For the recent development of the optimal stopping under model uncertainty/non-linear expectations and the closely related controller-stopper-games, see [34, 35, 28, 20, 22, 36, 54, 8, 9, 6, 19, 5, 26, 10, 47, 12, 11] among others.

As to the optimal stopping with expectation constraint, the Lagrange multiplier method introduced in [37] was later developed by many researches (see e.g. [52, 45, 41, 25, 4, 58, 42]), and has been applied to various economic and financial problems such as Markov decision processes with constrained stopping times [30, 29], non-exponential discounting and mean-variance portfolio optimization [48, 49] and quickest detection problem [50].

Our stochastic control approach in deriving the second DPP resembles those of two recent papers [2], [44]. By applying the martingale representation to the conditional expected cost, Ankirchner et al. [2] transformed the optimal stopping problem with expectation constraint to a stochastic optimization problem in which the stochastic integral of locally square-integrable controls is regarded as an additional state process. Miller [44] independently employed the same method to address the optimal stopping problem with first-moment constraint that is embedded in a time-inconsistent optimal stopping problem. The idea of expanding the state space by the conditional probability/expectation process has also appeared in the literature dealing with stochastic target problems, see e.g. [15, 17, 18, 16, 14].

Our paper is distinct from [2], [44] in four aspects: First, we first obtain the continuity of the value function $\mathcal{V}$, and using this establish the two DPPs (1.3) and (1.4), which were not addressed by them. Second, our value function $\mathcal{V}$ takes the starting moment t of the game as an input, so the related non-linear HJB equation (1.5) is of parabolic type rather than elliptic type. Third, we need the constraint $E_t[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr] \le y$ for the continuity and the DPPs of the value function, although the auxiliary optimal stopping problem considered in [44] is subject to constraint $E[\tau]=y$ and the dynamic programming equation studied by [2] is for the value function U of the optimal stopping with constraint $E[\int_0^\tau g(X_r^x)dr]=y$. See Remark 4.1 for a comparison of these two types of constraints. Fourth, our discussion of related non-linear HJB equations seems different from theirs. Our Theorem 5.1 obtains that the value function $\mathcal{V}$ is a viscosity supersolution of (1.5), and is only a viscosity subsolution of (1.5) with the upper semi-continuous envelope $\overline{\mathcal{H}}u$ of $\mathcal{H}u$. By assuming that the value U is a smooth function satisfying the DPP, Proposition 3.4 of [2] showed that U is a supersolution to a similar non-linear HJB equation to (1.5), and is further a subsolution if the Hamiltonian is continuous (see Subsection 6.1 of [2] for an example of discontinuous Hamiltonian). However, possible discontinuity of the Hamiltonian was not discussed in [44].

Lately, the optimal stopping with constraint on the distribution of stopping time has attracted a lot of research interests. Bayraktar and Miller [7] studied the optimal stopping of a Brownian motion with the restriction that the distribution of the stopping time must equal to a given measure consisting of finitely-many atoms. The applications of such a distribution-constrained optimal stopping problem in mathematical finance include model-free superhedging with an outlook on volatility and inverse first-passage-time problem. Within a weak formulation on the canonical path space, Kallblad [31] extended the distribution-constrained optimal stopping problem for a general target measure and for path-dependent cost functions. From the perspective of mass transport, Beiglboeck et al. [13] obtained a monotonicity principle for the optimal stopping of a Brownian motion under distribution constraint, and thus characterized the constrained optimal stopping rule as the first hitting time of a barrier in a suitable phase space. Very recently, Ankirchner et al. [1] showed that for optimally stopping a one-dimensional Markov process with first-moment constraint on stopping times, one only needs to consider those stopping times at which the law of the Markov process is a weighted sum of three Dirac measures. There are also some other types of optimal stopping problems with constraints: see [24] for an optimal stopping problem with a reward constraint; see [38, 39, 43, 40] for optimal stopping with information constraint.

The rest of the paper is organized as follows: In Subsection 1.1, we introduce notations and make standing assumptions on drift/diffusion coefficients and reward/cost functions. In Section 2, we set up the optimal stopping problem with expectation constraint over a general probability space and show the continuity of its value function in current state and budget constraint level. Section 3 explores the measurability/integrability properties of shifted processes and the flow property of shifted stochastic differential equations as technical preparation for proving our main result, two types of DPPs. Then in Subsection 4.1, we derive over the canonical space a DPP for the value function $\mathcal{V}$ of the optimal stopping with expectation constraint in which the conditional expected cost acts as an additional state process. In subsection 4.2, we transform the the optimal stopping problem with expectation constraint to a stochastic optimization problem with supermartingale controls and establish a second DPP for $\mathcal{V}$. Based on two DPPs, we characterize $\mathcal{V}$ as the viscosity solution to the related fully nonlinear parabolic HJB equation in Section 5. Section 6 contains proofs of our results while the demonstration of some auxiliary statements with starred labels in these proofs are relegated to the Appendix. We also include some technical lemmata in the appendix.

1.1 Notation and Preliminaries

For a generic Euclidian space $\mathbb{E}$, we denote its Borel sigma−field by $\mathscr{B}(\mathbb{E})$. For any $x \in \mathbb{E}$ and $\delta \in (0,\infty)$, $O_\delta(x) := \{x' \in \mathbb{E} \colon |x-x'|<\delta\}$ denotes the open ball centered at x with radius δ and its closure is $\overline{O}_\delta(x) := \{x' \in \mathbb{E} \colon |x-x'| \le \delta\}$.

Fix $l \in \mathbb{N}$ and $p \in [1,\infty)$. Let $c(t) \colon [0,\infty) \to (0,\infty)$ be a continuous function with $\int_0^\infty c(t)dt < \infty$, and let $\mathfrak{C}$ be a constant with $\mathfrak{C} \ge 1 + \int_0^\infty c(t)dt$. As $\lim\limits_{t\to\infty} c(t) = 0$, the continuity of $c(\cdot)$ implies that $\|c(\cdot)\| := \sup\limits_{t\in[0,\infty)} c(t) < \infty$. Also, let ρ be a modulus of continuity function and denote its inverse function by ρ^{-1}.

We shall consider the following drift/diffusion coefficients and reward/cost functions throughout the paper.

• Let $b \colon (0,\infty)\times\mathbb{R}^l \to \mathbb{R}^l$ be a $\mathscr{B}(0,\infty)\otimes\mathscr{B}(\mathbb{R}^l)/\mathscr{B}(\mathbb{R}^l)$−measurable function and let $\sigma \colon (0,\infty)\times\mathbb{R}^l \to \mathbb{R}^{l\times d}$ be a $\mathscr{B}(0,\infty)\otimes\mathscr{B}(\mathbb{R}^l)/\mathscr{B}(\mathbb{R}^{l\times d})$−measurable function such that for any $t \in (0,\infty)$ and $x_1, x_2 \in \mathbb{R}^l$

$$\big|b(t,x_1)-b(t,x_2)\big| \le c(t)|x_1-x_2|, \quad |b(t,0)| \le c(t), \tag{1.6}$$

$$\text{and} \quad \big|\sigma(t,x_1)-\sigma(t,x_2)\big| \le \sqrt{c(t)}|x_1-x_2|, \quad |\sigma(t,0)| \le \sqrt{c(t)}\,. \tag{1.7}$$

• The running reward function $f \colon (0,\infty)\times\mathbb{R}^l \to \mathbb{R}$ is a $\mathscr{B}(0,\infty)\otimes\mathscr{B}(\mathbb{R}^l)/\mathscr{B}(\mathbb{R})$−measurable function such that for any $t \in (0,\infty)$ and $x_1, x_2 \in \mathbb{R}^l$

$$\big|f(t,x_1)-f(t,x_2)\big| \le c(t)\big(|x_1-x_2| \vee |x_1-x_2|^p\big) \quad \text{and} \quad \big|f(t,0)\big| \le c(t). \tag{1.8}$$

• The terminal reward function $\pi \colon [0,\infty)\times\mathbb{R}^l \to \mathbb{R}$ is a continuous function such that for any $t, t' \in [0,\infty)$ and $x, x' \in \mathbb{R}^l$

$$|\pi(t,x)-\pi(t',x')| \le \rho\big(|t-t'|\big) + \mathfrak{C}\big(|x-x'| \vee |x-x'|^p\big) \quad \text{and} \quad \big|\pi(t,0)\big| \le \mathfrak{C}. \tag{1.9}$$

• The cost rate function $g \colon (0,\infty)\times\mathbb{R}^l \to (0,\infty)$ is a $\mathscr{B}(0,\infty)\otimes\mathscr{B}(\mathbb{R}^l)/\mathscr{B}(0,\infty)$−measurable function satisfying

(g1) $\big|g(t,x_1)-g(t,x_2)\big| \le c(t)\big(|x_1-x_2| \vee |x_1-x_2|^p\big)$, $\forall\, t \in (0,\infty)$, $\forall\, x_1, x_2 \in \mathbb{R}^l$;

(g2) $\int_0^t g(t,0)dr < \infty$, $\forall\, t \in (1,\infty)$;

(g3) For any $R \in (0,\infty)$, there exists $\kappa_R \in (0,\infty)$ such that $g(t,x) \ge \kappa_R$, $\forall\, t \in (0,\infty)$, $\forall\, x \in \mathbb{R}^l$ with $|x| \le R$. The constant κ_R can be regarded as the basic cost rate when the long-term state radius is R.

Moreover, we will use the convention $\inf\emptyset := \infty$ as well as the inequality

$$(1 \wedge n^{q-1}) \sum_{i=1}^n a_i^q \le \bigg(\sum_{i=1}^n a_i\bigg)^q \le (1 \vee n^{q-1}) \sum_{i=1}^n a_i^q \tag{1.10}$$

for any $q \in (0,\infty)$ and any finite subset $\{a_1, \cdots, a_n\}$ of $(0,\infty)$.

2 Continuity of Value Functions for General Optimal Stopping with Expectation Constraint

For an optimal stopping problem with expectation constraint, we first discuss the continuity of its value function over a general complete probability space $(\Omega, \mathcal{F}, P)$.

Let B be a d−dimensional standard Brownian motion on $(\Omega,\mathcal{F},P)$. The P−augmentation of its natural filtration $\mathbf{F} = \big\{\mathcal{F}_t := \sigma\big(\sigma(B_s; s\in[0,t]) \cup \mathscr{N}\big)\big\}_{t\in[0,\infty)}$ satisfies the *usual hypothesis*, where $\mathscr{N} := \big\{\mathcal{N} \subset \Omega \colon \mathcal{N} \subset A$ for some $A \in \mathcal{F}$ with $P(A)=0\big\}$ collects all P−null sets in $\mathcal{F}$. Let $\mathcal{T}$ stand for all $\mathbf{F}$−stopping times τ with $\tau < \infty$, P−a.s. For any $\mathbf{F}$−adapted continuous process X, we set $X_* := \sup\limits_{s\in[0,\infty)} |X_s|$.

2.1 Reward Processes

Let $(t,x) \in [0,\infty)\times\mathbb{R}^l$. It is well-known that under (1.6) and (1.7), the following stochastic differential equation (SDE) on Ω

$$X_s = x + \int_0^s b(t+r, X_r)dr + \int_0^s \sigma(t+r, X_r)dB_r, \quad s \in [0,\infty) \tag{2.1}$$

admits a unique solution $X^{t,x} = \{X_s^{t,x}\}_{s\in[0,\infty)}$, which is an $\mathbb{R}^l$−valued, $\mathbf{F}$−adapted continuous process satisfying

Lemma 2.1. *Let $q\in[1,\infty)$ and $(t,x)\in[0,\infty)\times\mathbb{R}^l$.*
(1) For some constant $C_q\ge 1$ depending on q and $\int_0^\infty c(s)ds$, we have

$$E\Big[\sup_{s\in[0,\infty)}\big|X_s^{t,x}\big|^q\Big]\le C_q\big(1+|x|^q\big);\quad E\Big[\sup_{s\in[0,\infty)}\big|X_s^{t,x'}-X_s^{t,x}\big|^q\Big]\le C_q|x'-x|^q,\ \ \forall\, x'\in\mathbb{R}^l;\quad \text{and} \tag{2.2}$$

$$E\Big[\sup_{\lambda\in(0,\delta]}\big|X_{\tau+\lambda}^{t,x}-X_\tau^{t,x}\big|^q\Big]\le C_q\big(1+|x|^q\big)\big(\|c(\cdot)\|^q\delta^q+\|c(\cdot)\|^{\frac{q}{2}}\delta^{\frac{q}{2}}\big),\ \ \forall\,\delta\in(0,\infty),\ \ \forall\,\tau\in\mathcal{T}. \tag{2.3}$$

(2) Given $\varpi\in[1,\infty)$, assume functions b and σ additionally satisfy that for any $0\le t_1<t_2<\infty$ and $x'\in\mathbb{R}^l$

$$\big|b(t_2,x')-b(t_1,x')\big|\le c(t_1)\rho(t_2-t_1)\big(1+|x'|^\varpi\big)\quad \text{and}\quad \big|\sigma(t_2,x')-\sigma(t_1,x')\big|\le\sqrt{c(t_1)}\rho(t_2-t_1)\big(1+|x'|^\varpi\big). \tag{2.4}$$

Then it holds for any $t'\in(t,\infty)$ that

$$E\Big[\sup_{s\in[0,\infty)}\big|X_s^{t',x}-X_s^{t,x}\big|^q\Big]\le C_{q,\varpi}\big(1+|x|^{q\varpi}\big)\big(\rho(t'-t)\big)^q, \tag{2.5}$$

where $C_{q,\varpi}\ge 1$ is some constant depending on q, ϖ and $\int_0^\infty c(s)ds$.

Given $t\in[0,\infty)$, let the state process evolve from position $x\in\mathbb{R}^l$ according to SDE (2.1). If the player chooses to exercise at time $\tau\in\mathcal{T}$, she will receive a running reward $\int_0^\tau f\big(t+s,X_s^{t,x}\big)\,ds$ and a terminal reward $\pi\big(t+\tau,X_\tau^{t,x}\big)$, whose totality is

$$R(t,x,\tau):=\int_0^\tau f\big(t+s,X_s^{t,x}\big)\,ds+\pi\big(t+\tau,X_\tau^{t,x}\big). \tag{2.6}$$

One can deduce from (1.8), (1.9) and the first inequality in (2.2) that

$$E\big[|R(t,x,\tau)|\big]\le 2\mathfrak{C}\big(2+C_p(1+|x|^p)\big):=\Psi(x). \tag{2.7}$$

Given another initial position $x'\in\mathbb{R}^l$, (1.8), (1.9), Hölder's inequality and the second inequality in (2.2) imply that

$$\begin{aligned}E\big[\big|R(t,x,\tau)-R(t,x',\tau)\big|\big]&\le E\bigg[\int_0^\tau\big|f(t+r,X_r^{t,x})-f(t+r,X_r^{t,x'})\big|dr+\big|\pi(t+\tau,X_\tau^{t,x})-\pi(t+\tau,X_\tau^{t,x'})\big|\bigg]\\&\le\Big(\int_0^\infty c(t+r)dr+\mathfrak{C}\Big)E\Big[\big(X^{t,x}-X^{t,x'}\big)_*+\big(X^{t,x}-X^{t,x'}\big)_*^p\Big]\le 2\mathfrak{C}\big((C_p)^{\frac{1}{p}}|x-x'|+C_p|x-x'|^p\big).\end{aligned} \tag{2.8}$$

2.2 Expectation Constraints

Let $(t,x)\in[0,\infty)\times\mathbb{R}^l$. As the first inequality in (2.2) shows that $(X_*^{t,x})^p<\infty$, $P-$a.s., (g1)$-$(g3) imply that $P-$a.s.

$$\int_0^s g\big(t+r,X_r^{t,x}\big)dr\le\int_0^s g(t+r,0)dr+\mathfrak{C}\big(X_*^{t,x}+(X_*^{t,x})^p\big)<\infty,\ \ \forall\, s\in(0,\infty)\quad\text{and}\quad\int_0^\infty g\big(t+r,X_r^{t,x}\big)dr=\infty. \tag{2.9}$$

Given $y\in[0,\infty)$, we try to maximize the player's expected total wealth $R(t,x,\tau)$ when her expected cost is subject to the following constraint:

$$E\bigg[\int_0^\tau g(t+r,X_r^{t,x})dr\bigg]\le y. \tag{2.10}$$

Like reward processes $\big\{\int_0^s f\big(t+r,X_r^{t,x}\big)dr\big\}_{s\in[0,\infty)}$ and $\big\{\pi(t+s,X_s^{t,x})\big\}_{s\in[0,\infty)}$, this expectation constraint is also state-related. Hence, starting from the initial state $x\in\mathbb{R}^l$, the value of the general optimal stopping problem with expectation constraint y is

$$V(t,x,y):=\sup_{\tau\in\mathcal{T}_{t,x}(y)}E\big[R(t,x,\tau)\big], \tag{2.11}$$

where $\mathcal{T}_{t,x}(y):=\big\{\mathbf{F}-\text{stopping time }\tau:\,E\big[\int_0^\tau g(t+r,X_r^{t,x})dr\big]\le y\big\}$.

For any $\tau\in\mathcal{T}_{t,x}(y)$, as $E\big[\int_0^\tau g(t+r,X_r^{t,x})dr\big]\le y<\infty$, one has $\int_0^\tau g(t+r,X_r^{t,x})dr<\infty$, $P-$a.s. The second part of (2.9) then implies that $\tau<\infty$, $P-$a.s. So $\mathcal{T}_{t,x}(y)=\big\{\tau\in\mathcal{T}:\,E\big[\int_0^\tau g(t+r,X_r^{t,x})dr\big]\le y\big\}$.

Example 2.1. *(Moment Constraints) For $q\in(1,\infty)$, $a\in[0,\infty)$ and $b\in(0,\infty)$, take $g(\mathfrak{t},\mathfrak{x}):=aq\mathfrak{t}^{q-1}+b$, $(\mathfrak{t},\mathfrak{x})\in(0,\infty)\times\mathbb{R}^l$. Then the constraint (2.10) for $t=0$ specify as the moment constraint $E[a\tau^q+b\tau]\le y$.*

Let $(t,x)\in[0,\infty)\times\mathbb{R}^l$. It is clear that

$$V(t,x,y) \text{ is increasing in } y. \tag{2.12}$$

As $\mathcal{T}_{t,x}(0)=\{0\}$, we see from (2.7) that

$$\Psi(x)\ge V(t,x,y)\ge V(t,x,0)=E\big[\pi(t,X_0^{t,x})\big]=\pi(t,x),\quad \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty). \tag{2.13}$$

When $y\in(0,\infty)$, we even have the following update of (2.11).

Lemma 2.2. *It holds for any $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty)$ that $V(t,x,y)=\sup\limits_{\tau\in\widehat{\mathcal{T}}_{t,x}(y)}E\big[R(t,x,\tau)\big]$, where $\widehat{\mathcal{T}}_{t,x}(y):=\big\{\tau\in\mathcal{T}_{t,x}(y)\colon \tau>0, P-a.s.\big\}$.*

The value function $V(t,x,y)$ of the general optimal stopping problem with expectation constraint is continuous in the following way:

Theorem 2.1. *(1) Given $t\in[0,\infty)$, $V(t,x,y)$ is continuous in $(x,y)\in\mathbb{R}^l\times[0,\infty)$ in the sense that for any $(x,\varepsilon)\in\mathbb{R}^l\times(0,1)$, there exists $\delta=\delta(t,x,\varepsilon)\in(0,1)$ such that for any $y\in[0,\infty)$*

$$\big|V(t,\mathfrak{x},\mathfrak{y})-V(t,x,y)\big|\le\varepsilon,\quad \forall\,(\mathfrak{x},\mathfrak{y})\in\overline{O}_\delta(x)\times\big[(y-\delta)^+,y+\delta\big].$$

(2) Given $\varpi\in[1,\infty)$, assume b,σ additionally satisfy (2.4) and f,g additionally satisfy that for any $0\le t_1<t_2<\infty$ and $x'\in\mathbb{R}^l$

$$\big|f(t_2,x')-f(t_1,x')\big|\vee\big|g(t_2,x')-g(t_1,x')\big|\le c(t_1)\rho(t_2-t_1)\big(1+|x'|^\varpi\big), \tag{2.14}$$

then $V(t,x,y)$ is continuous in $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty)$ in the sense that for any $(t,x,\varepsilon)\in[0,\infty)\times\mathbb{R}^l\times(0,1)$, there exists $\delta'=\delta'(t,x,\varepsilon)\in(0,1)$ such that for any $y\in[0,\infty)$

$$\big|V(\mathfrak{t},\mathfrak{x},\mathfrak{y})-V(t,x,y)\big|\le\varepsilon,\quad \forall\,(\mathfrak{t},\mathfrak{x},\mathfrak{y})\in\big[(t-\delta')^+,t+\delta'\big]\times\overline{O}_{\delta'}(x)\times\big[(y-\delta')^+,y+\delta'\big].$$

3 Shifted Processes

Let us review the properties of shifted processes on the canonical space so that we can study two types of dynamic programming principles of the optimal stopping problem with expectation constraint over the canonical space.

Fix $d\in\mathbb{N}$ and let $t\in[0,\infty)$. From now on, we consider the canonical space $\Omega^t:=\big\{\omega\in\mathbb{C}\big([t,\infty);\mathbb{R}^d\big):\omega(t)=0\big\}$ of continuous paths over period $[t,\infty)$, which is a separable complete metric space under the *uniform* norm $\|\omega\|_t:=\sup\limits_{s\in[t,\infty)}|\omega(s)|$. Let $\mathcal{F}^t:=\mathscr{B}(\Omega^t)$ be the Borel sigma field of Ω^t under $\|\cdot\|_t$. The canonical process $W^t=\{W^t_s\}_{s\in[t,\infty)}$ of Ω^t is a $d-$dimensional standard Brownian motion on $\big(\Omega^t,\mathcal{F}^t\big)$ under the Wiener measure P_t. Let $\mathscr{N}^t$ collect all P_t-null sets, i.e., $\mathscr{N}^t:=\big\{\mathcal{N}\subset\Omega^t:\mathcal{N}\subset A$ for some $A\in\mathcal{F}^t$ with $P_t(A)=0\big\}$, and set $\overline{\mathcal{F}}^t:=\sigma(\mathcal{F}^t\cup\mathscr{N}^t)$. The completion of $\big(\Omega^t,\mathcal{F}^t,P_t\big)$ is the probability space $\big(\Omega^t,\overline{\mathcal{F}}^t,\overline{P}_t\big)$ with $\overline{P}_t\Big|_{\mathcal{F}^t}=P_t$. For simplicity, we still write P_t for $\overline{P}_t$ and denote the expectation under $\overline{P}_t$ by $E_t[\cdot]$. For any sub sigma$-$field $\mathcal{G}$ of $\overline{\mathcal{F}}^t$, let $L^1(\mathcal{G})$ be the space of all real-valued, $\mathcal{G}-$measurable random variables ξ with $E_t\big[|\xi|\big]<\infty$.

We denote the natural filtration of W^t by $\mathbf{F}^t=\big\{\mathcal{F}^t_s:=\sigma\big(W^t_r;r\in[t,s]\big)\big\}_{s\in[t,\infty)}$. Its P_t-augmentation $\overline{\mathbf{F}}^t$ consists of $\overline{\mathcal{F}}^t_s:=\sigma\big(\mathcal{F}^t_s\cup\mathscr{N}^t\big)$, $s\in[t,\infty)$ and satisfies the *usual hypothesis*. Let $\overline{\mathcal{T}}^t$ stand for all stopping times τ with respect to the filtration $\overline{\mathbf{F}}^t$ such that $\tau<\infty$, P_t-a.s., and set $\overline{\mathcal{T}}^t_\sharp:=\big\{\tau\in\overline{\mathcal{T}}^t:\tau$ takes countably many values in $[t,\infty)\big\}$. For easy reference, we set $\mathcal{F}^t_\infty:=\mathcal{F}^t$ and $\overline{\mathcal{F}}^t_\infty:=\overline{\mathcal{F}}^t$.

The following spaces will be used in the sequel.

- For any $q\in[1,\infty)$, let $\mathbb{C}^q_t(\mathbb{E})=\mathbb{C}^q_{\overline{\mathbf{F}}^t}\big([t,\infty),\mathbb{E}\big)$ be the space of all $\mathbb{E}-$valued, $\overline{\mathbf{F}}^t-$adapted processes $\{X_s\}_{s\in[t,\infty)}$ with P_t-a.s. continuous paths such that $E_t\big[X^q_*\big]<\infty$ with $X_*:=\sup\limits_{s\in[t,\infty)}|X_s|$.
- Let $\mathbb{H}^{2,\text{loc}}_t$ denote all $\mathbb{R}^d-$valued, $\overline{\mathcal{F}}^t-$predictable processes $\{X_s\}_{s\in[t,\infty)}$ with $P_t\big\{\int_t^s|X_r|^2dr<\infty,\ \forall\,s\in[t,\infty)\big\}=1$.
- Let $\mathbb{M}_t$ denote all real-valued, uniformly integrable continuous martingales with respect to $\big(\overline{\mathbf{F}}^t,P_t\big)$.
- Set $\mathbb{K}_t:=\big\{K\in\mathbb{C}^1_t(\mathbb{R})\colon$ for P_t-a.s. $\omega\in\Omega^t$, $K_\cdot(\omega)$ is an continuous increasing path starting from $0\big\}$.

3.1 Concatenation of Sample Paths

Let $0\le t\le s<\infty$. We define a *translation* operator Π^t_s from Ω^t to Ω^s by

$$\big(\Pi^t_s(\omega)\big)(r):=\omega(r)-\omega(s), \quad \forall\,(r,\omega)\in[s,\infty)\times\Omega^t.$$

On the other hand, one can concatenate $\omega\in\Omega^t$ and $\widetilde{\omega}\in\Omega^s$ at time s by:

$$\big(\omega\otimes_s\widetilde{\omega}\big)(r):=\omega(r)\,\mathbf{1}_{\{r\in[t,s)\}}+\big(\omega(s)+\widetilde{\omega}(r)\big)\,\mathbf{1}_{\{r\in[s,\infty)\}}, \quad \forall\, r\in[t,\infty),$$

which is still of Ω^t.

Given $\omega\in\Omega^t$, we set $A^{s,\omega}:=\{\widetilde{\omega}\in\Omega^s:\omega\otimes_s\widetilde{\omega}\in A\}$ for any $A\subset\Omega^t$; and set $\omega\otimes_s\widetilde{A}:=\big\{\omega\otimes_s\widetilde{\omega}:\widetilde{\omega}\in\widetilde{A}\big\}$ for any $\widetilde{A}\subset\Omega^s$. In particular, $\emptyset^{s,\omega}:=\emptyset$ and $\omega\otimes_s\emptyset:=\emptyset$.

The next result shows that each $A\in\mathcal{F}^t_s$ consists of all branches $\omega\otimes_s\Omega^s$ with $\omega\in A$.

Lemma 3.1. *Let $0\le t\le s<\infty$ and $A\in\mathcal{F}^t_s$. It holds for any $\omega\in A$ that $\omega\otimes_s\Omega^s\subset A$ or $A^{s,\omega}=\Omega^s$.*

Let $\omega\in\Omega^t$. For any $\mathcal{F}^t_s-$measurable random variable ξ, since the set $\{\omega'\in\Omega^t:\xi(\omega')=\xi(\omega)\}=\xi^{-1}\big(\{\xi(\omega)\}\big)$ belongs to $\mathcal{F}^t_s$, Lemma 3.1 implies that

$$\omega\otimes_s\Omega^s\subset\{\omega'\in\Omega^t:\xi(\omega')=\xi(\omega)\} \quad \text{i.e.,} \quad \xi(\omega\otimes_s\widetilde{\omega})=\xi(\omega), \quad \forall\,\widetilde{\omega}\in\Omega^s. \tag{3.1}$$

To wit, the value $\xi(\omega)$ depends only on $\omega|_{[t,s]}$.

For any $r\in[s,\infty]$, the operation $()^{s,\omega}$ projects an $\mathcal{F}^t_r-$measurable set to an $\mathcal{F}^s_r-$measurable set while the operation $\omega\otimes_s\cdot$ transforms an $\mathcal{F}^s_r-$measurable set into an $\mathcal{F}^t_r-$measurable set.

Lemma 3.2. *Let $0\le t\le s<\infty$, $\omega\in\Omega^t$ and $r\in[s,\infty]$. We have $A^{s,\omega}\in\mathcal{F}^s_r$ for any $A\in\mathcal{F}^t_r$ and $\omega\otimes_s\widetilde{A}\in\mathcal{F}^t_r$ for any $\widetilde{A}\in\mathcal{F}^s_r$.*

3.2 Measurability and Integrability of Shifted Processes

Let $0\le t\le s<\infty$, let ξ be an $\mathbb{E}-$valued random variable on Ω^t and let $X=\{X_r\}_{r\in[t,\infty)}$ be an $\mathbb{E}-$valued process on Ω^t. For any $\omega\in\Omega^t$, we define the shifted random variable $\xi^{s,\omega}$ and the shifted process $X^{s,\omega}$ by

$$\xi^{s,\omega}(\widetilde{\omega}):=\xi(\omega\otimes_s\widetilde{\omega}) \quad \text{and} \quad X^{s,\omega}(r,\widetilde{\omega}):=X(r,\omega\otimes_s\widetilde{\omega}), \quad \forall\,(r,\widetilde{\omega})\in[s,\infty)\times\Omega^s.$$

By Lemma 3.2, shifted random variables and shifted processes inherit the measurability of original ones.

Proposition 3.1. *Let $0\le t\le s<\infty$ and let $\omega\in\Omega^t$.*
(1) Let ξ be an $\mathbb{E}-$valued random variable on Ω^t. If ξ is $\mathcal{F}^t_r-$measurable for some $r\in[s,\infty]$, the shifted random variable $\xi^{s,\omega}$ is $\mathcal{F}^s_r-$measurable.
(2) Let $X=\{X_r\}_{r\in[t,\infty)}$ be an $\mathbb{E}-$valued process on Ω^t. If X is $\mathbf{F}^t-$adapted, the shifted process $X^{s,\omega}=\big\{X^{s,\omega}_r\big\}_{r\in[s,\infty)}$ is $\mathbf{F}^s-$adapted.

In virtue of regular conditional probability distribution by [57], the shifted random variables carry on the integrability as follows:

Proposition 3.2. *Let $0\le t\le s<\infty$. If $\xi\in L^1(\mathcal{F}^t)$, then it holds for P_t-a.s. $\omega\in\Omega^t$ that $\xi^{s,\omega}\in L^1(\mathcal{F}^s)$ and*

$$E_t\big[\xi\big|\mathcal{F}^t_s\big](\omega)=E_s\big[\xi^{s,\omega}\big]\in\mathbb{R}. \tag{3.2}$$

Consequently, the shift of a P_t-null set still has zero P_s-probability.

Proposition 3.3. *Let $0\le t\le s<\infty$.*
(1) For any P_t-null set $\mathcal{N}\in\mathscr{N}^t$, it holds for P_t-a.s. $\omega\in\Omega^t$ that $\mathcal{N}^{s,\omega}\in\mathscr{N}^s$. Then for any two real-valued random variables ξ_1 and ξ_2 on Ω^t with $\xi_1\le\xi_2$, P_t-a.s., it holds for P_t-a.s. $\omega\in\Omega^t$ that $\xi^{s,\omega}_1\le\xi^{s,\omega}_2$, P_s-a.s.
(2) For any $\tau\in\overline{\mathcal{T}}^t$ with $\tau\ge s$, it holds for P_t-a.s. $\omega\in\Omega^t$ that $\tau^{s,\omega}\in\overline{\mathcal{T}}^s$.

Based on Proposition 3.3 (1) and Lemma A.4, we can extend Proposition 3.2 from raw filtration $\mathbf{F}^t$ to augmented filtration $\overline{\mathbf{F}}^t$, and can show that the shifted processes inherit the integrability of original ones.

Proposition 3.4. *Let* $0 \le t \le s < \infty$.
(1) For any $\overline{\mathcal{F}}^t_s$*−measurable random variable* ξ*, it holds for* P_t*−a.s.* $\omega \in \Omega^t$ *that* $\xi^{s,\omega} = \xi(\omega)$*,* P_s*−a.s.*
(2) For any $r \in [s,\infty]$ *and* $\overline{\mathcal{F}}^t_r$*−measurable random variable* ξ*, it holds for* P_t*−a.s.* $\omega \in \Omega^t$ *that* $\xi^{s,\omega}$ *is* $\overline{\mathcal{F}}^s_r$*−measurable. If* ξ *is integrable, then it holds for* P_t*−a.s.* $\omega \in \Omega^t$ *that* $\xi^{s,\omega}$ *is integrable and* $E_t\big[\xi\big|\overline{\mathcal{F}}^t_s\big](\omega) = E_s\big[\xi^{s,\omega}\big] \in \mathbb{R}$*.*
(3) Let $X = \{X_r\}_{r \in [t,\infty)}$ *be an* $\overline{\mathbf{F}}^t$*−adapted process with* P_t*−a.s. continuous paths. It holds for* P_t*−a.s.* $\omega \in \Omega^t$ *that the shifted process* $X^{s,\omega} = \big\{X^{s,\omega}_r\big\}_{r \in [s,\infty)}$ *is* $\overline{\mathbf{F}}^s$*−adapted with* P_s*−a.s. continuous paths. If* $X \in \mathbb{C}^q_t(\mathbb{E})$ *for some* $q \in [1,\infty)$*, then* $X^{s,\omega} \in \mathbb{C}^q_s(\mathbb{E})$ *for* P_t*−a.s.* $\omega \in \Omega^t$*.*

Moreover, the shift of a uniformly integrable martingale are still uniformly integrable martingales under the augmented filtrations.

Proposition 3.5. *Let* $0 \le t \le s < \infty$*. For any* $M = \{M_r\}_{r \in [t,\infty)} \in \mathbb{M}_t$*, it holds for* P_t*−a.s.* $\omega \in \Omega^t$ *that* $M^{s,\omega} = \{M^{s,\omega}_r\}_{r \in [s,\infty)}$ *is of* $\mathbb{M}_s$*.*

3.3 Shifted Stochastic Differential Equations

Let $(t,x) \in [0,\infty) \times \mathbb{R}^l$. The SDE (1.1) has a unique solution $\mathcal{X}^{t,x} = \{\mathcal{X}^{t,x}_s\}_{s \in [t,\infty)}$, which is an $\mathbb{R}^l$−valued, $\overline{\mathbf{F}}^t$−adapted continuous process. As it holds P_t−a.s. that

$$\mathcal{X}_{t+s} = x + \int_t^{t+s} b(r,\mathcal{X}_r)dr + \int_t^{t+s} \sigma(r,\mathcal{X}_r)\,dW^t_r = x + \int_0^s b(t+r,\mathcal{X}_{t+r})dr + \int_{r \in [0,s]} \sigma(t+r,\mathcal{X}_{t+r})\,dW^t_{t+r}, \quad s \in [0,\infty),$$

we see that $\big\{\mathcal{X}^{t,x}_{t+s}\big\}_{s \in [0,\infty)}$ is exactly the unique solution of (2.1) with the probabilistic specification

$$\big(\Omega,\mathcal{F},P,\mathscr{N},\{B_s\}_{s \in [0,\infty)},\{\mathcal{F}_s\}_{s \in [0,\infty)}\big) = \Big(\Omega^t,\mathcal{F}^t,P_t,\mathscr{N}^t,\{W^t_{t+s}\}_{s \in [0,\infty)},\big\{\overline{\mathcal{F}}^t_{t+s}\big\}_{s \in [0,\infty)}\Big). \tag{3.3}$$

Clearly, τ is an $\overline{\mathbf{F}}^t$−stopping time if and only if $\widetilde{\tau} := \tau - t$ is a stopping time with respect to the filtration $\big\{\overline{\mathcal{F}}^t_{t+s}\big\}_{s \in [0,\infty)}$. So the corresponding $\mathcal{T}$ under setting (3.3) is $\mathcal{T} = \big\{\widetilde{\tau} = \tau - t \colon \tau \in \overline{\mathcal{T}}^t\big\}$. It then follows from Lemma 2.1 that

Corollary 3.1. *Let* $q \in [1,\infty)$ *and* $(t,x) \in [0,\infty) \times \mathbb{R}^l$*. For the same constant* C_q *as in Lemma 2.1,*

$$E_t\bigg[\sup_{s \in [t,\infty)} \big|\mathcal{X}^{t,x}_s\big|^q\bigg] \le C_q\big(1+|x|^q\big); \quad E_t\bigg[\sup_{s \in [t,\infty)} \big|\mathcal{X}^{t,x'}_s - \mathcal{X}^{t,x}_s\big|^q\bigg] \le C_q|x'-x|^q, \ \ \forall\, x' \in \mathbb{R}^l; \quad \text{and} \tag{3.4}$$

$$E_t\bigg[\sup_{\lambda \in (0,\delta]} \big|\mathcal{X}^{t,x}_{\tau+\lambda} - \mathcal{X}^{t,x}_\tau\big|^q\bigg] \le C_q\big(1+|x|^q\big)\big(\|c(\cdot)\|^q\delta^q + \|c(\cdot)\|^{\frac{q}{2}}\delta^{\frac{q}{2}}\big), \ \ \forall\, \delta \in (0,\infty), \ \ \forall\, \tau \in \overline{\mathcal{T}}^t. \tag{3.5}$$

The shift of $\mathcal{X}^{t,x}$ given path $\omega|_{[t,s]}$ turns out to be the solution of the shifted stochastic differential equation (2.1) over period $[s,\infty)$ with initial state $\mathcal{X}^{t,x}_s(\omega)$:

Proposition 3.6. (*Flow Property*) *Let* $0 \le t \le s < \infty$*,* $x \in \mathbb{R}^l$ *and set* $\mathfrak{X} := \mathcal{X}^{t,x}$*. It holds for* P_t*−a.s.* $\omega \in \Omega^t$ *that* $P_s\big\{\widetilde{\omega} \in \Omega^s \colon \mathfrak{X}_r(\omega \otimes_s \widetilde{\omega}) = \mathcal{X}^{s,\mathfrak{X}_s(\omega)}_r(\widetilde{\omega}), \ \forall\, r \in [s,\infty)\big\} = 1$*.*

The proof of Proposition 3.6 depends on the following result about the convergence of shifted random variables in probability.

Lemma 3.3. *For any* $\{\xi_i\}_{i \in \mathbb{N}} \subset L^1\big(\overline{\mathcal{F}}^t\big)$ *that converges to 0 in probability* P_t*, we can find a subsequence* $\big\{\widehat{\xi}_i\big\}_{i \in \mathbb{N}}$ *of it such that for* P_t*−a.s.* $\omega \in \Omega^t$*,* $\big\{\widehat{\xi}^{s,\omega}_i\big\}_{i \in \mathbb{N}}$ *converges to 0 in probability* P_s*.*

4 Two Dynamic Programming Principle of Optimal Stopping with Expectation Constraint

In this section, we exploit the flow property of shifted stochastic differential equations to establish two types of dynamic programming principles (DPPs) of the optimal stopping problem with expectation constraint over the canonical space.

4.1 The First Dynamic Programming Principle for $\mathcal{V}$

Let the state process now evolve from time $t\in[0,\infty)$ and position $x\in\mathbb{R}^l$ according to SDE (1.1). If the player selects to exercise at time $\tau\in\overline{\mathcal{T}}^t$, she will receive a running reward $\int_t^\tau f(r,\mathcal{X}_r^{t,x})dr$ and a terminal reward $\pi\big(\tau,\mathcal{X}_\tau^{t,x}\big)$. So the player's total wealth is

$$\mathcal{R}(t,x,\tau):=\int_t^\tau f(s,\mathcal{X}_s^{t,x})ds+\pi\big(\tau,\mathcal{X}_\tau^{t,x}\big)=\int_0^{\widetilde{\tau}} f(t+s,\mathcal{X}_{t+s}^{t,x})ds+\pi\big(t+\widetilde{\tau},\mathcal{X}_{t+\widetilde{\tau}}^{t,x}\big),$$

which is the payment $R\big(t,x,\widetilde{\tau}\big)$ in (2.6) under the specification (3.3). By (2.7) and (2.8), one has

$$E_t\big[|\mathcal{R}(t,x,\tau)|\big]\le\Psi(x)\quad\text{and}\quad E_t\big[\big|\mathcal{R}(t,x,\tau)-\mathcal{R}(t,x',\tau)\big|\big]\le 2\mathfrak{C}\big((C_p)^{\frac{1}{p}}|x-x'|+C_p|x-x'|^p\big),\ \ \forall\, x'\in\mathbb{R}^l. \tag{4.1}$$

Given $y\in[0,\infty)$, set $\mathcal{T}_x^t(y):=\big\{\tau\in\overline{\mathcal{T}}^t: E_t[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr]\le y\big\}$. As $E_t\big[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\big]=E_t\big[\int_0^{\widetilde{\tau}} g(t+r,\mathcal{X}_{t+r}^{t,x})dr\big]$, we see that $\big\{\widetilde{\tau}=\tau-t:\tau\in\mathcal{T}_x^t(y)\big\}$ is the corresponding $T_{t,x}(y)$ under setting (3.3). Then the maximum of the player's expected wealth subject to the budget constraint $E_t\big[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\big]\le y$, i.e.,

$$\mathcal{V}(t,x,y):=\sup_{\tau\in\mathcal{T}_x^t(y)} E_t\big[\mathcal{R}(t,x,\tau)\big]=\sup_{\widetilde{\tau}\in T_{t,x}(y)} E_t\big[R\big(t,x,\widetilde{\tau}\big)\big] \tag{4.2}$$

is exactly the value function (2.11) of the constrained optimal stopping problem under the specification (3.3). Then (2.13) and Lemma 2.2 show that

$$\Psi(x)\ge\mathcal{V}(t,x,y)\ge\mathcal{V}(t,x,0)=\pi(t,x),\quad \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty) \tag{4.3}$$

$$\text{and}\quad \mathcal{V}(t,x,y)=\sup_{\tau\in\widehat{\mathcal{T}}_{t,x}(y)} E\big[R(t,x,\tau)\big],\quad \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty), \tag{4.4}$$

where $\widehat{\mathcal{T}}_x^t(y):=\big\{\tau\in\mathcal{T}_x^t(y):\tau>t,\,P_t\text{–a.s.}\big\}$. Also,

$$\text{Theorem 2.1 still holds for the value function } \mathcal{V}. \tag{4.5}$$

Now, let $(t,x)\in[0,\infty)\times\mathbb{R}^l$ and let $\tau\in\overline{\mathcal{T}}^t$ with $E_t\big[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\big]<\infty$. We define a real-valued, $\overline{\mathbf{F}}^t$–adapted continuous process:

$$\mathcal{Y}_s^{t,x,\tau}:=E_t\bigg[\int_t^\tau g\big(r,\mathcal{X}_r^{t,x}\big)dr\Big|\overline{\mathcal{F}}_s^t\bigg]-\int_t^{\tau\wedge s} g\big(r,\mathcal{X}_r^{t,x}\big)dr,\quad s\in[t,\infty).$$

Since it holds for any $s\in[t,\infty)$ that

$$\mathcal{Y}_s^{t,x,\tau}=E_t\bigg[\int_{\tau\wedge s}^\tau g\big(r,\mathcal{X}_r^{t,x}\big)dr\Big|\overline{\mathcal{F}}_s^t\bigg]=E_t\bigg[\int_s^{\tau\vee s} g\big(r,\mathcal{X}_r^{t,x}\big)dr\Big|\overline{\mathcal{F}}_s^t\bigg]\in[0,\infty),\quad P_t-\text{a.s.}, \tag{4.6}$$

the continuity of $\mathcal{Y}^{t,x,\tau}$ implies that

$$N_{t,x,\tau}:=\big\{\mathcal{Y}_s^{t,x,\tau}\notin[0,\infty)\text{ for some } s\in[t,\infty)\big\}\in\mathscr{N}^t. \tag{4.7}$$

Then we have the first dynamic programming principle for the value function $\mathcal{V}$ in which the conditional expected cost $\mathcal{Y}^{t,x,\tau}$ acts as an additional state process.

Theorem 4.1. *Let* $t\in[0,\infty)$.
(1) For any $(x,y)\in\mathbb{R}^l\times[0,\infty)$*, let* $\{\zeta(\tau)\}_{\tau\in\mathcal{T}_x^t(y)}$ *be a family of* $\overline{\mathcal{T}}_\sharp^t$*–stopping times. Then we have the DPP (1.3), where* $\sup_{\tau\in\mathcal{T}_x^t(y)} E_t[\cdot]$ *can be replaced by* $\sup_{\tau\in\widehat{\mathcal{T}}_x^t(y)} E_t[\cdot]$ *if* $y>0$.
(2) If $\mathcal{V}(s,x,y)$ *is continuous in* $(s,x,y)\in[t,\infty)\times\mathbb{R}^l\times(0,\infty)$*, then* (1.3) *holds for any* $(x,y)\in\mathbb{R}^l\times[0,\infty)$ *and any family* $\{\zeta(\tau)\}_{\tau\in\mathcal{T}_x^t(y)}$ *of* $\overline{\mathcal{T}}^t$*–stopping times.*

4.2 An Alternative Stochastic Control Problem and the Second Dynamic Programming Principle for $\mathcal{V}$

Fix $t\in[0,\infty)$ and set $\mathfrak{A}_t:=\{\alpha=M-K:(M,K)\in\mathbb{M}_t\times\mathbb{K}_t\}$. Clearly, each $\alpha\in\mathfrak{A}_t$ is a uniformly integrable continuous supermartingales with respect to $\big(\overline{\mathbf{F}}^t,P_t\big)$.

Let $x\in\mathbb{R}^l$ and $\alpha\in\mathfrak{A}_t$. We define a continuous supermartingale with respect to $\big(\overline{\mathbf{F}}^t,P_t\big)$

$$Y^{t,x,\alpha}_s:=\alpha_s-\int_t^s g\big(r,\mathcal{X}^{t,x}_r\big)dr,\quad s\in[t,\infty),$$

and define an $\overline{\mathbf{F}}^t-$stopping time

$$\tau(t,x,\alpha):=\inf\big\{s\in[t,\infty):Y^{t,x,\alpha}_s=0\big\}. \tag{4.8}$$

The uniform integrability of α implies that the limit $\lim\limits_{s\to\infty}\alpha_s$ exists in $\mathbb{R}$, P_t-a.s. Since $\int_t^\infty g\big(r,\mathcal{X}^{t,x}_r\big)dr=\int_0^\infty g\big(t+r,\mathcal{X}^{t,x}_{t+r}\big)dr=\infty$, P_t-a.s. by (2.9), one can deduce that

$$\tau(t,x,\alpha)<\infty,\quad P_t-\text{a.s.} \tag{4.9}$$

Namely, $\tau(t,x,\alpha)\in\overline{\mathcal{T}}^t$.

Given $\alpha\in\mathfrak{A}_t$, the expected wealth $E_t\big[\mathcal{R}\big(t,x',\tau(t,x',\alpha)\big)\big]$ is continuous in $x\in\mathbb{R}^l$, which will play an important role in the demonstration of the second DPP for $\mathcal{V}$ (Theorem 4.2).

Proposition 4.1. *Let $(t,x)\in[0,\infty)\times\mathbb{R}^l$ and let $\alpha\in\mathfrak{A}_t$. For any $\varepsilon\in(0,1)$, there exists $\delta=\delta(t,x,\varepsilon)\in(0,1)$ such that*

$$E_t\Big[\big|\mathcal{R}\big(t,x',\tau(t,x',\alpha)\big)-\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big|\Big]\le\varepsilon,\quad \forall\, x'\in\overline{O}_\delta(x).$$

For any $y\in(0,\infty)$, we set $\mathfrak{A}_t(y):=\big\{\alpha\in\mathfrak{A}_t:\alpha_t=y,\ P_t-\text{a.s.}\big\}$.

Proposition 4.2. *Given $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty)$, $\alpha\to\tau(t,x,\alpha)$ is a surjective mapping from $\mathfrak{A}_t(y)$ to $\widehat{\mathcal{T}}^t_x(y)$.*

Remark 4.1. *Let $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty)$.*
1) Let $\tau\in\widehat{\mathcal{T}}^t_x(y)$. Proposition 4.2 shows that $\tau=\tau(t,x,\alpha)$ for some $\alpha\in\mathfrak{A}_t(y)$. In particular, we see from (6.87) *of its proof that α is a martingale (resp. supermartingale) if $E_t\big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\big]-y=0$ (resp. ≤ 0). To wit, the constraint $E_t\big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\big]=y$ (resp. $\le y$) corresponds to martingale (resp. supermartingale) controls in the alternative stochastic optimization problem.*

In case that α is a martingale, we know from the martingale representation theorem that $\alpha_s=y+\int_t^s\mathfrak{q}_r dW^t_r$, $s\in[t,\infty)$ for some $\mathfrak{q}\in\mathbb{H}^{2,\text{loc}}_t$. However reversely, for a $\widetilde{\mathfrak{q}}\in\mathbb{H}^{2,\text{loc}}_t$, $\widetilde{\alpha}_s:=y+\int_t^s\widetilde{\mathfrak{q}}_r dW^t_r$, $s\in[t,\infty)$ could be a strict local martingale with $E_t\big[\int_t^{\tau(t,x,\widetilde{\alpha})}g(r,\mathcal{X}^{t,x}_r)dr\big]<y$, see Example A.1 in the appendix. This is the reason why [44] requires $E[\tau^2]<\infty$ (see line -4 in page 3 therein) for the one-to-one correspondence between constrained stopping rules and squarely-integrable controls.
2) Define the value of the optimal stopping under the constraint $E_t\big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\big]=y$ by

$$\mathcal{U}(t,x,y):=\sup\Big\{E_t\big[\mathcal{R}(t,x,\tau)\big]:\tau\in\overline{\mathcal{T}}^t\ \text{with}\ E_t\Big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\Big]=y\Big\}.$$

Clearly, $\mathcal{U}(t,x,y)\le\mathcal{V}(t,x,y)$. However, we do not know whether they are equal since $\mathcal{U}(t,x,y)$ may not be increasing in y (cf. line 5 of Lemma 1.1 of [2]).
3) The constraint $E_t\big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\big]\le y$ is necessary for proving the continuity and the first DPP of the value function $\mathcal{V}$: Even if τ_1 in (6.18) *has $E\big[\int_0^{\tau_1}g(t+r,X^{t,x}_r)dr\big]=y$, the approximately optimal stopping time $\widehat{\tau}_1$ constructed in the case* (6.20) *may satisfy $E\big[\int_0^{\widehat{\tau}_1}g(t+r,X^{t,\mathfrak{x}}_r)dr\big]<(y-\delta)^+$ rather than $E\big[\int_0^{\widehat{\tau}_1}g(t+r,X^{t,\mathfrak{x}}_r)dr\big]=(y-\delta)^+$. Even if the $\tau\in\mathcal{T}^t_x(y)$ given in Lemma 6.1 reaches $E_t\big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\big]=y$, the pasting $\overline{\tau}$ of τ with the locally $\varepsilon-$optimal stopping times τ^i_n's in* (6.72) *satisfies $E_t\big[\int_t^{\overline{\tau}}g(r,\mathcal{X}^{t,x}_r)dr\big]<y+\varepsilon$ but $E_t\big[\int_t^{\overline{\tau}}g(r,\mathcal{X}^{t,x}_r)dr\big]=y+\varepsilon$ after a series of estimations in* (A.2).

By Proposition 4.2 and (4.4), we can alternatively express the optimal stopping problem with expectation constraints (2.10) as a stochastic control problem:

$$\mathcal{V}(t,x,y)=\sup_{\alpha\in\mathfrak{A}_t(y)} E_t\big[\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big],\quad \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty). \tag{4.10}$$

Moreover, we have the second dynamic programming principle for the value function $\mathcal{V}$ in which the controlled supermartingale $Y^{t,x,\alpha}$ serves as an additional state process.

Theorem 4.2. *Let* $t\in[0,\infty)$.
(1) For any $(x,y)\in\mathbb{R}^l\times[0,\infty)$, *let* $\{\zeta(\alpha)\}_{\alpha\in\mathfrak{A}_t(y)}$ *be a family of* $\overline{\mathcal{T}}^{\,t}_{\sharp}-$*stopping times. Then we have the DPP* (1.4).
(2) If $\mathcal{V}(s,x,y)$ *is continuous in* $(s,x,y)\in[t,\infty)\times\mathbb{R}^l\times(0,\infty)$, *then* (1.4) *holds for any* $(x,y)\in\mathbb{R}^l\times[0,\infty)$ *and any family* $\{\zeta(\alpha)\}_{\alpha\in\mathfrak{A}_t(y)}$ *of* $\overline{\mathcal{T}}^{\,t}-$*stopping times.*

5 Related Fully Non-linear Parabolic HJB Equations

In this section, we show that the value function of the optimal stopping problem with expectation constraint is the viscosity solution to a related fully non-linear parabolic Hamilton-Jacobi-Bellman (HJB) equation.

For any $\phi(t,x,y)\in C^{1,2,2}\big([0,\infty)\times\mathbb{R}^l\times[0,\infty)\big)$, we set

$$\mathscr{D}\phi(t,x,y):=\big(D_x\phi, D_x^2\phi, \partial_y\phi, \partial_y^2\phi, D_x(\partial_y\phi)\big)(t,x,y)\in\mathbb{R}^l\times\mathbb{S}_l\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^l,\quad \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty),$$

where $\mathbb{S}_l$ denotes the set of all $\mathbb{R}^{l\times l}-$valued symmetric matrices.

Recall the definition of viscosity solutions to a parabolic equation with a general (non-linear) Hamiltonian $H: [0,\infty)\times\mathbb{R}^l\times\mathbb{R}\times\mathbb{R}^l\times\mathbb{S}_l\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^l\to[-\infty,\infty]$.

Definition 5.1. *An upper (resp. lower) semi-continuous function* $u:[0,\infty)\times\mathbb{R}^l\times[0,\infty)\to\mathbb{R}$ *is called a viscosity subsolution (resp. supersolution) of*

$$\begin{cases}-\partial_t u(t,x,y)-H\big(t,x,u(t,x,y),\mathscr{D}u(t,x,y)\big)=0, & \forall\,(t,x,y)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty),\\ u(t,x,0)=\pi(t,x), \quad \forall\,(t,x)\in[0,\infty)\times\mathbb{R}^l\end{cases}$$

if $u(t,x,0)\le$ *(resp.* $\ge$*)* $\pi(t,x)$, $\forall\,(t,x)\in[0,\infty)\times\mathbb{R}^l$, *and if for any* $(t_o,x_o,y_o)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty)$ *and* $\phi\in C^{1,2,2}\big([0,\infty)\times\mathbb{R}^l\times[0,\infty)\big)$ *such that* $u-\phi$ *attains a strict local maximum* 0 *(resp. strict local minimum* 0*) at* (t_o,x_o,y_o), *one has*

$$-\partial_t\phi(t_o,x_o,y_o)-H\big(t_o,x_o,\phi(t_o,x_o,y_o),\mathscr{D}\phi(t_o,x_o,y_o)\big)\le (\textit{resp. } \ge)\ 0.$$

For any $\phi\in C^{1,2,2}\big([0,\infty)\times\mathbb{R}^l\times[0,\infty)\big)$, we also define

$$\begin{aligned}\mathcal{L}_x\phi(t,x,y) &:= \frac{1}{2}trace\big(\sigma(t,x)\cdot\sigma^T(t,x)\cdot D_x^2\phi(t,x,y)\big)+b^T(t,x)\cdot D_x\phi(t,x,y),\\ \mathcal{H}\phi(t,x,y) &:= \sup_{a\in\mathbb{R}^d}\Big\{\frac{1}{2}|a|^2\partial_y^2\phi(t,x,y)+\big(D_x(\partial_y\phi(t,x,y))\big)^T\cdot\sigma(t,x)\cdot a\Big\}\ge 0,\ (t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty),\end{aligned}$$

as well as the upper semi-continuous envelope of $\mathcal{H}\phi$ (the smallest upper semi-continuous function above $\mathcal{H}\phi$)

$$\overline{\mathcal{H}}\phi(t,x,y):=\varlimsup_{(t',x',y')\to(t,x,y)}\mathcal{H}\phi(t',x',y')=\lim_{\delta\to 0}\downarrow\sup_{(t',x',y')\in\mathscr{O}_\delta(t,x,y)}\mathcal{H}\phi(t',x',y'),\ (t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty), \tag{5.1}$$

where $\mathscr{O}_\delta(t,x,y):=\big[(t-\delta)^+,t+\delta\big]\times\overline{O}_\delta(x)\times\big[(y-\delta)^+,y+\delta\big]$.

Theorem 5.1. *Assume that* b,σ *additionally satisfy* (2.4) *and* f,g *additionally satisfy* (2.14). *Then the value function* $\mathcal{V}$ *in* (4.2) *is a viscosity supersolution of*

$$\begin{cases}-\partial_t u(t,x,y)-\mathcal{L}_x u(t,x,y)+g(t,x)\partial_y u(t,x,y)-\mathcal{H}u(t,x,y)-f(t,x)=0, & \forall\,(t,x,y)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty),\\ u(t,x,0)=\pi(t,x), \quad \forall\,(t,x)\in[0,\infty)\times\mathbb{R}^l,\end{cases} \tag{5.2}$$

and is a viscosity subsolution of

$$\begin{cases}-\partial_t u(t,x,y)-\mathcal{L}_x u(t,x,y)+g(t,x)\partial_y u(t,x,y)-\overline{\mathcal{H}}u(t,x,y)-f(t,x)=0, & \forall\,(t,x,y)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty),\\ u(t,x,0)=\pi(t,x), \quad \forall\,(t,x)\in[0,\infty)\times\mathbb{R}^l.\end{cases} \tag{5.3}$$

Remark 5.1. *See Section 5.2 of [44] for the connection between the fully non-linear parabolic HJB equation* (5.2) *and generalized Monge-Ampère equations.*

6 Proofs

6.1 Proofs of Section 2

Proof of Lemma 2.1: In this proof, we set $\overline{c}:=\int_0^\infty c(s)ds$ and let $\mathfrak{c}_q$ denote a generic constant depending only on q, whose form may vary from line to line.
1) Let $T\in(0,\infty)$ and set $\widetilde{q}:=q\vee 2$. Given $s\in[0,T]$, we set $\Phi_s:=\sup\limits_{r\in[0,s]}\big|X^{t,x}_r\big|$, (2.1) and (1.6) show that

$$\begin{aligned}\Phi_s &\le |x|+\int_0^s\big(|b(t+r,0)|+\big|b\big(t+r,X^{t,x}_r\big)-b(t+r,0)\big|\big)dr+\sup_{s'\in[0,s]}\Big|\int_0^{s'}\sigma(t+r,X^{t,x}_r)dB_r\Big|\\ &\le |x|+\int_0^s c(t+r)dr+\int_0^s c(t+r)|X^{t,x}_r|dr+\sup_{s'\in[0,s]}\Big|\int_0^{s'}\sigma(t+r,X^{t,x}_r)dB_r\Big|,\quad P-\text{a.s.}\end{aligned}\tag{6.1}$$

Taking $\widetilde{q}-$th power of (6.1), we can deduce from Hölder's inequality, the Burkholder-Davis-Gundy inequality, (1.7) and Fubini's Theorem that

$$\begin{aligned}E\big[\Phi_s^{\widetilde{q}}\big]&\le 4^{\widetilde{q}-1}|x|^{\widetilde{q}}+4^{\widetilde{q}-1}\overline{c}^{\,\widetilde{q}}+4^{\widetilde{q}-1}\Big(\int_0^s c^{\frac{\widetilde{q}}{\widetilde{q}-1}}(t+r)dr\Big)^{\widetilde{q}-1}E\bigg[\int_0^s\big|X^{t,x}_r\big|^{\widetilde{q}}dr\bigg]+\mathfrak{c}_qE\bigg[\Big(\int_0^s c(t+r)\big(1+|X^{t,x}_r|\big)^2dr\Big)^{\frac{\widetilde{q}}{2}}\bigg]\\ &\le 4^{\widetilde{q}-1}|x|^{\widetilde{q}}+4^{\widetilde{q}-1}\overline{c}^{\,\widetilde{q}}+4^{\widetilde{q}-1}\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-1}}(t+r)dr\Big)^{\widetilde{q}-1}\int_0^s E\big[\Phi_r^{\widetilde{q}}\big]dr+\mathfrak{c}_q\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-2}}(t+r)dr\Big)^{\frac{\widetilde{q}}{2}-1}\int_0^s E\big[(1+\Phi_r)^{\widetilde{q}}\big]dr\\ &\le \mathfrak{c}_q\bigg[|x|^{\widetilde{q}}+\overline{c}^{\,\widetilde{q}}+T\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-2}}(t+r)dr\Big)^{\frac{\widetilde{q}}{2}-1}\bigg]+\bigg[4^{\widetilde{q}-1}\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-1}}(t+r)dr\Big)^{\widetilde{q}-1}+\mathfrak{c}_q\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-2}}(t+r)dr\Big)^{\frac{\widetilde{q}}{2}-1}\bigg]\int_0^s E\big[\Phi_r^{\widetilde{q}}\big]dr.\end{aligned}$$

An application of Gronwall's inequality then gives that

$$\begin{aligned}E\big[\Phi_s^q\big]\,&\le\,1+E\big[\Phi_s^{\widetilde{q}}\big]\le 1+\mathfrak{c}_q\bigg[|x|^{\widetilde{q}}+\overline{c}^{\,\widetilde{q}}+T\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-2}}(t+r)dr\Big)^{\frac{\widetilde{q}}{2}-1}\bigg]\\ &\quad\times\exp\bigg\{4^{\widetilde{q}-1}\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-1}}(t+r)dr\Big)^{\widetilde{q}-1}s+\mathfrak{c}_q\Big(\int_0^T c^{\frac{\widetilde{q}}{\widetilde{q}-2}}(t+r)dr\Big)^{\frac{\widetilde{q}}{2}-1}s\bigg\}<\infty,\quad\forall\, s\in[0,T].\end{aligned}\tag{6.2}$$

Let $s\in[0,T]$. Since the Burkholder-Davis-Gundy inequality and (1.6) also show that

$$\begin{aligned}&E\bigg[\sup_{s'\in[0,s]}\Big|\int_0^{s'}\sigma(t+r,X^{t,x}_r)dB_r\Big|^q\bigg]\le\mathfrak{c}_qE\bigg[\Big(\int_0^s\big|\sigma(t+r,X^{t,x}_r)\big|^2dr\Big)^{\frac{q}{2}}\bigg]\le\mathfrak{c}_qE\bigg[\Big(\int_0^s c(t+r)\big(1+|X^{t,x}_r|\big)^2dr\Big)^{\frac{q}{2}}\bigg]\\ &\quad\le\mathfrak{c}_qE\bigg[(1+\Phi_s)^{\frac{q}{2}}\Big(\int_0^s c(t+r)\big(1+|X^{t,x}_r|\big)dr\Big)^{\frac{q}{2}}\bigg]\le E\bigg[\frac12 8^{1-q}(1+\Phi_s)^q+\mathfrak{c}_q\Big(\int_0^s c(t+r)\big(1+|X^{t,x}_r|\big)dr\Big)^q\bigg]\\ &\quad\le\frac12 4^{1-q}\big(1+E\big[\Phi_s^q\big]\big)+\mathfrak{c}_q\Big(\int_0^s c(t+r)dr\Big)^q+\mathfrak{c}_qE\bigg[\Big(\int_0^s c(t+r)\Phi_rdr\Big)^q\bigg],\end{aligned}$$

taking $q-$th power of (6.1) and using Fubini's Theorem yield that

$$\begin{aligned}4^{1-q}E\big[\Phi_s^q\big]\,&\le\,|x|^q+\overline{c}^q+E\bigg[\Big(\int_0^s c(t+r)\Phi_rdr\Big)^q\bigg]+E\bigg[\sup_{s'\in[0,s]}\Big|\int_0^{s'}\sigma(t+r,X^{t,x}_r)dB_r\Big|^q\bigg]\\ &\le\,|x|^q+\mathfrak{c}_q\overline{c}^q+\frac12 4^{1-q}\big(1+E\big[\Phi_s^q\big]\big)+\mathfrak{c}_q\Big(\int_0^s c(t+r)dr\Big)^{q-1}E\bigg[\int_0^s c(t+r)\Phi_r^qdr\bigg].\end{aligned}\tag{6.3}$$

Here, we applied Hölder's inequality $\big|\int_0^s\mathfrak{a}_r\mathfrak{b}_rdr\big|\le\big(\int_0^s|\mathfrak{a}_r|^qdr\big)^{\frac1q}\big(\int_0^s|\mathfrak{b}_r|^{\frac{q}{q-1}}dr\big)^{\frac{q-1}{q}}$ with $(\mathfrak{a}_r,\mathfrak{b}_r)=\Big(c^{\frac1q}(t+r)\Phi_r,c^{\frac{q-1}{q}}(t+r)\Big)$. As $E\bigg[\sup\limits_{r\in[0,s]}\big|X^{t,x}_r\big|^q\bigg]<\infty$ by (6.2), it follows from (6.3) that for any $s\in[0,T]$

$$E\big[\Phi_s^q\big]\le 1+2\times 4^{q-1}|x|^q+\mathfrak{c}_q\overline{c}^q+\mathfrak{c}_q\overline{c}^{q-1}\int_0^s c(t+r)E\big[\Phi_r^q\big]dr.\tag{6.4}$$

Applying Gronwall's inequality again yields that $E\big[\Phi^q_s\big] \le \big(1+2\times 4^{q-1}|x|^q+\mathfrak{c}_q\overline{c}^q\big)\exp\big\{\mathfrak{c}_q\overline{c}^{q-1}\int_0^s c(t+r)dr\big\}$, $\forall\, s\in[0,T]$. In particular, taking $s=T$ and then letting $T\to\infty$, one can deduce from the monotone convergence theorem that

$$E\bigg[\sup_{r\in[0,\infty)}\big|X^{t,x}_r\big|^q\bigg] \le \big(1+2\times 4^{q-1}|x|^q+\mathfrak{c}_q\overline{c}^q\big)\exp\big\{\mathfrak{c}_q\overline{c}^q\big\}. \tag{6.5}$$

2) Let $\mathscr{X}_s := X^{t,x}_s - X^{t,x'}_s$, $\forall\, s\in[0,\infty)$. Given $s\in[0,\infty)$, we set $\widetilde{\Phi}_s := \sup\limits_{r\in[0,s]}\big|\mathscr{X}_r\big|$. Since an analogy to (6.1) shows that

$$\widetilde{\Phi}_s \le |x'-x| + \int_0^s c(t+r)|\mathscr{X}_r|dr + \sup_{s'\in[0,s]}\bigg|\int_0^{s'}\big(\sigma(t+r,X^{t,x}_r)-\sigma(t+r,X^{t,x'}_r)\big)dB_r\bigg|, \quad P-\text{a.s.},$$

the Burkholder-Davis-Gundy inequality and (1.7) imply that

$$\begin{aligned} 3^{1-q}E\big[\widetilde{\Phi}^q_s\big] &\le |x'-x|^q + E\bigg[\Big(\int_0^s c(t+r)|\mathscr{X}_r|dr\Big)^q\bigg] + \mathfrak{c}_q E\bigg[\Big(\int_0^s c(t+r)|\mathscr{X}_r|^2dr\Big)^{\frac{q}{2}}\bigg] \\ &\le |x'-x|^q + E\bigg[\Big(\int_0^s c(t+r)|\mathscr{X}_r|dr\Big)^q\bigg] + \mathfrak{c}_q E\bigg[\widetilde{\Phi}^{q/2}_s\Big(\int_0^s c(t+r)|\mathscr{X}_r|dr\Big)^{\frac{q}{2}}\bigg] \\ &\le |x'-x|^q + \frac{1}{2}3^{1-q}E\big[\widetilde{\Phi}^q_s\big] + \mathfrak{c}_q E\bigg[\Big(\int_0^s c(t+r)|\mathscr{X}_r|dr\Big)^q\bigg]. \end{aligned}$$

Since $E\big[\widetilde{\Phi}^q_s\big] \le 2^{q-1}E\Big[\big(X^{t,x}_*\big)^q + \big(X^{t,x'}_*\big)^q\Big] < \infty$ by Part 1, an analogy to (6.4) shows that

$$E\big[\widetilde{\Phi}^q_s\big] \le 2\times 3^{q-1}|x'-x|^q + \mathfrak{c}_q\overline{c}^{q-1}\int_0^s c(t+r)E\big[\widetilde{\Phi}^q_r\big]dr, \quad \forall\, s\in[0,\infty).$$

Then we see from Gronwall's inequality that $E\big[\widetilde{\Phi}^q_s\big] \le 2\times 3^{q-1}|x'-x|^q\exp\big\{\mathfrak{c}_q\overline{c}^{q-1}\int_0^s c(t+r)dr\big\}$, $\forall\, s\in[0,\infty)$. As $s\to\infty$, the monotone convergence theorem implies that $E\bigg[\sup\limits_{r\in[0,\infty)}\big|X^{t,x'}_r - X^{t,x}_r\big|^q\bigg] \le 2\times 3^{q-1}|x'-x|^q\exp\big\{\mathfrak{c}_q\overline{c}^q\big\}$.

3) Let $\delta\in(0,\infty)$ and $\tau\in\mathcal{T}$. For any $\lambda\in(0,\delta]$, since it holds $P-$a.s. that

$$X^{t,x}_{\tau+\lambda} - X^{t,x}_\tau = \int_\tau^{\tau+\lambda} b(t+r,X^{t,x}_r)dr + \int_\tau^{\tau+\lambda}\sigma(t+r,X^{t,x}_r)dB_r = \int_\tau^{\tau+\lambda} b(t+r,X^{t,x}_r)dr + \int_0^{\tau+\lambda}\mathbf{1}_{\{\tau<r<\tau+\delta\}}\sigma(t+r,X^{t,x}_r)dB_r,$$

taking $q-$th power and using (1.6) yield that

$$\big|X^{t,x}_{\tau+\lambda} - X^{t,x}_\tau\big|^q \le 2^{q-1}\Big(\int_\tau^{\tau+\lambda} c(t+r)\big(1+|X^{t,x}_r|\big)dr\Big)^q + 2^{q-1}\sup_{s\in[0,\infty)}\bigg|\int_0^s \mathbf{1}_{\{\tau<r<\tau+\delta\}}\sigma(t+r,X^{t,x}_r)dB_r\bigg|^q, \quad P-\text{a.s.}$$

Then the Burkholder-Davis-Gundy inequality shows that

$$\begin{aligned} E\bigg[\sup_{\lambda\in(0,\delta]}\big|X^{t,x}_{\tau+\lambda} - X^{t,x}_\tau\big|^q\bigg] &\le 2^{q-1}E\bigg[\Big(\int_\tau^{\tau+\delta} c(t+r)\big(1+|X^{t,x}_r|\big)dr\Big)^q\bigg] + \mathfrak{c}_q E\bigg[\Big(\int_\tau^{\tau+\delta}\big|\sigma(t+r,X^{t,x}_r)\big|^2dr\Big)^{\frac{q}{2}}\bigg] \\ &\le 2^{q-1}\delta^q\|c(\cdot)\|^q E\bigg[\Big(1+\sup_{r\in[0,\infty)}|X^{t,x}_r|\Big)^q\bigg] + \mathfrak{c}_q E\bigg[\Big(\int_\tau^{\tau+\delta} c(t+r)\big(1+|X^{t,x}_r|\big)^2dr\Big)^{\frac{q}{2}}\bigg] \\ &\le \mathfrak{c}_q\big(\delta^q\|c(\cdot)\|^q + \delta^{\frac{q}{2}}\|c(\cdot)\|^{\frac{q}{2}}\big)\bigg(1+E\bigg[\sup_{r\in[0,\infty)}|X^{t,x}_r|^q\bigg]\bigg), \end{aligned}$$

which together with (6.5) leads to (2.3).

4) Now, we assume functions b and σ satisfy (2.4) for some $\varpi\in[1,\infty)$. Let $t'\in(t,\infty)$ and define $\widehat{\mathscr{X}_s} := X^{t',x}_s - X^{t,x}_s$, $\forall\, s\in[0,T]$. By (2.4), it holds $P-$a.s. that

$$\begin{aligned} \big|b(t'+r,X^{t',x}_r)-b(t+r,X^{t,x}_r)\big| &\le \big|b(t'+r,X^{t',x}_r)-b(t'+r,X^{t,x}_r)\big| + \big|b(t'+r,X^{t,x}_r)-b(t+r,X^{t,x}_r)\big| \\ &\le c(t'+r)\big|\widehat{\mathscr{X}_r}\big| + c(t+r)\rho(t'-t)\big(1+|X^{t,x}_r|^\varpi\big), \quad \forall\, r\in[0,\infty), \end{aligned} \tag{6.6}$$

and similarly that

$$\big|\sigma(t'+r,X_r^{t',x})-\sigma(t+r,X_r^{t,x})\big|\le\sqrt{c(t'+r)}\big|\widehat{\mathscr{X}_r}\big|+\sqrt{c(t+r)}\rho(t'-t)\big(1+|X_r^{t,x}|^{\varpi}\big),\quad\forall\, r\in[0,\infty). \tag{6.7}$$

Given $s\in[0,\infty)$, we set $\widehat{\Phi}_s:=\sup\limits_{r\in[0,s]}\big|\widehat{\mathscr{X}_r}\big|$, (6.6) shows that $P-$a.s.

$$\widehat{\Phi}_s\le\int_0^s c(t'+r)\big|\widehat{\mathscr{X}_r}\big|dr+\rho(t'-t)\int_0^s c(t+r)\big(1+|X_r^{t,x}|^{\varpi}\big)dr+\sup_{s'\in[0,s]}\Big|\int_0^{s'}\big(\sigma(t'+r,X_r^{t',x})-\sigma(t+r,X_r^{t,x})\big)dB_r\Big|. \tag{6.8}$$

The Burkholder-Davis-Gundy inequality, (1.10) and (6.7) imply that

$$\begin{aligned}
E\bigg[\sup_{s'\in[0,s]}\Big|\int_0^{s'}\big(\sigma(t'+r,X_r^{t',x})-\sigma(t+r,X_r^{t,x})\big)dB_r\Big|^q\bigg]&\le\mathfrak{c}_q E\bigg[\Big(\int_0^s\big|\sigma(t'+r,X_r^{t',x})-\sigma(t+r,X_r^{t,x})\big|^2dr\Big)^{\frac q2}\bigg]\\
&\le\mathfrak{c}_q E\bigg[\widehat{\Phi}_s^{q/2}\Big(\int_0^s c(t'+r)|\widehat{\mathscr{X}_r}|dr\Big)^{\frac q2}\bigg]+\mathfrak{c}_q\big(\rho(t'-t)\big)^qE\bigg[\Big(\int_0^s c(t+r)\big(1+|X_r^{t,x}|^{\varpi}\big)^2dr\Big)^{\frac q2}\bigg]\\
&\le\frac12 3^{1-q}E\big[\widehat{\Phi}_s^q\big]+\mathfrak{c}_qE\bigg[\Big(\int_0^s c(t'+r)|\widehat{\mathscr{X}_r}|dr\Big)^q\bigg]+\mathfrak{c}_q\overline{c}^{\frac q2}\big(\rho(t'-t)\big)^qE\bigg[\Big(1+\sup_{r\in[0,s]}|X_r^{t,x}|^{\varpi}\Big)^q\bigg].
\end{aligned}$$

Taking $q-$th power in (6.8) and using an analogy to (6.4) yield that

$$\begin{aligned}
3^{1-q}E\big[\widehat{\Phi}_s^q\big]\;&\le\;E\bigg[\Big(\int_0^s c(t'+r)|\widehat{\mathscr{X}_r}|dr\Big)^q\bigg]+\big(\rho(t'-t)\big)^qE\bigg[\Big(\int_0^s c(t+r)\big(1+|X_r^{t,x}|^{\varpi}\big)dr\Big)^q\bigg]\\
&\quad+E\bigg[\sup_{s'\in[0,s]}\Big|\int_0^{s'}\big(\sigma(t'+r,X_r^{t',x})-\sigma(t+r,X_r^{t,x})\big)dB_r\Big|^q\bigg]\\
&\le\;\frac12 3^{1-q}E\big[\widehat{\Phi}_s^q\big]+\mathfrak{c}_q\overline{c}^{q-1}\int_0^s c(t'+r)E\big[\widehat{\Phi}_r^q\big]dr+\mathfrak{c}_q\big(\overline{c}^{\frac q2}+\overline{c}^q\big)\big(\rho(t'-t)\big)^qE\bigg[1+\sup_{r\in[0,s]}|X_r^{t,x}|^{q\varpi}\bigg].
\end{aligned}$$

As $E\big[\widehat{\Phi}_s^q\big]\le 2^{q-1}E\Big[\big(X_*^{t,x}\big)^q+\big(X_*^{t',x}\big)^q\Big]<\infty$ by Part 1, it then follows from Gronwall's inequality that

$$E\big[\widehat{\Phi}_s^q\big]\le\mathfrak{c}_q\big(\overline{c}^{\frac q2}+\overline{c}^q\big)\big(\rho(t'-t)\big)^qE\bigg[1+\sup_{r\in[0,\infty)}|X_r^{t,x}|^{q\varpi}\bigg]\exp\Big\{\mathfrak{c}_q\overline{c}^{q-1}\int_0^s c(t'+r)dr\Big\},\quad\forall\, s\in[0,\infty).$$

Letting $s\to\infty$, we can deduce from the monotone convergence theorem that

$$E\bigg[\sup_{r\in[0,\infty)}\big|X_r^{t',x}-X_r^{t,x}\big|^q\bigg]\le\mathfrak{c}_q\big(\overline{c}^{\frac q2}+\overline{c}^q\big)\big(\rho(t'-t)\big)^qE\bigg[1+\sup_{r\in[0,\infty)}|X_r^{t,x}|^{q\varpi}\bigg]\exp\big\{\mathfrak{c}_q\overline{c}^q\big\},$$

which together with (6.5) proves (2.5). □

Proof of (2.7)**:** We see from (1.8) that

$$|f(t',x')|\le|f(t',x')-f(t',0)|+|f(t',0)|\le c(t')\big(1+|x'|\vee|x'|^p\big)\le c(t')\big(2+|x'|^p\big),\quad\forall\,(t',x')\in(0,\infty)\times\mathbb{R}^l. \tag{6.9}$$

Similarly, (1.9) shows that

$$|\pi(t',x')|\le\mathfrak{C}\big(2+|x'|^p\big),\quad\forall\,(t',x')\in(0,\infty)\times\mathbb{R}^l. \tag{6.10}$$

Given $\tau\in\mathcal{T}$, Since (6.9), (6.10) show that

$$\big|R(t,x,\tau)\big|\le\big(2+(X_*^{t,x})^p\big)\int_0^\infty c(t+r)dr+\mathfrak{C}\big(2+(X_*^{t,x})^p\big)\le 2\mathfrak{C}\big(2+(X_*^{t,x})^p\big), \tag{6.11}$$

the first inequality in (2.2) implies that $E\big[\big|R(t,x,\tau)\big|\big]\le 2\mathfrak{C}\big(2+C_p(1+|x|^p)\big)=\Psi(x)$. □

Proof of Lemma 2.2: Let $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty)$. Since $\mathcal{F}_0$ consist of $\mathcal{F}-$measurable sets A with $P(A)=0$ or $P(A)=1$, it holds for any $\tau\in\mathcal{T}$ that

$$P\{\tau=0\}=1 \text{ or } P\{\tau>0\}=1. \tag{6.12}$$

It follows that $V(t,x,y) = E\big[R(t,x,0)\big] \vee \Big(\sup\limits_{\tau \in \widehat{\mathcal{T}}_{t,x}(y)} E\big[R\big(t,x,\tau\big)\big]\Big)$. So it suffices to show that $E\big[R(t,x,0)\big] \le \sup\limits_{\tau \in \widehat{\mathcal{T}}_{t,x}(y)} E\big[R\big(t,x,\tau\big)\big]$.

We arbitrarily pick up τ from $\widehat{\mathcal{T}}_{t,x}(y)$. Given $n \in \mathbb{N}$, it is clear that $\tau_n := \tau \wedge (1/n)$ also belongs to $\widehat{\mathcal{T}}_{t,x}(y)$, so

$$\sup_{\tau \in \widehat{\mathcal{T}}_{t,x}(y)} E\big[R\big(t,x,\tau\big)\big] \ge E\big[R\big(t,x,\tau_n\big)\big] = E\bigg[\int_0^{\tau_n} f(t+s, X_s^{t,x})ds + \pi\big(t+\tau_n, X_{\tau_n}^{t,x}\big)\bigg]. \tag{6.13}$$

An analogy to (6.11) shows that $\big|R(t,x,\tau_n)\big| \le \big(2+(X_*^{t,x})^p\big)\int_t^\infty c(r)dr + \mathfrak{C}\big(2+(X_*^{t,x})^p\big) \le 2\mathfrak{C}\big(2+(X_*^{t,x})^p\big)$, whose $E-$expectation equals to $2\mathfrak{C}\big(2+C_p(1+|x|^p)\big) = \Psi(x)$ by the first inequality in (3.4). Then letting $n \to \infty$ in (6.13), we can deduce from (1.9), the continuity of process $X^{t,x}$ and the dominated convergence theorem that

$$\sup_{\tau \in \widehat{\mathcal{T}}_{t,x}(y)} E\big[R\big(t,x,\tau\big)\big] \ge \lim_{n\to\infty} E\bigg[\int_0^{\tau_n} f(t+s, X_s^{t,x})ds + \pi\big(t+\tau_n, X_{\tau_n}^{t,x}\big)\bigg] = E\big[\pi(t, X_0^{t,x})\big] = \pi(t,x) = E\big[R(t,x,0)\big]. \quad \square$$

Proof of Theorem 2.1: 1) Fix $t \in [0,\infty)$. We let $(x,\varepsilon) \in \mathbb{R}^l \times (0,1)$ and set $\varepsilon_o := (5+10\mathfrak{C})^{-1}\varepsilon$. Since $\mathfrak{M} := E\big[(X_*^{t,x})^p\big] < \infty$ by the first inequality in (2.2), we can find $\lambda_o = \lambda_o(t,x,\varepsilon) \in \big(0,\varepsilon_o\big)$ such that

$$E\big[\mathbf{1}_A\,(X_*^{t,x})^p\big] < \varepsilon_o \quad \text{for any } A \in \mathcal{F} \text{ with } P(A) < \lambda_o\,. \tag{6.14}$$

There exists $R = R(t,x,\varepsilon) \in (0,\infty)$ such that the set $A_R := \big\{X_*^{t,x} > R\big\} \in \mathcal{F}$ satisfies $P(A_R) < \lambda_o/2$.

Let $\lambda = \lambda(t,x,\varepsilon) \in (0,1)$ satisfy that

$$\sqrt{\lambda} \le \Big(\frac{1}{6}\lambda_o \kappa_R\Big) \wedge \frac{\varepsilon_o}{(2+\mathfrak{M})\|c(\cdot)\|} \wedge \rho^{-1}(\varepsilon_o) \quad \text{and} \tag{6.15}$$

$$(C_p)^{\frac{1}{p}}\big(1+|x|\big)\big(\|c(\cdot)\|\lambda^{\frac{1}{2}} + \|c(\cdot)\|^{\frac{1}{2}}\lambda^{\frac{1}{4}}\big) + C_p\big(1+|x|^p\big)\big(\|c(\cdot)\|^p\lambda^{\frac{p}{2}} + \|c(\cdot)\|^{\frac{p}{2}}\lambda^{\frac{p}{4}}\big) \le \varepsilon_o\,. \tag{6.16}$$

We pick up $\delta = \delta(t,x,\varepsilon) \in (0,1)$ such that

$$\mathfrak{C}(C_p)^{\frac{1}{p}}\delta + \mathfrak{C}C_p\delta^p \le \lambda \wedge \varepsilon_o\,, \tag{6.17}$$

and fix $y \in [0,\infty)$.

1a) *We first demonstrate that* $V(t,\mathfrak{x},\mathfrak{y}) \ge V(t,x,y) - \varepsilon,\ \forall\,(\mathfrak{x},\mathfrak{y}) \in \overline{O}_\delta(x) \times [(y-\delta)^+, \infty)$.

Let $\tau_1 = \tau_1(t,x,y,\varepsilon) \in \mathcal{T}_{t,x}(y)$ such that

$$E\big[R(t,x,\tau_1)\big] \ge V(t,x,y) - \varepsilon_o, \tag{6.18}$$

and let $\mathfrak{x} \in \overline{O}_\delta(x)$.

We claim that there exists a stopping time $\widehat{\tau}_1 = \widehat{\tau}_1(t,x,\mathfrak{x},y,\varepsilon) \in \mathcal{T}_{t,\mathfrak{x}}\big((y-\delta)^+\big)$ satisfying

$$\widehat{\tau}_1 \le \tau_1 \quad \text{and} \quad P\big(A_R^c \cap \{\tau_1 > \widehat{\tau}_1 + \sqrt{\lambda}\,\}\big) < \lambda_o/2. \tag{6.19}$$

Set $\delta_y := \delta \wedge y$, which satisfies $y - \delta_y = (y-\delta) \vee (y-y) = (y-\delta)^+$.

If $E\big[\int_0^{\tau_1} g(t+r, X_r^{t,\mathfrak{x}})dr\big] \le y - \delta_y$ $\big($i.e. $\tau_1 \in \mathcal{T}_{t,\mathfrak{x}}((y-\delta)^+)\big)$, we directly set $\widehat{\tau}_1 := \tau_1$.

Otherwise, set $\mathfrak{a} := E\big[\int_0^{\tau_1} g(t+r, X_r^{t,\mathfrak{x}})dr\big] - y + \delta_y > 0$ (In this case, one must have $y > 0$). Since both $\big\{E\big[\int_0^{\tau_1} g(t+r, X_r^{t,\mathfrak{x}})dr\big|\mathcal{F}_s\big]\big\}_{s\in[0,\infty)}$ and $\big\{\int_0^s g(t+r, X_r^{t,\mathfrak{x}})dr\big\}_{s\in[0,\infty)}$ are $\mathbf{F}-$adapted continuous processes,

$$\widehat{\tau}_1 = \widehat{\tau}_1(t,x,\mathfrak{x},y,\varepsilon) := \inf\bigg\{s \in [0,\infty) : E\bigg[\int_0^{\tau_1} g(t+r, X_r^{t,\mathfrak{x}})dr\bigg|\mathcal{F}_s\bigg] - \int_0^s g(t+r, X_r^{t,\mathfrak{x}})dr \le \mathfrak{a}\bigg\} \tag{6.20}$$

defines an $\mathbf{F}-$stopping time which satisfies $E\big[\int_0^{\tau_1} g(t+r, X_r^{t,\mathfrak{x}})dr\big|\mathcal{F}_{\widehat{\tau}_1}\big] - \int_0^{\widehat{\tau}_1} g(t+r, X_r^{t,\mathfrak{x}})dr = \mathfrak{a}$. Taking expectation $E[\cdot]$ yields that

$$E\bigg[\int_0^{\widehat{\tau}_1} g(t+r, X_r^{t,\mathfrak{x}})dr\bigg] = E\bigg[\int_0^{\tau_1} g(t+r, X_r^{t,\mathfrak{x}})dr\bigg] - \mathfrak{a} = y - \delta_y = (y-\delta)^+, \quad \text{so } \widehat{\tau}_1 \in \mathcal{T}_{t,\mathfrak{x}}\big((y-\delta)^+\big). \tag{6.21}$$

As $E\big[\int_0^{\tau_1} g(t+r,X_r^{t,\mathfrak{x}})dr\big|\mathcal{F}_{\tau_1}\big]-\int_0^{\tau_1} g(t+r,X_r^{t,\mathfrak{x}})dr=0<\mathfrak{a}$, we also see that $\widehat{\tau}_1\le\tau_1$.

The condition (g1), Hölder's inequality, the second inequality in (2.2) and (6.17) show that

$$E\Big[\int_0^\infty \big|g(t+r,X_r^{t,\mathfrak{x}})-g(t+r,X_r^{t,x})\big|dr\Big] \le E\bigg[\Big((X^{t,\mathfrak{x}}-X^{t,x})_*+(X^{t,\mathfrak{x}}-X^{t,x})_*^p\Big)\int_0^\infty c(t+r)dr\bigg]\le\mathfrak{C}(C_p)^{\frac{1}{p}}|\mathfrak{x}-x|+\mathfrak{C}C_p|\mathfrak{x}-x|^p\le\lambda. \tag{6.22}$$

Since $E\big[\int_0^{\tau_1} g(t+r,X_r^{t,x})dr\big]\le y$ and since $\lambda\ge\lambda\wedge\varepsilon_o>\mathfrak{C}(C_p)^{\frac{1}{p}}\delta\ge\delta\ge\delta_y$ by (6.17), one has

$$\begin{aligned}\mathfrak{a} &= E\Big[\int_0^{\tau_1} g(t+r,X_r^{t,\mathfrak{x}})dr\Big]-y+\delta_y<E\Big[\int_0^{\tau_1}\big(g(t+r,X_r^{t,\mathfrak{x}})-g(t+r,X_r^{t,x})\big)dr\Big]+\lambda\\ &\le E\Big[\int_0^\infty \big|g(t+r,X_r^{t,\mathfrak{x}})-g(t+r,X_r^{t,x})\big|dr\Big]+\lambda\le 2\lambda.\end{aligned}$$

Using (6.22) again, we can deduce from (6.21) that

$$\begin{aligned}2\lambda &> \mathfrak{a}=E\Big[\int_{\widehat{\tau}_1}^{\tau_1} g(t+r,X_r^{t,\mathfrak{x}})dr\Big]\ge E\Big[\int_{\widehat{\tau}_1}^{\tau_1} g(t+r,X_r^{t,x})dr\Big]-E\Big[\int_0^\infty\big|g(t+r,X_r^{t,\mathfrak{x}})-g(t+r,X_r^{t,x})\big|dr\Big]\\ &\ge E\bigg[\mathbf{1}_{A_R^c\cap\{\tau_1>\widehat{\tau}_1+\sqrt{\lambda}\}}\int_{\widehat{\tau}_1}^{\tau_1} g(t+r,X_r^{t,x})dr\bigg]-\lambda\ge\kappa_R\sqrt{\lambda}\,P\big(A_R^c\cap\{\tau_1>\widehat{\tau}_1+\sqrt{\lambda}\,\}\big)-\lambda. \end{aligned}\tag{6.23}$$

It follows from (6.15) that $P\big(A_R^c\cap\{\tau_1>\widehat{\tau}_1+\sqrt{\lambda}\,\}\big)<\frac{3\sqrt{\lambda}}{\kappa_R}\le\lambda_o/2$, proving the claim (6.19).

Set $\mathcal{A}:=\{\tau_1\le\widehat{\tau}_1+\sqrt{\lambda}\}=\{\widehat{\tau}_1\le\tau_1\le\widehat{\tau}_1+\sqrt{\lambda}\}$. Since (6.19) shows that

$$P(\mathcal{A}^c)=P\{\tau_1>\widehat{\tau}_1+\sqrt{\lambda}\}\le P(A_R)+P\big(A_R^c\cap\big\{\tau_1>\widehat{\tau}_1+\sqrt{\lambda}\,\big\}\big)<\lambda_o<\varepsilon_o,$$

(6.9)$-$(6.15) imply that

$$\begin{aligned}E\bigg[\Big|\int_0^{\widehat{\tau}_1} f(t+r,X_r^{t,x})dr-\int_0^{\tau_1} f(t+r,X_r^{t,x})dr\Big|\bigg] &\le E\bigg[\big(2+(X_*^{t,x})^p\big)\Big(\mathbf{1}_{\mathcal{A}^c}\int_0^\infty c(t+r)dr+\mathbf{1}_{\mathcal{A}}\|c(\cdot)\|(\tau_1-\widehat{\tau}_1)\Big)\bigg]\\ &< \mathfrak{C}\big(2P(\mathcal{A}^c)+\varepsilon_o\big)+\sqrt{\lambda}(2+\mathfrak{M})\|c(\cdot)\|<(1+3\mathfrak{C})\varepsilon_o\,, \end{aligned}\tag{6.24}$$

$$\text{and}\quad E\big[\mathbf{1}_{\mathcal{A}^c}\big|\pi\big(\widehat{\tau}_1,X_{\widehat{\tau}_1}^{t,x}\big)-\pi\big(\tau_1,X_{\tau_1}^{t,x}\big)\big|\big] \le 2\mathfrak{C}E\big[\mathbf{1}_{\mathcal{A}^c}\big(2+(X_*^{t,x})^p\big)\big]<2\mathfrak{C}\big(2P(\mathcal{A}^c)+\varepsilon_o\big)<6\mathfrak{C}\varepsilon_o\,. \tag{6.25}$$

Also, we can deduce from (1.9), (6.15), Hölder's inequality, (2.3) and (6.16) that

$$\begin{aligned}&E\big[\mathbf{1}_{\mathcal{A}}\big|\pi\big(\widehat{\tau}_1,X_{\widehat{\tau}_1}^{t,x}\big)-\pi\big(\tau_1,X_{\tau_1}^{t,x}\big)\big|\big]\le E\big[\mathbf{1}_{\mathcal{A}}\rho\big(\tau_1-\widehat{\tau}_1\big)\big]+\mathfrak{C}E\Big[\mathbf{1}_{\mathcal{A}}\Big(\big|X_{\widehat{\tau}_1}^{t,x}-X_{\tau_1}^{t,x}\big|+\big|X_{\widehat{\tau}_1}^{t,x}-X_{\tau_1}^{t,x}\big|^p\Big)\Big]\\ &\le\rho\big(\sqrt{\lambda}\big)+\mathfrak{C}\bigg\{E\bigg[\mathbf{1}_{\mathcal{A}}\sup_{r\in(0,\sqrt{\lambda}\,]}\big|X_{\widehat{\tau}_1+r}^{t,x}-X_{\widehat{\tau}_1}^{t,x}\big|^p\bigg]\bigg\}^{\frac{1}{p}}+\mathfrak{C}E\bigg[\mathbf{1}_{\mathcal{A}}\sup_{r\in(0,\sqrt{\lambda}\,]}\big|X_{\widehat{\tau}_1+r}^{t,x}-X_{\widehat{\tau}_1}^{t,x}\big|^p\bigg]\\ &\le\varepsilon_o+\mathfrak{C}(C_p)^{\frac{1}{p}}\big(1+|x|\big)\big(\|c(\cdot)\|\lambda^{\frac{1}{2}}+\|c(\cdot)\|^{\frac{1}{2}}\lambda^{\frac{1}{4}}\big)+\mathfrak{C}C_p\big(1+|x|^p\big)\big(\|c(\cdot)\|^p\lambda^{\frac{p}{2}}+\|c(\cdot)\|^{\frac{p}{2}}\lambda^{\frac{p}{4}}\big)\le(1+\mathfrak{C})\varepsilon_o\,. \end{aligned}\tag{6.26}$$

Combining (6.24), (6.25) and (6.26) yields that

$$E\big[\big|R(t,x,\widehat{\tau}_1)-R(t,x,\tau_1)\big|\big]<(2+10\mathfrak{C})\varepsilon_o, \tag{6.27}$$

which together with (2.8) and (6.17) show that

$$E\big[\big|R(t,\mathfrak{x},\widehat{\tau}_1)-R(t,x,\tau_1)\big|\big]\le E\big[\big|R(t,\mathfrak{x},\widehat{\tau}_1)-R(t,x,\widehat{\tau}_1)\big|\big]+E\big[\big|R(t,x,\widehat{\tau}_1)-R(t,x,\tau_1)\big|\big]<(4+10\mathfrak{C})\varepsilon_o=\varepsilon-\varepsilon_o.$$

Then it follows from (2.12) and (6.18) that for any $(\mathfrak{x},\mathfrak{y})\in\overline{O}_\delta(x)\times[(y-\delta)^+,\infty)$,

$$V(t,\mathfrak{x},\mathfrak{y})\ge V\big(t,\mathfrak{x},(y-\delta)^+\big)\ge E\big[R(t,\mathfrak{x},\widehat{\tau}_1)\big]>E\big[R(t,x,\tau_1)\big]-\varepsilon+\varepsilon_o\ge V(t,x,y)-\varepsilon. \tag{6.28}$$

1b) *To show* $V(t,\mathfrak{x},\mathfrak{y})\le V(t,x,y)+\varepsilon,\ \forall\,(\mathfrak{x},\mathfrak{y})\in\overline{O}_\delta(x)\times[0,y+\delta]$, *we let* $\mathfrak{x}\in\overline{O}_\delta(x)$.

There exists $\tau_2=\tau_2(t,\mathfrak{x},y,\varepsilon)\in\mathcal{T}_{t,\mathfrak{x}}(y+\delta)$ such that

$$E\big[R(t,\mathfrak{x},\tau_2)\big]\ge V\big(t,\mathfrak{x},y+\delta\big)-\varepsilon_o. \tag{6.29}$$

We claim that we can also construct a stopping time $\widehat{\tau}_2=\widehat{\tau}_2(t,x,\mathfrak{x},y,\varepsilon)\in\mathcal{T}_{t,x}(y)$ satisfying

$$\widehat{\tau}_2\le\tau_2 \quad \text{and} \quad P\big(A_R^c\cap\{\tau_2>\widehat{\tau}_2+\sqrt{\lambda}\,\}\big)<\lambda_o/2. \tag{6.30}$$

If $E\big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\big]\le y$ $\big($i.e. $\tau_2\in\mathcal{T}_{t,x}(y)\big)$, we directly set $\widehat{\tau}_2:=\tau_2$. Otherwise, set $\mathfrak{b}:=E\big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\big]-y>0$. Similar to (6.20), $\widehat{\tau}_2=\widehat{\tau}_2(t,x,\mathfrak{x},y,\varepsilon):=\inf\big\{s\in[0,\infty):\ E\big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\big|\mathcal{F}_s\big]-\int_0^s g(t+r,X_r^{t,x})dr\le\mathfrak{b}\big\}$ is an $\mathbf{F}-$stopping time satisfying $E\big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\big|\mathcal{F}_{\widehat{\tau}_2}\big]-\int_0^{\widehat{\tau}_2}g(t+r,X_r^{t,x})dr=\mathfrak{b}$. Taking expectation $E[\cdot]$ yields that

$$E\Big[\int_0^{\widehat{\tau}_2}g(t+r,X_r^{t,x})dr\Big]=E\Big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\Big]-\mathfrak{b}=y, \quad \text{so } \widehat{\tau}_2\in\mathcal{T}_{t,x}(y). \tag{6.31}$$

As $E\big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\big|\mathcal{F}_{\tau_2}\big]-\int_0^{\tau_2}g(t+r,X_r^{t,x})dr=0<\mathfrak{b}$, we also see that $\widehat{\tau}_2\le\tau_2$.

Since $E\big[\int_0^{\tau_2}g(t+r,X_r^{t,\mathfrak{x}})dr\big]\le y+\delta<y+\lambda$, we can deduce from (6.22) and (6.31) that

$$\begin{aligned}2\lambda &\ge E\Big[\int_0^\infty\big|g(t+r,X_r^{t,\mathfrak{x}})-g(t+r,X_r^{t,x})\big|dr\Big]+\lambda\ge E\Big[\int_0^{\tau_2}\big(g(t+r,X_r^{t,x})-g(t+r,X_r^{t,\mathfrak{x}})\big)dr\Big]+\lambda\\ &> E\Big[\int_0^{\tau_2}g(t+r,X_r^{t,x})dr\Big]-y=\mathfrak{b}=E\Big[\int_{\widehat{\tau}_2}^{\tau_2}g(t+r,X_r^{t,x})dr\Big]\ge E\Big[\mathbf{1}_{A_R^c\cap\{\tau_2>\widehat{\tau}_2+\sqrt{\lambda}\}}\int_{\widehat{\tau}_2}^{\tau_2}g(t+r,X_r^{t,x})dr\Big]\\ &\ge \kappa_R\sqrt{\lambda}\,P\big(A_R^c\cap\{\tau_2>\widehat{\tau}_2+\sqrt{\lambda}\,\}\big).\end{aligned}$$

By (6.15), $P\big(A_R^c\cap\{\tau_2>\widehat{\tau}_2+\sqrt{\lambda}\,\}\big)<\frac{2\sqrt{\lambda}}{\kappa_R}<\lambda_o/2$, proving the claim (6.30).

An analogy to (6.24)$-$(6.26) yields that $E\big[\big|R(t,x,\widehat{\tau}_2)-R(t,x,\tau_2)\big|\big]<(2+10\mathfrak{C})\varepsilon_o$, so we see from (2.8) and (6.17)

$$E\big[\big|R(t,x,\widehat{\tau}_2)-R(t,\mathfrak{x},\tau_2)\big|\big]\le E\big[\big|R(t,x,\widehat{\tau}_2)-R(t,x,\tau_2)\big|\big]+E\big[\big|R(t,x,\tau_2)-R(t,\mathfrak{x},\tau_2)\big|\big]<(4+10\mathfrak{C})\varepsilon_o=\varepsilon-\varepsilon_o.$$

It then follows from (2.12) and (6.29) that for any $(\mathfrak{x},\mathfrak{y})\in\overline{O}_\delta(x)\times[0,y+\delta]$,

$$V(t,\mathfrak{x},\mathfrak{y})\le V\big(t,\mathfrak{x},y+\delta\big)\le E\big[R(t,\mathfrak{x},\tau_2)\big]+\varepsilon_o<E\big[R(t,x,\widehat{\tau}_2)\big]+\varepsilon\le V(t,x,y)+\varepsilon,$$

which together with (6.28) leads to that $\big|V(t,\mathfrak{x},\mathfrak{y})-V(t,x,y)\big|\le\varepsilon,\ \forall\,(\mathfrak{x},\mathfrak{y})\in\overline{O}_\delta(x)\times\big[(y-\delta)^+,y+\delta\big]$.

2) Next, let $\varpi\in[1,\infty)$, we further assume that b,σ additionally satisfy (2.4) and f,g additionally satisfy (2.14).

Fix $(t,x,\varepsilon)\in[0,\infty)\times\mathbb{R}^l\times(0,1)$. Given $\mathfrak{t}\in[0,\infty)$ and $\zeta\in\mathcal{T}$, (1.8), (1.9), (2.14), Hölder's inequality, (2.5), (1.10) and the first inequality in (2.2) imply that

$$\begin{aligned}&E\big[|R(\mathfrak{t},x,\zeta)-R(t,x,\zeta)|\big]\\ &\le E\Big[\int_0^\zeta\Big(\big|f(\mathfrak{t}+r,X_r^{\mathfrak{t},x})-f(\mathfrak{t}+r,X_r^{t,x})\big|+\big|f(\mathfrak{t}+r,X_r^{t,x})-f(t+r,X_r^{t,x})\big|\Big)dr+\Big|\pi\big(\mathfrak{t}+\zeta,X_\zeta^{\mathfrak{t},x}\big)-\pi\big(t+\zeta,X_\zeta^{t,x}\big)\Big|\Big]\\ &\le E\Big[\big((X^{\mathfrak{t},x}-X^{t,x})_*+(X^{\mathfrak{t},x}-X^{t,x})_*^p\big)\Big(\int_0^\infty c(\mathfrak{t}+r)dr+\mathfrak{C}\Big)\Big]+\rho(|\mathfrak{t}-t|)+\rho(|\mathfrak{t}-t|)E\Big[\big(1+|X_*^{t,x}|^\varpi\big)\int_0^\infty c(\mathfrak{t}\wedge t+r)dr\Big]\\ &\le 2\mathfrak{C}C_{p,\varpi}^{1/p}(1+|x|^\varpi)\rho(|\mathfrak{t}-t|)+2\mathfrak{C}C_{p,\varpi}(1+|x|^{p\varpi})\big(\rho(|\mathfrak{t}-t|)\big)^p+\rho(|\mathfrak{t}-t|)+\mathfrak{C}\rho(|\mathfrak{t}-t|)\big(1+C_\varpi(1+|x|^\varpi)\big).\end{aligned} \tag{6.32}$$

Let us still set ε_o, $\mathfrak{M}$ and take $\lambda_o=\lambda_o(t,x,\varepsilon)$, $R=R(t,x,\varepsilon)$, $\lambda=\lambda(t,x,\varepsilon)$ as in Part 1. We now choose $\delta'=\delta'(t,x,\varepsilon)\in(0,1)$ such that

$$(C_p)^{\frac{1}{p}}\delta'+C_p(\delta')^p+C_{p,\varpi}^{1/p}\big(1+|x|^\varpi\big)\rho(\delta')+C_{p,\varpi}\big(1+|x|^{p\varpi}\big)\big(\rho(\delta')\big)^p+\rho(\delta')+\rho(\delta')\big(1+C_\varpi(1+|x|^\varpi)\big)\le\frac{\lambda\wedge\varepsilon_o}{\mathfrak{C}}, \tag{6.33}$$

and fix $y\in[0,\infty)$.

2a) *To show that $V(\mathfrak{t},\mathfrak{x},\mathfrak{y})\ge V(t,x,y)-\varepsilon,\ \forall\,(\mathfrak{t},\mathfrak{x},\mathfrak{y})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)\times[(y-\delta')^+,\infty)$, we let $(\mathfrak{t},\mathfrak{x})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)$.*

The condition (g1), (2.14), Hölder's inequality, (2.2), (2.5), (1.10) and (6.33) show that

$$E\Big[\int_0^\infty \big|g(\mathfrak{t}+r,X_r^{\mathfrak{t},\mathfrak{x}})-g(t+r,X_r^{t,x})\big|dr\Big]$$
$$\le E\bigg[\int_0^\infty\Big(\big|g(\mathfrak{t}+r,X_r^{\mathfrak{t},\mathfrak{x}})-g(\mathfrak{t}+r,X_r^{\mathfrak{t},x})\big|+\big|g(\mathfrak{t}+r,X_r^{\mathfrak{t},x})-g(\mathfrak{t}+r,X_r^{t,x})\big|+\big|g(\mathfrak{t}+r,X_r^{t,x})-g(t+r,X_r^{t,x})\big|\Big)dr\bigg]$$
$$\le E\bigg[\Big((X^{\mathfrak{t},\mathfrak{x}}-X^{\mathfrak{t},x})_*+(X^{\mathfrak{t},\mathfrak{x}}-X^{\mathfrak{t},x})_*^p+(X^{\mathfrak{t},x}-X^{t,x})_*+(X^{\mathfrak{t},x}-X^{t,x})_*^p\Big)\int_0^\infty c(\mathfrak{t}+r)dr\bigg]$$
$$+\rho(|\mathfrak{t}-t|)E\bigg[\big(1+(X_*^{t,x})^\varpi\big)\int_0^\infty c(\mathfrak{t}\wedge t+r)dr\bigg]$$
$$\le\mathfrak{C}(C_p)^{\frac1p}|x-\mathfrak{x}|+\mathfrak{C}C_p|x-\mathfrak{x}|^p+\mathfrak{C}C_{p,\varpi}^{1/p}\big(1+|x|^\varpi\big)\rho(|\mathfrak{t}-t|)+\mathfrak{C}C_{p,\varpi}\big(1+|x|^{p\varpi}\big)\big(\rho(|\mathfrak{t}-t|)\big)^p+\mathfrak{C}\rho(|\mathfrak{t}-t|)\big(1+C_\varpi(1+|x|^\varpi)\big)$$
$$\le\mathfrak{C}(C_p)^{\frac1p}\delta'+\mathfrak{C}C_p(\delta')^p+\mathfrak{C}C_{p,\varpi}^{1/p}\big(1+|x|^\varpi\big)\rho(\delta')+\mathfrak{C}C_{p,\varpi}\big(1+|x|^{p\varpi}\big)\big(\rho(\delta')\big)^p+\mathfrak{C}\rho(\delta')\big(1+C_\varpi(1+|x|^\varpi)\big)\le\lambda. \tag{6.34}$$

Let $\tau_3=\tau_3(t,x,y,\varepsilon)\in\mathcal{T}_{t,x}(y)$ such that

$$E\big[R(t,x,\tau_3)\big]\ge V(t,x,y)-\varepsilon_o. \tag{6.35}$$

If $E\big[\int_0^{\tau_3}g(\mathfrak{t}+r,X_r^{\mathfrak{t},\mathfrak{x}})dr\big]\le(y-\delta')^+$, we directly set $\widehat\tau_3:=\tau_3$. Otherwise, we define $\widehat\tau_3=\widehat\tau_3(t,\mathfrak{t},x,\mathfrak{x},y,\varepsilon):=\inf\big\{s\in[0,\infty):E\big[\int_0^{\tau_3}g(\mathfrak{t}+r,X_r^{\mathfrak{t},\mathfrak{x}})dr\big|\mathcal{F}_s\big]-\int_0^s g(\mathfrak{t}+r,X_r^{\mathfrak{t},\mathfrak{x}})dr\le\mathfrak{a}'\big\}$ with $\mathfrak{a}':=E\big[\int_0^{\tau_3}g(\mathfrak{t}+r,X_r^{\mathfrak{t},\mathfrak{x}})dr\big]-(y-\delta')^+>0$. Similar to (6.19), one can deduce from (6.34) that $\widehat\tau_3$ is a $\mathcal{T}_{\mathfrak{t},\mathfrak{x}}\big((y-\delta')^+\big)-$stopping time satisfying

$$\widehat\tau_3\le\tau_3\quad\text{and}\quad P\big(A_R^c\cap\big\{\tau_3>\widehat\tau_3+\sqrt{\lambda}\,\big\}\big)<\lambda_o/2.$$

Using similar arguments to those that lead to (6.27), one can deduce from (6.14)$-$(6.16) that $E\big[\big|R(t,x,\widehat\tau_3)-R(t,x,\tau_3)\big|\big]<(2+10\mathfrak{C})\varepsilon_o$. Then applying (2.8) with $(t,x,x',\tau)=\big(\mathfrak{t},\mathfrak{x},x,\widehat\tau_3\big)$ and applying (6.32) with $\zeta=\widehat\tau_3$, we see from (6.33) that

$$\begin{aligned}E\big[\big|R(\mathfrak{t},\mathfrak{x},\widehat\tau_3)-R(t,x,\tau_3)\big|\big]\;&\le\;E\big[\big|R(\mathfrak{t},\mathfrak{x},\widehat\tau_3)-R(\mathfrak{t},x,\widehat\tau_3)\big|\big]+E\big[\big|R(\mathfrak{t},x,\widehat\tau_3)-R(t,x,\widehat\tau_3)\big|\big]+E\big[\big|R(t,x,\widehat\tau_3)-R(t,x,\tau_3)\big|\big]\\&\le\;2\mathfrak{C}(C_p)^{\frac1p}\delta'+2\mathfrak{C}C_p(\delta')^p+2\mathfrak{C}C_{p,\varpi}^{1/p}(1+|x|^\varpi)\rho(\delta')+2\mathfrak{C}C_{p,\varpi}(1+|x|^{p\varpi})\big(\rho(\delta')\big)^p\\&\quad+\rho(\delta')+\mathfrak{C}\rho(\delta')\big(1+C_\varpi(1+|x|^\varpi)\big)+(2+10\mathfrak{C})\varepsilon_o<(4+10\mathfrak{C})\varepsilon_o=\varepsilon-\varepsilon_o.\end{aligned}$$

It follows from (2.12) and (6.35) that for any $(\mathfrak{t},\mathfrak{x},\mathfrak{y})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)\times[(y-\delta')^+,\infty)$,

$$V(\mathfrak{t},\mathfrak{x},\mathfrak{y})\ge V\big(\mathfrak{t},\mathfrak{x},(y-\delta')^+\big)\ge E\big[R(\mathfrak{t},\mathfrak{x},\widehat\tau_3)\big]>E\big[R(t,x,\tau_3)\big]-\varepsilon+\varepsilon_o\ge V(t,x,y)-\varepsilon. \tag{6.36}$$

2b) *We next show that* $V(\mathfrak{t},\mathfrak{x},\mathfrak{y})\le V(t,x,y)+\varepsilon,\ \forall\,(\mathfrak{t},\mathfrak{x},\mathfrak{y})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)\times[0,y+\delta']$.

Let $(\mathfrak{t},\mathfrak{x})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)$. There exists $\tau_4=\tau_4(t,\mathfrak{t},x,\mathfrak{x},y,\varepsilon)\in\mathcal{T}_{\mathfrak{t},\mathfrak{x}}(y+\delta')$ such that

$$E\big[R(\mathfrak{t},\mathfrak{x},\tau_4)\big]\ge V\big(\mathfrak{t},\mathfrak{x},y+\delta'\big)-\varepsilon_o. \tag{6.37}$$

If $E\big[\int_0^{\tau_4}g(t+r,X_r^{t,x})dr\big]\le y$, we directly set $\widehat\tau_4:=\tau_4$. Otherwise, we define $\widehat\tau_4=\widehat\tau_4(t,\mathfrak{t},x,\mathfrak{x},y,\varepsilon):=\inf\big\{s\in[0,\infty):E\big[\int_0^{\tau_4}g(t+r,X_r^{t,x})dr\big|\mathcal{F}_s\big]-\int_0^s g(t+r,X_r^{t,x})dr\le\mathfrak{b}'\big\}$ with $\mathfrak{b}':=E\big[\int_0^{\tau_4}g(t+r,X_r^{t,x})dr\big]-y>0$. Analogous to (6.30), we can deduce from (6.34) that $\widehat\tau_4$ is a $\mathcal{T}_{t,x}(y)-$stopping time satisfying

$$\widehat\tau_4\le\tau_4\quad\text{and}\quad P\big(A_R^c\cap\{\tau_4>\widehat\tau_4+\sqrt{\lambda}\,\}\big)<\lambda_o/2.$$

Since an analogy to (6.24)$-$(6.26) gives that $E\big[\big|R(t,x,\widehat\tau_4)-R(t,x,\tau_4)\big|\big]<(2+10\mathfrak{C})\varepsilon_o$, applying (6.32) with $\zeta=\tau_4$ and applying (2.8) with $(t,x,x',\tau)=\big(\mathfrak{t},x,\mathfrak{x},\tau_4\big)$, we see from (6.33) that

$$\begin{aligned}E\big[\big|R(t,x,\widehat\tau_4)-R(\mathfrak{t},\mathfrak{x},\tau_4)\big|\big]\;&\le\;E\big[\big|R(t,x,\widehat\tau_4)-R(t,x,\tau_4)\big|\big]+E\big[\big|R(t,x,\tau_4)-R(\mathfrak{t},x,\tau_4)\big|\big]+E\big[\big|R(\mathfrak{t},x,\tau_4)-R(\mathfrak{t},\mathfrak{x},\tau_4)\big|\big]\\&\le\;(2+10\mathfrak{C})\varepsilon_o+2\mathfrak{C}C_{p,\varpi}^{1/p}(1+|x|^\varpi)\rho(\delta')+2\mathfrak{C}C_{p,\varpi}(1+|x|^{p\varpi})\big(\rho(\delta')\big)^p+\rho(\delta')\\&\quad+\mathfrak{C}\rho(\delta')\big(1+C_\varpi(1+|x|^\varpi)\big)+2\mathfrak{C}(C_p)^{\frac1p}\delta'+2\mathfrak{C}C_p(\delta')^p<(4+10\mathfrak{C})\varepsilon_o=\varepsilon-\varepsilon_o.\end{aligned}$$

It then follows from (2.12) and (6.37) that for any $(\mathfrak{t},\mathfrak{x},\mathfrak{y})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)\times[0,y+\delta']$

$$V(\mathfrak{t},\mathfrak{x},\mathfrak{y})\le V\big(\mathfrak{t},\mathfrak{x},y+\delta'\big)\le E\big[R(\mathfrak{t},\mathfrak{x},\tau_4)\big]+\varepsilon_o<E\big[R(t,x,\widehat\tau_4)\big]+\varepsilon\le V(t,x,y)+\varepsilon,$$

which together with (6.36) yields $\big|V(\mathfrak{t},\mathfrak{x},\mathfrak{y})-V(t,x,y)\big|\le\varepsilon,\ \forall\,(\mathfrak{t},\mathfrak{x},\mathfrak{y})\in[(t-\delta')^+,t+\delta']\times\overline{O}_{\delta'}(x)\times\big[(y-\delta')^+,y+\delta'\big]$. □

6.2 Proofs of Section 3

Proof of Lemma 3.1: Set $\Lambda := \Big\{A \subset \Omega^t : A = \underset{\omega \in A}{\cup} \big(\omega \otimes_s \Omega^s\big)\Big\}$. Clearly, $\emptyset, \Omega^t \in \Lambda$. For any $A \in \Lambda$, we claim that

$$\omega \otimes_s \Omega^s \subset A^c \text{ for any } \omega \in A^c. \tag{6.38}$$

Assume not, there exist an $\omega \in A^c$ and an $\widetilde{\omega} \in \Omega^s$ such that $\omega \otimes_s \widetilde{\omega} \in A$. Then $\big(\omega \otimes_s \widetilde{\omega}\big) \otimes_s \Omega^s \subset A$ and it follows that $\omega \in \omega \otimes_s \Omega^s = \big(\omega \otimes_s \widetilde{\omega}\big) \otimes_s \Omega^s \subset A$. A contradiction appear. So (6.38) holds, which shows that $A^c \in \Lambda$.

For any $\{A_n\}_{n \in \mathbb{N}} \subset \Lambda$, one can deduce that $\underset{n \in \mathbb{N}}{\cup} A_n = \underset{n \in \mathbb{N}}{\cup} \Big(\underset{\omega \in A_n}{\cup} \big(\omega \otimes_s \Omega^s\big)\Big) = \underset{\omega \in \underset{n \in \mathbb{N}}{\cup} A_n}{\cup} \big(\omega \otimes_s \Omega^s\big)$, namely, $\underset{n \in \mathbb{N}}{\cup} A_n \in \Lambda$. Given $r \in [t,s]$ and $\mathcal{E} \in \mathscr{B}(\mathbb{R}^d)$, if $\omega \in (W^t_r)^{-1}(\mathcal{E})$, it holds for any $\widetilde{\omega} \in \Omega^s$ that $\big(\omega \otimes_s \widetilde{\omega}\big)(r) = \omega(r) \in \mathcal{E}$ or $\omega \otimes_s \widetilde{\omega} \in (W^t_r)^{-1}(\mathcal{E})$, which implies that $(W^t_r)^{-1}(\mathcal{E}) \in \Lambda$. Hence, Λ is a sigma$-$field of Ω^t containing all generating sets of $\mathcal{F}^t_s$. It follows that $\mathcal{F}^t_s \subset \Lambda$, proving the lemma. □

Proof of Lemma 3.2: Let us regard $\omega \otimes_s \cdot$ as a mapping Γ from Ω^s to Ω^t, i.e., $\Gamma(\widetilde{\omega}) := \omega \otimes_s \widetilde{\omega},\ \forall\, \widetilde{\omega} \in \Omega^s$. So $A^{s,\omega} = \Gamma^{-1}(A)$ for any $A \subset \Omega^t$.

1) Assume first that $r \in [s,\infty)$. Given $t' \in [t,r]$ and $\mathcal{E} \in \mathscr{B}(\mathbb{R}^d)$, we can deduce that

$$\Gamma^{-1}\big((W^t_{t'})^{-1}(\mathcal{E})\big) = \{\widetilde{\omega} \in \Omega^s : W^t_{t'}(\omega \otimes_s \widetilde{\omega}) \in \mathcal{E}\} = \begin{cases} \Omega^s, & \text{if } t' \in [t,s) \text{ and } \omega(t') \in \mathcal{E}; \\ \emptyset, & \text{if } t' \in [t,s) \text{ and } \omega(t') \notin \mathcal{E}; \\ \big\{\widetilde{\omega} \in \Omega^s : \omega(s) + \widetilde{\omega}(t') \in \mathcal{E}\big\} = (W^s_{t'})^{-1}(\mathcal{E}') \in \mathcal{F}^s_r, & \text{if } t' \in [s,r]; \end{cases}$$

where $\mathcal{E}' := \mathcal{E} - \omega(s) = \{x - \omega(s) : x \in \mathcal{E}\} \in \mathscr{B}(\mathbb{R}^d)$. So all generating sets of $\mathcal{F}^t_r$ belong to $\Lambda_r := \big\{A \subset \Omega^t : \Gamma^{-1}(A) \in \mathcal{F}^s_r\big\}$, which is clearly a sigma$-$field of Ω^t. It follows that $\mathcal{F}^t_r \subset \Lambda_r$, or $A^{s,\omega} = \Gamma^{-1}(A) \in \mathcal{F}^s_r$ for any $A \in \mathcal{F}^t_r$.

On the other hand, let $\widetilde{A} \in \mathcal{F}^s_r$. We know from Lemma A.2 (1) that $(\Pi^t_s)^{-1}\big(\widetilde{A}\big) \in \mathcal{F}^t_r$. Since the continuity of paths in Ω^t shows that $\omega \otimes_s \Omega^s = \big\{\omega' \in \Omega^t : \omega'(t') = \omega(t'),\ \forall\, t' \in (t,s) \cap \mathbb{Q}\big\} = \underset{t' \in (t,s) \cap \mathbb{Q}}{\cap} (W^t_{t'})^{-1}\big(\{\omega(t')\}\big) \in \mathcal{F}^t_s \subset \mathcal{F}^t_r$, one can deduce that $\omega \otimes_s \widetilde{A} = (\Pi^t_s)^{-1}\big(\widetilde{A}\big) \cap \big(\omega \otimes_s \Omega^s\big) \in \mathcal{F}^t_r$.

2) Next, we consider the case of $r = \infty$. Given $r' \in [s,\infty)$, since $\Gamma^{-1}(A) \in \mathcal{F}^s_{r'} \subset \mathcal{F}^s$ for any $A \in \mathcal{F}^t_{r'}$, we see that $\mathcal{F}^t_{r'} \subset \Lambda := \big\{A \subset \Omega^t : \Gamma^{-1}(A) \in \mathcal{F}^s\big\}$, which is clearly a sigma$-$field of Ω^t. It follows from Lemma A.1 (1) that $\mathcal{F}^t = \sigma\Big(\underset{r' \in [t,\infty)}{\cup} \mathcal{F}^t_{r'}\Big) = \sigma\Big(\underset{r' \in [s,\infty)}{\cup} \mathcal{F}^t_{r'}\Big) \subset \Lambda$. So $A^{s,\omega} = \Gamma^{-1}(A) \in \mathcal{F}^s$ for any $A \in \mathcal{F}^t$.

On the other hand, let $r' \in [s,\infty)$. Since $\Gamma\big(\widetilde{A}\big) = \omega \otimes_s \widetilde{A} \in \mathcal{F}^t_{r'} \subset \mathcal{F}^t$ for any $\widetilde{A} \in \mathcal{F}^s_{r'}$, one has $\mathcal{F}^s_{r'} \subset \widetilde{\Lambda} := \big\{\widetilde{A} \subset \Omega^s : \Gamma\big(\widetilde{A}\big) \in \mathcal{F}^t\big\}$. Given $\widetilde{A} \in \widetilde{\Lambda}$, it is clear that $\Gamma\big(\widetilde{A}\big) \cup \Gamma\big(\widetilde{A}^c\big)$ is a disjoint union of $\Gamma(\Omega^s) = \omega \otimes_s \Omega^s \in \mathcal{F}^t_s \subset \mathcal{F}^t$. It follows that $\Gamma\big(\widetilde{A}^c\big) = (\omega \otimes_s \Omega^s) \backslash \Gamma\big(\widetilde{A}\big) \in \mathcal{F}^t$. Also, it holds for any $\{\widetilde{A}_n\}_{n \in \mathbb{N}} \subset \widetilde{\Lambda}$ that $\Gamma\Big(\underset{n \in \mathbb{N}}{\cup} \widetilde{A}_n\Big) = \underset{n \in \mathbb{N}}{\cup} \Gamma\big(\widetilde{A}_n\big) \in \mathcal{F}^t$. So $\widetilde{\Lambda}$ is a sigma$-$field of Ω^s that contains all $\mathcal{F}^s_{r'}$, $r' \in [s,\infty)$. Then Lemma A.1 (1) implies that $\mathcal{F}^s = \sigma\Big(\underset{r' \in [s,\infty)}{\cup} \mathcal{F}^s_{r'}\Big) \subset \widetilde{\Lambda}$, or $\omega \otimes_s \widetilde{A} = \Gamma\big(\widetilde{A}\big) \in \mathcal{F}^t$ for any $\widetilde{A} \in \mathcal{F}^s$. □

Proof of Proposition 3.1: **1)** Let ξ be an $\mathbb{E}-$valued random variable on Ω^t that is $\mathcal{F}^t_r-$measurable for some $r \in [s,\infty]$. For any $\mathcal{E} \in \mathscr{B}(\mathbb{E})$, since $\xi^{-1}(\mathcal{E}) \in \mathcal{F}^t_r$, Lemma 3.2 shows that $\big(\xi^{s,\omega}\big)^{-1}(\mathcal{E}) = \big\{\widetilde{\omega} \in \Omega^s : \xi(\omega \otimes_s \widetilde{\omega}) \in \mathcal{E}\big\} = \big\{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in \xi^{-1}(\mathcal{E})\big\} = \big(\xi^{-1}(\mathcal{E})\big)^{s,\omega} \in \mathcal{F}^s_r$. So $\xi^{s,\omega}$ is $\mathcal{F}^s_r-$measurable.

2) Let $\{X_r\}_{r \in [t,\infty)}$ be an $\mathbb{E}-$valued, $\mathbf{F}^t-$adapted process. For any $r \in [s,\infty)$ and $\mathcal{E} \in \mathscr{B}(\mathbb{E})$, since $X_r \in \mathcal{F}^t_r$, one can deduce from Lemma 3.2 that $\big(X^{s,\omega}_r\big)^{-1}(\mathcal{E}) = \big\{\widetilde{\omega} \in \Omega^s : X\big(r, \omega \otimes_s \widetilde{\omega}\big) \in \mathcal{E}\big\} = \big\{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in X^{-1}_r(\mathcal{E})\big\} = \big(X^{-1}_s(\mathcal{E})\big)^{s,\omega} \in \mathcal{F}^s_r$, which shows that $\big\{X^{s,\omega}_r\big\}_{r \in [s,\infty)}$ is $\mathbf{F}^s-$adapted. □

Proof of Proposition 3.2: In virtue of Theorem 1.3.4 and (1.3.15) of [57], there exists a family $\{P^\omega_s\}_{\omega \in \Omega^t}$ of probabilities on $(\Omega^t, \mathcal{F}^t)$, called the regular conditional probability distribution of P_t with respect to the sigma-field $\mathcal{F}^t_s$, such that

(i) For any $A \in \mathcal{F}^t$, the mapping $\omega \to P^\omega_s(A)$ is $\mathcal{F}^t_s-$measurable;

(ii) For any $\xi \in L^1(\mathcal{F}^t)$, $E_{P^\omega_s}[\xi] = E_t\big[\xi \big| \mathcal{F}^t_s\big](\omega)$ for P_t-a.s. $\omega \in \Omega^t$; (6.39)

(iii) For any $\omega \in \Omega^t$, $P^\omega_s\big(\omega \otimes_s \Omega^s\big) = 1$. (6.40)

1) Given $\omega \in \Omega^t$, Lemma 3.2 shows that $\omega \otimes_s \widetilde{A} \in \mathcal{F}^t$ for any $\widetilde{A} \in \mathcal{F}^s$. Then one can deduce from (6.40) that $P^{s,\omega}\big(\widetilde{A}\big) := P^\omega_s\big(\omega \otimes_s \widetilde{A}\big)$, $\forall\, \widetilde{A} \in \mathcal{F}^s$ defines a probability measure on $(\Omega^s, \mathcal{F}^s)$. We claim that for P_t-a.s. $\omega \in \Omega^t$

$$P^{s,\omega}\big(\widetilde{A}\big) = P_s\big(\widetilde{A}\big), \quad \forall\, \widetilde{A} \in \mathcal{F}^s. \tag{6.41}$$

To see this, we let $\widetilde{A} \in \mathcal{F}^s$. Since $(\Pi^t_s)^{-1}\big(\widetilde{A}\big) \in \mathcal{F}^t$ by Lemma A.2 (1), (6.40) and (6.39) imply that for P_t-a.s. $\omega \in \Omega^t$

$$P^{s,\omega}\big(\widetilde{A}\big) = P^\omega_s\big(\omega \otimes_s \widetilde{A}\big) = P^\omega_s\big((\Pi^t_s)^{-1}\big(\widetilde{A}\big) \cap (\omega \otimes_s \Omega^s)\big) = P^\omega_s\big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\big) = E_t\Big[\mathbf{1}_{(\Pi^t_s)^{-1}(\widetilde{A})}\big|\mathcal{F}^t_s\Big](\omega). \tag{6.42}$$

We can deduce from Lemma A.1 (1) that

$$\begin{aligned}(\Pi^t_s)^{-1}(\mathcal{F}^s) &= (\Pi^t_s)^{-1}\big(\sigma\big\{(W^s_r)^{-1}(\mathcal{E}) : r \in [s,\infty),\ \mathcal{E} \in \mathscr{B}(\mathbb{R}^d)\big\}\big) = \sigma\big\{(\Pi^t_s)^{-1}\big((W^s_r)^{-1}(\mathcal{E})\big) : r \in [s,\infty),\ \mathcal{E} \in \mathscr{B}(\mathbb{R}^d)\big\} \\ &= \sigma\big\{(W^t_r - W^t_s)^{-1}(\mathcal{E}) : r \in [s,\infty),\ \mathcal{E} \in \mathscr{B}(\mathbb{R}^d)\big\} = \sigma\big(W^t_r - W^t_s; r \in [s,\infty)\big),\end{aligned}$$

which is independent of $\mathcal{F}^t_s$ under P_t. Then (6.42) and Lemma A.2 (2) show that for P_t-a.s. $\omega \in \Omega^t$,

$$P^{s,\omega}\big(\widetilde{A}\big) = E_t\Big[\mathbf{1}_{(\Pi^t_s)^{-1}(\widetilde{A})}\big|\mathcal{F}^t_s\Big](\omega) = E_t\Big[\mathbf{1}_{(\Pi^t_s)^{-1}(\widetilde{A})}\Big] = P_t\big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\big) = P_s\big(\widetilde{A}\big).$$

As $\mathscr{C}^s_\infty := \Big\{ \underset{i=1}{\overset{m}{\cap}} (W^s_{s_i})^{-1}\big(O_{\delta_i}(x_i)\big) : m \in \mathbb{N},\ s_i \in \mathbb{Q} \cup \{s\}$ with $s \le s_1 \le \cdots \le s_m,\ x_i \in \mathbb{Q}^d,\ \delta_i \in \mathbb{Q}_+ \Big\}$ is a countable set, we can find a $\mathcal{N} \in \mathscr{N}^s$ such that for any $\omega \in \mathcal{N}^c$, $P^{s,\omega}\big(\widetilde{A}\big) = P_s\big(\widetilde{A}\big)$ holds for each $\widetilde{A} \in \mathscr{C}^s_\infty$. To wit, $\mathscr{C}^s_\infty \subset \Lambda := \big\{\widetilde{A} \in \Omega^s : P^{s,\omega}\big(\widetilde{A}\big) = P_s\big(\widetilde{A}\big),\ \forall\, \omega \in \mathcal{N}^c\big\}$. It is easy to see that Λ is a Dynkin system. As $\mathscr{C}^s_\infty$ is closed under intersection, Lemma A.1 (2) and Dynkin System Theorem show that $\mathcal{F}^s = \sigma(\mathscr{C}^s_\infty) \subset \Lambda$. Namely, it holds for any $\omega \in \mathcal{N}^c$ that $P^{s,\omega}\big(\widetilde{A}\big) = P_s\big(\widetilde{A}\big)$, $\forall\, \widetilde{A} \in \mathcal{F}^s$, proving (6.41).

2) Now, let $\xi \in L^1(\mathcal{F}^t)$. Proposition 3.1 (1) shows that $\xi^{s,\omega}$ is $\mathcal{F}^s-$measurable for any $\omega \in \Omega^t$. Also, we can deduce from (6.39)$-$(6.41) that for P_t-a.s. $\omega \in \Omega^t$

$$\begin{aligned}E_s\big[|\xi^{s,\omega}|\big] &= \int_{\widetilde{\omega} \in \Omega^s} \big|\xi^{s,\omega}(\widetilde{\omega})\big|\, dP^{s,\omega}(\widetilde{\omega}) = \int_{\widetilde{\omega} \in \Omega^s} \big|\xi\big(\omega \otimes_s \widetilde{\omega}\big)\big|\, dP^\omega_s\big(\omega \otimes_s \widetilde{\omega}\big) = \int_{\omega' \in \omega \otimes_s \Omega^s} \big|\xi(\omega')\big|\, dP^\omega_s(\omega') \\ &= \int_{\omega' \in \Omega^t} \big|\xi(\omega')\big|\, dP^\omega_s(\omega') = E_{P^\omega_s}\big[|\xi|\big] = E_t\big[|\xi|\big|\mathcal{F}^t_s\big](\omega) < \infty,\end{aligned}$$

thus $\xi^{s,\omega} \in L^1\big(\mathcal{F}^s\big)$. Similarly, it holds for P_t-a.s. $\omega \in \Omega^t$ that $E_s\big[\xi^{s,\omega}\big] = E_t\big[\xi\big|\mathcal{F}^t_s\big](\omega) \in \mathbb{R}$. □

Proof of Proposition 3.3: 1) Let $\mathcal{N}$ be a P_t-null set, so there exists an $A \in \mathcal{F}^t$ with $P_t(A) = 0$ such that $\mathcal{N} \subset A$. For any $\omega \in \Omega^t$, Lemma 3.2 shows that $\mathcal{N}^{s,\omega} = \{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in \mathcal{N}\} \subset \{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in A\} = A^{s,\omega} \in \mathcal{F}$, and we see that $(\mathbf{1}_A)^{s,\omega}(\widetilde{\omega}) = \mathbf{1}_{\{\omega \otimes_s \widetilde{\omega} \in A\}} = \mathbf{1}_{\{\widetilde{\omega} \in A^{s,\omega}\}} = \mathbf{1}_{A^{s,\omega}}(\widetilde{\omega})$, $\forall\, \widetilde{\omega} \in \Omega^s$. Then (3.2) implies that for P_t-a.s. $\omega \in \Omega^t$

$$P_s\big(A^{s,\omega}\big) = E_s\big[\mathbf{1}_{A^{s,\omega}}\big] = E_s\big[(\mathbf{1}_A)^{s,\omega}\big] = E_t\big[\mathbf{1}_A\big|\mathcal{F}^t_s\big](\omega) = 0, \quad \text{and thus} \quad \mathcal{N}^{s,\omega} \in \mathscr{N}^s. \tag{6.43}$$

Next, let ξ_1 and ξ_2 be two real-valued random variables with $\xi_1 \le \xi_2$, P_t-a.s. Since $\mathcal{N} := \{\omega \in \Omega^t : \xi_1(\omega) > \xi_2(\omega)\} \in \mathscr{N}^t$, (6.43) leads to that for P_t-a.s. $\omega \in \Omega^t$,

$$0 = P_s\big(\mathcal{N}^{s,\omega}\big) = P_s\big\{\widetilde{\omega} \in \Omega^s : \xi_1\big(\omega \otimes_s \widetilde{\omega}\big) > \xi_2\big(\omega \otimes_s \widetilde{\omega}\big)\big\} = P_s\big\{\widetilde{\omega} \in \Omega^s : \xi^{s,\omega}_1(\widetilde{\omega}) > \xi^{s,\omega}_2(\widetilde{\omega})\big\}.$$

2) Let $\tau \in \overline{\mathcal{T}}^t$ with $\tau \ge s$ and let $r \in [s,\infty)$. As $A_r := \{\tau \le r\} \in \overline{\mathcal{F}}^t_r$, there exists an $\widetilde{A}_r \in \mathcal{F}^t_r$ such that $\mathcal{N}_r := A_r \,\Delta\, \widetilde{A}_r \in \mathscr{N}^t$ (see e.g. Problem 2.7.3 of [32]). By Part (1), it holds for all $\omega \in \Omega^t$ except on a P_t-null set $\widehat{\mathcal{N}}_r$ that $\mathcal{N}^{s,\omega}_r \in \mathscr{N}^s$. Given $\omega \in \widehat{\mathcal{N}}^c_r$, since $A^{s,\omega}_r \,\Delta\, \widetilde{A}^{s,\omega}_r = \big(A_r \,\Delta\, \widetilde{A}_r\big)^{s,\omega} = \mathcal{N}^{s,\omega}_r \in \mathscr{N}^s$ and since $\widetilde{A}^{s,\omega}_r \in \mathcal{F}^s_r$ by Lemma 3.2, we can deduce that $A^{s,\omega}_r \in \overline{\mathcal{F}}^s_r$ and it follows that

$$\{\tau^{s,\omega} \le r\} = \{\widetilde{\omega} \in \Omega^s : \tau^{s,\omega}(\widetilde{\omega}) \le r\} = \{\widetilde{\omega} \in \Omega^s : \tau(\omega \otimes_s \widetilde{\omega}) \le r\} = \{\widetilde{\omega} \in \Omega^s : \omega \otimes_s \widetilde{\omega} \in A_r\} = A^{s,\omega}_r \in \overline{\mathcal{F}}^s_r. \tag{6.44}$$

Let $\omega \in \underset{r \in (s,\infty) \cap \mathbb{Q}}{\cap} \widehat{\mathcal{N}}^c_r$. For any $r \in [s,\infty)$, there exists a sequence $\{r_n\}_{n \in \mathbb{N}}$ in $(s,\infty) \cap \mathbb{Q}$ such that $\lim_{n \to \infty} \downarrow r_n = r$. Then (6.44) and the right-continuity of Brownian filtration $\overline{\mathbf{F}}^s$ (under P_s) imply that $\{\tau^{s,\omega} \le r\} = \underset{n \in \mathbb{N}}{\cap} \{\tau^{s,\omega} \le r_n\} \in \overline{\mathcal{F}}^s_{r+} = \overline{\mathcal{F}}^s_r$. Hence $\tau^{s,\omega} \in \overline{\mathcal{T}}^s$. □

Proof of Proposition 3.4: 1) Let $r\in[s,\infty]$ and ξ be an $\overline{\mathcal{F}}^t_r$−measurable random variable. By Lemma A.4 (2), there exists an $\mathcal{F}^t_r$−measurable random variable $\widetilde{\xi}$ that equals to ξ except on a $\mathcal{N}\in\mathscr{N}^t$. Proposition 3.1 (1) shows that $\widetilde{\xi}^{s,\omega}$ is $\mathcal{F}^s_r$−measurable for any $\omega\in\Omega^t$. Also, we see from Proposition 3.3 (1) that for P_t−a.s. $\omega\in\Omega^t$,

$$\big\{\widetilde{\omega}\in\Omega^s:\widetilde{\xi}^{s,\omega}(\widetilde{\omega})\neq\xi^{s,\omega}(\widetilde{\omega})\big\}=\big\{\widetilde{\omega}\in\Omega^s:\omega\otimes_s\widetilde{\omega}\in\mathcal{N}\big\}=\mathcal{N}^{s,\omega}\in\mathscr{N}^s \tag{6.45}$$

and thus $\xi^{s,\omega}\in\overline{\mathcal{F}}^s_r$. In particular, if ξ is an $\overline{\mathcal{F}}^t_s$−measurable and $\widetilde{\xi}$ is $\mathcal{F}^t_s$−measurable, then (6.45) and (3.1) imply that P_t−a.s. $\omega\in\Omega^t$, $\xi^{s,\omega}=\widetilde{\xi}^{s,\omega}=\widetilde{\xi}(\omega)=\xi(\omega)$, P_s−a.s.

Suppose next that ξ is integrable (so is $\widetilde{\xi}$). Proposition 3.2 and Lemma A.4 (1) show that for P_t−a.s. $\omega\in\Omega^t$, $\widetilde{\xi}^{s,\omega}$ is integrable (so is $\xi^{s,\omega}$) and $E_t\big[\xi\big|\overline{\mathcal{F}}^t_s\big](\omega)=E_t\big[\xi\big|\mathcal{F}^t_s\big](\omega)=E_t\big[\,\widetilde{\xi}\,\big|\mathcal{F}^t_s\big](\omega)=E_s\big[\,\widetilde{\xi}^{s,\omega}\big]=E_s\big[\xi^{s,\omega}\big]\in\mathbb{R}$.

2a) Let $X=\{X_r\}_{r\in[t,\infty)}$ be an $\overline{\mathbf{F}}^t$−adapted process with P_t−a.s. continuous paths and set $\mathcal{N}_1:=\big\{\omega\in\Omega^t:$ the path $X_\cdot(\omega)$ is not continuous$\big\}\in\mathscr{N}^t$. In light of Lemma A.4 (3), we can find an $\mathbb{E}$−valued, $\mathbf{F}^t$−predictable process $\widetilde{X}=\big\{\widetilde{X}_r\big\}_{r\in[t,\infty)}$ such that $\mathcal{N}_2:=\{\omega\in\Omega^t:\widetilde{X}_r(\omega)\neq X_r(\omega)$ for some $r\in[t,\infty)\}\in\mathscr{N}^t$. In particular, $\widetilde{X}$ is an $\mathbf{F}^t$−adapted process.

Proposition 3.1 (2) shows that the shifted process $\widetilde{X}^{s,\omega}$ is $\mathbf{F}^s$−adapted for any $\omega\in\Omega^t$, and Proposition 3.3 (1) implies that for any $\omega\in\Omega^t$ except on a P_t−null set $\mathcal{N}_3$ that $\big(\mathcal{N}_1\cup\mathcal{N}_2\big)^{s,\omega}\in\mathscr{N}^s$. Let $\omega\in\mathcal{N}_3^c$. Since

$$\big\{\widetilde{\omega}\in\Omega^s:X^{s,\omega}_\cdot(\widetilde{\omega})\text{ is not continuous}\big\}\cup\big\{\widetilde{\omega}\in\Omega^s:\widetilde{X}^{s,\omega}_r(\widetilde{\omega})\neq X^{s,\omega}_r(\widetilde{\omega})\text{ for some }r\in[s,\infty)\big\}\subset\big(\mathcal{N}_1\cup\mathcal{N}_2\big)^{s,\omega}\in\mathscr{N}^s,$$

one can deduce that $X^{s,\omega}$ is an $\overline{\mathbf{F}}^s$−adapted process with P_s−a.s. continuous paths.

2b) Next, let us further assume that $X\in\mathbb{C}^q_t(\mathbb{E})$ for some $q\in[1,\infty)$. Define $\xi:=\sup\limits_{r\in[t,\infty)\cap\mathbb{Q}}\big|\widetilde{X}_r\big|^q\in\mathcal{F}^t$. As ξ equals to X^q_* on $(\mathcal{N}_1\cup\mathcal{N}_2)^c$, one has $X^q_*\in\overline{\mathcal{F}}^t$ and thus $E_t[\xi]=E_t\big[X^q_*\big]<\infty$. According to Part (1), it holds for all $\omega\in\Omega^t$ except on a P_t−null set $\mathcal{N}_4$ that $\xi^{s,\omega}$ is $\overline{\mathcal{F}}^s$−measurable and P_s−integrable.

Let $\omega\in(\mathcal{N}_3\cup\mathcal{N}_4)^c$. For any $\widetilde{\omega}\in\big((\mathcal{N}_1\cup\mathcal{N}_2)^{s,\omega}\big)^c=\big((\mathcal{N}_1\cup\mathcal{N}_2)^c\big)^{s,\omega}$, the continuity of the path $X^{s,\omega}_\cdot(\widetilde{\omega})=X_\cdot(\omega\otimes_s\widetilde{\omega})$ implies that $\sup\limits_{r\in[s,\infty)}\big|X^{s,\omega}_r(\widetilde{\omega})\big|^q=\sup\limits_{r\in[s,\infty)\cap\mathbb{Q}}\big|X_r(\omega\otimes_s\widetilde{\omega})\big|^q=\sup\limits_{r\in[s,\infty)\cap\mathbb{Q}}\big|\widetilde{X}_r(\omega\otimes_s\widetilde{\omega})\big|^q\le\xi(\omega\otimes_s\widetilde{\omega})$. It follows that $E_s\Big[\sup\limits_{r\in[s,\infty)}|X^{s,\omega}_r|^q\Big]\le E_s\big[\xi^{s,\omega}\big]<\infty$. Hence, $X^{s,\omega}\in\mathbb{C}^q_s(\mathbb{E})$ for any $\omega\in(\mathcal{N}_3\cup\mathcal{N}_4)^c$. □

Proof of Proposition 3.5: Let $M=\{M_r\}_{r\in[t,\infty)}\in\mathbb{M}_t$. By Proposition 3.4 (3), it holds for P_t−a.s. $\omega\in\Omega^t$ that $M^{s,\omega}$ is an $\overline{\mathbf{F}}^s$−adapted process with P_s−a.s. continuous paths. So we only need to show that $M^{s,\omega}$ is a uniformly integrable martingale with respect to $\big(\overline{\mathbf{F}}^s,P_s\big)$ for P_t−a.s. $\omega\in\Omega^t$.

By the uniform integrability of M, there exists $\xi\in L^1\big(\overline{\mathcal{F}}^t\big)$ such that for any $r\in[s,\infty)$,

$$M_r=E_t\big[\xi\big|\overline{\mathcal{F}}^t_r\big],\quad P_t\text{−a.s.} \tag{6.46}$$

Set $\mathcal{N}:=\{\omega\in\Omega^t:$ the path $M_\cdot(\omega)$ is not continuous$\}\in\mathscr{N}^t$. Proposition 3.3 (1) and Proposition 3.4 (2) imply that for all $\omega\in\Omega^t$ except on a $\mathcal{N}_o\in\mathscr{N}^t$, one has $\mathcal{N}^{s,\omega}\in\mathscr{N}^s$ and $\xi^{s,\omega}\in L^1\big(\overline{\mathcal{F}}^s\big)$.

Fix $r\in[s,\infty)$. As $M_r\in L^1\big(\overline{\mathcal{F}}^t_r\big)$, Proposition 3.4 (2) shows that for all $\omega\in\Omega^t$ except on a P_t−null set $\mathcal{N}^1_r$, $M^{s,\omega}_r\in L^1\big(\overline{\mathcal{F}}^s_r\big)$.

Let $\widetilde{A}\in\overline{\mathcal{F}}^s_r$. By Lemma A.3 (2), the set $\mathcal{A}:=(\Pi^t_s)^{-1}\big(\widetilde{A}\big)$ belongs to $\overline{\mathcal{F}}^t_r$, so $\mathbf{1}_{\mathcal{A}}M_r\in L^1\big(\overline{\mathcal{F}}^t_r\big)$ and $\mathbf{1}_{\mathcal{A}}\xi\in L^1\big(\overline{\mathcal{F}}^t\big)$. Since it holds for any $\omega\in\Omega^t$ and $\widetilde{\omega}\in\Omega^s$ that $(\mathbf{1}_{\mathcal{A}})^{s,\omega}(\widetilde{\omega})=\mathbf{1}_{\{\omega\otimes_s\widetilde{\omega}\in\mathcal{A}\}}=\mathbf{1}_{\{\Pi^t_s(\omega\otimes_s\widetilde{\omega})\in\widetilde{A}\}}=\mathbf{1}_{\{\widetilde{\omega}\in\widetilde{A}\}}=\mathbf{1}_{\widetilde{A}}(\widetilde{\omega})$, Proposition 3.4 (2) and (6.46) yield that for P_t−a.s. $\omega\in\Omega^t$

$$E_s\big[\mathbf{1}_{\widetilde{A}}M^{s,\omega}_r\big]=E_t\big[\mathbf{1}_{\mathcal{A}}M_r|\overline{\mathcal{F}}^t_s\big](\omega)=E_t\Big[\mathbf{1}_{\mathcal{A}}E_t\big[\xi|\overline{\mathcal{F}}^t_r\big]\Big|\overline{\mathcal{F}}^t_s\Big](\omega)=E_t\Big[E_t\big[\mathbf{1}_{\mathcal{A}}\xi|\overline{\mathcal{F}}^t_r\big]\Big|\overline{\mathcal{F}}^t_s\Big](\omega)=E_t\big[\mathbf{1}_{\mathcal{A}}\xi|\overline{\mathcal{F}}^t_s\big](\omega)=E_s\big[\mathbf{1}_{\widetilde{A}}\xi^{s,\omega}\big].$$

As $\mathscr{C}^s_r:=\Big\{\bigcap\limits_{i=1}^m(W^s_{s_i})^{-1}\big(O_{\delta_i}(x_i)\big):\ m\in\mathbb{N},\ s_i\in\mathbb{Q}_+\cup\{s\}\text{ with }s\le s_1\le\cdots\le s_m\le r,\ x_i\in\mathbb{Q}^d,\ \delta_i\in\mathbb{Q}_+\Big\}$ is a countable set, there exists a $\mathcal{N}^2_r\in\mathscr{N}^s$ such that for any $\omega\in(\mathcal{N}^2_r)^c$, $E_s\big[\mathbf{1}_{\widetilde{A}}M^{s,\omega}_r\big]=E_s\big[\mathbf{1}_{\widetilde{A}}\xi^{s,\omega}\big]$ holds for each $\widetilde{A}\in\mathscr{C}^s_r$. To wit, $\mathscr{C}^s_r\subset\Lambda_r:=\big\{\widetilde{A}\subset\Omega^s:E_s\big[\mathbf{1}_{\widetilde{A}}M^{s,\omega}_r\big]=E_s\big[\mathbf{1}_{\widetilde{A}}\xi^{s,\omega}\big],\ \forall\,\omega\in(\mathcal{N}^2_r)^c\big\}$. It is easy to see that $\mathscr{C}^s_r$ is closed under intersection and Λ is a Dynkin system. Then Lemma A.1 (2) and Dynkin System Theorem show that $\mathcal{F}^s_r=\sigma(\mathscr{C}^s_r)\subset\Lambda_r$. Clearly, $\mathscr{N}^s$ also belongs to Λ_r, so

$$\overline{\mathcal{F}}^s_r=\sigma(\mathcal{F}^s_r\cup\mathscr{N}^s)\subset\Lambda_r. \tag{6.47}$$

Now, let $\omega \in \mathcal{N}_o^c \cap \Big(\underset{r\in[s,\infty)\cap\mathbb{Q}}{\cup} (\mathcal{N}_r^1 \cup \mathcal{N}_r^2)\Big)^c$. For any $r \in [s,\infty)$, (6.47) shows that $E_s\big[\mathbf{1}_{\widetilde{A}} M_r^{s,\omega}\big] = E_s\big[\mathbf{1}_{\widetilde{A}} \xi^{s,\omega}\big]$, $\forall\, \widetilde{A} \in \overline{\mathcal{F}}^s_r$ and thus $E_s[\xi^{s,\omega}|\overline{\mathcal{F}}^s_r] = M_r^{s,\omega}$, P_s-a.s. Since $\{\widetilde{\omega} \in \Omega^s : \text{path } M^{s,\omega}_\cdot(\widetilde{\omega}) \text{ is not continuous}\} \subset \mathcal{N}^{s,\omega} \in \mathscr{N}^s$, we can deduce from the continuity of process $\big\{E_s[\xi^{s,\omega}|\overline{\mathcal{F}}^s_r]\big\}_{r\in[s,\infty)}$ that $P_s\big\{M_r^{s,\omega} = E_s\big[\xi^{s,\omega}|\overline{\mathcal{F}}^s_r\big],\ \forall\, r\in[s,\infty)\big\} = 1$. Therefore, $M^{s,\omega}$ is a uniformly integrable continuous martingale with respect to $\big(\overline{\mathbf{F}}^s, P_s\big)$. □

Proof of Lemma 3.3: Let $\{\xi_i\}_{i\in\mathbb{N}}$ be a sequence of $L^1\big(\overline{\mathcal{F}}^t\big)$ that converges to 0 in probability P_t, i.e.

$$\lim_{i\to\infty}\downarrow E_t\big[\mathbf{1}_{\{|\xi_i|>1/n\}}\big] = \lim_{i\to\infty}\downarrow P_t\big(|\xi_i| > 1/n\big) = 0, \quad \forall\, n \in \mathbb{N}. \tag{6.48}$$

In particular, $\lim\limits_{i\to\infty}\downarrow E_t\big[\mathbf{1}_{\{|\xi_i|>1\}}\big] = 0$ allows us to extract a subsequence $S_1 = \big\{\xi_i^1\big\}_{i\in\mathbb{N}}$ from $\{\xi_i\}_{i\in\mathbb{N}}$ such that $\lim\limits_{i\to\infty}\mathbf{1}_{\{|\xi_i^1|>1\}} = 0$, P_t-a.s. Clearly, S_1 also satisfies (6.48). Then by $\lim\limits_{i\to\infty}\downarrow E_t\big[\mathbf{1}_{\{|\xi_i^1|>1/2\}}\big] = 0$, we can find a subsequence $S_2 = \big\{\xi_i^2\big\}_{i\in\mathbb{N}}$ of S_1 such that $\lim\limits_{i\to\infty}\mathbf{1}_{\{|\xi_i^2|>1/2\}} = 0$, P_t-a.s. Inductively, for each $n \in \mathbb{N}$ we can select a subsequence $S_{n+1} = \{\xi_i^{n+1}\}_{i\in\mathbb{N}}$ of $S_n = \{\xi_i^n\}_{i\in\mathbb{N}}$ such that $\lim\limits_{i\to\infty}\mathbf{1}_{\{|\xi_i^{n+1}|>\frac{1}{n+1}\}} = 0$, P_t-a.s.

For any $i \in \mathbb{N}$, we set $\widetilde{\xi}_i := \xi_i^i$, which belongs to S_n for $n = 1,\cdots,i$. Given $n \in \mathbb{N}$, since $\{\widetilde{\xi}_i\}_{i=n}^\infty \subset S_n$, it holds P_t-a.s. that $\lim\limits_{i\to\infty}\mathbf{1}_{\{|\widetilde{\xi}_i|>\frac{1}{n}\}} = 0$. Then a conditional-expectation version of the bound convergence theorem and Proposition 3.4 (2) imply that for all $\omega \in \Omega^t$ except on a P_t-null set $\mathcal{N}_n$, $\widetilde{\xi}_i$ is $\overline{\mathcal{F}}^s-$measurable and

$$0 = \lim_{i\to\infty} E_t\Big[\mathbf{1}_{\{|\widetilde{\xi}_i|>1/n\}}\big|\overline{\mathcal{F}}^t_s\Big](\omega) = \lim_{i\to\infty} E_s\Big[\big(\mathbf{1}_{\{|\widetilde{\xi}_i|>1/n\}}\big)^{s,\omega}\Big]. \tag{6.49}$$

Let $\omega \in \Big(\underset{n\in\mathbb{N}}{\cup}\mathcal{N}_n\Big)^c$. For any $n \in \mathbb{N}$, one can deduce that

$$\big(\mathbf{1}_{\{|\widetilde{\xi}_i|>1/n\}}\big)^{s,\omega}(\widetilde{\omega}) = \mathbf{1}_{\{|\widetilde{\xi}_i(\omega\otimes_s\widetilde{\omega})|>1/n\}} = \mathbf{1}_{\{|\widetilde{\xi}_i^{s,\omega}(\widetilde{\omega})|>1/n\}} = \Big(\mathbf{1}_{\{|\widetilde{\xi}_i^{s,\omega}|>1/n\}}\Big)(\widetilde{\omega}), \quad \forall\, \widetilde{\omega} \in \Omega^s,$$

which together with (6.49) leads to that $\lim\limits_{i\to\infty} P_s\Big(|\widetilde{\xi}_i^{s,\omega}| > 1/n\Big) = \lim\limits_{i\to\infty} E_s\Big[\big(\mathbf{1}_{\{|\widetilde{\xi}_i|>1/n\}}\big)^{s,\omega}\Big] = 0.$ □

Proof of Proposition 3.6: As $\mathfrak{X} \in \mathbb{C}^2_t(\mathbb{R}^l)$ by Corollary 3.1, we know from Proposition 3.4 (3) that for P_t-a.s. $\omega \in \Omega^t$, $\big\{\mathfrak{X}_r^{s,\omega}\big\}_{r\in[s,\infty)} \in \mathbb{C}^2_s(\mathbb{R}^l)$.

To show that for P_t-a.s. $\omega\in\Omega^t$, $\mathfrak{X}^{s,\omega}$ solves (1.1) over $[s,\infty)$ with initial state $\mathfrak{X}_s(\omega)$, we let $\mathcal{N}_1$ be the P_t-null set such that $\mathfrak{X}$ satisfies (1.1) on $\mathcal{N}_1^c$. Define $M_{s'} := \int_t^{s'} \mathbf{1}_{\{r>s\}}\sigma(r,\mathfrak{X}_r)dW_r^t$, $s' \in [t,\infty)$.

1) By Proposition 3.4 (1), there exists a P_t-null set $\mathcal{N}_2$ such that for any $\omega \in \mathcal{N}_2^c$, $\mathfrak{X}_s(\omega\otimes_s\widetilde{\omega}) = \mathfrak{X}_s(\omega)$ holds for all $\widetilde{\omega} \in \Omega^s$ except on a $\mathcal{N}_\omega \in \mathscr{N}^s$.

Let $\omega \in \mathcal{N}_1^c \cap \mathcal{N}_2^c$ and $\widetilde{\omega} \in \mathcal{N}_\omega^c$. Implementing (1.1) on the path $\omega\otimes_s\widetilde{\omega}$ over period $[s,\infty)$ yields that

$$\begin{aligned} \mathfrak{X}_{s'}^{s,\omega}(\widetilde{\omega}) \;&=\; \mathfrak{X}_{s'}(\omega\otimes_s\widetilde{\omega}) = \mathfrak{X}_s(\omega\otimes_s\widetilde{\omega}) + \int_s^{s'} b\big(r,\mathfrak{X}_r(\omega\otimes_s\widetilde{\omega})\big)dr + \Big(\int_s^{s'}\sigma(r,\mathfrak{X}_r)dW_r^t\Big)(\omega\otimes_s\widetilde{\omega}) \\ &=\; \mathfrak{X}_s(\omega) + \int_s^{s'} b\big(r,\mathfrak{X}_r^{s,\omega}(\widetilde{\omega})\big)dr + M_{s'}^{s,\omega}(\widetilde{\omega}), \quad s' \in [s,\infty). \end{aligned} \tag{6.50}$$

So it remains to show that for P_t-a.s. $\omega \in \Omega^t$, it holds P_s-a.s. that

$$M_{s'}^{s,\omega} = \int_s^{s'} \sigma\big(r,\mathfrak{X}_r^{s,\omega}\big)\,dW_r^s, \quad s' \in [s,\infty). \tag{6.51}$$

2) Since $\{M_{s'}\}_{s'\in[t,\infty)}$ is a square-integrable martingale with respect to $\big(\overline{\mathbf{F}}^t, P_t\big)$ by (1.7) and Corollary 3.1, we know that (see e.g. Problem 3.2.27 of [32]) there is a sequence of $\mathbb{R}^{l\times d}-$valued, $\overline{\mathbf{F}}^t-$simple processes $\Big\{\Phi_r^n = \sum_{i\in\mathbb{N}}\eta_i^n\mathbf{1}_{\{r\in(t_i^n,t_{i+1}^n]\}},\ r \in [t,\infty)\Big\}_{n\in\mathbb{N}}$ (where $\{t_i^n\}_{i\in\mathbb{N}}$ is an increasing sequence in $[t,\infty)$ and $\eta_i^n \in \overline{\mathcal{F}}^t_{t_i^n}$ for $i \in \mathbb{N}$) such that

$$P_t-\lim_{n\to\infty}\int_t^\infty trace\Big\{\big(\Phi_r^n - \sigma(r,\mathfrak{X}_r)\big)\big(\Phi_r^n - \sigma(r,\mathfrak{X}_r)\big)^T\Big\}dr = 0 \quad \text{and} \quad P_t-\lim_{n\to\infty}\sup_{s'\in[t,\infty)}\big|M_{s'}^n - M_{s'}\big| = 0,$$

where $M^n_{s'} := \int_t^{s'} \Phi^n_r dW^t_r = \sum_{i\in\mathbb{N}} \eta^n_i \big(W^t_{s'\wedge t^n_{i+1}} - W^t_{s'\wedge t^n_i}\big)$. Then it directly follows that

$$P_t - \lim_{n\to\infty} \int_s^\infty trace\Big\{\big(\Phi^n_r - \sigma(r,\mathfrak{X}_r)\big)\big(\Phi^n_r - \sigma(r,\mathfrak{X}_r)\big)^T\Big\} dr = 0 \quad \text{and} \quad P_t - \lim_{n\to\infty} \sup_{s'\in[s,\infty)} \big|M^n_{s'} - M_{s'}\big| = 0.$$

By Lemma 3.3, $\{\Phi^n\}_{n\in\mathbb{N}}$ has a subsequence $\Big\{\widehat{\Phi}^n_r = \sum_{i\in\mathbb{N}} \widehat{\eta}^n_i \mathbf{1}_{\{r\in(\widehat{t}^n_i, \widehat{t}^n_{i+1}]\}},\ r\in[t,\infty)\Big\}_{n\in\mathbb{N}}$ such that for any $\omega\in\Omega^t$ except on a P_t-null set $\mathcal{N}_4$

$$\begin{aligned} 0 &= P_s - \lim_{n\to\infty} \bigg(\int_s^\infty trace\Big\{\big(\widehat{\Phi}^n_r - \sigma(r,\mathfrak{X}_r)\big)\big(\widehat{\Phi}^n_r - \sigma(r,\mathfrak{X}_r)\big)^T\Big\} dr\bigg)^{s,\omega} \\ &= P_s - \lim_{n\to\infty} \int_s^\infty trace\Big\{\Big(\big(\widehat{\Phi}^n\big)^{s,\omega}_r - \sigma(r,\mathfrak{X}^{s,\omega}_r)\Big)\Big(\big(\widehat{\Phi}^n\big)^{s,\omega}_r - \sigma(r,\mathfrak{X}^{s,\omega}_r)\Big)^T\Big\} dr \qquad (6.52) \\ \text{and} \quad 0 &= P_s - \lim_{n\to\infty} \Big(\sup_{s'\in[s,\infty)} \big|\widehat{M}^n_{s'} - \widehat{M}^n_s - M_{s'}\big|\Big)^{s,\omega} \\ &= P_s - \lim_{n\to\infty} \sup_{s'\in[s,\infty)} \Big|\big(\widehat{M}^n\big)^{s,\omega}_{s'} - \big(\widehat{M}^n\big)^{s,\omega}_s - M^{s,\omega}_{s'}\Big|, \qquad (6.53) \end{aligned}$$

where $\widehat{M}^n_{s'} := \int_t^{s'} \widehat{\Phi}^n_r dW^t_r = \sum\limits_{i\in\mathbb{N}} \widehat{\eta}^n_i \Big(W^t_{s'\wedge \widehat{t}^n_{i+1}} - W^t_{s'\wedge \widehat{t}^n_i}\Big)$.

Given $n\in\mathbb{N}$, let ℓ_n be the largest integer such that $\widehat{t}^n_{\ell_n} < s$. For any $i = \ell_n, \ell_n+1, \cdots$, we set $s^n_i := \widehat{t}^n_i \vee s$. Since $\widehat{\eta}^n_i \in \overline{\mathcal{F}}^t_{\widehat{t}^n_i} \subset \overline{\mathcal{F}}^t_{s^n_i}$. Proposition 3.4 (2) shows that $\big(\widehat{\eta}^n_i\big)^{s,\omega} \in \overline{\mathcal{F}}^s_{s^n_i}$ holds for any $\omega\in\Omega^t$ except on a P_t-null set $\mathcal{N}^n_i$. Let $\omega \in \widehat{\Omega} := \mathcal{N}^c_4 \cap \Big(\bigcap\limits_{n\in\mathbb{N}} \bigcap\limits_{i=\ell_n}^\infty (\mathcal{N}^n_i)^c\Big)$. As $s^n_{\ell_n} = s$, one has $\big(\widehat{\eta}^n_{\ell_n}\big)^{s,\omega} \in \overline{\mathcal{F}}^s_s$. For any $s'\in[s,\infty)$ and $\widetilde{\omega}\in\Omega^s$,

$$\big(\widehat{\Phi}^n\big)^{s,\omega}_{s'}(\widetilde{\omega}) = \widehat{\Phi}^n_{s'}(\omega\otimes_s\widetilde{\omega}) = \sum_{i\in\mathbb{N}} \widehat{\eta}^n_i(\omega\otimes_s\widetilde{\omega})\,\mathbf{1}_{\{s'\in(\widehat{t}^n_i,\widehat{t}^n_{i+1}]\}} = \big(\widehat{\eta}^n_{\ell_n}\big)^{s,\omega}(\widetilde{\omega})\mathbf{1}_{\{s'\in[s,s^n_{\ell_n+1}]\}} + \sum_{i=\ell_n+1}^\infty \big(\widehat{\eta}^n_i\big)^{s,\omega}(\widetilde{\omega})\,\mathbf{1}_{\{s'\in(s^n_i,s^n_{i+1}]\}}.$$

So $\big\{\big(\widehat{\Phi}^n\big)^{s,\omega}_{s'}\big\}_{s'\in[s,\infty)}$ is an $\mathbb{R}^{l\times d}-$valued, $\overline{\mathbf{F}}^s-$simple process. Applying Proposition 3.2.26 of [32] and using (6.52) yield that

$$0 = P_s - \lim_{n\to\infty} \sup_{s'\in[s,\infty)} \bigg|\int_s^{s'} \big(\widehat{\Phi}^n\big)^{s,\omega}_r dW^s_r - \int_s^{s'} \sigma(r,\mathfrak{X}^{s,\omega}_r) dW^s_r\bigg|. \qquad (6.54)$$

For any $\widetilde{\omega}\in\Omega^s$, one can deduce that

$$\begin{aligned} \big(\widehat{M}^n\big)^{s,\omega}_{s'}(\widetilde{\omega}) - \big(\widehat{M}^n\big)^{s,\omega}_s(\widetilde{\omega}) &= \sum_{i=\ell}^\infty \widehat{\eta}^n_i(\omega\otimes_s\widetilde{\omega})\Big(\big(\omega\otimes_s\widetilde{\omega}\big)\big(s'\wedge s^n_{i+1}\big) - \big(\omega\otimes_s\widetilde{\omega}\big)\big(s'\wedge s^n_i\big)\Big) = \sum_{i=\ell}^\infty \big(\widehat{\eta}^n_i\big)^{s,\omega}(\widetilde{\omega})\Big(\widetilde{\omega}\big(s'\wedge s^n_{i+1}\big) - \widetilde{\omega}\big(s'\wedge s^n_i\big)\Big) \\ &= \sum_{i=\ell}^\infty \big(\widehat{\eta}^n_i\big)^{s,\omega}(\widetilde{\omega})\Big(W^s_{s'\wedge s^n_{i+1}} - W^s_{s'\wedge s^n_i}\Big)(\widetilde{\omega}) = \bigg(\int_s^{s'} \big(\widehat{\Phi}^n\big)^{s,\omega}_r dW^s_r\bigg)(\widetilde{\omega}), \quad s'\in[s,\infty), \end{aligned}$$

which together with (6.53) and (6.54) shows that (6.51) holds P_s-a.s. for any $\omega\in\widehat{\Omega}$. Eventually, we see from (6.50) that $P_s\big\{\widetilde{\omega}\in\Omega^s : \mathfrak{X}_r(\omega\otimes_s\widetilde{\omega}) = \mathfrak{X}^{s,\omega}_r(\widetilde{\omega}) = \mathcal{X}^{s,\mathfrak{X}_s(\omega)}_r(\widetilde{\omega}),\ \forall\, r\in[s,\infty)\big\} = 1$ for any $\omega\in\widehat{\Omega}$. □

6.3 Proof of Section 4

The proof of the first DPP (Theorem 4.1) is based on the following auxiliary result.

Lemma 6.1. *Given $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty)$, let $\tau\in\mathcal{T}^t_x(y)$ and $\zeta\in\overline{\mathcal{T}}^t_\sharp$. Then*

$$E_t\big[\mathcal{R}(t,x,\tau)\big] \le E_t\bigg[\mathbf{1}_{\{\tau\le\zeta\}}\mathcal{R}(t,x,\tau) + \mathbf{1}_{\{\tau>\zeta\}}\Big(\mathcal{V}(\zeta,\mathcal{X}^{t,x}_\zeta,\mathcal{Y}^{t,x,\tau}_\zeta) + \int_t^\zeta f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg] \le \mathcal{V}(t,x,y). \qquad (6.55)$$

Proof: 1) *Let us start with some basic settings.*

Denote $(\mathfrak{X},\mathfrak{Y}):=(\mathcal{X}^{t,x},\mathcal{Y}^{t,x,\tau})$ and let ζ take values in a countable subset $\{t_i\}_{i\in\mathbb{N}}$ of $[t,\infty)$. In light of Lemma A.4 (3), there exists an $\mathbb{R}^l-$valued, $\mathbf{F}^t-$predictable process $\widetilde{\mathfrak{X}}=\big\{\widetilde{\mathfrak{X}}_r\big\}_{r\in[t,\infty)}$ such that $\mathcal{N}:=\{\omega\in\Omega^t:\widetilde{\mathfrak{X}}_r(\omega)\neq\mathfrak{X}_r(\omega)$ for some $r\in[t,\infty)\}\in\mathscr{N}^t$.

Let $i\in\mathbb{N}$. By Proposition 3.3 (1), we can find a P_t-null set $\mathcal{N}_i$ such that for any $\omega\in\mathcal{N}_i^c$, $\mathcal{N}^{t_i,\omega}$ is a $P_{t_i}-$null set. For any $r\in[t,t_i]$, since $\widetilde{\mathfrak{X}}_r\in\mathcal{F}^t_r\subset\mathcal{F}^t_{t_i}$, (3.1) implies that

$$\mathfrak{X}_r(\omega\otimes_{t_i}\widetilde{\omega})=\widetilde{\mathfrak{X}}_r(\omega\otimes_{t_i}\widetilde{\omega})=\widetilde{\mathfrak{X}}_r(\omega)=\mathfrak{X}_r(\omega),\quad\forall\,\omega\in\mathcal{N}^c\cap\mathcal{N}_i^c,\quad\forall\,\widetilde{\omega}\in(\mathcal{N}^c)^{t_i,\omega}.\tag{6.56}$$

Also Proposition 3.6 shows that for all $\omega\in\Omega^t$ except on a P_t-null set $\widetilde{\mathcal{N}}_i$,

$$\mathcal{N}^i_\omega:=\Big\{\widetilde{\omega}\in\Omega^{t_i}:\mathfrak{X}^{t_i,\omega}_r(\widetilde{\omega})\neq\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r(\widetilde{\omega}),\ \text{for some } r\in[t_i,\infty)\Big\}\in\mathscr{N}^{t_i}.\tag{6.57}$$

Let τ_i be a $\overline{\mathcal{T}}^t-$stopping time with $\tau_i\ge t_i$. According to Proposition 3.3 (2) and Proposition 3.4 (2), it holds for all $\omega\in\Omega^t$ except on a P_t-null set $\widehat{\mathcal{N}}_i$ that $\tau^i_\omega:=\tau_i^{t_i,\omega}\in\overline{\mathcal{T}}^{t_i}$,

$$E_t\big[\mathcal{R}(t,x,\tau_i)\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega)=E_{t_i}\Big[\big(\mathcal{R}(t,x,\tau_i)\big)^{t_i,\omega}\Big]\ \text{ and }\ E_t\bigg[\int_t^{\tau_i}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg](\omega)=E_{t_i}\bigg[\Big(\int_t^{\tau_i}g(r,\mathfrak{X}_r)dr\Big)^{t_i,\omega}\bigg].\tag{6.58}$$

Let $\omega\in\mathcal{N}^c\cap\mathcal{N}_i^c\cap\widetilde{\mathcal{N}}_i^c\cap\widehat{\mathcal{N}}_i^c$. Given $\widetilde{\omega}\in\big(\mathcal{N}^{t_i,\omega}\cup\mathcal{N}^i_\omega\big)^c=(\mathcal{N}^c)^{t_i,\omega}\cap\big(\mathcal{N}^i_\omega\big)^c$, (6.57) shows $\mathfrak{X}_r(\omega\otimes_{t_i}\widetilde{\omega})=\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r(\widetilde{\omega})$ for any $r\in[t_i,\infty)$. In particular, taking $r=\tau^i_\omega(\widetilde{\omega})$ yields that $\mathfrak{X}\big(\tau_i(\omega\otimes_{t_i}\widetilde{\omega}),\omega\otimes_{t_i}\widetilde{\omega}\big)=\mathfrak{X}\big(\tau^i_\omega(\widetilde{\omega}),\omega\otimes_{t_i}\widetilde{\omega}\big)=\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_{\tau^i_\omega}(\widetilde{\omega})$, which together with (6.56) leads to that

$$\begin{aligned}\big(\mathcal{R}(t,x,\tau_i)\big)^{t_i,\omega}(\widetilde{\omega})&=\int_t^{\tau_i(\omega\otimes_{t_i}\widetilde{\omega})}f\big(r,\mathfrak{X}_r(\omega\otimes_{t_i}\widetilde{\omega})\big)dr+\pi\big(\tau_i(\omega\otimes_{t_i}\widetilde{\omega}),\mathfrak{X}_{\tau_i}(\omega\otimes_{t_i}\widetilde{\omega})\big)\\&=\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr+\int_{t_i}^{\tau^i_\omega(\widetilde{\omega})}f\Big(r,\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r(\widetilde{\omega})\Big)dr+\pi\Big(\tau^i_\omega(\widetilde{\omega}),\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_{\tau^i_\omega}(\widetilde{\omega})\Big)\\&=\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr+\big(\mathcal{R}(t_i,\mathfrak{X}_{t_i}(\omega),\tau^i_\omega)\big)(\widetilde{\omega}),\end{aligned}$$

and similarly, $\big(\int_t^{\tau_i}g(r,\mathfrak{X}_r)dr\big)^{t_i,\omega}(\widetilde{\omega})=\Big(\int_{t_i}^{\tau^i_\omega}g\Big(r,\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r\Big)dr\Big)(\widetilde{\omega})+\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr$. Taking expectation $E_t[\cdot]$, we see from (6.58) that for P_t-a.s. $\omega\in\Omega^t$, τ^i_ω is a $\overline{\mathcal{T}}^{t_i}-$stopping time satisfying

$$E_t\big[\mathcal{R}(t,x,\tau_i)\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega)\ =\ E_{t_i}\big[\mathcal{R}(t_i,\mathfrak{X}_{t_i}(\omega),\tau^i_\omega)\big]+\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr,\tag{6.59}$$

$$\text{and}\quad E_t\bigg[\int_t^{\tau_i}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg](\omega)\ =\ E_{t_i}\bigg[\int_{t_i}^{\tau^i_\omega}g\Big(r,\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r\Big)dr\bigg]+\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr.\tag{6.60}$$

2) *We next show the first inequality in* (6.55).

Let $i\in\mathbb{N}$ and set $\tau_i:=\tau\vee t_i\in\overline{\mathcal{T}}^t$. We can deduce from (6.60), (4.6), (4.7) and (6.59) that for P_t-a.s. $\omega\in\Omega^t$, $\tau^i_\omega:=\tau_i^{t_i,\omega}$ is a $\overline{\mathcal{T}}^{t_i}-$stopping time satisfying

$$E_{t_i}\bigg[\int_{t_i}^{\tau^i_\omega}g\Big(r,\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r\Big)dr\bigg]=E_t\bigg[\int_t^{\tau\vee t_i}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg](\omega)-\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr=\mathfrak{Y}_{t_i}(\omega)\in[0,\infty),$$

and

$$E_t\big[\mathcal{R}(t,x,\tau_i)\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega)=E_{t_i}\big[R\big(t_i,\mathfrak{X}_{t_i}(\omega),\tau^i_\omega\big)\big]+\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr\le\mathcal{V}\big(t_i,\mathfrak{X}_{t_i}(\omega),\mathfrak{Y}_{t_i}(\omega)\big)+\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr.\tag{6.61}$$

As $\{\tau>\zeta\}\in\overline{\mathcal{F}}^t_{\tau\wedge\zeta}\subset\overline{\mathcal{F}}^t_\zeta$ (see e.g. Lemma 1.2.16 of [32]), one has $\{\tau>\zeta=t_i\}=\{\tau>\zeta\}\cap\{\zeta=t_i\}\in\overline{\mathcal{F}}^t_{t_i}$. Then (6.61) shows that

$$\begin{aligned}&E_t\big[\mathbf{1}_{\{\tau>\zeta=t_i\}}\mathcal{R}(t,x,\tau)\big]=E_t\big[\mathbf{1}_{\{\tau>\zeta=t_i\}}\mathcal{R}(t,x,\tau_i)\big]=E_t\Big[\mathbf{1}_{\{\tau>\zeta=t_i\}}E_t\big[\mathcal{R}(t,x,\tau_i)\big|\overline{\mathcal{F}}^t_{t_i}\big]\Big]\\&\quad\le E_t\bigg[\mathbf{1}_{\{\tau>\zeta=t_i\}}\Big(\mathcal{V}\big(t_i,\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i}\big)+\int_t^{t_i}f\big(r,\mathfrak{X}_r\big)dr\Big)\bigg]=E_t\bigg[\mathbf{1}_{\{\tau>\zeta=t_i\}}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big)+\int_t^{\zeta}f\big(r,\mathfrak{X}_r\big)dr\Big)\bigg].\end{aligned}\tag{6.62}$$

Since (4.3), (6.9), (6.10) and the first inequality in (3.4) imply that

$$E_t\Big[\big|\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)\big|+\int_t^\zeta \big|f(r,\mathfrak{X}_r)\big|dr\Big] \le E_t\Big[2\mathfrak{C}\big(2+C_p(1+|\mathfrak{X}_\zeta|^p)\big)+\int_t^\infty c(r)\big(2+|\mathfrak{X}_\zeta|^p\big)dr\Big]$$
$$\le 2\mathfrak{C}(3+C_p)+\mathfrak{C}(1+2C_p)E_t\big[\mathfrak{X}_*^p\big]<\infty, \tag{6.63}$$

taking summation over $i\in\mathbb{N}$ in (6.62), we can deduce from the first inequality in (4.1) and the dominated convergence theorem that

$$E_t\big[\mathbf{1}_{\{\tau>\zeta\}}\mathcal{R}(t,x,\tau)\big]=E_t\Big[\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\tau>\zeta=t_i\}}\mathcal{R}(t,x,\tau)\Big]=\sum_{i\in\mathbb{N}}E_t\big[\mathbf{1}_{\{\tau>\zeta=t_i\}}\mathcal{R}(t,x,\tau)\big]$$
$$\le\sum_{i\in\mathbb{N}}E_t\Big[\mathbf{1}_{\{\tau>\zeta=t_i\}}\Big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)+\int_t^\zeta f\big(r,\mathfrak{X}_r\big)dr\Big)\Big]=E_t\Big[\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\tau>\zeta=t_i\}}\Big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)+\int_t^\zeta f\big(r,\mathfrak{X}_r\big)dr\Big)\Big]$$
$$=E_t\Big[\mathbf{1}_{\{\tau>\zeta\}}\Big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)+\int_t^\zeta f\big(r,\mathfrak{X}_r\big)dr\Big)\Big]. \tag{6.64}$$

It follows that $E_t\big[\mathcal{R}(t,x,\tau)\big]\le E_t\Big[\mathbf{1}_{\{\tau\le\zeta\}}\mathcal{R}(t,x,\tau)+\mathbf{1}_{\{\tau>\zeta\}}\big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)+\int_t^\zeta f(r,\mathfrak{X}_r)dr\big)\Big]$.

3) *Now, we demonstrate the second inequality in* (6.55).

Fix $\varepsilon\in(0,1)$ and let $i\in\mathbb{N}$, $\mathfrak{x}\in\mathbb{R}^l$. In light of (4.5) and Theorem 2.1 (1), there exists $\delta_i(\mathfrak{x})\in(0,\varepsilon/2)$ such that

$$\mathfrak{C}(C_p)^{\frac{1}{p}}\delta_i(\mathfrak{x})+\mathfrak{C}C_p\big(\delta_i(\mathfrak{x})\big)^p<\varepsilon/4, \tag{6.65}$$

and that for any $\mathfrak{y}\in[0,\infty)$,

$$\big|\mathcal{V}(t_i,\mathfrak{x}',\mathfrak{y}')-\mathcal{V}(t_i,\mathfrak{x},\mathfrak{y})\big|\le\varepsilon/4,\quad \forall\,(\mathfrak{x}',\mathfrak{y}')\in\overline{O}_{\delta_i(\mathfrak{x})}(\mathfrak{x})\times\big[(\mathfrak{y}-\delta_i(\mathfrak{x}))^+,\mathfrak{y}+\delta_i(\mathfrak{x})\big]. \tag{6.66}$$

Then (g1), Hölder's inequality and the second inequality in (3.4) imply that

$$E_{t_i}\Big[\int_{t_i}^\varsigma\big|g\big(r,\mathcal{X}_r^{t_i,\mathfrak{x}}\big)-g\big(r,\mathcal{X}_r^{t_i,\mathfrak{x}'}\big)\big|dr\Big]\le E_{t_i}\Big[\int_{t_i}^\infty c(r)\big(\big|\mathcal{X}_r^{t_i,\mathfrak{x}}-\mathcal{X}_r^{t_i,\mathfrak{x}'}\big|+\big|\mathcal{X}_r^{t_i,\mathfrak{x}}-\mathcal{X}_r^{t_i,\mathfrak{x}'}\big|^p\big)dr\Big]$$
$$\le\Big(\int_0^\infty c(r)dr\Big)E_{t_i}\Big[\big(\mathcal{X}^{t_i,\mathfrak{x}}-\mathcal{X}^{t_i,\mathfrak{x}'}\big)_*+\big(\mathcal{X}^{t_i,\mathfrak{x}}-\mathcal{X}^{t_i,\mathfrak{x}'}\big)_*^p\Big]\le\mathfrak{C}(C_p)^{\frac{1}{p}}|\mathfrak{x}-\mathfrak{x}'|+\mathfrak{C}C_p|\mathfrak{x}-\mathfrak{x}'|^p$$
$$\le\mathfrak{C}(C_p)^{\frac{1}{p}}\delta_i(\mathfrak{x})+\mathfrak{C}C_p\big(\delta_i(\mathfrak{x})\big)^p<\varepsilon/4,\quad\forall\,\varsigma\in\overline{\mathcal{T}}^{t_i},\ \forall\,\mathfrak{x}'\in\overline{O}_{\delta_i(\mathfrak{x})}(\mathfrak{x}). \tag{6.67}$$

We can find a sequence $\big\{(x_n^i,y_n^i)\big\}_{n\in\mathbb{N}}$ in $\mathbb{R}^l\times[0,\infty)$ such that $\mathbb{R}^l\times[0,\infty)=\underset{n\in\mathbb{N}}{\cup}\mathcal{O}_n^i\times\mathcal{D}_n^i$ with $\mathcal{O}_n^i:=O_{\delta_i(x_n^i)}(x_n^i)$ and $\mathcal{D}_n^i:=\begin{cases}\big((y_n^i-\delta_i(x_n^i))^+,\ y_n^i+\delta_i(x_n^i)\big), & \text{if } y_n^i>0,\\ \big[0,\delta_i(x_n^i)\big), & \text{if } y_n^i=0.\end{cases}$

Let $n\in\mathbb{N}$. We set $A_n^i:=\{\tau>\zeta=t_i\}\cap\big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\in\mathcal{O}_n^i\times\mathcal{D}_n^i\big\}\cap\mathcal{N}_{t,x,\tau}^c\in\overline{\mathcal{F}}_{t_i}^t$ and $\mathcal{A}_n^i:=A_n^i\Big\backslash\Big(\underset{n'<n}{\cup}A_{n'}^i\Big)\in\overline{\mathcal{F}}_{t_i}^t$. There exists a $\tau_n^i\in\mathcal{T}_{x_n^i}^{t_i}\big(y_n^i\big)$ such that

$$E_{t_i}\big[\mathcal{R}(t_i,x_n^i,\tau_n^i)\big]\ge\mathcal{V}\big(t_i,x_n^i,y_n^i\big)-\varepsilon/4. \tag{6.68}$$

Lemma A.3 shows that $\tau_n^i(\Pi_{t_i}^t)$ is a $\overline{\mathcal{T}}^t$−stopping time with values in $[t_i,\infty]$ such that $\big(\tau_n^i(\Pi_{t_i}^t)\big)^{t_i,\omega}(\widetilde{\omega})=\tau_n^i\big(\Pi_{t_i}^t(\omega\otimes_{t_i}\widetilde{\omega})\big)=\tau_n^i(\widetilde{\omega})$ for any $\omega\in\Omega^t$ and $\widetilde{\omega}\in\Omega^{t_i}$. Also, by (6.59) and (6.60), it holds for any $\omega\in\Omega^t$ except on a P_t−null set $\mathcal{N}^{i,n}$ that

$$E_t\Big[R\big(t,x,\tau_n^i(\Pi_{t_i}^t)\big)\big|\overline{\mathcal{F}}_{t_i}^t\Big](\omega)=E_{t_i}\big[\mathcal{R}(t_i,\mathfrak{X}_{t_i}(\omega),\tau_n^i)\big]+\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr, \tag{6.69}$$

$$\text{and}\quad E_t\Big[\int_t^{\tau_n^i(\Pi_{t_i}^t)}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}_{t_i}^t\Big](\omega)=E_{t_i}\Big[\int_{t_i}^{\tau_n^i}g\Big(r,\mathcal{X}_r^{t_i,\mathfrak{X}_{t_i}(\omega)}\Big)dr\Big]+\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr. \tag{6.70}$$

Clearly, the disjoint union $\underset{i,n\in\mathbb{N}}{\cup}\mathcal{A}_n^i$ satisfies that

$$\underset{i,n\in\mathbb{N}}{\cup}\mathcal{A}_n^i=\underset{i,n\in\mathbb{N}}{\cup}A_n^i=\Big(\underset{i\in\mathbb{N}}{\cup}\{\tau>\zeta=t_i\}\cap\Big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\in\underset{n\in\mathbb{N}}{\cup}\mathcal{O}_n^i\times\mathcal{D}_n^i\Big\}\Big)\cap\mathcal{N}_{t,x,\tau}^c$$
$$=\Big(\underset{i\in\mathbb{N}}{\cup}\{\tau>\zeta=t_i\}\Big)\cap\mathcal{N}_{t,x,\tau}^c=\{\tau>\zeta\}\cap\mathcal{N}_{t,x,\tau}^c. \tag{6.71}$$

We claim that

$$\overline{\tau}:=\mathbf{1}_{\{\tau\le\zeta\}}\tau+\sum_{i,n\in\mathbb{N}}\mathbf{1}_{\mathcal{A}^i_n}\tau^i_n(\Pi^t_{t_i})+\mathbf{1}_{\{\tau>\zeta\}\cap\mathcal{N}_{t,x,\tau}}t \text{ belongs to } \mathcal{T}^t_x(y+\varepsilon). \tag{6.72*}$$

Let $i,n\in\mathbb{N}$ and $\omega\in\mathcal{A}^i_n\cap\big(\mathcal{N}^{i,n}\big)^c$. As $\mathfrak{X}_{t_i}(\omega)\in\mathcal{O}^i_n=O_{\delta_i(x^i_n)}(x^i_n)$, (6.65) and the second inequality in (4.1) imply

$$E_{t_i}\Big[\big|\mathcal{R}(t_i,\mathfrak{X}_{t_i}(\omega),\tau^i_n)-\mathcal{R}(t_i,x^i_n,\tau^i_n)\big|\Big]\le 2\mathfrak{C}\Big((C_p)^{\frac{1}{p}}\delta_i(x^i_n)+C_p\big(\delta_i(x^i_n)\big)^p\Big)<\varepsilon/2. \tag{6.73}$$

Since $\big|\mathfrak{X}_{t_i}(\omega)-x^i_n\big|\vee\big|\mathfrak{Y}_{t_i}(\omega)-y^i_n\big|<\delta_i(x^i_n)$, applying (6.66) with $(\mathfrak{x},\mathfrak{y})=(x^i_n,y^i_n)$ and $(\mathfrak{x}',\mathfrak{y}')=\big(\mathfrak{X}_{t_i}(\omega),\mathfrak{Y}_{t_i}(\omega)\big)$, we can deduce from (6.69), (6.73) and (6.68) that

$$\begin{aligned}E_t\Big[R\big(t,x,\tau^i_n(\Pi^t_{t_i})\big)\big|\overline{\mathcal{F}}^t_{t_i}\Big](\omega) &> E_{t_i}\big[\mathcal{R}(t_i,x^i_n,\tau^i_n)\big]+\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr-\varepsilon/2\ge\mathcal{V}\big(t_i,x^i_n,y^i_n\big)+\int_t^{t_i}f\big(r,\mathfrak{X}_r(\omega)\big)dr-\frac{3}{4}\varepsilon\\ &\ge \mathcal{V}\big(t_i,\mathfrak{X}_{t_i}(\omega),\mathfrak{Y}_{t_i}(\omega)\big)+\int_t^{t_i}f(r,\mathfrak{X}_r(\omega))dr-\varepsilon.\end{aligned}$$

Taking expectation $E_t[\cdot]$ over $\mathcal{A}^i_n$ yields that

$$\begin{aligned}E_t\big[\mathbf{1}_{\mathcal{A}^i_n}\mathcal{R}(t,x,\overline{\tau})\big] &= E_t\Big[\mathbf{1}_{\mathcal{A}^i_n}R\big(t,x,\tau^i_n(\Pi^t_{t_i})\big)\Big]=E_t\Big[\mathbf{1}_{\mathcal{A}^i_n}E_t\Big[R\big(t,x,\tau^i_n(\Pi^t_{t_i})\big)\big|\overline{\mathcal{F}}^t_{t_i}\Big]\Big]\\ &\ge E_t\Big[\mathbf{1}_{\mathcal{A}^i_n}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big)+\int_t^\zeta f(r,\mathfrak{X}_r)dr-\varepsilon\Big)\Big].\end{aligned}$$

Similar to (6.64), taking summation up over $i,n\in\mathbb{N}$, we can deduce from (6.71), (6.63), the first inequality in (4.1) and the dominated convergence theorem that

$$E_t\big[\mathbf{1}_{\{\tau>\zeta\}}\mathcal{R}(t,x,\overline{\tau})\big]\ge E_t\Big[\mathbf{1}_{\{\tau>\zeta\}}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big)+\int_t^\zeta f(r,\mathfrak{X}_r)dr-\varepsilon\Big)\Big].$$

It thus follows that $\mathcal{V}(t,x,y+\varepsilon)\ge E_t\big[\mathcal{R}(t,x,\overline{\tau})\big]\ge E_t\Big[\mathbf{1}_{\{\tau\le\zeta\}}\mathcal{R}(t,x,\tau)+\mathbf{1}_{\{\tau>\zeta\}}\big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)+\int_t^\zeta f(r,\mathfrak{X}_r)dr\big)\Big]-\varepsilon$. As $\varepsilon\to\infty$, the second inequality in (6.55) follows from the continuity of $\mathcal{V}$ in y $\big($i.e. (4.5) and Theorem 2.1 (1)$\big)$. □

Proof of Theorem 4.1: Fix $t\in[0,\infty)$.
1) Let $(x,y)\in\mathbb{R}^l\times[0,\infty)$ and let $\{\zeta(\tau)\}_{\tau\in\mathcal{T}^t_x(y)}$ be a family of $\overline{\mathcal{T}}^t_\sharp$−stopping times. For any $\tau\in\mathcal{T}^t_x(y)$, taking $\zeta=\zeta(\tau)$ in (6.55) yields that

$$E_t\big[\mathcal{R}(t,x,\tau)\big]\le E_t\Big[\mathbf{1}_{\{\tau\le\zeta(\tau)\}}\mathcal{R}(t,x,\tau)+\mathbf{1}_{\{\tau>\zeta(\tau)\}}\Big(\mathcal{V}(\zeta(\tau),\mathcal{X}^{t,x}_{\zeta(\tau)},\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)})+\int_t^{\zeta(\tau)}f(r,\mathcal{X}^{t,x}_r)dr\Big)\Big]\le\mathcal{V}(t,x,y).$$

Taking supremum over $\tau\in\mathcal{T}^t_x(y)$ $\big($or taking supremum over $\widehat{\tau}\in\mathcal{T}^t_x(y)$ if $y>0\big)$, we can deduce (1.3) from (4.4).
2) Next, assume that $\mathcal{V}(s,x,y)$ is continuous in $(s,x,y)\in[t,\infty)\times\mathbb{R}^l\times(0,\infty)$.

We fix $(x,y)\in\mathbb{R}^l\times[0,\infty)$ and a family $\{\zeta(\tau)\}_{\tau\in\mathcal{T}^t_x(y)}$ of $\overline{\mathcal{T}}^t$−stopping times. Let $\tau\in\mathcal{T}^t_x(y)$, $n\in\mathbb{N}$ and define

$$\zeta_n=\zeta_n(\tau):=\mathbf{1}_{\{\zeta(\tau)=t\}}t+\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\zeta(\tau)\in(t+(i-1)2^{-n},t+i2^{-n}]\}}(t+i2^{-n})\in\overline{\mathcal{T}}^t.$$

Applying (6.55) with $\zeta=\zeta_n$ yields that

$$E_t\big[\mathcal{R}(t,x,\tau)\big]\le E_t\Big[\mathbf{1}_{\{\tau\le\zeta_n\}}\mathcal{R}(t,x,\tau)+\mathbf{1}_{\{\tau>\zeta_n\}}\Big(\mathcal{V}\big(\zeta_n,\mathcal{X}^{t,x}_{\zeta_n},\mathcal{Y}^{t,x,\tau}_{\zeta_n}\big)+\int_t^{\zeta_n}f(r,\mathcal{X}^{t,x}_r)dr\Big)\Big]\le\mathcal{V}(t,x,y). \tag{6.74}$$

An analogy to (6.63) shows that

$$\big|\mathcal{V}\big(\zeta_n,\mathcal{X}^{t,x}_{\zeta_n},\mathcal{Y}^{t,x,\tau}_{\zeta_n}\big)\big|+\int_t^{\zeta_n}\big|f(r,\mathcal{X}^{t,x}_r)\big|dr\le 2\mathfrak{C}(3+C_p)+\mathfrak{C}(1+2C_p)\mathfrak{X}^p_*\in L^1\big(\overline{\mathcal{F}}^t\big). \tag{6.75}$$

We claim that $\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)}>0$, P_t-a.s. on $\{\tau>\zeta(\tau)\}$. To see it, we set $A:=\{\tau>\zeta(\tau)\}\cap\big\{\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)}=0\big\}\in\overline{\mathcal{F}}^t_{\zeta(\tau)}$ and can deduce that

$$\begin{aligned}0&=E_t\Big[\mathbf{1}_A\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)}\Big]=E_t\bigg[\mathbf{1}_A\Big(E_t\Big[\int_t^\tau g(r,\mathcal{X}^{t,x}_r)dr\Big|\overline{\mathcal{F}}^t_{\zeta(\tau)}\Big]-\int_t^{\zeta(\tau)}g(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg]\\&=E_t\bigg[E_t\Big[\mathbf{1}_A\int_{\zeta(\tau)}^\tau g(r,\mathcal{X}^{t,x}_r)dr\Big|\overline{\mathcal{F}}^t_{\zeta(\tau)}\Big]\bigg]=E_t\Big[\mathbf{1}_A\int_{\zeta(\tau)}^\tau g(r,\mathcal{X}^{t,x}_r)dr\Big],\end{aligned}$$

which implies that $\mathbf{1}_A\int_{\zeta(\tau)}^\tau g(r,\mathcal{X}^{t,x}_r)dr=0$, P_t-a.s. It follows from the strict positivity of function g that $P_t(A)=0$, proving the claim. As $\lim\limits_{n\to\mathbb{N}}\downarrow\zeta_n=\zeta(\tau)$, one has $\lim\limits_{n\to\mathbb{N}}\downarrow\mathbf{1}_{\{\tau\le\zeta_n\}}=\mathbf{1}_{\{\tau\le\zeta(\tau)\}}$. The continuity of function $\mathcal{V}$ in $(s,\mathfrak{x},\mathfrak{y})\in[t,\infty)\times\mathbb{R}^l\times(0,\infty)$ and the continuity of processes $\big(\mathcal{X}^{t,x},\mathcal{Y}^{t,x,\tau}\big)$ then show that $\lim\limits_{n\to\mathbb{N}}\mathbf{1}_{\{\tau\le\zeta_n\}}\mathcal{V}\big(\zeta_n,\mathcal{X}^{t,x}_{\zeta_n},\mathcal{Y}^{t,x,\tau}_{\zeta_n}\big)=\mathbf{1}_{\{\tau\le\zeta(\tau)\}}\mathcal{V}\big(\zeta(\tau),\mathcal{X}^{t,x}_{\zeta(\tau)},\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)}\big)$, P_t-a.s.

Letting $n\to\infty$ in (6.74), we can deduce from (6.75), the first inequality in (4.1) and the dominated convergence theorem that

$$E_t\big[\mathcal{R}(t,x,\tau)\big]\le E_t\bigg[\mathbf{1}_{\{\tau\le\zeta(\tau)\}}\mathcal{R}(t,x,\tau)+\mathbf{1}_{\{\tau>\zeta(\tau)\}}\Big(\mathcal{V}\big(\zeta(\tau),\mathcal{X}^{t,x}_{\zeta(\tau)},\mathcal{Y}^{t,x,\tau}_{\zeta(\tau)}\big)+\int_t^{\zeta(\tau)}f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg]\le\mathcal{V}(t,x,y).\quad(6.76)$$

Taking supremum over $\tau\in\mathcal{T}^t_x(y)$ $\big($or taking supremum over $\widehat{\tau}\in\mathcal{T}^t_x(y)$ if $y>0\big)$, we obtain (1.3) again from (4.4). □

Proof of Proposition 4.1: Let us simply denote $\tau(t,x,\alpha)$ by τ_o. For $n\in\mathbb{N}$, an analogy to (4.9) shows that

$$\underline{\tau}_n:=\inf\big\{s\in[t,\infty):Y^{t,x,\alpha}_s=1/n\big\}\quad\text{and}\quad\overline{\tau}_n:=\inf\big\{s\in[t,\infty):Y^{t,x,\alpha}_s=-1/n\big\}$$

define two $\overline{\mathcal{T}}^t-$stopping times.

By definition, $\alpha=M-K$ for some $(M,K)\in\mathbb{M}_t\times\mathbb{K}_t$. It holds for all $\omega\in\Omega^t$ except on a P_t-null set $\mathcal{N}$ that $M_\cdot(\omega)$ is a continuous path, that $K_\cdot(\omega)$ is an continuous increasing path and that $\overline{\tau}_n(\omega)<\infty$ for any $n\in\mathbb{N}$.

1) *We first show that*

$$\lim_{n\to\infty}\uparrow\underline{\tau}_n=\lim_{n\to\infty}\downarrow\overline{\tau}_n=\tau_o\quad P_t-\text{a.s.}\qquad(6.77)$$

Let $\omega\in\mathcal{N}^c$ and set $\underline{\tau}(\omega):=\lim\limits_{n\to\infty}\uparrow\underline{\tau}_n(\omega)\le\tau_o(\omega)$. The continuity of path $Y^{t,x,\alpha}_\cdot(\omega)$ implies that $Y^{t,x,\alpha}\big(\underline{\tau}_n(\omega),\omega\big)=1/n$, $\forall\,n\in\mathbb{N}$ and thus $Y^{t,x,\alpha}\big(\underline{\tau}(\omega),\omega\big)=\lim\limits_{n\to\infty}Y^{t,x,\alpha}\big(\underline{\tau}_n(\omega),\omega\big)=0$. It follows that $\tau_o(\omega)=\underline{\tau}(\omega)=\lim\limits_{n\to\infty}\uparrow\underline{\tau}_n(\omega)$.

On the other hand, we define a $\overline{\mathcal{T}}^t-$stopping time $\overline{\tau}:=\lim\limits_{n\to\infty}\downarrow\overline{\tau}_n\ge\tau_o$ and let $\omega\in\mathcal{N}^c$. For any $n\in\mathbb{N}$, as $\overline{\tau}_n(\omega)<\infty$, the continuity of path $Y^{t,x,\alpha}_\cdot(\omega)$ again gives that $Y^{t,x,\alpha}\big(\overline{\tau}_n(\omega),\omega\big)=-1/n$. Letting $n\to\infty$ yields that $Y^{t,x,\alpha}\big(\overline{\tau}(\omega),\omega\big)=\lim\limits_{n\to\infty}Y^{t,x,\alpha}\big(\overline{\tau}_n(\omega),\omega\big)=0$.

Since M is a uniformly integrable martingale, we know from the optional sampling theorem that

$$E_t\bigg[K_{\overline{\tau}}-K_{\tau_o}+\int_{\tau_o}^{\overline{\tau}}g(r,\mathcal{X}^{t,x}_r)dr\bigg]=E_t\Big[M_{\overline{\tau}}-M_{\tau_o}-Y^{t,x,\alpha}_{\overline{\tau}}+Y^{t,x,\alpha}_{\tau_o}\Big]=0,$$

which implies $K_{\overline{\tau}}-K_{\tau_o}+\int_{\tau_o}^{\overline{\tau}}g(r,\mathcal{X}^{t,x}_r)dr=0$, P_t-a.s. Then one can deduce from the strict positivity of function g that $\tau_o=\overline{\tau}=\lim\limits_{n\to\infty}\downarrow\overline{\tau}_n$, P_t-a.s., proving (6.77).

2) Next, let $\varepsilon\in(0,1)$ and set $\varepsilon_o:=(4+10\mathfrak{C})^{-1}\varepsilon$. As $\mathfrak{M}:=E_t\big[(\mathcal{X}^{t,x}_*)^p\big]<\infty$ by the first inequality in (3.4), we can find $\lambda_o=\lambda_o(t,x,\varepsilon)\in\big(0,\varepsilon_o\big)$ such that

$$E_t\big[\mathbf{1}_A\,(\mathcal{X}^{t,x}_*)^p\big]<\varepsilon_o\ \text{ for any }A\in\overline{\mathcal{F}}^t\text{ with }P_t(A)<\lambda_o\,.\qquad(6.78)$$

There exists $R=R(t,x,\varepsilon)\in(0,\infty)$ such that the set $A_R:=\big\{\mathcal{X}^{t,x}_*>R\big\}\in\overline{\mathcal{F}}^t$ satisfies $P_t(A_R)<\lambda_o/2$.

Let $\lambda=\lambda(t,x,\varepsilon)\in(0,1)$ satisfy that

$$\lambda\le\frac{\varepsilon_o}{(2+\mathfrak{M})\|c(\cdot)\|}\wedge\rho^{-1}(\varepsilon_o)\quad\text{and}\qquad(6.79)$$

$$(C_p)^{\frac1p}\big(1+|x|\big)\big(\|c(\cdot)\|\lambda+\|c(\cdot)\|^{\frac12}\lambda^{\frac12}\big)+C_p\big(1+|x|^p\big)\big(\|c(\cdot)\|^p\lambda^p+\|c(\cdot)\|^{\frac p2}\lambda^{\frac p2}\big)\le\varepsilon_o.\qquad(6.80)$$

We pick up $\delta=\delta(t,x,\varepsilon)\in\Big(0,\frac{1}{\mathfrak{C}\mathfrak{n}}\big(\frac{\lambda_o}{2C_p}\big)^{\frac{1}{p}}\Big)$ such that

$$\mathfrak{C}(C_p)^{\frac{1}{p}}\delta+\mathfrak{C}C_p\delta^p\le\lambda\wedge\varepsilon_o, \tag{6.81}$$

Set $\Omega_n:=\{\tau_o-\lambda\le\underline{\tau}_n\le\overline{\tau}_n\le\tau_o+\lambda\}\in\overline{\mathcal{F}}^t$ for any $n\in\mathbb{N}$. As (6.77) implies that $P_t\big(\underset{n\in\mathbb{N}}{\cup}\Omega_n\big)=1$, there exists $\mathfrak{n}\in\mathbb{N}$ such that $P_t(\Omega_{\mathfrak{n}})>1-\lambda_o/2$.

Now, fix $x'\in\overline{O}_\delta(x)$ and simply denote $\tau(t,x',\alpha)$ by τ'. We define $A':=\big\{(\mathcal{X}^{t,x'}-\mathcal{X}^{t,x})_*\le(\mathfrak{C}\mathfrak{n})^{-1}\big\}\in\overline{\mathcal{F}}^t$. The second inequality in (3.4) shows that

$$P_t\big((A')^c\big)=\mathfrak{C}^p\mathfrak{n}^pE_t\big[(\mathcal{X}^{t,x'}-\mathcal{X}^{t,x})_*^p\big]\le C_p\mathfrak{C}^p\mathfrak{n}^p|x'-x|^p\le C_p\mathfrak{C}^p\mathfrak{n}^p\delta^p<\lambda_o/2.$$

So the set $\mathcal{A}:=A'\cap\Omega_{\mathfrak{n}}\in\overline{\mathcal{F}}^t$ satisfies that $P_t(\mathcal{A}^c)=P_t\big((A')^c\cup\Omega_{\mathfrak{n}}^c\big)\le P_t\big((A')^c\big)+P_t\big(\Omega_{\mathfrak{n}}^c\big)<\lambda_o<\varepsilon_o$.

Let $\omega\in\mathcal{A}$. Since it holds for any $s\in[t,\infty)$ that

$$\begin{aligned}\big|Y_s^{t,x',\alpha}(\omega)-Y_s^{t,x,\alpha}(\omega)\big| &\le \int_t^s\big|g\big(r,\mathcal{X}_r^{t,x'}(\omega)\big)-g\big(r,\mathcal{X}_r^{t,x}(\omega)\big)\big|dr\le\int_t^s c(r)\Big(\big|\mathcal{X}_r^{t,x'}-\mathcal{X}_r^{t,x}\big|\vee\big|\mathcal{X}_r^{t,x'}-\mathcal{X}_r^{t,x}\big|^p\Big)(\omega)dr\\ &\le \big(\mathfrak{C}\mathfrak{n}\big)^{-1}\int_t^\infty c(r)dr\le 1/\mathfrak{n},\end{aligned}$$

we see that

$$Y_s^{t,x',\alpha}(\omega)\ge Y_s^{t,x,\alpha}(\omega)-1/\mathfrak{n}>0,\ \ \forall\,s\in\big[t,\underline{\tau}_n(\omega)\big)\quad\text{and}\quad Y_s^{t,x,\alpha}(\omega)\ge Y_s^{t,x',\alpha}(\omega)-1/\mathfrak{n}>-1/\mathfrak{n},\ \ \forall\,s\in\big[t,\tau'(\omega)\big).$$

The former implies that $\tau'(\omega)\ge\underline{\tau}_n(\omega)$ while the latter means that $\overline{\tau}_n(\omega)\ge\tau'(\omega)$. In summary,

$$\tau_o-\lambda\le\underline{\tau}_{\mathfrak{n}}\le\tau'\le\overline{\tau}_{\mathfrak{n}}\le\tau_o+\lambda\ \text{ on }\ \mathcal{A}. \tag{6.82}$$

By an analogy to (6.24) and (6.25), we can deduce from (6.9), (6.10), (6.78), (6.82) and (6.79) that

$$\begin{aligned}E_t\bigg[\int_{\tau_o\wedge\tau'}^{\tau_o\vee\tau'}\big|f(r,\mathcal{X}_r^{t,x})\big|dr\bigg] &\le E_t\bigg[\big(2+(\mathcal{X}_*^{t,x})^p\big)\Big(\mathbf{1}_{\mathcal{A}^c}\int_t^\infty c(r)dr+\mathbf{1}_{\mathcal{A}}\|c(\cdot)\|\big|\tau'-\tau_o\big|\Big)\bigg]\\ &<\mathfrak{C}\big(2P_t(\mathcal{A}^c)+\varepsilon_o\big)+\lambda(2+\mathfrak{M})\|c(\cdot)\|<(1+3\mathfrak{C})\varepsilon_o\,, \end{aligned}\tag{6.83}$$

$$\text{and}\quad E_t\big[\mathbf{1}_{\mathcal{A}^c}\big|\pi\big(\tau',\mathcal{X}_{\tau'}^{t,x}\big)-\pi\big(\tau_o,\mathcal{X}_{\tau_o}^{t,x}\big)\big|\big]\le 2\mathfrak{C}E_t\big[\mathbf{1}_{\mathcal{A}^c}\big(2+(\mathcal{X}_*^{t,x})^p\big)\big]<2\mathfrak{C}\big(2P_t(\mathcal{A}^c)+\varepsilon_o\big)<6\mathfrak{C}\varepsilon_o\,. \tag{6.84}$$

And similar to (6.26), Hölder's inequality, (1.9), (6.79), (6.82), (3.5) and (6.80) imply that

$$\begin{aligned}&E_t\big[\mathbf{1}_{\mathcal{A}}\big|\pi\big(\tau',\mathcal{X}_{\tau'}^{t,x}\big)-\pi\big(\tau_o,\mathcal{X}_{\tau_o}^{t,x}\big)\big|\big]\le E_t\big[\mathbf{1}_{\mathcal{A}}\rho\big(|\tau'-\tau_o|\big)\big]+\mathfrak{C}E_t\Big[\mathbf{1}_{\mathcal{A}}\Big(\big|\mathcal{X}_{\tau'}^{t,x}-\mathcal{X}_{\tau_o}^{t,x}\big|+\big|\mathcal{X}_{\tau'}^{t,x}-\mathcal{X}_{\tau_o}^{t,x}\big|^p\Big)\Big]\\ &\le\rho(\lambda)+\mathfrak{C}\bigg\{E_t\bigg[\mathbf{1}_{\mathcal{A}}\sup_{r\in(0,\lambda]}\big|\mathcal{X}_{\tau'\wedge\tau_o+r}^{t,x}-\mathcal{X}_{\tau'\wedge\tau_o}^{t,x}\big|^p\bigg]\bigg\}^{\frac{1}{p}}+\mathfrak{C}E_t\bigg[\mathbf{1}_{\mathcal{A}}\sup_{r\in(0,\lambda]}\big|\mathcal{X}_{\tau'\wedge\tau_o+r}^{t,x}-\mathcal{X}_{\tau'\wedge\tau_o}^{t,x}\big|^p\bigg]\\ &\le\varepsilon_o+\mathfrak{C}(C_p)^{\frac{1}{p}}\big(1+|x|\big)\big(\|c(\cdot)\|\lambda+\|c(\cdot)\|^{\frac{1}{2}}\lambda^{\frac{1}{2}}\big)+\mathfrak{C}C_p\big(1+|x|^p\big)\big(\|c(\cdot)\|^p\lambda^p+\|c(\cdot)\|^{\frac{p}{2}}\lambda^{\frac{p}{2}}\big)\le(1+\mathfrak{C})\varepsilon_o\,. \end{aligned}\tag{6.85}$$

Combining (6.83), (6.84) and (6.85) yields that

$$E_t\big[\big|\mathcal{R}(t,x,\tau')-\mathcal{R}(t,x,\tau_o)\big|\big]\le E_t\bigg[\int_{\tau_o\wedge\tau'}^{\tau_o\vee\tau'}|f(r,\mathcal{X}_r^{t,x})|dr+\Big|\pi\big(\tau_o,\mathcal{X}_{\tau_o}^{t,x}\big)-\pi\big(\tau',\mathcal{X}_{\tau'}^{t,x}\big)\Big|\bigg]<(2+10\mathfrak{C})\varepsilon_o\,,$$

which together with (2.8) and (6.81) leads to that

$$E_t\big[\big|\mathcal{R}(t,x',\tau')-\mathcal{R}(t,x,\tau_o)\big|\big]\le E_t\big[\big|\mathcal{R}(t,x',\tau')-\mathcal{R}(t,x,\tau')\big|\big]+E_t\big[\big|\mathcal{R}(t,x,\tau')-\mathcal{R}(t,x,\tau_o)\big|\big]<(4+10\mathfrak{C})\varepsilon_o=\varepsilon\,. \qquad \square$$

Proof of Proposition 4.2: Let $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty)$.

1) Let $\alpha\in\mathfrak{A}_t(y)$. Since $\tau(t,x,\alpha)<\infty$, P_t-a.s. by (4.9), the continuity of process $Y^{t,x,\alpha}$ implies that

$$\alpha_{\tau(t,x,\alpha)}=\int_t^{\tau(t,x,\alpha)}g(r,\mathcal{X}_r^{t,x})dr,\quad P_t-\text{a.s.} \tag{6.86}$$

One can then deduce from the uniform integrability of the $(\overline{\mathbf{F}}^t, P_t)-$supermartingale α and the optional sampling theorem that $E_t\Big[\int_t^{\tau(t,x,\alpha)} g(r,\mathcal{X}_r^{t,x})dr\Big]=E_t\big[\alpha_{\tau(t,x,\alpha)}\big]\le E_t[\alpha_t]=y$, namely, $\tau(t,x,\alpha)\in\mathcal{T}_x^t(y)$. As $Y_t^{t,x,\alpha}=\alpha_t=y>0$, P_t-a.s., we also derive from the continuity of process $Y^{t,x,\alpha}$ that $\tau(t,x,\alpha)>t$, P_t-a.s. Thus $\alpha\to\tau(t,x,\alpha)$ is a mapping from $\mathfrak{A}_t(y)$ to $\widehat{\mathcal{T}}_x^t(y)$.

2) Next, let $\tau\in\widehat{\mathcal{T}}_x^t(y)$ and set $\delta:=y-E_t\big[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\big]\ge 0$. Clearly, $M_s:=\delta+E_t\Big[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\Big|\overline{\mathcal{F}}_s^t\Big]\ge 0$, $s\in[t,\infty)$ is a uniformly integrable continuous martingale with respect to $(\overline{\mathbf{F}}^t,P_t)$, i.e., $M\in\mathbb{M}_t$.

Define $J_s:=\inf\limits_{s'\in[t,s]}E_t\big[\tau-t|\overline{\mathcal{F}}_{s'}^t\big]$, $s\in[t,\infty)$ and let $\mathcal{N}$ be the $P-$null set such that for any $\omega\in\mathcal{N}^c$, the path $E_t\big[\tau-t|\overline{\mathcal{F}}_\cdot^t\big](\omega)$ is continuous and $E_t\big[\tau-t|\overline{\mathcal{F}}_s^t\big](\omega)\ge 0$, $\forall\, s\in[t,s]\cap\mathbb{Q}$. For any $\omega\in\mathcal{N}^c$, we can deduce that $J_\cdot(\omega)$ is a nonnegative, continuous decreasing process. Given $s\in[t,\infty)$, set $\xi_s:=\inf\limits_{s'\in[t,s]\cap\mathbb{Q}}E_t\big[\tau-t|\overline{\mathcal{F}}_{s'}^t\big]$, which is $\overline{\mathcal{F}}_s^t-$measurable random variable. The continuity of process $E_t\big[\tau-t|\overline{\mathcal{F}}_s^t\big]$, $s\in[t,\infty)$ shows that $J_s=\xi_s$ on $\mathcal{N}^c$, so J_s is also $\overline{\mathcal{F}}_s^t-$measurable. It follows that

$$K_s:=\delta\bigg[1\wedge\Big(\frac{s-t}{J_s}\Big)^+\bigg]\in[0,\delta],\quad s\in[t,\infty) \tag{6.87}$$

is an $\overline{\mathbf{F}}^t-$adapted continuous increasing process. Since $\tau>t$, P_t-a.s., one has $J_t=E_t\big[\tau-t|\overline{\mathcal{F}}_t^t\big]=E_t[\tau-t]>0$, P_t-a.s. and thus $K_t=0$, P_t-a.s. To wit, $K\in\mathbb{K}_t$.

Set $\alpha:=M-K$. It is clear that

$$\alpha_s=M_s-K_s\ge\delta+E_t\bigg[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\Big|\overline{\mathcal{F}}_s^t\bigg]-\delta=E_t\bigg[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\Big|\overline{\mathcal{F}}_s^t\bigg],\quad\forall\, s\in[t,\infty).$$

As $\alpha_t=M_t-K_t=\delta+E_t\big[\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr\big]+0=y$, P_t-a.s., we see that $\alpha\in\mathfrak{A}_t(y)$. Since $J_\tau\le E_t\big[(\tau-t)|\overline{\mathcal{F}}_\tau^t\big]=\tau-t$, P_t-a.s., one has $K_\tau=\delta$, P_t-a.s. and thus

$$\alpha_\tau=M_\tau-\delta=\int_t^\tau g(r,\mathcal{X}_r^{t,x})dr,\quad P_t-\text{a.s.} \tag{6.88}$$

This shows $\tau(t,x,\alpha)\le\tau$, P_t-a.s. On the other hand, subtracting (6.86) from (6.88) and applying the optional sampling theorem to α again yield that $0\le E_t\big[\int_{\tau(t,x,\alpha)}^\tau g(r,\mathcal{X}_r^{t,x})dr\big]=E_t\big[\alpha_\tau-\alpha_{\tau(t,x,\alpha)}\big]\le 0$. The strict positivity of function g then implies that $\tau(t,x,\alpha)=\tau$, P_t-a.s. □

Similar to Lemma 6.1, the following auxiliary result is crucial for proving the second DPP of $\mathcal{V}$ (Theorem 4.2).

Lemma 6.2. *Given $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times(0,\infty)$, let $\alpha\in\mathfrak{A}_t(y)$ and let $\zeta\in\overline{\mathcal{T}}_\sharp^t$. Then*

$$\begin{aligned}E_t\big[\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big]&\le E_t\bigg[\mathbf{1}_{\{\tau(t,x,\alpha)\le\zeta\}}\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)+\mathbf{1}_{\{\tau(t,x,\alpha)>\zeta\}}\Big(\mathcal{V}\big(\zeta,\mathcal{X}_\zeta^{t,x},Y_\zeta^{t,x,\alpha}\big)+\int_t^\zeta f(r,\mathcal{X}_r^{t,x})dr\Big)\bigg]\\&\le\mathcal{V}(t,x,y).\end{aligned} \tag{6.89}$$

Proof: Suppose that $\alpha=M-K$ for some $(M,K)\in\mathbb{M}_t\times\mathbb{K}_t$. We denote $(\mathfrak{X},\mathfrak{Y},\widehat{\tau}):=\big(\mathcal{X}^{t,x},Y^{t,x,\alpha},\tau(t,x,\alpha)\big)$ and let ζ take values in a countable subset $\{t_i\}_{i\in\mathbb{N}}$ of $[t,\infty)$.

1) *Let us start with the first inequality in* (6.89).

Since α is a uniformly integrable continuous supermartingales with respect to $(\overline{\mathbf{F}}^t,P_t)$, one has $\alpha_{\widehat{\tau}}=\int_t^{\widehat{\tau}}(r,\mathfrak{X}_r)dr$ and the optional sampling theorem implies that

$$\begin{aligned}\mathfrak{Y}_{\widehat{\tau}\wedge\zeta}&=\alpha_{\widehat{\tau}\wedge\zeta}-\int_t^{\widehat{\tau}\wedge\zeta}g(r,\mathfrak{X}_r)dr\ge E_t\big[\alpha_{\widehat{\tau}}|\overline{\mathcal{F}}_{\widehat{\tau}\wedge\zeta}^t\big]-\int_t^{\widehat{\tau}\wedge\zeta}g(r,\mathfrak{X}_r)dr\\&=E_t\Big[\int_t^{\widehat{\tau}}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}_{\widehat{\tau}\wedge\zeta}^t\Big]-\int_t^{\widehat{\tau}\wedge\zeta}g(r,\mathfrak{X}_r)dr=\mathcal{Y}_{\widehat{\tau}\wedge\zeta}^{t,x,\widehat{\tau}},\quad P_t-\text{a.s.}\end{aligned} \tag{6.90}$$

As $\widehat{\tau}\in\widehat{\mathcal{T}}_x^t(y)$ by Proposition 4.2, we see from (6.55) that

$$\begin{aligned}&E_t\bigg[\mathbf{1}_{\{\widehat{\tau}\le\zeta\}}\mathcal{R}\big(t,x,\widehat{\tau}\big)+\mathbf{1}_{\{\widehat{\tau}>\zeta\}}\Big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)+\int_t^\zeta f(r,\mathfrak{X}_r)dr\Big)\bigg]\\&\qquad\ge E_t\bigg[\mathbf{1}_{\{\widehat{\tau}\le\zeta\}}\mathcal{R}\big(t,x,\widehat{\tau}\big)+\mathbf{1}_{\{\widehat{\tau}>\zeta\}}\Big(\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathcal{Y}_\zeta^{t,x,\widehat{\tau}})+\int_t^\zeta f(r,\mathfrak{X}_r)dr\Big)\bigg]\ge E_t\big[\mathcal{R}(t,x,\widehat{\tau})\big],\end{aligned}$$

proving the first inequality in (6.89).

2) *The proof of the second inequality in* (6.89) *is relatively lengthy, we split it into several steps.*

By an analogy to (6.12), we must have either $P_t\{\zeta=t\}=1$ or $P_t\{\zeta>t\}=1$. If $P_t\{\zeta=t\}=1$, as $\mathfrak{Y}_t=\alpha_t=y>0$, P_t-a.s., one has $\widehat{\tau}=\tau(t,x,\alpha)>t=\zeta$, P_t-a.s. Then

$$E_t\bigg[\mathbf{1}_{\{\widehat{\tau}\le\zeta\}}\mathcal{R}\big(t,x,\widehat{\tau}\big)+\mathbf{1}_{\{\widehat{\tau}>\zeta\}}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big)+\int_t^\zeta f(r,\mathfrak{X}_r)dr\Big)\bigg]=E_t\big[\mathcal{V}(t,\mathfrak{X}_t,\mathfrak{Y}_t)\big]=E_t\big[\mathcal{V}(t,x,y)\big]=\mathcal{V}(t,x,y).$$

So let us suppose that $t_1>t$ in the rest of this proof. There exists a P_t-null set $\mathcal{N}$ such that for any $\omega\in\mathcal{N}^c$, $M_\cdot(\omega)$ is a continuous path and $K_\cdot(\omega)$ is an continuous increasing path. By the uniform integrability of M, there exists $\xi\in L^1\big(\overline{\mathcal{F}}^t\big)$ such that P_t-a.s.

$$M_s=E_t\big[\xi\big|\overline{\mathcal{F}}^t_s\big],\quad \forall\, s\in[t,\infty). \tag{6.91}$$

For any $i\in\mathbb{N}$, similar to (6.56) and (6.57), it holds for all $\omega\in\Omega^t$ except on a P_t-null set $\mathcal{N}_i$ that

$$\mathcal{N}^i_\omega:=\Big\{\widetilde{\omega}\in\Omega^{t_i}\colon \mathfrak{X}_s(\omega\otimes_{t_i}\widetilde{\omega})\neq\mathfrak{X}_s(\omega)\text{ for some } s\in[t,t_i] \text{ or } \mathfrak{X}_r(\omega\otimes_{t_i}\widetilde{\omega})\neq\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r(\widetilde{\omega})\text{ for some } r\in[t_i,\infty)\Big\}\in\mathscr{N}^{t_i}. \tag{6.92}$$

2a) Fix $\varepsilon\in(0,1)$. The first inequality in (4.1) and an analogy to (6.63) show that

$$E_t\Big[\big|\mathcal{R}(t,x,\widehat{\tau})\big|+\big|\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)\big|+\int_t^\zeta\big|f(r,\mathfrak{X}_r)\big|dr\Big]\le\Psi(x)+2\mathfrak{C}(3+C_p)+\mathfrak{C}(1+2C_p)E_t\big[\mathfrak{X}^p_*\big]<\infty.$$

So there exists $\lambda=\lambda(t,x,\alpha,\varepsilon)\in(0,1)$ such that

$$E_t\bigg[\mathbf{1}_A\Big(\big|\mathcal{R}(t,x,\widehat{\tau})\big|+\big|\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)\big|+\int_t^\zeta\big|f(r,\mathfrak{X}_r)\big|dr\Big)\bigg]<\varepsilon/5\ \text{ for any } A\in\overline{\mathcal{F}}^t \text{ with } P_t(A)<\lambda\,. \tag{6.93}$$

We can find $\mathcal{I}_o\in\mathbb{N}$ such that $P_t\{\zeta>t_{\mathcal{I}_o}\}<\lambda/2$.

Let $i=1,\cdots,\mathcal{I}_o$ and $(\mathfrak{x},\mathfrak{y})\in\mathbb{R}^l\times(0,\infty)$. In light of (4.5) and Theorem 2.1 (1), there exists $\delta_i(\mathfrak{x},\mathfrak{y})\in\big(0,1\wedge\mathfrak{y}\wedge\varepsilon\big)$ such that

$$\big|\mathcal{V}(t_i,\mathfrak{x}',\mathfrak{y}')-\mathcal{V}(t_i,\mathfrak{x},\mathfrak{y})\big|\le\varepsilon/5,\quad \forall\,(\mathfrak{x}',\mathfrak{y}')\in\overline{O}_{\delta_i(\mathfrak{x},\mathfrak{y})}(\mathfrak{x})\times\big[\mathfrak{y}-\delta_i(\mathfrak{x},\mathfrak{y}),\mathfrak{y}+\delta_i(\mathfrak{x},\mathfrak{y})\big]. \tag{6.94}$$

By (4.10), there exists $\alpha(t_i,\mathfrak{x},\mathfrak{y})\in\mathfrak{A}_{t_i}\big(\mathfrak{y}-\delta_i(\mathfrak{x},\mathfrak{y})\big)$ such that

$$\mathcal{V}\big(t_i,\mathfrak{x},\mathfrak{y}-\delta_i(\mathfrak{x},\mathfrak{y})\big)=\sup_{\widetilde{\alpha}\in\mathfrak{A}_{t_i}(\mathfrak{y}-\delta_i(\mathfrak{x},\mathfrak{y}))}E_{t_i}\Big[\mathcal{R}\big(t_i,\mathfrak{x},\tau(t_i,\mathfrak{x},\widetilde{\alpha})\big)\Big]\le E_{t_i}\Big[\mathcal{R}\big(t_i,\mathfrak{x},\tau(t_i,\mathfrak{x},\alpha(t_i,\mathfrak{x},\mathfrak{y}))\big)\Big]+\varepsilon/5, \tag{6.95}$$

and Proposition 4.1 shows that for some $\widehat{\delta}_i(\mathfrak{x},\mathfrak{y})\in\big(0,\delta_i(\mathfrak{x},\mathfrak{y})\big]$

$$E_{t_i}\Big[\Big|\mathcal{R}\big(t_i,\mathfrak{x}',\tau(t_i,\mathfrak{x}',\alpha(t_i,\mathfrak{x},\mathfrak{y}))\big)-\mathcal{R}\big(t_i,\mathfrak{x},\tau(t_i,\mathfrak{x},\alpha(t_i,\mathfrak{x},\mathfrak{y}))\big)\Big|\Big]\le\varepsilon/5,\quad \forall\,\mathfrak{x}'\in\overline{O}_{\widehat{\delta}_i(\mathfrak{x},\mathfrak{y})}(\mathfrak{x}). \tag{6.96}$$

Let us simply write $\mathcal{O}_i(\mathfrak{x},\mathfrak{y})$ for the open set $O_{\widehat{\delta}_i(\mathfrak{x},\mathfrak{y})}(\mathfrak{x})\times\big(\mathfrak{y}-\widehat{\delta}_i(\mathfrak{x},\mathfrak{y}),\mathfrak{y}+\widehat{\delta}_i(\mathfrak{x},\mathfrak{y})\big)$.

Since (4.8) implies that $\mathfrak{Y}_{t_i}(\omega)>0$ for any $\omega\in\{\widehat{\tau}>t_i\}$, one has

$$P_t\{\widehat{\tau}>t_i\}=P_t\big(\{\widehat{\tau}>t_i\}\cap\big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\in\mathbb{R}^l\times(0,\infty)\big\}\big)=\lim_{R\to\infty}\uparrow P_t\big(\{\widehat{\tau}>t_i\}\cap\big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\in\overline{O}_R(0)\times[R^{-1},R]\big\}\big).$$

So there exists $R_i\in(0,\infty)$ such that

$$P_t\big(\{\widehat{\tau}>t_i\}\cap\big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\notin\overline{O}_{R_i}(0)\times[R_i^{-1},R_i]\big\}\big)\le\frac{\lambda}{2^{i+1}}, \tag{6.97}$$

and we can find a finite subset $\big\{(x^i_n,y^i_n)\big\}_{n=1}^{\mathfrak{n}_i}$ of $\overline{O}_{R_i}(0)\times[R_i^{-1},R_i]$ such that $\bigcup\limits_{n=1}^{\mathfrak{n}_i}\mathcal{O}_i(x^i_n,y^i_n)\supset\overline{O}_{R_i}(0)\times[R_i^{-1},R_i]$.

Let $n=1,\cdots,\mathfrak{n}_i$ and define $A^i_n:=\{\widehat{\tau}>\zeta=t_i\}\cap\big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\in\mathcal{O}_i(x^i_n,y^i_n)\big\}\in\overline{\mathcal{F}}^t_{t_i}$. Clearly,

$$\mathfrak{Y}_{t_i}(\omega)-y^i_n\in\big(-\widehat{\delta}_i(x^i_n,y^i_n),\widehat{\delta}_i(x^i_n,y^i_n)\big)\subset\big(-\delta_i(x^i_n,y^i_n),\delta_i(x^i_n,y^i_n)\big),\quad \forall\,\omega\in A^i_n.$$

We also set $\mathcal{A}^i_n := A^i_n \Big\backslash \Big(\underset{n'<n}{\cup} A^i_{n'} \Big) \in \overline{\mathcal{F}}^t_{t_i}$ and define a $\overline{\mathcal{F}}^t_{t_i}-$measurable random variable $\eta^i_n := \mathbf{1}_{\mathcal{A}^i_n} \big(\mathfrak{Y}_{t_i} - y^i_n + \delta_i(x^i_n, y^i_n) \big) \in \big[0, 2\delta_i(x^i_n, y^i_n)\big)$. Suppose that $\alpha^{i,n} := \alpha(t_i, x^i_n, y^i_n)$ equals to $M^{i,n} - K^{i,n}$ for some $\big(M^{i,n}, K^{i,n}\big) \in \mathbb{M}_{t_i} \times \mathbb{K}_{t_i}$. By the uniform integrability of $M^{i,n}$, there exists $\xi^{i,n} \in L^1\big(\overline{\mathcal{F}}^{t_i}\big)$ such that $P_{t_i}-$a.s.

$$M^{i,n}_s = E_{t_i}\big[\xi^{i,n} \big| \overline{\mathcal{F}}^{t_i}_s\big], \quad \forall\, s \in [t_i, \infty). \tag{6.98}$$

Let $\mathcal{N}^{i,n}$ be the $P_{t_i}-$null set such that for any $\widetilde{\omega} \in (\mathcal{N}^{i,n})^c$, $M^{i,n}_\cdot(\widetilde{\omega})$ is a continuous path; $K^{i,n}_\cdot(\widetilde{\omega})$ is an continuous increasing path; and

$$\alpha^{i,n}_{t_i}(\widetilde{\omega}) = M^{i,n}_{t_i}(\widetilde{\omega}) = y^i_n - \delta_i(x^i_n, y^i_n) > 0. \tag{6.99}$$

As $(\Pi^t_{t_i})^{-1}(\mathcal{N}^{i,n})$ is a P_t-null set by Lemma A.3 (1), one can deduce from Lemma A.3 (2) that

$$\begin{aligned} &M^{i,n}_s(\Pi^t_{t_i}),\ s \in [t_i, \infty) \text{ is an } \overline{\mathbf{F}}^t\text{-adapted continuous process with } M^{i,n}_{t_i}(\Pi^t_{t_i}) = y^i_n - \delta_i(x^i_n, y^i_n),\ P_t\text{-a.s. and} \\ &K^{i,n}_s(\Pi^t_{t_i}),\ s \in [t_i, \infty) \text{ is an } \overline{\mathbf{F}}^t\text{-adapted, continuous increasing process with } K^{i,n}_{t_i}(\Pi^t_{t_i}) = 0,\ P_t\text{-a.s.} \end{aligned} \tag{6.100}$$

An analogy to (4.8) and (4.9) shows that $\nu^i_n := \inf\big\{ s \in [t_i, \infty) : \alpha^{i,n}_s(\Pi^t_{t_i}) - \int_{t_i}^s g(r, \mathfrak{X}_r) dr = 0 \big\}$ defines a $\overline{\mathcal{T}}^t-$stopping time. Since $\alpha^{i,n}_{t_i}(\Pi^t_{t_i}) > 0$, P_t-a.s. by (6.99), we see that $\nu^i_n > t_i$, P_t-a.s. and thus $E_t\big[\nu^i_n - t_i \big| \overline{\mathcal{F}}^t_{t_i}\big] > 0$, P_t-a.s.

Similar to the proof of Proposition 4.2, $J^{i,n}_s := \underset{s' \in [t_i, s]}{\inf} E_t\big[\nu^i_n - t_i \big| \overline{\mathcal{F}}^t_{s'}\big]$, $s \in [t_i, \infty)$ is an $\overline{\mathbf{F}}^t-$adapted, non-negative, continuous decreasing process such that $J^{i,n}_{t_i} = E_t\big[\nu^i_n - t_i \big| \overline{\mathcal{F}}^t_{t_i}\big] > 0$, P_t-a.s. and that $J^{i,n}_{\nu^i_n} \le E_t\Big[\nu^i_n - t_i \big| \overline{\mathcal{F}}^t_{\nu^i_n}\Big] = \nu^i_n - t_i$, P_t-a.s. Then $\mathcal{K}^{i,n}_s := \eta^i_n \Big[1 \wedge \big(\frac{s - t_i}{J^{i,n}_s} \big)^+ \Big] \ge 0$, $s \in [t_i, \infty)$ defines an $\overline{\mathbf{F}}^t-$adapted, continuous increasing process over period $[t_i, \infty)$ such that

$$\mathcal{K}^{i,n}_{t_i} = 0 \quad \text{and} \quad \mathcal{K}^{i,n}_{\nu^i_n} = \eta^i_n \ \text{ holds except on a } P_t\text{-null set } \mathcal{N}^{i,n}_K. \tag{6.101}$$

Set $\mathcal{A}_\sharp := \overset{\mathcal{I}_o}{\underset{i=1}{\cup}} \overset{\mathfrak{n}_i}{\underset{n=1}{\cup}} \mathcal{A}^i_n \in \overline{\mathcal{F}}^t_{t_{\mathcal{I}_o}}$ and $\mathcal{N}_\sharp := \mathcal{N} \cup \Big(\overset{\mathcal{I}_o}{\underset{i=1}{\cup}} \overset{\mathfrak{n}_i}{\underset{n=1}{\cup}} (\Pi^t_{t_i})^{-1}(\mathcal{N}^{i,n}) \Big) \in \mathscr{N}^t$. We claim that

$$\begin{aligned} \overline{M}_s &:= M_s + \textstyle\sum_{i=1}^{\mathcal{I}_o} \sum_{n=1}^{\mathfrak{n}_i} \mathbf{1}_{\{s \ge t_i\} \cap \mathcal{A}^i_n} \big(M^{i,n}_s(\Pi^t_{t_i}) - M_s + M_{t_i} - y^i_n + \delta_i(x^i_n, y^i_n) \big),\ s \in [t, \infty) \text{ is of } \mathbb{M}_t, \\ \text{and } \ \overline{K}_s &:= K_s + \textstyle\sum_{i=1}^{\mathcal{I}_o} \sum_{n=1}^{\mathfrak{n}_i} \mathbf{1}_{\{s \ge t_i\} \cap \mathcal{A}^i_n} \big(K^{i,n}_s(\Pi^t_{t_i}) - K_s + K_{t_i} + \mathcal{K}^{i,n}_s \big),\ s \in [t, \infty) \text{ is of } \mathbb{K}_t. \end{aligned} \tag{6.102*}$$

As $t_1 > t$ by assumption, it holds P_t-a.s. that $\overline{M}_t = M_t = y$. So $\overline{\alpha} := \overline{M} - \overline{K} \in \mathfrak{A}_t(y)$.

2b) *Setting $\overline{\tau} := \tau(t, x, \overline{\alpha})$, we next show that $\overline{\tau} = \widehat{\tau}$, P_t-a.s. on $\{\widehat{\tau} \le \zeta\} \cup \big(\{\widehat{\tau} > \zeta\} \cap \mathcal{A}^c_\sharp \big)$.*

Since (6.102) shows that

$$\big(\overline{M}_s(\omega), \overline{K}_s(\omega) \big) = \big(M_s(\omega), K_s(\omega) \big),\ \forall\, (s, \omega) \in \big([t, \infty) \times \mathcal{A}^c_\sharp \big) \cup [\![t, \zeta [\![, \tag{6.103}$$

we obtain that

$$\overline{\alpha}_s(\omega) = \alpha_s(\omega), \quad \forall\, (s, \omega) \in \big([t, \infty) \times \mathcal{A}^c_\sharp \big) \cup [\![t, \zeta [\![. \tag{6.104}$$

So for any $\omega \in \mathcal{A}^c_\sharp$, one has $\widehat{\tau}(\omega) = \big(\tau(t, x, \alpha) \big)(\omega) = \inf\Big\{ s \in [t, \infty) : \alpha_s(\omega) - \int_t^s g\big(r, \mathfrak{X}_r(\omega)\big) dr = 0 \Big\} = \inf\Big\{ s \in [t, \infty) : \overline{\alpha}_s(\omega) - \int_t^s g\big(r, \mathfrak{X}_r(\omega)\big) dr = 0 \Big\} = \big(\tau(t, x, \overline{\alpha}) \big)(\omega) = \overline{\tau}(\omega)$.

Let $\omega \in \{\widehat{\tau} \le \zeta \le \mathcal{I}_o\} \cap \mathcal{N}^c_\sharp$. By (6.104),

$$\overline{\alpha}(s, \omega) = \alpha(s, \omega), \quad \forall\, s \in [t, \zeta(\omega)). \tag{6.105}$$

If $\widehat{\tau}(\omega) < \zeta(\omega)$, one can deduce from (6.105) that

$$\begin{aligned} \widehat{\tau}(\omega) &= \inf\Big\{ s \in [t, \infty) : \alpha_s(\omega) - \int_t^s g\big(r, \mathfrak{X}_r(\omega)\big) dr = 0 \Big\} = \inf\Big\{ s \in [t, \zeta(\omega)) : \alpha_s(\omega) - \int_t^s g\big(r, \mathfrak{X}_r(\omega)\big) dr = 0 \Big\} \\ &= \inf\Big\{ s \in [t, \zeta(\omega)) : \overline{\alpha}_s(\omega) - \int_t^s g\big(r, \mathfrak{X}_r(\omega)\big) dr = 0 \Big\}, \end{aligned}$$

which implies that $\overline{\tau}(\omega)=\inf\big\{s\in[t,\infty):\overline{\alpha}_s(\omega)-\int_t^s g\big(r,\mathfrak{X}_r(\omega)\big)dr=0\big\}=\widehat{\tau}(\omega)$.

Otherwise, suppose that $\widehat{\tau}(\omega)=\zeta(\omega)$. The definition of $\tau(t,x,\alpha)$ and (6.105) show that

$$\overline{\alpha}(s,\omega)=\alpha(s,\omega)>\int_t^s g\big(r,\mathfrak{X}_r(\omega)\big)dr,\ \ \forall\, s\in[t,\zeta(\omega)) \quad \text{and} \quad \alpha\big(\zeta(\omega),\omega\big)=\int_t^{\zeta(\omega)} g\big(r,\mathfrak{X}_r(\omega)\big)dr. \tag{6.106}$$

As $\overline{M}.(\omega), M.(\omega), \overline{K}.(\omega), K.(\omega)$ are all continuous paths by the proof of (6.102), we see from (6.103) and (6.106) that

$$\overline{\alpha}\big(\zeta(\omega),\omega\big)=\big(\overline{M}-\overline{K}\big)\big(\zeta(\omega),\omega\big)=(M-K)\big(\zeta(\omega),\omega\big)=\alpha\big(\zeta(\omega),\omega\big)=\int_t^{\zeta(\omega)} g\big(r,\mathfrak{X}_r(\omega)\big)dr,$$

which means that $\overline{\tau}(\omega)=\big(\tau(t,x,\overline{\alpha})\big)(\omega)=\zeta(\omega)=\widehat{\tau}(\omega)$. Hence, we have verified that

$$\overline{\tau}=\widehat{\tau},\quad P_t\text{-a.s. on } \mathcal{A}_\sharp^c\cup\{\widehat{\tau}\le\zeta\le\mathcal{I}_o\}=\{\widehat{\tau}\le\zeta\}\cup\big(\{\widehat{\tau}>\zeta\}\cap\mathcal{A}_\sharp^c\big). \tag{6.107}$$

2c) *Let $i=1,\cdots,\mathcal{I}_o$ and $n\in 1,\cdots,\mathfrak{n}_i$. In this step, we demonstrate that*

$$E_t\big[\mathbf{1}_{\mathcal{A}_n^i}\mathcal{R}(t,x,\overline{\tau})\big]\ge E_t\Big[\mathbf{1}_{\mathcal{A}_n^i}\Big(\mathcal{V}\big(t_i,\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i}\big)+\int_t^{t_i} f(r,\mathfrak{X}_r)dr-4\varepsilon/5\Big)\Big].$$

Set $\widehat{\mathcal{N}}^{i,n}:=\{\omega\in\Omega^t:\nu_n^i(\omega)=\infty\}\in\mathscr{N}^t$ and $G_n^i:=\mathcal{A}_n^i\cap\big(\mathcal{N}_i\cup\widehat{\mathcal{N}}^{i,n}\cup\mathcal{N}_K^{i,n}\cup(\Pi_{t_i}^t)^{-1}(\mathcal{N}^{i,n})\big)^c\in\overline{\mathcal{F}}_{t_i}^t$. Let $\omega\in G_n^i$. The definition of ν_n^i shows that

$$\alpha_s^{i,n}(\Pi_{t_i}^t(\omega))>\int_{t_i}^s g\big(r,\mathfrak{X}_r(\omega)\big)dr,\quad \forall\, s\in\big[t_i,\nu_n^i(\omega)\big) \quad \text{and} \quad \alpha^{i,n}\big(\nu_n^i(\omega),\Pi_{t_i}^t(\omega)\big)=\int_{t_i}^{\nu_n^i(\omega)} g\big(r,\mathfrak{X}_r(\omega)\big)dr. \tag{6.108}$$

Since $\omega\in\mathcal{A}_n^i\subset\{\widehat{\tau}>t_i\}$ and since

$$\overline{\alpha}_s(\omega)\,=\,\mathbf{1}_{\{s<t_i\}}\alpha_s(\omega)+\mathbf{1}_{\{s\ge t_i\}}\Big(\alpha_s^{i,n}\big(\Pi_{t_i}^t(\omega)\big)+\int_t^{t_i} g\big(r,\mathfrak{X}_r(\omega)\big)dr+\eta_n^i(\omega)-\mathcal{K}_s^{i,n}(\omega)\Big),\quad s\in[t,\infty),$$

we can deduce from (6.101) and (6.108) that

$$\begin{aligned}
\overline{\alpha}_s(\omega)\,&=\,\alpha_s(\omega)>\int_t^s g\big(r,\mathfrak{X}_r(\omega)\big)dr,\quad \forall\, s\in[t,t_i),\\
\overline{\alpha}_s(\omega)\,&\ge\,\alpha_s^{i,n}\big(\Pi_{t_i}^t(\omega)\big)+\int_t^{t_i} g\big(r,\mathfrak{X}_r(\omega)\big)dr>\int_t^s g\big(r,\mathfrak{X}_r(\omega)\big)dr,\quad \forall\, s\in\big[t_i,\nu_n^i(\omega)\big),\\
\text{and}\quad \overline{\alpha}\big(\nu_n^i(\omega),\omega\big)\,&=\,\alpha^{i,n}\big(\nu_n^i(\omega),\Pi_{t_i}^t(\omega)\big)+\int_t^{t_i} g\big(r,\mathfrak{X}_r(\omega)\big)dr=\int_t^{\nu_n^i(\omega)} g\big(r,\mathfrak{X}_r(\omega)\big)dr,
\end{aligned}$$

which implies that

$$\overline{\tau}(\omega)=\big(\tau(t,x,\overline{\alpha})\big)(\omega)=\nu_n^i(\omega),\quad \forall\,\omega\in G_n^i. \tag{6.109}$$

Similar to Problem 2.7.3 of [32], there exists $\widetilde{G}_n^i\in\mathcal{F}_{t_i}^t$ such that $\mathcal{N}_G^{i,n}:=G_n^i\Delta\widetilde{G}_n^i\in\mathscr{N}^t$. By Proposition 3.3 (1), it holds for all $\omega\in\Omega^t$ except on a P_t-null set $\widehat{\mathcal{N}}_G^{i,n}$ that $\big(\mathcal{N}_G^{i,n}\big)^{t_i,\omega}\in\mathscr{N}^{t_i}$.

Now, let $\omega\in G_n^i\cap\widetilde{G}_n^i\cap\big(\widehat{\mathcal{N}}_G^{i,n}\big)^c$ and $\widetilde{\omega}\in\big((\mathcal{N}_G^{i,n})^{t_i,\omega}\cup\mathcal{N}_\omega^i\big)^c$. As $\omega\in\widetilde{G}_n^i$ and $\widetilde{\omega}\in\big((\mathcal{N}_G^{i,n})^{t_i,\omega}\big)^c=\big((\mathcal{N}_G^{i,n})^c\big)^{t_i,\omega}$, Lemma 3.1 shows that $\omega\otimes_{t_i}\widetilde{\omega}\in\widetilde{G}_n^i$ and thus $\omega\otimes_{t_i}\widetilde{\omega}\in\widetilde{G}_n^i\cap\big(\mathcal{N}_G^{i,n}\big)^c=G_n^i\cap\widetilde{G}_n^i\subset G_n^i$. Applying (6.109) with $\omega=\omega\otimes_{t_i}\widetilde{\omega}$, we see from (6.92) that

$$\begin{aligned}
\overline{\tau}(\omega\otimes_{t_i}\widetilde{\omega})\,&=\,\nu_n^i(\omega\otimes_{t_i}\widetilde{\omega})=\inf\big\{s\in[t_i,\infty):\alpha_s^{i,n}(\widetilde{\omega})-\int_{t_i}^s g\big(r,\mathfrak{X}_r(\omega\otimes_{t_i}\widetilde{\omega})\big)dr=0\big\}\\
&=\,\inf\Big\{s\in[t_i,\infty):\alpha_s^{i,n}(\widetilde{\omega})-\int_{t_i}^s g\Big(r,\mathcal{X}_r^{t_i,\mathfrak{X}_{t_i}(\omega)}(\widetilde{\omega})\Big)dr=0\Big\}=\big(\tau\big(t_i,\mathfrak{X}_{t_i}(\omega),\alpha^{i,n}\big)\big)(\widetilde{\omega})=:\tau_\omega^{i,n}(\widetilde{\omega}).
\end{aligned}$$

Then (6.92) again shows that

$$\big(\mathcal{R}(t,x,\overline{\tau})\big)^{t_i,\omega}(\widetilde{\omega}) = \big(\mathcal{R}(t,x,\overline{\tau})\big)(\omega\otimes_{t_i}\widetilde{\omega}) = \int_t^{\overline{\tau}(\omega\otimes_{t_i}\widetilde{\omega})} f\big(r,\mathfrak{X}_r(\omega\otimes_{t_i}\widetilde{\omega})\big)dr + \pi\Big(\overline{\tau}(\omega\otimes_{t_i}\widetilde{\omega}),\mathfrak{X}\big(\overline{\tau}(\omega\otimes_{t_i}\widetilde{\omega}),\omega\otimes_{t_i}\widetilde{\omega}\big)\Big)$$
$$= \int_t^{t_i} f\big(r,\mathfrak{X}_r(\omega)\big)dr + \int_{t_i}^{\tau^{i,n}_\omega(\widetilde{\omega})} f\Big(r,\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}_r(\widetilde{\omega})\Big)dr + \pi\Big(\tau^{i,n}_\omega(\widetilde{\omega}),\mathcal{X}^{t_i,\mathfrak{X}_{t_i}(\omega)}\big(\tau^{i,n}_\omega(\widetilde{\omega}),\widetilde{\omega}\big)\Big)$$
$$= \int_t^{t_i} f\big(r,\mathfrak{X}_r(\omega)\big)dr + \big(\mathcal{R}(t_i,\mathfrak{X}_{t_i}(\omega),\tau^{i,n}_\omega)\big)(\widetilde{\omega}).$$

Taking expectation $E_{t_i}[\cdot]$ over $\widetilde{\omega}\in\Omega^{t_i}$ except the $P_{t_i}-$null set $(\mathcal{N}^{i,n}_G)^{t_i,\omega}\cup\mathcal{N}^i_\omega$ yields that

$$E_{t_i}\Big[\big(\mathcal{R}(t,x,\overline{\tau})\big)^{t_i,\omega}\Big] = E_{t_i}\big[\mathcal{R}\big(t_i,\mathfrak{X}_{t_i}(\omega),\tau(t_i,\mathfrak{X}_{t_i}(\omega),\alpha^{i,n})\big)\big] + \int_t^{t_i} f\big(r,\mathfrak{X}_r(\omega)\big)dr.$$

Since $\big(\mathfrak{X}_{t_i}(\omega),\mathfrak{Y}_{t_i}(\omega)\big)\in O_{\widehat{\delta}_i(x^i_n,y^i_n)}(x^i_n)\times\big(y^i_n-\widehat{\delta}_i(x^i_n,y^i_n),y^i_n+\widehat{\delta}_i(x^i_n,y^i_n)\big)$, using (6.96) with $(\mathfrak{x},\mathfrak{y},\mathfrak{x}')=\big(x^i_n,y^i_n,\mathfrak{X}_{t_i}(\omega)\big)$ and applying (6.94) with $(\mathfrak{x},\mathfrak{y},\mathfrak{x}',\mathfrak{y}')=\big(x^i_n,y^i_n,x^i_n,y^i_n-\delta_i(x^i_n,y^i_n)\big)$ and $(\mathfrak{x},\mathfrak{y},\mathfrak{x}',\mathfrak{y}')=\big(x^i_n,y^i_n,\mathfrak{X}_{t_i}(\omega),\mathfrak{Y}_{t_i}(\omega)\big)$ respectively, we can deduce from (6.95) that

$$E_{t_i}\Big[\big(\mathcal{R}(t,x,\overline{\tau})\big)^{t_i,\omega}\Big] - \int_t^{t_i} f(r,\mathfrak{X}_r(\omega))dr \ge E_{t_i}\Big[\mathcal{R}\big(t_i,x^i_n,\tau(t_i,x^i_n,\alpha^{i,n})\big)\Big] - \varepsilon/5 \ge \mathcal{V}\big(t_i,x^i_n,y^i_n-\delta_i(x^i_n,y^i_n)\big) - 2\varepsilon/5$$
$$\ge \mathcal{V}\big(t_i,x^i_n,y^i_n\big) - 3\varepsilon/5 \ge \mathcal{V}\big(t_i,\mathfrak{X}_{t_i}(\omega),\mathfrak{Y}_{t_i}(\omega)\big) - 4\varepsilon/5, \quad \forall\,\omega\in G^i_n\cap\widetilde{G}^i_n\cap\big(\widehat{\mathcal{N}}^{i,n}_G\big)^c. \tag{6.110}$$

The first inequality in (4.1) and Proposition 3.4 (2) imply that $E_t\big[\mathcal{R}(t,x,\overline{\tau})\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega) = E_{t_i}\Big[\big(\mathcal{R}(t,x,\overline{\tau})\big)^{t_i,\omega}\Big]$ for P_t-a.s. $\omega\in\Omega^t$. As $\mathbf{1}_{G^i_n\cap\widetilde{G}^i_n} = \mathbf{1}_{G^i_n}\mathbf{1}_{\widetilde{G}^i_n} = \mathbf{1}_{G^i_n} = \mathbf{1}_{\mathcal{A}^i_n}$, P_t-a.s., we can derive from (6.110) that

$$E_t\big[\mathbf{1}_{\mathcal{A}^i_n}\mathcal{R}(t,x,\overline{\tau})\big] = E_t\Big[\mathbf{1}_{\mathcal{A}^i_n}E_t\big[\mathcal{R}(t,x,\overline{\tau})\big|\overline{\mathcal{F}}^t_{t_i}\big]\Big] = E_t\Big[\mathbf{1}_{G^i_n\cap\widetilde{G}^i_n}E_{t_i}\big[(\mathcal{R}(t,x,\overline{\tau}))^{t_i,\omega}\big]\Big]$$
$$\ge E_t\Big[\mathbf{1}_{G^i_n\cap\widetilde{G}^i_n}\Big(\mathcal{V}\big(t_i,\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i}\big) + \int_t^{t_i} f(r,\mathfrak{X}_r)dr - 4\varepsilon/5\Big)\Big] = E_t\Big[\mathbf{1}_{\mathcal{A}^i_n}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big) + \int_t^{\zeta} f(r,\mathfrak{X}_r)dr - 4\varepsilon/5\Big)\Big].$$

Taking summation over $n\in 1,\cdots,\mathfrak{n}_i$ and $i=1,\cdots,\mathcal{I}_o$ and using the conclusion of Part 2 yield that

$$E_t\big[\mathcal{R}(t,x,\overline{\tau})\big] \ge E_t\Big[\mathbf{1}_{\{\widehat{\tau}\le\zeta\}\cup(\{\widehat{\tau}>\zeta\}\cap\mathcal{A}^c_\sharp)}\mathcal{R}(t,x,\widehat{\tau}) + \mathbf{1}_{\mathcal{A}_\sharp}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big) + \int_t^{\zeta} f(r,\mathfrak{X}_r)dr\Big)\Big] - 4\varepsilon/5. \tag{6.111}$$

2d) Since $\underset{n=1}{\overset{\mathfrak{n}_i}{\cup}}\mathcal{A}^i_n = \underset{n=1}{\overset{\mathfrak{n}_i}{\cup}}A^i_n = \{\widehat{\tau}>\zeta=t_i\}\cap\Big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\in\underset{n=1}{\overset{\mathfrak{n}_i}{\cup}}\mathcal{O}_i(x^i_n,y^i_n)\Big\}$ for $i=1,\cdots,\mathcal{I}_o$, one can deduce that

$$\{\widehat{\tau}>\zeta\}\cap\mathcal{A}^c_\sharp = \{\widehat{\tau}>\zeta>\mathcal{I}_o\}\cup\bigg(\underset{i=1}{\overset{\mathcal{I}_o}{\cup}}\Big(\{\widehat{\tau}>\zeta=t_i\}\cap\Big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\notin\underset{n=1}{\overset{\mathfrak{n}_i}{\cup}}\mathcal{O}_i(x^i_n,y^i_n)\Big\}\Big)\bigg),$$

and (6.97) implies that $P_t\big(\{\widehat{\tau}>\zeta\}\cap\mathcal{A}^c_\sharp\big) \le P_t\{\zeta>\mathcal{I}_o\} + \sum_{i=1}^{\mathcal{I}_o}P_t\big(\{\widehat{\tau}>t_i\}\cap\big\{(\mathfrak{X}_{t_i},\mathfrak{Y}_{t_i})\notin\overline{O}_{R_i}(0)\times[R_i^{-1},R_i]\big\}\big) < \lambda$. It then follows from (6.93) that

$$\bigg|E_t\Big[\mathbf{1}_{\{\widehat{\tau}>\zeta\}\cap\mathcal{A}^c_\sharp}\Big(\mathcal{R}(t,x,\widehat{\tau}) - \mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big) - \int_t^{\zeta} f(r,\mathfrak{X}_r)dr\Big)\Big]\bigg|$$
$$\le E_t\Big[\mathbf{1}_{\{\widehat{\tau}>\zeta\}\cap\mathcal{A}^c_\sharp}\Big(\big|\mathcal{R}(t,x,\widehat{\tau})\big| + \big|\mathcal{V}(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta)\big| + \int_t^{\zeta}\big|f(r,\mathfrak{X}_r)\big|dr\Big)\Big] < \varepsilon/5,$$

which together with (6.111) and (4.10) leads to that

$$\mathcal{V}(t,x,y) \ge E_t\big[\mathcal{R}(t,x,\overline{\tau})\big] \ge E_t\Big[\mathbf{1}_{\{\widehat{\tau}\le\zeta\}}\mathcal{R}(t,x,\widehat{\tau}) + \mathbf{1}_{\{\widehat{\tau}>\zeta\}}\Big(\mathcal{V}\big(\zeta,\mathfrak{X}_\zeta,\mathfrak{Y}_\zeta\big) + \int_t^{\zeta} f(r,\mathfrak{X}_r)dr\Big)\Big] - \varepsilon.$$

Letting $\varepsilon\to\infty$ yields the second inequality in (6.89). □

Proof of Theorem 4.2: Fix $t\in[0,\infty)$.

1) Let $(x,y)\in\mathbb{R}^l\times(0,\infty)$ and $\{\zeta(\alpha)\}_{\alpha\in\mathfrak{A}_t(y)}$ be a family of $\overline{\mathcal{T}}^t_\sharp-$stopping times. For any $\alpha\in\mathfrak{A}_t(y)$, taking $\zeta=\zeta(\alpha)$ in (6.89) yields that

$$E_t\big[\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big] \le E_t\bigg[\mathbf{1}_{\{\tau(t,x,\alpha)\le\zeta(\alpha)\}}\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big) +\mathbf{1}_{\{\tau(t,x,\alpha)>\zeta(\alpha)\}}\Big(\mathcal{V}\big(\zeta(\alpha),\mathcal{X}^{t,x}_{\zeta(\alpha)},Y^{t,x,\alpha}_{\zeta(\alpha)}\big)+\int_t^{\zeta(\alpha)}f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg]\le\mathcal{V}(t,x,y).$$

Taking supremum over $\alpha\in\mathfrak{A}_t(y)$, we obtain (1.4) from (4.10).

2) Next, suppose that $\mathcal{V}(s,x,y)$ is continuous in $(s,x,y)\in[t,\infty)\times\mathbb{R}^l\times(0,\infty)$.

We fix $(x,y)\in\mathbb{R}^l\times(0,\infty)$ and a family $\{\zeta(\alpha)\}_{\alpha\in\mathfrak{A}_t(y)}$ of $\overline{\mathcal{T}}^t-$stopping times. Let $\alpha\in\mathfrak{A}_t(y)$, $n\in\mathbb{N}$ and define

$$\zeta_n=\zeta_n(\alpha):=\mathbf{1}_{\{\zeta(\alpha)=t\}}t+\sum_{i\in\mathbb{N}}\mathbf{1}_{\{\zeta(\alpha)\in(t+(i-1)2^{-n},t+i2^{-n}]\}}(t+i2^{-n})\in\overline{\mathcal{T}}^t.$$

Applying (6.89) with $\zeta=\zeta_n$ yields that

$$E_t\big[\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big] \le E_t\bigg[\mathbf{1}_{\{\tau(t,x,\alpha)\le\zeta_n\}}\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big) +\mathbf{1}_{\{\tau(t,x,\alpha)>\zeta_n\}}\Big(\mathcal{V}\big(\zeta_n,\mathcal{X}^{t,x}_{\zeta_n},Y^{t,x,\alpha}_{\zeta_n}\big)+\int_t^{\zeta_n}f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg]\le\mathcal{V}(t,x,y).$$

As $n\to\infty$, using similar arguments to those that lead to (6.76) we can deduce from the continuity of function $\mathcal{V}$ in $(s,\mathfrak{x},\mathfrak{y})\in[t,\infty)\times\mathbb{R}^l\times(0,\infty)$, the continuity of processes $\big(\mathcal{X}^{t,x},Y^{t,x,\alpha}\big)$, and the dominated convergence theorem that

$$E_t\big[\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big)\big] \le E_t\bigg[\mathbf{1}_{\{\tau(t,x,\alpha)\le\zeta(\alpha)\}}\mathcal{R}\big(t,x,\tau(t,x,\alpha)\big) +\mathbf{1}_{\{\tau(t,x,\alpha)>\zeta(\alpha)\}}\Big(\mathcal{V}\big(\zeta(\alpha),\mathcal{X}^{t,x}_{\zeta(\alpha)},Y^{t,x,\alpha}_{\zeta(\alpha)}\big)+\int_t^{\zeta(\alpha)}f(r,\mathcal{X}^{t,x}_r)dr\Big)\bigg]\le\mathcal{V}(t,x,y).$$

Taking supremum over $\alpha\in\mathfrak{A}_t(y)$ and using (4.10) yield (1.4) again. □

6.4 Proof of Section 5

Proof of Theorem 5.1: Under (2.4) and (2.14), Theorem 2.1 (2) and (4.5) show that $\mathcal{V}$ is continuous in $(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty)$. By (4.3), $\mathcal{V}(t,x,0)=\pi(t,x)$ for any $(t,x)\in[0,\infty)\times\mathbb{R}^l$.

1) We first show that $\mathcal{V}$ is a viscosity supersolution of (5.2).

Let $(t_o,x_o,y_o)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty)$ and let $\phi\in C^{1,2,2}\big([0,\infty)\times\mathbb{R}^l\times[0,\infty)\big)$ such that $\mathcal{V}-\phi$ attains a strict local minimum 0 at (t_o,x_o,y_o). So there exists a $\delta_o\in\big(0,t_o\wedge y_o\big)$ such that for any $(t,x,y)\in O_{\delta_o}(t_o,x_o,y_o)\backslash\big\{(t_o,x_o,y_o)\big\}$

$$(\mathcal{V}-\phi)(t,x,y)>(\mathcal{V}-\phi)(t_o,x_o,y_o)=0 \text{ and } \big|D_x\phi(t,x,y)-D_x\phi(t_o,x_o,y_o)\big|\vee\big|\partial_y\phi(t,x,y)-\partial_y\phi(t_o,x_o,y_o)\big|<1. \quad (6.112)$$

According to (2.4) and (2.14), the functions b,σ,f,g are continuous in (t,x). Then

$$\widehat{\phi}\,(t,x,y):=-\partial_t\phi(t,x,y)-\mathcal{L}_x\phi(t,x,y)+g(t,x)\partial_y\phi(t,x,y)-f(t,x),\quad \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty) \quad (6.113)$$

is also a continuous function.

To show $\widehat{\phi}\,(t_o,x_o,y_o)-\mathcal{H}\phi(t_o,x_o,y_o)\ge0$, it suffices to verify that for any $a\in\mathbb{R}^d$

$$\widehat{\phi}\,(t_o,x_o,y_o)-\tfrac{1}{2}|a|^2\partial_y^2\phi(t_o,x_o,y_o)-\big(D_x(\partial_y\phi(t_o,x_o,y_o))\big)^T\cdot\sigma(t_o,x_o)\cdot a\ge0.$$

Assume not, i.e. there exists an $\mathfrak{a}\in\mathbb{R}^d$ such that

$$\varepsilon:=\frac{1}{2}|\mathfrak{a}|^2\partial_y^2\phi(t_o,x_o,y_o)+\big(D_x(\partial_y\phi(t_o,x_o,y_o))\big)^T\cdot\sigma(t_o,x_o)\cdot\mathfrak{a}-\widehat{\phi}\,(t_o,x_o,y_o)>0.$$

Using the continuity of σ, ϕ and $\widehat{\phi}$, we can find some $\delta\in(0,\delta_o)$ such that

$$\frac{1}{2}|\mathfrak{a}|^2\partial_y^2\phi(t,x,y)+\big(D_x(\partial_y\phi(t,x,y))\big)^T\cdot\sigma(t,x)\cdot\mathfrak{a}-\widehat{\phi}\,(t,x,y)\ge\frac{1}{2}\varepsilon>0,\quad \forall\,(t,x,y)\in\overline{O}_\delta(t_o,x_o,y_o). \tag{6.114}$$

Clearly, $M_s:=y_o+\mathfrak{a}^T\cdot W_s^{t_o}$, $s\in[t_o,\infty)$ is a uniformly integrable continuous martingale with respect to $\big(\overline{\mathbf{F}}^{t_o},P_{t_o}\big)$. By taking $K\equiv 0$, we have $\alpha^o:=M\in\mathfrak{A}_{t_o}(y_o)$. As $\Theta_s:=\big(s,\mathcal{X}_s^{t_o,x_o},Y_s^{t_o,x_o,\alpha^o}\big)$, $s\in[t_o,\infty)$ are $\overline{\mathbf{F}}^{t_o}-$adapted continuous processes with $\Theta_{t_o}=(t_o,x_o,y_o)$, P_t-a.s., $\zeta:=\inf\big\{s\in[t_o,\infty)\colon\Theta_s\notin\overline{O}_\delta(t_o,x_o,y_o)\big\}$ defines an $\overline{\mathbf{F}}^{t_o}$-stopping time with $t_o<\zeta\le t_o+\delta$, $P_{t_o}-$a.s. Since

$$\Theta_s\in\overline{O}_\delta(t_o,x_o,y_o)\text{ on the stochastic interval }[\![t_o,\zeta[\![, \tag{6.115}$$

(6.114), (6.112), (1.7) and (2.4) imply that

$$\frac{1}{2}|\mathfrak{a}|^2\partial_y^2\phi(\Theta_r)+\big(D_x(\partial_y\phi(\Theta_r))\big)^T\cdot\sigma(r,\mathcal{X}_r^{t_o,x_o})\cdot\mathfrak{a}-\widehat{\phi}\,(\Theta_r)\ge\frac{1}{2}\varepsilon>0$$

$$\text{and}\quad \big|D_x\phi(\Theta_r)\big|\big|\sigma(r,\mathcal{X}_r^{t_o,x_o})\big|+\big|\partial_y\phi(\Theta_r)\big||\mathfrak{a}|\le\big(1+|D_x\phi(t_o,x_o,y_o)|\big)\big(|\sigma(t_o,x_o)|+\sqrt{\|c(\cdot)\|}\,\delta+\sqrt{\|c(\cdot)\|}\rho(\delta)(1+|x_o|^\varpi)\big)$$
$$+\big(1+|\partial_y\phi(t_o,x_o,y_o)|\big)|\mathfrak{a}|<\infty \tag{6.116}$$

holds on $[\![t_o,\zeta[\![$. Applying Itô's formula to process $\big\{\phi(\Theta_s)\big\}_{s\in[t_o,\infty)}$ then yields that

$$\begin{aligned}\phi(\Theta_\zeta)-\phi(t_o,x_o,y_o)&=\int_{t_o}^{\zeta}\Big(\partial_t\phi(\Theta_r)-g(r,\mathcal{X}_r^{t_o,x_o})\partial_y\phi(\Theta_r)+\mathcal{L}_x\phi(\Theta_r)+\frac{1}{2}|\mathfrak{a}|^2\partial_y^2\phi(\Theta_r)+\big(D_x(\partial_y\phi(\Theta_r))\big)^T\cdot\sigma(r,\mathcal{X}_r^{t_o,x_o})\cdot\mathfrak{a}\Big)dr\\&\quad+\int_{t_o}^{\zeta}\big((D_x\phi(\Theta_r))^T\cdot\sigma(r,\mathcal{X}_r^{t_o,x_o})+\partial_y\phi(\Theta_r)\cdot\mathfrak{a}^T\big)dW_r^{t_o},\\&\ge-\int_{t_o}^{\zeta}f(r,\mathcal{X}_r^{t_o,x_o})dr+\int_{t_o}^{\zeta}\big((D_x\phi(\Theta_r))^T\cdot\sigma(r,\mathcal{X}_r^{t_o,x_o})+\partial_y\phi(\Theta_r)\cdot\mathfrak{a}^T\big)dW_r^{t_o},\quad P_{t_o}-a.s.\end{aligned} \tag{6.117}$$

Set $\mathfrak{m}_1:=\min\limits_{(t,x,y)\in\partial O_\delta(t_o,x_o,y_o)}(\mathcal{V}-\phi)(t,x,y)>0$ by (6.112). The continuity of process Θ and (6.115) show that

$$P_{t_o}\big\{\Theta_\zeta\in\partial O_\delta(t_o,x_o,y_o)\big\}=P_{t_o}\big\{Y_s^{t_o,x_o,\alpha^o}\ge y_o-\delta>0,\ \forall\,s\in[t_o,\zeta]\big\}=1, \tag{6.118}$$

the latter of which implies that

$$\tau(t_o,x_o,\alpha^o)>\zeta>t_o,\quad P_{t_o}-a.s. \tag{6.119}$$

Taking expectation $E_{t_o}[\cdot]$ in (6.117) and applying Theorem 4.2 (2) with $\zeta(\alpha)\equiv\zeta$, we can derive from (6.116), (6.118) that

$$\begin{aligned}\phi(t_o,x_o,y_o)+\mathfrak{m}_1&\le E_{t_o}\Big[\phi(\Theta_\zeta)+\int_t^\zeta f(r,\mathcal{X}_r^{t_o,x_o})dr\Big]+\mathfrak{m}_1\le E_{t_o}\Big[\mathcal{V}(\Theta_\zeta)+\int_t^\zeta f(r,\mathcal{X}_r^{t_o,x_o})dr\Big]\\&=E_{t_o}\Big[\mathbf{1}_{\{\tau(t_o,x_o,\alpha^o)\le\zeta\}}\mathcal{R}\big(t_o,x_o,\tau(t_o,x_o,\alpha^o)\big)+\mathbf{1}_{\{\tau(t_o,x_o,\alpha^o)>\zeta\}}\Big(\mathcal{V}(\Theta_\zeta)+\int_t^\zeta f(r,\mathcal{X}_r^{t_o,x_o})dr\Big)\Big]\\&\le\sup_{\alpha\in\mathfrak{A}_{t_o}(y_o)}E_{t_o}\Big[\mathbf{1}_{\{\tau(t_o,x_o,\alpha)\le\zeta\}}\mathcal{R}\big(t_o,x_o,\tau(t_o,x_o,\alpha)\big)+\mathbf{1}_{\{\tau(t_o,x_o,\alpha)>\zeta\}}\Big(\mathcal{V}\big(\zeta,\mathcal{X}_\zeta^{t_o,x_o},Y_\zeta^{t_o,x_o,\alpha}\big)+\int_t^\zeta f(r,\mathcal{X}_r^{t_o,x_o})dr\Big)\Big]\\&=\mathcal{V}(t_o,x_o,y_o)=\phi(t_o,x_o,y_o).\end{aligned} \tag{6.120}$$

A contradiction appears.

We can also employ the first DPP (Theorem 4.1) to induce the incongruity: Denote $\tau_o:=\tau(t_o,x_o,\alpha^o)$. By the continuity of process Y^{t_o,x_o,α^o},

$$y_o+\mathfrak{a}^T\cdot W_{\tau_o}^{t_o}=\int_{t_o}^{\tau_o}g\big(r,\mathcal{X}_r^{t_o,x_o}\big)dr,\quad P_{t_o}-a.s. \tag{6.121}$$

So $E_{t_o}\big[\int_{t_o}^{\tau_o} g\big(r,\mathcal{X}_r^{t_o,x_o}\big)dr\big]=y_o$, which together with (6.119) shows $\tau_o\in\widehat{\mathcal{T}}_{x_o}^{t_o}(y_o)$. On the other hand, taking conditional expectation $E_{t_o}\big[\cdot\,\big|\overline{\mathcal{F}}_\zeta^{t_o}\big]$ in (6.121), one can deduce from (6.119) and the optional sampling theorem that

$$\begin{aligned} Y_\zeta^{t_o,x_o,\alpha^o} &= y_o+\mathfrak{a}^T\cdot W_\zeta^{t_o}-\int_{t_o}^\zeta g\big(r,\mathcal{X}_r^{t_o,x_o}\big)dr=E_{t_o}\big[y_o+\mathfrak{a}^T\cdot W_{\tau_o}^{t_o}\big|\overline{\mathcal{F}}_\zeta^{t_o}\big]-\int_{t_o}^\zeta g\big(r,\mathcal{X}_r^{t_o,x_o}\big)dr \\ &= E_{t_o}\Big[\int_{t_o}^{\tau_o} g\big(r,\mathcal{X}_r^{t_o,x_o}\big)dr\Big|\overline{\mathcal{F}}_\zeta^{t_o}\Big]-\int_{t_o}^\zeta g\big(r,\mathcal{X}_r^{t_o,x_o}\big)dr=\mathcal{Y}_\zeta^{t_o,x_o,\tau_o}, \quad P_{t_o}-a.s. \end{aligned}$$

Then we can apply Theorem 4.1 (2) with $\zeta(\alpha)\equiv\zeta$ to continue the deduction in (6.120)

$$\begin{aligned} \phi(t_o,x_o,y_o)+\mathfrak{m}_1 &\le E_{t_o}\bigg[\mathbf{1}_{\{\tau_o\le\zeta\}}\mathcal{R}\big(t_o,x_o,\tau_o\big)+\mathbf{1}_{\{\tau_o>\zeta\}}\Big(\mathcal{V}(\zeta,\mathcal{X}_\zeta^{t_o,x_o},\mathcal{Y}_\zeta^{t_o,x_o,\tau_o})+\int_t^\zeta f(r,\mathcal{X}_r^{t_o,x_o})dr\Big)\bigg] \\ &\le \sup_{\tau\in\widehat{\mathcal{T}}_{x_o}^{t_o}(y_o)} E_{t_o}\bigg[\mathbf{1}_{\{\tau\le\zeta\}}\mathcal{R}\big(t_o,x_o,\tau_o\big)+\mathbf{1}_{\{\tau>\zeta\}}\Big(\mathcal{V}\big(\zeta,\mathcal{X}_\zeta^{t_o,x_o},\mathcal{Y}_\zeta^{t_o,x_o,\tau}\big)+\int_t^\zeta f(r,\mathcal{X}_r^{t_o,x_o})dr\Big)\bigg] \\ &= \mathcal{V}(t_o,x_o,y_o)=\phi(t_o,x_o,y_o). \end{aligned}$$

The contradiction recurs. Therefore, $\mathcal{V}$ is a viscosity supersolution of (5.2).

2) Next, we demonstrate that $\mathcal{V}$ is also a viscosity subsolution of (5.3).

Let $(t_o,x_o,y_o)\in(0,\infty)\times\mathbb{R}^l\times(0,\infty)$ and let $\varphi\in C^{1,2,2}\big([0,\infty)\times\mathbb{R}^l\times[0,\infty)\big)$ such that $\mathcal{V}-\varphi$ attains a strict local maximum 0 at (t_o,x_o,y_o). So there exists a $\lambda_o\in\big(0,t_o\wedge y_o\big)$ such that for any $(t,x,y)\in O_{\lambda_o}(t_o,x_o,y_o)\backslash\big\{(t_o,x_o,y_o)\big\}$

$$(\mathcal{V}-\varphi)(t,x,y)<(\mathcal{V}-\varphi)(t_o,x_o,y_o)=0 \text{ and } \big|D_x\varphi(t,x,y)-D_x\varphi(t_o,x_o,y_o)\big|\vee\big|\partial_y\varphi(t,x,y)-\partial_y\varphi(t_o,x_o,y_o)\big|<1. \quad (6.122)$$

Similar to (6.113), $\widehat{\varphi}\,(t,x,y):=-\partial_t\varphi(t,x,y)-\mathcal{L}_x\varphi(t,x,y)+g(t,x)\partial_y\varphi(t,x,y)-f(t,x),\ \forall\,(t,x,y)\in[0,\infty)\times\mathbb{R}^l\times[0,\infty)$ defines a continuous function. If $\overline{\mathcal{H}}\varphi(t_o,x_o,y_o)=\infty$, then $\widehat{\varphi}\,(t_o,x_o,y_o)-\overline{\mathcal{H}}\varphi(t_o,x_o,y_o)\le 0$ holds automatically.

So let us just consider the case $\overline{\mathcal{H}}\varphi(t_o,x_o,y_o)<\infty$. By (5.1), there exists $\widetilde{\lambda}_o\in(0,\lambda_o)$ such that $\mathcal{H}\varphi(t,x,y)\le\overline{\mathcal{H}}\varphi(t_o,x_o,y_o)+1<\infty$ and thus $\partial_y^2\varphi(t,x,y)\le 0,\ \forall\,(t,x,y)\in O_{\widetilde{\lambda}_o}(t_o,x_o,y_o)$. If one had $\partial_y\varphi(t_o,x_o,y_o)\le 0$, (2.12) and (6.122) would imply that

$$\begin{aligned} \varphi(t_o,x_o,y) &= \varphi(t_o,x_o,y_o)+\int_{y_o}^y\varphi_y(t_o,x_o,s)ds=\varphi(t_o,x_o,y_o)+(y-y_o)\cdot\partial_y\varphi(t_o,x_o,y_o)+\int_{y_o}^y\int_{y_o}^s\varphi_y^2(t_o,x_o,r)drds \\ &\le \varphi(t_o,x_o,y_o)=\mathcal{V}(t_o,x_o,y_o)\le\mathcal{V}(t_o,x_o,y), \quad \forall\,y\in\big(y_o,y_o+\widetilde{\lambda}_o\big). \end{aligned}$$

which contradicts with the strict local maximum of $\mathcal{V}-\varphi$ at (t_o,x_o,y_o). Hence we must have

$$\partial_y\varphi(t_o,x_o,y_o)>0. \quad (6.123)$$

To draw a contradiction, we assume that

$$\epsilon:=\widehat{\varphi}\,(t_o,x_o,y_o)-\overline{\mathcal{H}}\varphi(t_o,x_o,y_o)>0.$$

According to (6.123) and the continuity of $\widehat{\varphi}$, there exists $\lambda\in(0,\lambda_o)$ such that for any $(t,x,y)\in\overline{O}_\lambda(t_o,x_o,y_o)$

$$\partial_y\varphi(t,x,y)\ge 0 \quad\text{and}\quad \mathcal{H}\varphi(t,x,y)\le\overline{\mathcal{H}}\varphi(t_o,x_o,y_o)+\epsilon/2=\widehat{\varphi}\,(t_o,x_o,y_o)-\epsilon/2\le\widehat{\varphi}(t,x,y). \quad (6.124)$$

Fix $\alpha\in\mathfrak{A}_{t_o}(y_o)$, so $\alpha=M^\alpha-K^\alpha$ for some $(M^\alpha,K^\alpha)\in\mathbb{M}_{t_o}\times\mathbb{K}_{t_o}$. In light of the martingale representation theorem, one can find $\mathfrak{q}^\alpha\in\mathbb{H}_{t_o}^{2,\text{loc}}$ such that

$$P_{t_o}\Big\{\int_{t_o}^s|\mathfrak{q}_r^\alpha|^2dr<\infty,\ \forall\,s\in[t_o,\infty)\Big\}=P_{t_o}\Big\{M_s^\alpha=\int_{t_o}^s(\mathfrak{q}_r^\alpha)^TdW_r^{t_o},\ \forall\,s\in[t_o,\infty)\Big\}=1. \quad (6.125)$$

As $\Theta_s^\alpha:=\big(s,\mathcal{X}_s^{t_o,x_o},Y_s^{t_o,x_o,\alpha}\big)$ $s\in[t_o,\infty)$ are $\overline{\mathbf{F}}^{t_o}-$adapted continuous processes with $\Theta_{t_o}^\alpha=(t_o,x_o,y_o)$, $P_{t_o}-$a.s.,

$$\zeta^\alpha:=\inf\big\{s\in[t_o,\infty):\Theta_s^\alpha\notin\overline{O}_\lambda(t_o,x_o,y_o)\big\} \quad (6.126)$$

defines an $\overline{\mathbf{F}}^{t_o}$−stopping time with $t_o<\zeta^\alpha\le t_o+\lambda$, P_{t_o}−a.s. The continuity of processes Θ^α implies that P_{t_o}−a.s.

$$\Theta^\alpha_s\in\overline{O}_\lambda(t_o,x_o,y_o),\quad \forall\, s\in[t_o,\zeta^\alpha]. \tag{6.127}$$

Similar to (6.116), we can deduce from (6.127), (6.124) and (6.122) that for P_{t_o}−a.s.

$$\partial_y\varphi(\Theta^\alpha_r)\ge 0,\ \frac{1}{2}\big|\mathfrak{q}^\alpha_r\big|^2\partial^2_y\varphi(\Theta^\alpha_r)+\big(D_x(\partial_y\varphi(\Theta^\alpha_r))\big)^T\cdot\sigma(r,\mathcal{X}^{t_o,x_o}_r)\cdot\mathfrak{q}^\alpha_r-\widehat{\varphi}\,(\Theta^\alpha_r)\le\mathcal{H}\varphi(\Theta^\alpha_r)-\widehat{\varphi}\,(\Theta^\alpha_r)\le 0\quad\text{and} \tag{6.128}$$

$$\big|D_x\varphi(\Theta^\alpha_r)\big|\big|\sigma(r,\mathcal{X}^{t_o,x_o}_r)\big|+\big|\partial_y\varphi(\Theta^\alpha_r)\big|\big|\mathfrak{q}^\alpha_r\big|\le\big(1+|D_x\varphi(t_o,x_o,y_o)|\big)\big(|\sigma(t_o,x_o)|+\sqrt{\|c(\cdot)\|}\,\delta+\sqrt{\|c(\cdot)\|}\rho(\delta)(1+|x_o|^{\varpi})\big)$$
$$+\big(1+|\partial_y\varphi(t_o,x_o,y_o)|\big)\big|\mathfrak{q}^\alpha_r\big|<\infty,\quad \forall\, s\in[t_o,\zeta^\alpha]. \tag{6.129}$$

Let $n\in\mathbb{N}$ and define $\zeta^\alpha_n:=\inf\big\{s\in[t_o,\infty):\int_{t_o}^s|\mathfrak{q}^\alpha_r|^2dr>n\big\}\wedge\zeta^\alpha\in\overline{\mathcal{T}}^{t_o}$. Applying Itô's formula to process $\big\{\varphi(\Theta^\alpha_s)\big\}_{s\in[t_o,\infty)}$, and using (6.128) yield that

$$\begin{aligned}
&\varphi\big(\Theta^\alpha_{\zeta^\alpha_n}\big)-\varphi(t_o,x_o,y_o)\\
&\quad=\int_{t_o}^{\zeta^\alpha_n}\Big(\partial_t\varphi(\Theta^\alpha_r)-g(r,\mathcal{X}^{t_o,x_o}_r)\partial_y\varphi(\Theta^\alpha_r)+\mathcal{L}_x\varphi(\Theta^\alpha_r)+\frac{1}{2}\big|\mathfrak{q}^\alpha_r\big|^2\partial^2_y\varphi(\Theta^\alpha_r)+\big(D_x(\partial_y\varphi(\Theta^\alpha_r))\big)^T\cdot\sigma(r,\mathcal{X}^{t_o,x_o}_r)\cdot\mathfrak{q}^\alpha_r\Big)dr\\
&\qquad-\int_{t_o}^{\zeta^\alpha_n}\partial_y\varphi(\Theta^\alpha_r)dK^\alpha_r+\int_{t_o}^{\zeta^\alpha_n}\big((D_x\varphi(\Theta^\alpha_r))^T\cdot\sigma(r,\mathcal{X}^{t_o,x_o}_r)+\partial_y\varphi(\Theta^\alpha_r)\cdot(\mathfrak{q}^\alpha_r)^T\big)dW^{t_o}_r,\\
&\quad\le-\int_{t_o}^{\zeta^\alpha_n}f(r,\mathcal{X}^{t_o,x_o}_r)dr+\int_{t_o}^{\zeta^\alpha_n}\big((D_x\varphi(\Theta^\alpha_r))^T\cdot\sigma(r,\mathcal{X}^{t_o,x_o}_r)+\partial_y\varphi(\Theta^\alpha_r)\cdot(\mathfrak{q}^\alpha_r)^T\big)dW^{t_o}_r,\quad P_{t_o}-a.s.
\end{aligned}$$

Taking expectation $E_{t_o}[\cdot]$, we see from (6.129) that $\varphi(t_o,x_o,y_o)\ge E_{t_o}\Big[\varphi\big(\Theta^\alpha_{\zeta^\alpha_n}\big)+\int_{t_o}^{\zeta^\alpha_n}f(r,\mathcal{X}^{t_o,x_o}_r)dr\Big]$. Since (6.127) and the continuity of f show that $\big|\varphi\big(\Theta^\alpha_{\zeta^\alpha_n}\big)\big|+\int_{t_o}^{\zeta^\alpha_n}\big|f(r,\mathcal{X}^{t_o,x_o}_r)\big|dr\le\max\limits_{(t,x,y)\in\overline{O}_\lambda(t_o,x_o,y_o)}|\varphi(t,x,y)|+\lambda\max\limits_{(t,x)\in\overline{O}_\lambda(t_o,x_o)}|f(t,x)|$ and since $\lim\limits_{n\to\infty}\uparrow\zeta^\alpha_n=\zeta^\alpha$, P_{t_o}−a.s. by (6.125), we can derive from the dominated convergence theorem that

$$\varphi(t_o,x_o,y_o)\ge\lim_{n\to\infty}E_{t_o}\bigg[\varphi\big(\Theta^\alpha_{\zeta^\alpha_n}\big)+\int_{t_o}^{\zeta^\alpha_n}f\big(r,\mathcal{X}^{t_o,x_o}_r\big)dr\bigg]=E_{t_o}\bigg[\varphi\big(\Theta^\alpha_{\zeta^\alpha}\big)+\int_{t_o}^{\zeta^\alpha}f\big(r,\mathcal{X}^{t_o,x_o}_r\big)dr\bigg]. \tag{6.130}$$

Set $\mathfrak{m}_2:=\min\limits_{(t,x,y)\in\partial O_\lambda(t_o,x_o,y_o)}(\varphi-\mathcal{V})(t,x,y)>0$ by (6.122). The continuity of Θ^α and (6.127) show that

$$P_{t_o}\big\{\Theta^\alpha_{\zeta^\alpha}\in\partial O_\lambda(t_o,x_o,y_o)\big\}=P_{t_o}\big\{Y^{t_o,x_o,\alpha}_s\ge y_o-\lambda>0,\ \forall\, s\in[t_o,\zeta^\alpha]\big\}=1.$$

The latter of which implies that $\tau(t_o,x_o,\alpha)>\zeta^\alpha>t_o$, P_{t_o}−a.s., which together with (6.130) leads to that

$$\begin{aligned}
\varphi(t_o,x_o,y_o)-\mathfrak{m}_2&\ge E_{t_o}\bigg[\varphi(\Theta^\alpha_{\zeta^\alpha})-\mathfrak{m}_2+\int_t^{\zeta^\alpha}f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg]\ge E_{t_o}\bigg[\mathcal{V}(\Theta^\alpha_{\zeta^\alpha})+\int_t^{\zeta^\alpha}f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg]\\
&\ge E_{t_o}\bigg[\mathbf{1}_{\{\tau(t_o,x_o,\alpha)\le\zeta^\alpha\}}\mathcal{R}\big(t_o,x_o,\tau(t_o,x_o,\alpha)\big)+\mathbf{1}_{\{\tau(t_o,x_o,\alpha)>\zeta^\alpha\}}\bigg(\mathcal{V}\big(\Theta^\alpha_{\zeta^\alpha}\big)+\int_t^{\zeta^\alpha}f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg)\bigg].
\end{aligned} \tag{6.131}$$

Taking supremum over $\alpha\in\mathfrak{A}_{t_o}(y_o)$, we can deduce from Theorem 4.2 that

$$\begin{aligned}
\varphi(t_o,x_o,y_o)-\mathfrak{m}_2&\ge\sup_{\alpha\in\mathfrak{A}_{t_o}(y_o)}E_{t_o}\bigg[\mathbf{1}_{\{\tau(t_o,x_o,\alpha)\le\zeta^\alpha\}}\mathcal{R}\big(t_o,x_o,\tau(t_o,x_o,\alpha)\big)+\mathbf{1}_{\{\tau(t_o,x_o,\alpha)>\zeta^\alpha\}}\bigg(\mathcal{V}\big(\Theta^\alpha_{\zeta^\alpha}\big)+\int_t^{\zeta^\alpha}f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg)\bigg]\\
&=\mathcal{V}(t_o,x_o,y_o)=\varphi(t_o,x_o,y_o).
\end{aligned}$$

A contradiction appears.

We can also use the first DPP (Theorem 4.1) to get the incongruity: Let $\tau\in\widehat{\mathcal{T}}^{t_o}_{x_o}(y_o)$. By Proposition 4.2, $\tau=\tau(t_o,x_o,\alpha)$ for some $\alpha\in\mathfrak{A}_{t_o}(y_o)$. Let ζ^α be the $\overline{\mathbf{F}}^{t_o}$−stopping time defined in (6.126). Similar to (6.90), one has $Y^{t_o,x_o,\alpha}_{\tau\wedge\zeta^\alpha}\ge\mathcal{Y}^{t_o,x_o,\tau}_{\tau\wedge\zeta^\alpha}$, P_o−a.s. It follows from (6.131) that

$$\begin{aligned}
\varphi(t_o,x_o,y_o)-\mathfrak{m}_2\ &\ge\ E_{t_o}\bigg[\mathbf{1}_{\{\tau\le\zeta^\alpha\}}\mathcal{R}(t_o,x_o,\tau)+\mathbf{1}_{\{\tau>\zeta^\alpha\}}\bigg(\mathcal{V}\big(\zeta^\alpha,\mathcal{X}^{t_o,x_o}_{\zeta^\alpha},Y^{t_o,x_o,\alpha}_{\zeta^\alpha}\big)+\int_t^{\zeta^\alpha}f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg)\bigg]\\
&\ge\ E_{t_o}\bigg[\mathbf{1}_{\{\tau\le\zeta^\alpha\}}\mathcal{R}(t_o,x_o,\tau)+\mathbf{1}_{\{\tau>\zeta^\alpha\}}\bigg(\mathcal{V}\big(\zeta^\alpha,\mathcal{X}^{t_o,x_o}_{\zeta^\alpha},\mathcal{Y}^{t_o,x_o,\tau}_{\zeta^\alpha}\big)+\int_t^{\zeta^\alpha}f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg)\bigg].
\end{aligned}$$

Taking supremum over $\tau\in\widehat{\mathcal{T}}^{t_o}_{x_o}(y_o)$ and applying Theorem 4.1 (2) yield that

$$\begin{aligned}\varphi(t_o,x_o,y_o)-\mathfrak{m}_2 &\ge \sup_{\tau\in\widehat{\mathcal{T}}^{t_o}_{x_o}(y_o)} E_{t_o}\bigg[\mathbf{1}_{\{\tau\le\zeta^\alpha\}}\mathcal{R}(t_o,x_o,\tau)+\mathbf{1}_{\{\tau>\zeta^\alpha\}}\bigg(\mathcal{V}\big(\zeta^\alpha,\mathcal{X}^{t_o,x_o}_{\zeta^\alpha},\mathcal{Y}^{t_o,x_o,\tau}_{\zeta^\alpha}\big)+\int_t^{\zeta^\alpha} f(r,\mathcal{X}^{t_o,x_o}_r)dr\bigg)\bigg]\\ &= \mathcal{V}(t_o,x_o,y_o)=\varphi(t_o,x_o,y_o).\end{aligned}$$

The contradiction appears again. □

A Appendix

Lemma A.1. *Let $t\in[0,\infty)$. (1) The sigma−field $\mathcal{F}^t$ satisfies $\mathscr{B}(\Omega^t)=\sigma\big(W^t_s;s\in[t,\infty)\big)=\sigma\Big(\underset{s\in[t,\infty)}{\cup}\mathcal{F}^t_s\Big)$.*

(2) For any $s\in[t,\infty]$, the sigma−field $\mathcal{F}^t_s$ can be countably generated by $\mathscr{C}^t_s:=\Big\{\bigcap_{i=1}^m (W^t_{t_i})^{-1}\big(O_{\delta_i}(x_i)\big):\ m\in\mathbb{N},\ t_i\in\mathbb{Q}_+\cup\{t\}$ with $t\le t_1\le\cdots\le t_m\le s,\ x_i\in\mathbb{Q}^d,\ \delta_i\in\mathbb{Q}_+\Big\}$.

Proof: 1a) Let $\omega\in\Omega^t$ and $\delta\in(0,\infty)$. For any $n\in\mathbb{N}$ with $n>1/\delta$, since all paths in Ω^t are continuous, we can deduce that

$$\begin{aligned}\overline{O}_{\delta-1/n}(\omega) &= \{\omega'\in\Omega^t:|\omega'(s)-\omega(s)|\le\delta-1/n,\ \forall\, s\in[t,\infty)\}=\{\omega'\in\Omega^t:|\omega'(s)-\omega(s)|\le\delta-1/n,\ \forall\, s\in[t,\infty)\cap\mathbb{Q}\}\\ &= \bigcap_{s\in[t,\infty)\cap\mathbb{Q}}\{\omega'\in\Omega^t: W^t_s(\omega')\in\overline{O}_{\delta-1/n}(\omega(s))\}=\bigcap_{s\in[t,\infty)\cap\mathbb{Q}}(W^t_s)^{-1}\big(\overline{O}_{\delta-1/n}(\omega(s))\big)\in\sigma\big(W^t_s;s\in[t,\infty)\big).\end{aligned}$$

It follows that $O_\delta(\omega)=\underset{n\in\mathbb{N}}{\cup}\overline{O}_{\delta-1/n}(\omega)\in\sigma\big(W^t_s;s\in[t,\infty)\big)$. As $\mathscr{B}(\Omega^t)$ is generated by open sets $\{O_\delta(\omega):\omega\in\Omega^t,\delta\in(0,\infty)\}$, one thus has $\mathscr{B}(\Omega^t)\subset\sigma\big(W^t_s;s\in[t,\infty)\big)$.

Next, let $s\in[t,\infty)$, $x\in\mathbb{R}^d$ and $\delta\in(0,\infty)$. Given $\omega\in(W^t_s)^{-1}(O_\delta(x))$, set $\lambda=\lambda(s,x,\omega):=\delta-|W^t_s(\omega)-x|>0$. Since

$$|W^t_s(\omega')-x|\le|\omega'(s)-\omega(s)|+|\omega(s)-x|\le\|\omega'-\omega\|_t+|\omega(s)-x|<\lambda+|\omega(s)-x|=\delta,\qquad \forall\,\omega'\in O_\lambda(\omega),$$

we see that $O_\lambda(\omega)\subset(W^t_s)^{-1}\big(O_\delta(x)\big)$ and thus $(W^t_s)^{-1}\big(O_\delta(x)\big)$ is an open set under the uniform norm $\|\cdot\|_t$. Then $O_\delta(x)\in\Lambda_s:=\{\mathcal{E}\subset\mathbb{R}^d:(W^t_s)^{-1}(\mathcal{E})\in\mathscr{B}(\Omega^t)\}$, which is a sigma−field of $\mathbb{R}^d$. As $\mathscr{B}(\mathbb{R}^d)$ is generated by open sets $\{O_\delta(x):x\in\mathbb{R}^d,\delta\in(0,\infty)\}$, one has $\mathscr{B}(\mathbb{R}^d)\subset\Lambda_s$, which implies that $\sigma\big(W^t_s;s\in[t,\infty)\big)=\sigma\big\{(W^t_s)^{-1}(\mathcal{E}):s\in[t,\infty),\ \mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\big\}\subset\mathscr{B}(\Omega^t)$.

1b) Clearly, $\mathcal{F}^t_s=\sigma\big(W^t_r;r\in[t,s]\big)\subset\sigma\big(W^t_r;r\in[t,\infty)\big)$, $\forall\,s\in[t,\infty)$. It follows that $\sigma\Big(\underset{s\in[t,\infty)}{\cup}\mathcal{F}^t_s\Big)\subset\sigma\big(W^t_s;s\in[t,\infty)\big)$. On the other hand, since $(W^t_s)^{-1}(\mathcal{E})\in\mathcal{F}^t_s\subset\sigma\Big(\underset{r\in[t,\infty)}{\cup}\mathcal{F}^t_r\Big)$ for any $s\in[t,\infty)$ and $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, we have $\sigma\big(W^t_s;s\in[t,\infty)\big)=\sigma\big\{(W^t_s)^{-1}(\mathcal{E}):s\in[t,\infty),\ \mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\big\}\subset\sigma\Big(\underset{s\in[t,\infty)}{\cup}\mathcal{F}^t_s\Big)$.

2) Fix $s\in[t,\infty]$. Define $[t,s\rangle:=[t,s]$ if $s<\infty$ and $[t,s\rangle:=[t,\infty)$ if $s=\infty$. Let $r\in[t,s\rangle$ and let $\{r_i\}_{i\in\mathbb{N}}\subset\{t\}\cup\big((t,r)\cap\mathbb{Q}_+\big)$ with $\lim_{i\to\infty}\uparrow r_i=r$. For any $x\in\mathbb{Q}^d$ and $\delta\in\mathbb{Q}_+$, the continuity of paths in Ω^t implies that $(W^t_r)^{-1}\big(O_\delta(x)\big)=\bigcup_{n=\lceil 2/\delta\rceil}^{\infty}\ \bigcup_{m\in\mathbb{N}}\ \bigcap_{i>m}\Big((W^t_{r_i})^{-1}\big(\overline{O}_{\delta-\frac{1}{n}}(x)\big)\Big)\in\sigma(\mathscr{C}^t_s)$. Thus $O_\delta(x)\in\widehat{\Lambda}_r:=\big\{\mathcal{E}\subset\mathbb{R}^d:(W^t_r)^{-1}(\mathcal{E})\in\sigma(\mathscr{C}^t_s)\big\}$, which is a sigma−field of $\mathbb{R}^d$. Since $\mathscr{B}(\mathbb{R}^d)$ can also be generated by $\{O_\delta(x):x\in\mathbb{Q}^d,\ \delta\in\mathbb{Q}_+\}$, we see that $\mathscr{B}(\mathbb{R}^d)\subset\widehat{\Lambda}_r$. It follows that $\mathcal{F}^t_s=\sigma\big(W^t_r;r\in[t,s\rangle\big)=\sigma\big\{(W^t_r)^{-1}(\mathcal{E}):r\in[t,s\rangle,\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\big\}\subset\sigma(\mathscr{C}^t_s)$. On the other hand, it is clear that $\sigma(\mathscr{C}^t_s)\subset\sigma\big\{(W^t_r)^{-1}(\mathcal{E}):r\in[t,s\rangle,\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)\big\}=\sigma\big(W^t_r;r\in[t,s\rangle\big)=\mathcal{F}^t_s$. □

Lemma A.2. *Let $0\le t\le s<\infty$.*
(1) The mapping Π^t_s is $\mathcal{F}^t_r/\mathcal{F}^s_r$−measurable for any $r\in[s,\infty]$. Then for each $\mathbf{F}^s$−stopping time τ, $\tau(\Pi^t_s)$ is a $\mathbf{F}^t$−stopping time with values in $[s,\infty]$.
(2) The law of Π^t_s under P_t is P_s: i.e., $P_t\circ(\Pi^t_s)^{-1}\big(\widetilde{A}\big)=P_s\big(\widetilde{A}\big)$, $\forall\,\widetilde{A}\in\mathcal{F}^s$.

Proof: 1) For any $r\in[s,\infty)$, an analogy to Lemma A.1 of [10] shows that $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}^t_r\subset\mathcal{F}^t$ for any $\widetilde{A}\in\mathcal{F}^s_r$. Set $\widetilde{\Lambda}:=\big\{\widetilde{A}\subset\Omega^s:(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}^t\big\}$, which is a sigma−field of Ω^s. As $\mathcal{F}^s_r\subset\widetilde{\Lambda}$ for any $r\in[s,\infty)$, we see from Lemma A.1 (1) that $\mathcal{F}^s=\sigma\Big(\underset{r\in[s,\infty)}{\cup}\mathcal{F}^s_r\Big)\subset\widetilde{\Lambda}$, i.e., $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}^t$ for any $\widetilde{A}\in\mathcal{F}^s$.

Let τ be an $\mathbf{F}^s-$stopping time. For any $r\in[s,\infty)$, since $\{\tau\le r\}\in\mathcal{F}^s_r$, we see that $\big\{\tau(\Pi^t_s)\le r\big\}=(\Pi^t_s)^{-1}\big(\{\tau\le r\}\big)\in\mathcal{F}^t_r$. Thus $\tau(\Pi^t_s)$ is a $\mathbf{F}^t-$stopping time with values in $[s,\infty]$.

2) Next, let us demonstrate that the induced probability $\widetilde{P}:=P_t\circ(\Pi^t_s)^{-1}$ equals to P_s on $\mathcal{F}^s$. Since the Wiener measure P_s on $(\Omega^s,\mathcal{F}^s)$ is unique (see e.g. Proposition I.3.3 of [53]), it suffices to show that the canonical process W^s is a Brownian motion on Ω^s under $\widetilde{P}$: Let $s\le r<r'<\infty$. For any $\mathcal{E}\in\mathscr{B}(\mathbb{R}^d)$, one can deduce that

$$\begin{aligned}(\Pi^t_s)^{-1}\Big(\big(W^s_{r'}-W^s_r\big)^{-1}(\mathcal{E})\Big)&=\big\{\omega\in\Omega^t: W^s_{r'}\big((\Pi^t_s)(\omega)\big)-W^s_r\big((\Pi^t_s)(\omega)\big)\in\mathcal{E}\big\}\\&=\big\{\omega\in\Omega^t:\omega(r')-\omega(s)-\big(\omega(r)-\omega(s)\big)\in\mathcal{E}\big\}=(W^t_{r'}-W^t_r)^{-1}(\mathcal{E}).\end{aligned}\tag{A.1}$$

So $\widetilde{P}\Big(\big(W^s_{r'}-W^s_r\big)^{-1}(\mathcal{E})\Big)=P_t\big((W^t_{r'}-W^t_r)^{-1}(\mathcal{E})\big)$, which shows that the distribution of $W^s_{r'}-W^s_r$ under $\widetilde{P}$ is the same as that of $W^t_{r'}-W^t_r$ under P_t (a $d-$dimensional normal distribution with mean 0 and variance matrix $(r'-r)I_{d\times d}$).

On the other hand, for any $\widetilde{A}\in\mathcal{F}^s_r$, since $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}^t_r$ is independent of $W^t_{r'}-W^t_r$ under P_t, (A.1) implies that

$$\begin{aligned}&\widetilde{P}\Big(\widetilde{A}\cap\big(W^s_{r'}-W^s_r\big)^{-1}(\mathcal{E})\Big)=P_t\Big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\cap(\Pi^t_s)^{-1}\Big(\big(W^s_{r'}-W^s_r\big)^{-1}(\mathcal{E})\Big)\Big)\\&\quad=P_t\Big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\Big)\cdot P_t\Big((\Pi^t_s)^{-1}\Big(\big(W^s_{r'}-W^s_r\big)^{-1}(\mathcal{E})\Big)\Big)=\widetilde{P}\big(\widetilde{A}\big)\cdot\widetilde{P}\Big(\big(W^s_{r'}-W^s_r\big)^{-1}(\mathcal{E})\Big),\quad\forall\,\mathcal{E}\in\mathscr{B}(\mathbb{R}^d),\end{aligned}$$

which shows that $W^s_{r'}-W^s_r$ is independent of $\mathcal{F}^s_r$ under $\widetilde{P}$. Hence, W^s is a Brownian motion on Ω^s under $\widetilde{P}$. □

We have the following extension of Lemma A.2.

Lemma A.3. *Let $0\le t\le s<\infty$.*
(1) For any P_s-null set $\widetilde{\mathcal{N}}$, $(\Pi^t_s)^{-1}\big(\widetilde{\mathcal{N}}\big)$ is a P_t-null set.
(2) For any $r\in[s,\infty]$, the mapping Π^t_s is $\overline{\mathcal{F}}^t_r/\overline{\mathcal{F}}^s_r-$measurable. Then for each $\tau\in\overline{\mathcal{T}}^s$, $\tau(\Pi^t_s)$ is a $\overline{\mathcal{T}}^t-$stopping time with values in $[s,\infty]$.
(3) $P_t\circ(\Pi^t_s)^{-1}\big(\widetilde{A}\big)=P_s\big(\widetilde{A}\big)$ holds for any $\widetilde{A}\in\overline{\mathcal{F}}^s$.

Proof: 1) Let $\widetilde{\mathcal{N}}\in\mathscr{N}^s$, so there exists an $\widetilde{A}\in\mathcal{F}^s$ such that $\widetilde{\mathcal{N}}\subset\widetilde{A}$ and $P_s\big(\widetilde{A}\big)=0$. Lemma A.2 implies that $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}^t$ and that $P_t\big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\big)=P_s\big(\widetilde{A}\big)=0$. As $(\Pi^t_s)^{-1}\big(\widetilde{\mathcal{N}}\big)\subset(\Pi^t_s)^{-1}\big(\widetilde{A}\big)$, we see that $(\Pi^t_s)^{-1}\big(\widetilde{\mathcal{N}}\big)\in\mathscr{N}^t$.
2) Given $r\in[s,\infty]$, Lemma A.2 (1) shows that $\mathcal{F}^s_r\subset\Lambda_r:=\big\{\widetilde{A}\subset\Omega^s:(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\mathcal{F}^t_r\big\}\subset\overline{\Lambda}_r:=\big\{\widetilde{A}\subset\Omega^s:(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\overline{\mathcal{F}}^t_r\big\}$, which is clearly a sigma$-$field of Ω^s. Since $\mathscr{N}^s\subset\overline{\Lambda}_r$ by Part (1), it follows that $\overline{\mathcal{F}}^s_r=\sigma(\mathcal{F}^s_r\cup\mathscr{N}^s)\subset\overline{\Lambda}_t$, i.e. $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\in\overline{\mathcal{F}}^t_r$ for any $\widetilde{A}\in\overline{\mathcal{F}}^s_r$.

Let $\tau\in\overline{\mathcal{T}}^s$. For any $r\in[s,\infty)$, since $\{\tau\le r\}\in\overline{\mathcal{F}}^s_r$, we see that $\big\{\tau(\Pi^t_s)\le r\big\}=(\Pi^t_s)^{-1}\big(\{\tau\le r\}\big)\in\overline{\mathcal{F}}^t_r$. Thus $\tau(\Pi^t_s)$ is a $\overline{\mathcal{T}}^t-$stopping time with values in $[s,\infty]$.
3) Let $\widetilde{A}\in\overline{\mathcal{F}}^s$. Similar to Problem 2.7.3 of [32], there exists an $\mathcal{A}\in\mathcal{F}^s$ such that $\widetilde{A}\Delta\mathcal{A}\in\mathscr{N}^s$. Since

$$\begin{aligned}(\Pi^t_s)^{-1}\big(\widetilde{A}\Delta\mathcal{A}\big)&=(\Pi^t_s)^{-1}\big((\widetilde{A}\cap\mathcal{A}^c)\cup(\mathcal{A}\cap\widetilde{A}^c)\big)=(\Pi^t_s)^{-1}(\widetilde{A}\cap\mathcal{A}^c)\cup(\Pi^t_s)^{-1}(\mathcal{A}\cap\widetilde{A}^c)\\&=\Big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\cap\big((\Pi^t_s)^{-1}(\mathcal{A})\big)^c\Big)\cup\Big((\Pi^t_s)^{-1}(\mathcal{A})\cap\big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\big)^c\Big)=(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\Delta(\Pi^t_s)^{-1}\big(\mathcal{A}\big),\end{aligned}$$

we know from Part (1) that $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)\Delta(\Pi^t_s)^{-1}\big(\mathcal{A}\big)$ is a P_t-null set. So by Part (2) and Lemma A.2 (1), the $\overline{\mathcal{F}}^t-$measurable random variable $(\Pi^t_s)^{-1}\big(\widetilde{A}\big)$ equals to the $\mathcal{F}^t-$measurable random variable $(\Pi^t_s)^{-1}\big(\mathcal{A}\big)$, P_t-a.s. Then Lemma A.2 (2) yields that $P_t\big((\Pi^t_s)^{-1}\big(\widetilde{A}\big)\big)=P_t\big((\Pi^t_s)^{-1}(\mathcal{A})\big)=P_s(\mathcal{A})=P_s(\widetilde{A})$. □

Lemma A.4. *Let $t\in[0,\infty)$.*
(1) For any $\xi\in L^1\big(\overline{\mathcal{F}}^t,\mathbb{E}\big)$ and $s\in[t,\infty]$, $E_t\big[\xi\big|\overline{\mathcal{F}}^t_s\big]=E_t[\xi|\mathcal{F}^t_s]$, P_t-a.s. Consequently, an $\mathbb{E}-$valued martingale (resp. local martingale or semi-martingale) with respect to $(\mathbf{F}^t,P_t)$ is also a martingale (resp. local martingale or semi-martingale) with respect to $\big(\overline{\mathbf{F}}^t,P_t\big)$.
(2) For any $s\in[t,\infty]$ and any $\mathbb{E}-$valued, $\overline{\mathcal{F}}^t_s-$measurable random variable ξ, there exists an $\mathbb{E}-$valued, $\mathcal{F}^t_s-$measurable random variable $\widetilde{\xi}$ such that $\widetilde{\xi}=\xi$, P_t-a.s.
(3) For any $\mathbb{E}-$valued, $\overline{\mathbf{F}}^t-$adapted process $X=\{X_s\}_{s\in[t,\infty)}$ with P_t-a.s. left-continuous paths, there exists an $\mathbb{E}-$valued, $\mathbf{F}^t-$predictable process $\widetilde{X}=\big\{\widetilde{X}_s\big\}_{s\in[t,\infty)}$ such that $\big\{\omega\in\Omega^t:\widetilde{X}_s(\omega)\neq X_s(\omega)\text{ for some }s\in[t,\infty)\big\}\in\mathscr{N}^t$.

Proof: 1) Let $\xi\in L^1\big(\overline{\mathcal{F}}^t,\mathbb{E}\big)$ and $s\in[t,\infty]$. For any $A\in\overline{\mathcal{F}}^t_s=\sigma\big(\mathcal{F}^t_s\cup\mathscr{N}^t\big)$, similar to Problem 2.7.3 of [32], there exists an $\widetilde{A}\in\mathcal{F}^t_s$ such that $A\Delta\widetilde{A}\in\mathscr{N}^t$. Then we can deduce that $\int_A\xi dP_t=\int_{\widetilde{A}}\xi dP_t=\int_{\widetilde{A}}E_t\big[\xi\big|\mathcal{F}^t_s\big]dP_t=\int_A E_t\big[\xi\big|\mathcal{F}^t_s\big]dP_t$, which implies that $E_t\big[\xi\big|\overline{\mathcal{F}}^t_s\big]=E_t\big[\xi\big|\mathcal{F}^t_s\big]$, P_t-a.s.

2) Let $s\in[t,\infty]$ and let ξ be an $\mathbb{E}-$valued, $\overline{\mathcal{F}}^t_s-$measurable random variable. We first assume $\mathbb{E}=\mathbb{R}$. For any $n\in\mathbb{N}$, we set $\xi_n:=(\xi\wedge n)\vee(-n)\in\overline{\mathcal{F}}^t_s$ and see from Part (1) that $\widetilde{\xi}_n:=E_t\big[\xi_n\big|\mathcal{F}^t_s\big]=E_t\big[\xi_n\big|\overline{\mathcal{F}}^t_s\big]=\xi_n$, P_t-a.s. Clearly, the random variable $\widetilde{\xi}:=\Big(\overline{\lim\limits_{n\to\infty}}\widetilde{\xi}_n\Big)\mathbf{1}_{\big\{\overline{\lim\limits_{n\to\infty}}\widetilde{\xi}_n<\infty\big\}}$ is $\mathcal{F}^t_s-$measurable and satisfies

$$\widetilde{\xi}=\Big(\overline{\lim_{n\to\infty}}\xi_n\Big)\mathbf{1}_{\big\{\overline{\lim\limits_{n\to\infty}}\xi_n<\infty\big\}}=\xi\mathbf{1}_{\{\xi<\infty\}}=\xi,\quad P_t-\text{a.s.}$$

When $\mathbb{E}=\mathbb{R}^k$ for some $k>1$, let ξ^i be the i-th component of ξ, $i=1,\cdots,k$. We denote by $\widetilde{\xi}^i$ the real-valued, $\mathcal{F}^t_s-$measurable random variable such that $\widetilde{\xi}^i=\xi^i$, P_t-a.s. Then $\widetilde{\xi}=(\widetilde{\xi}^1,\cdots,\widetilde{\xi}^k)$ is an $\mathbb{E}-$valued, $\mathcal{F}^t_s-$measurable random variable such that $\widetilde{\xi}=\xi$, P_t-a.s.

3) Let $X=\{X_s\}_{s\in[t,\infty)}$ be an $\mathbb{E}-$valued, $\overline{\mathbf{F}}^t-$adapted process with P_t-a.s. left-continuous paths. Like Part (2), it suffices to discuss the case of $\mathbb{E}=\mathbb{R}$. For any $s\in[t,\infty)\cap\mathbb{Q}$, Part (2) shows that there exists a real-valued, $\mathcal{F}^t_s-$measurable random variable $\mathscr{X}_s$ such that $\mathscr{X}_s=X_s$, P_t-a.s. Define $\mathcal{N}:=\big\{\omega\in\Omega^t:\text{the path }X_\cdot(\omega)\text{ is not left-continuous}\big\}\cup\Big(\underset{s\in[t,\infty)\cap\mathbb{Q}}{\cup}\{X_s\neq\mathscr{X}_s\}\Big)\in\mathscr{N}^t$.

For any $n\in\mathbb{N}$, set $t^n_i=t+i/n$, $\forall\,i\in\mathbb{N}\cup\{0\}$. Since $X^n_s:=\mathscr{X}_t\mathbf{1}_{\{s=t\}}+\sum_{i=1}^{n^2}\mathscr{X}_{t^n_{i-1}}\mathbf{1}_{\{s\in(t^n_{i-1},t^n_i]\}}$, $s\in[t,\infty)$ is a real$-$valued, $\mathbf{F}^t-$predictable process, we see that $\widetilde{X}_s:=\Big(\overline{\lim\limits_{n\to\infty}}X^n_s\Big)\mathbf{1}_{\big\{\overline{\lim\limits_{n\to\infty}}X^n_s<\infty\big\}}$, $s\in[t,\infty)$ also defines a real$-$valued, $\mathbf{F}^t-$predictable process.

Let $\omega\in\mathcal{N}^c$ and $s\in(t,\infty)$. For any $n\in\mathbb{N}$ with $n\ge s-t$, since $s\in\big(s_n-\frac{1}{n},s_n\big]$ with $s_n:=t+\frac{\lceil n(s-t)\rceil}{n}$, one has $X^n_s(\omega)=\mathscr{X}_{s_n-\frac{1}{n}}(\omega)=X_{s_n-\frac{1}{n}}(\omega)$. Clearly, $\lim\limits_{n\to\infty}(s_n-\frac{1}{n})=s$. As $n\to\infty$, the left-continuity of X shows that $\lim\limits_{n\to\infty}X^n_s(\omega)=\lim\limits_{n\to\infty}X_{s_n-\frac{1}{n}}(\omega)=X_s(\omega)$, which implies that $\mathcal{N}^c\subset\big\{\omega\in\Omega^t:\widetilde{X}_s(\omega)=X_s(\omega),\ \forall\,s\in[t,\infty)\big\}$. □

Example A.1. *Suppose that $d=1$ and $\mathfrak{g}:=\sup\limits_{(t,x)\in(0,\infty)\times\mathbb{R}^l}g(t,x)<\infty$. Given $(t,x)\in[0,\infty)\times\mathbb{R}^l$, there exist $y\in(0,\infty)$ and $\mathfrak{q}\in\mathbb{H}^{2,\text{loc}}_t(\mathbb{R})$ such that $\alpha_s:=y+\int_t^s\mathfrak{q}_r dW^t_r$, $s\in[t,\infty)$ is a positive strict local martingale with respect to $\big(\overline{\mathbf{F}}^t,P_t\big)$ that satisfies $E_t\big[\int_t^{\tau(t,x,\alpha)}g(r,\mathcal{X}^{t,x}_r)dr\big]<y$.*

Proof: Let $q\in(1,\infty)$. In light of [23], the solution $\{\Upsilon_s\}_{s\in[t,\infty)}$ to

$$\Upsilon_s=1+\int_t^s(\Upsilon_r)^q dW^t_r,\quad s\in[t,\infty)$$

is positive strict local martingale with respect to $(\overline{\mathbf{F}}^t,P_t)$, So there exists a $\mathfrak{s}\in(0,\infty)$ such that $E_t[\Upsilon_{\mathfrak{s}}]<1$.

Let $y\in[1+\mathfrak{g}(\mathfrak{s}-t),\infty)$ and set $\mathfrak{q}^o_s:=(\Upsilon_s)^q>0$, $s\in[t,\infty)$. For any $n\in\mathbb{N}$, the $\overline{\mathbf{F}}^t-$stopping times $\zeta_n:=\inf\{s\in[t,\infty):|\Upsilon_s-1|>n\}$ satisfies that $E_t\big[\int_t^{\zeta_n}(\mathfrak{q}^o_r)^2dr\big]=E_t\big[|\Upsilon_{\zeta_n}-1|^2\big]\le n^2$. So it holds except on a P_t-null set $\mathcal{N}_n$ that $\int_t^{\zeta_n}(\mathfrak{q}^o_r)^2dr<\infty$. Since Υ is also a supermartingale such that $\Upsilon_\infty:=\lim\limits_{s\to\infty}\Upsilon_s$ exists in $[0,\infty)$, P_t-a.s., the continuity of process Υ implies that for all $\omega\in\Omega^t$ except on a P_t-null set $\widetilde{\mathcal{N}}$, $\zeta_{\mathfrak{n}}(\omega)=\infty$ for some $\mathfrak{n}=\mathfrak{n}(\omega)\in\mathbb{N}$. Given $\omega\in\Big(\underset{n\in\mathbb{N}}{\cap}\mathcal{N}^c_n\Big)\cap\widetilde{\mathcal{N}}^c$, one has $\int_t^\infty|\mathfrak{q}^o_r(\omega)|^2dr=\int_t^{\zeta_{\mathfrak{n}}(\omega)}|\mathfrak{q}^o_r(\omega)|^2dr<\infty$. Thus, $\mathfrak{q}^o\in L^{2,\text{loc}}_t(\mathbb{R})$.

Set $\alpha^o_s:=y+\int_t^s\mathfrak{q}^o_r dW^t_r=\Upsilon_s+y-1>0$, $s\in[t,\infty)$. As it holds P_t-a.s. that

$$\int_t^s g(r,\mathcal{X}^{t,x}_r)dr\le\mathfrak{g}(\mathfrak{s}-t)<y-1+\Upsilon_s=\alpha^o_s,\quad\forall\,s\in[t,\mathfrak{s}],$$

we see that $\mathfrak{s}<\tau_o:=\tau(t,x,\alpha^o)$, P_t-a.s.

Next, let us define $\mathfrak{q}_s:=\mathbf{1}_{\{s\le\tau_o\}}\mathfrak{q}^o_s$, $s\in[t,\infty)$, which is clearly of $L^{2,\text{loc}}_t(\mathbb{R})$. Then $\alpha_s:=y+\int_t^s\mathfrak{q}_r dW^t_r=\alpha^o_{\tau_o\wedge s}$, $\forall\,s\in[t,\infty)$ and it follows that $\tau(t,x,\alpha)=\tau_o>\mathfrak{s}$, P_t-a.s. Since $\alpha^o=\Upsilon+y-1$ is a positive continuous supermartingale, we can deduce from the continuity of α and the optional sampling theorem that

$$E_t\Big[\int_t^{\tau(t,x,\alpha)}g(r,\mathcal{X}^{t,x}_r)dr\Big]=E_t\big[\alpha_{\tau(t,x,\alpha)}\big]=E_t\big[\alpha^o_{\tau_o}\big]\le E_t\big[\alpha^o_{\mathfrak{s}}\big]=E[\Upsilon_{\mathfrak{s}}+y-1]<y.$$ □

A.1 Proofs of Starred Statements in Section 6

Proof of (6.72)**:** Given $s\in[t,\infty)$, let i_o be the largest integer such that $t_{i_o}\le s$. For any $i=1,\cdots,i_o$ and $n\in\mathbb{N}$, since $\{\tau\le\zeta\}\in\overline{\mathcal{F}}^t_{\tau\land\zeta}\subset\overline{\mathcal{F}}^t_\tau$, one can deduce that $\{\overline{\tau}\le s\}=\big(\{\tau\le\zeta\}\cap\{\tau\le s\}\big)\cup\Big(\underset{i\le i_o}{\cup}\ \underset{n\in\mathbb{N}}{\cup}\mathcal{A}^i_n\cap\big\{\tau^i_n(\Pi^t_{t_i})\le s\big\}\Big)\cup\big(\{\tau>\zeta\}\cap\mathcal{N}_{t,x,\tau}\big)\in\overline{\mathcal{F}}^t_s$. So $\overline{\tau}\in\overline{\mathcal{T}}^t$.

For $i,n\in\mathbb{N}$ and $\omega\in\mathcal{A}^i_n\cap\big(\mathcal{N}^{i,n}\big)^c\subset\{\tau>\zeta=t_i\}$, since $\mathfrak{X}_{t_i}(\omega)\in\mathcal{O}^i_n=O_{\delta_i(x^i_n)}(x^i_n)$ and $\mathfrak{Y}_{t_i}(\omega)\in\mathcal{D}^i_n\subset\big(y^i_n-\varepsilon/2,\infty\big)$, applying (6.67) with $(\mathfrak{x},\mathfrak{x}',\varsigma)=\big(x^i_n,\mathfrak{X}_{t_i}(\omega),\tau^i_n\big)$, we see from (6.70) that

$$E_t\bigg[\int_t^{\tau^i_n(\Pi^t_{t_i})}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg](\omega)<E_{t_i}\bigg[\int_{t_i}^{\tau^i_n}g\big(r,\mathcal{X}^{t_i,x^i_n}_r\big)dr\bigg]+\varepsilon/2+\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr\le y^i_n+\varepsilon/2+\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr$$
$$<\mathfrak{Y}_{t_i}(\omega)+\varepsilon+\int_t^{t_i}g\big(r,\mathfrak{X}_r(\omega)\big)dr=E_t\bigg[\int_t^\tau g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg](\omega)+\varepsilon. \tag{A.2}$$

Taking summation over $i,n\in\mathbb{N}$, one can deduce from (6.71) and the monotone convergence theorem that

$$E_t\bigg[\mathbf{1}_{\{\tau>\zeta\}}\int_t^{\overline{\tau}}g(r,\mathfrak{X}_r)dr\bigg]=E_t\bigg[\sum_{i,n\in\mathbb{N}}\mathbf{1}_{\mathcal{A}^i_n}\int_t^{\tau^i_n(\Pi^t_{t_i})}g(r,\mathfrak{X}_r)dr\bigg]=\sum_{i,n\in\mathbb{N}}E_t\bigg[\mathbf{1}_{\mathcal{A}^i_n}\int_t^{\tau^i_n(\Pi^t_{t_i})}g(r,\mathfrak{X}_r)dr\bigg]$$
$$=\sum_{i,n\in\mathbb{N}}E_t\bigg[\mathbf{1}_{\mathcal{A}^i_n}E_t\bigg[\int_t^{\tau^i_n(\Pi^t_{t_i})}g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg]\bigg]\le\sum_{i,n\in\mathbb{N}}E_t\bigg[\mathbf{1}_{\mathcal{A}^i_n}E_t\bigg[\int_t^\tau g(r,\mathfrak{X}_r)dr\Big|\overline{\mathcal{F}}^t_{t_i}\bigg]\bigg]+\varepsilon$$
$$=\sum_{i,n\in\mathbb{N}}E_t\bigg[\mathbf{1}_{\mathcal{A}^i_n}\int_t^\tau g(r,\mathfrak{X}_r)dr\bigg]+\varepsilon=E_t\bigg[\sum_{i,n\in\mathbb{N}}\mathbf{1}_{\mathcal{A}^i_n}\int_t^\tau g(r,\mathfrak{X}_r)dr\bigg]+\varepsilon=E_t\bigg[\mathbf{1}_{\{\tau>\zeta\}}\int_t^\tau g(r,\mathfrak{X}_r)dr\bigg]+\varepsilon.$$

It follows that $E_t\Big[\int_t^{\overline{\tau}}g(r,\mathfrak{X}_r)dr\Big]\le E_t\big[\int_t^\tau g(r,\mathfrak{X}_r)dr\big]+\varepsilon\le y+\varepsilon$. Thus $\overline{\tau}\in\mathcal{T}^t_x(y+\varepsilon)$. □

Proof of (6.102)**: 1)** Given $i=1,\cdots,\mathcal{I}_o$, if $A\in\overline{\mathcal{F}}^t_{t_i}$ and if $\{\Upsilon_s\}_{s\in[t_i,\infty)}$ is an $\overline{\mathbf{F}}^t-$adapted continuous process over period $[t_i,\infty)$ with $\Upsilon_{t_i}=0$, P_t-a.s., one can easily deduce that $\big\{\mathbf{1}_{\{s\ge t_i\}\cap A}\Upsilon_s\big\}_{s\in[t,\infty)}$ is an $\overline{\mathbf{F}}^t-$adapted continuous process starting from 0. Then we see from (6.100) that $\overline{M}$ is an $\overline{\mathbf{F}}^t-$adapted continuous process and $\overline{K}$ is an $\overline{\mathbf{F}}^t-$adapted continuous increasing process with $\overline{K}_t=K_t=0$, P_t-a.s.

For any $\omega\in\mathcal{A}^c_\sharp\cap\mathcal{N}^c$, $\overline{K}_\cdot(\omega)=K_\cdot(\omega)$ is an increasing path; for $i=1,\cdots,\mathcal{I}_o$ and $n=1,\cdots,\mathfrak{n}_i$, it holds for any $\omega\in\mathcal{A}^i_n\cap(\Pi^t_{t_i})^{-1}\big((\widetilde{\mathcal{N}}^{i,n})^c\big)$ that

$$\overline{K}_s(\omega)=\mathbf{1}_{\{s<t_i\}}K_s(\omega)+\mathbf{1}_{\{s\ge t_i\}}\big(K^{i,n}_s(\Pi^t_{t_i}(\omega))+K_{t_i}(\omega)+\mathcal{K}^{i,n}_s(\omega)\big),\quad s\in[t,\infty)$$

is an also increasing path. Thus, $\overline{K}$ has increasing paths except the P_t-null set $\mathcal{N}_\sharp$.

2) *To show $\overline{M}$ is a uniformly integrable martingale with respect to $(\overline{\mathbf{F}}^t,P_t)$, we define*

$$\overline{\xi}:=\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\big(\xi^{i,n}(\Pi^t_{t_i})-\xi+M_{t_i}-y^i_n+\delta_i(x^i_n,y^i_n)\big)=\mathbf{1}_{\mathcal{A}_\sharp}(M_\zeta-\xi)+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\big(\xi^{i,n}(\Pi^t_{t_i})-y^i_n+\delta_i(x^i_n,y^i_n)\big).$$

Since M is a uniformly integrable continuous $\big(\overline{\mathbf{F}}^t,P_t\big)-$martingale, we know from the optional sampling theorem (e.g. Theorem II.3.2 of [53]) that $M_\zeta=E_t\big[\xi\big|\overline{\mathcal{F}}^t_\zeta\big]$, P_t-a.s. It follows that

$$E_t\big[|M_\zeta|\big]=E_t\Big[\big|E_t\big[\xi\big|\overline{\mathcal{F}}^t_\zeta\big]\big|\Big]\le E_t\Big[E_t\big[|\xi|\big|\overline{\mathcal{F}}^t_\zeta\big]\Big]=E_t\big[|\xi|\big]<\infty. \tag{A.3}$$

Given $i=1,\cdots,\mathcal{I}_o$ and $n=1,\cdots,\mathfrak{n}_i$, as $\xi^{i,n}$ is $\overline{\mathcal{F}}^{t_i}-$measurable, Lemma A.3 (2) implies that $\xi^{i,n}(\Pi^t_{t_i})$ is $\overline{\mathcal{F}}^t-$measurable. By Proposition 3.4 (2), it holds for P_t-a.s. $\omega\in\Omega^t$ that

$$E_t\big[|\xi^{i,n}(\Pi^t_{t_i})|\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega)=E_{t_i}\Big[\big(\big|\xi^{i,n}(\Pi^t_{t_i})\big|\big)^{t_i,\omega}\Big]=E_{t_i}\big[|\xi^{i,n}|\big]<\infty. \tag{A.4}$$

Taking expectation $E_t[\cdot]$ and using (A.3), one can deduce that

$$E_t\big[\,|\overline{\xi}\,|\big]\le E_t\big[|\xi|+|M_\zeta|\big]+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\big(E_t\big[|\xi^{i,n}(\Pi^t_{t_i})|\big]+y^i_n-\delta_i(x^i_n,y^i_n)\big)=2E_t\big[|\xi|\big]+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\big(E_{t_i}\big[|\xi^{i,n}|\big]+y^i_n\big)<\infty,$$

which shows that $\overline{\xi}\in L^1\big(\overline{\mathcal{F}}^t\big)$.

Fix $s\in[t,\infty)$. We denote by i_o the largest integer such that $t_{i_o}\le s$. Let $i=1,\cdots,i_o$ and $n=1,\cdots,\mathfrak{n}_i$. In light of Proposition 3.4 (2), there exists $\mathfrak{N}^{i,n}\in\mathscr{N}^t$ such that

$$E_t\big[\xi^{i,n}(\Pi^t_{t_i})\big|\overline{\mathcal{F}}^t_s\big](\omega)=E_s\Big[\big(\xi^{i,n}(\Pi^t_{t_i})\big)^{s,\omega}\Big]=E_s\Big[(\xi^{i,n})^{s,\Pi^t_{t_i}(\omega)}\Big],\quad\forall\,\omega\in(\mathfrak{N}^{i,n})^c. \tag{A.5}$$

The last equality uses the fact that $\Pi^t_{t_i}(\omega\otimes_s\widehat{\omega})=\Pi^t_{t_i}(\omega)\otimes_s\widehat{\omega}$ for all $\widehat{\omega}\in\Omega^s$. Applying Proposition 3.4 (2) again and using (6.98), we can find $\widetilde{\mathcal{N}}^{i,n}\in\mathscr{N}^{t_i}$ such that

$$M^{i,n}_s(\widetilde{\omega})=E_{t_i}\big[\xi^{i,n}\big|\overline{\mathcal{F}}^{t_i}_s\big](\widetilde{\omega})=E_s\big[(\xi^{i,n})^{s,\widetilde{\omega}}\big],\quad\forall\,\widetilde{\omega}\in\big(\widetilde{\mathcal{N}}^{i,n}\big)^c. \tag{A.6}$$

By Lemma A.3 (1), $\overline{\mathfrak{N}}^{i,n}:=\mathfrak{N}^{i,n}\cup\big((\Pi^t_{t_i})^{-1}\big(\widetilde{\mathcal{N}}^{i,n}\big)\big)$ is a P_t-null set. For any $\omega\in\big(\overline{\mathfrak{N}}^{i,n}\big)^c=(\mathfrak{N}^{i,n})^c\cap\big((\Pi^t_{t_i})^{-1}\big(\widetilde{\mathcal{N}}^{i,n}\big)\big)^c=(\mathfrak{N}^{i,n})^c\cap\Big((\Pi^t_{t_i})^{-1}\big(\big(\widetilde{\mathcal{N}}^{i,n}\big)^c\big)\Big)$, using (A.5) and taking $\widetilde{\omega}=\Pi^t_{t_i}(\omega)$ in (A.6) yield that

$$E_t\big[\xi^{i,n}(\Pi^t_{t_i})\big|\overline{\mathcal{F}}^t_s\big](\omega)=E_s\Big[(\xi^{i,n})^{s,\Pi^t_{t_i}(\omega)}\Big]=M^{i,n}_s\big(\Pi^t_{t_i}(\omega)\big).$$

Then we see from (6.91) that

$$\begin{aligned}E_t\Bigg[\sum_{i=1}^{i_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\big(\xi^{i,n}(\Pi^t_{t_i})-\xi+M_{t_i}-y^i_n+\delta_i(x^i_n,y^i_n)\big)\Big|\overline{\mathcal{F}}^t_s\Bigg]&=\sum_{i=1}^{i_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\Big(E_t\big[\xi^{i,n}(\Pi^t_{t_i})-\xi\big|\overline{\mathcal{F}}^t_s\big]+M_{t_i}-y^i_n+\delta_i(x^i_n,y^i_n)\Big)\\&=\sum_{i=1}^{i_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\big(M^{i,n}_s\big(\Pi^t_{t_i}\big)-M_s+M_{t_i}-y^i_n+\delta_i(x^i_n,y^i_n)\big)=\overline{M}_s-M_s,\quad P_t-\text{a.s.}\end{aligned} \tag{A.7}$$

If $i_o=\mathcal{I}_o$, (A.7) just shows that $E_t\big[\,\overline{\xi}\,\big|\overline{\mathcal{F}}^t_s\big]=\overline{M}_s-M_s$, P_t-a.s. and thus $\overline{M}_s=E_t\big[\,\overline{\xi}+\xi\big|\overline{\mathcal{F}}^t_s\big]$. Suppose next that $i_o<\mathcal{I}_o$. For $i=i_o+1,\cdots,\mathcal{I}_o$ and $n=1,\cdots,\mathfrak{n}_i$, using an analogy to (A.4), we can deduce from (6.98) and (6.99) that for P_t-a.s. $\omega\in\Omega^t$, $E_t\big[\xi^{i,n}(\Pi^t_{t_i})\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega)=E_{t_i}\big[\xi^{i,n}\big]=y^i_n-\delta_i(x^i_n,y^i_n)$. It follows that

$$\begin{aligned}&E_t\Bigg[\sum_{i=i_o+1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\big(\xi^{i,n}(\Pi^t_{t_i})-\xi+M_{t_i}-y^i_n+\delta_i(x^i_n,y^i_n)\big)\Big|\overline{\mathcal{F}}^t_s\Bigg]\\&\qquad=\sum_{i=i_o+1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}E_t\Big[\mathbf{1}_{\mathcal{A}^i_n}\Big(E_t\big[\xi^{i,n}(\Pi^t_{t_i})-\xi\big|\overline{\mathcal{F}}^t_{t_i}\big]+M_{t_i}-y^i_n+\delta_i(x^i_n,y^i_n)\Big)\Big|\overline{\mathcal{F}}^t_s\Big]=0,\quad P_t-\text{a.s.},\end{aligned}$$

which together with (A.7) yields $\overline{M}_s=E_t\big[\,\overline{\xi}+\xi\big|\overline{\mathcal{F}}^t_s\big]$, P_t-a.s. again. Therefore, $\overline{M}$ is a uniformly integrable martingale with respect to $(\overline{\mathbf{F}}^t,P_t)$.

3) *We now prove that* $E_t\big[\,\overline{K}_*\big]<\infty$.

Let $i=1,\cdots,\mathcal{I}_o$ and $n=1,\cdots,\mathfrak{n}_i$. It is clear that $\sup\limits_{s\in[t_i,\infty)\cap\mathbb{Q}}K^{i,n}_s$ is $\mathcal{F}^{t_i}-$measurable. Since the continuity of $K^{i,n}$ implies that $K^{i,n}_*(\widetilde{\omega})=\sup\limits_{s\in[t_i,\infty)\cap\mathbb{Q}}K^{i,n}_s(\widetilde{\omega})$ for any $\widetilde{\omega}\in(\mathcal{N}^{i,n})^c$, the random variable $K^{i,n}_*$ is $\overline{\mathcal{F}}^{t_i}-$measurable and we thus know from Lemma A.3 (2) that $K^{i,n}_*(\Pi^t_{t_i})$ is $\overline{\mathcal{F}}^t-$measurable. An analogy to (A.4) then shows that for P_t-a.s. $\omega\in\Omega^t$

$$E_t\big[K^{i,n}_*(\Pi^t_{t_i})\big|\overline{\mathcal{F}}^t_{t_i}\big](\omega)=E_{t_i}\big[\big(K^{i,n}_*(\Pi^t_{t_i})\big)^{t_i,\omega}\big]=E_{t_i}\big[K^{i,n}_*\big]<\infty. \tag{A.8}$$

As $\eta^i_n\le2\delta_i(x^i_n,y^i_n)$ on $\mathcal{A}^i_n$, it holds for any $s\in[t,\infty)$ that

$$\overline{K}_s=\mathbf{1}_{\mathcal{A}^c_\sharp}K_s+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\mathbf{1}_{\mathcal{A}^i_n}\Big(\mathbf{1}_{\{s<t_i\}}K_s+\mathbf{1}_{\{s\ge t_i\}}\big(K^{i,n}_s(\Pi^t_{t_i})+K_{t_i}+\mathcal{K}^{i,n}_s\big)\Big)\le K_*+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\big(K^{i,n}_*(\Pi^t_{t_i})+2\delta_i(x^i_n,y^i_n)\big).$$

Taking supremum over $s\in[t,\infty)$ and taking expectation $E_t[\cdot]$, we see from (A.8) that

$$E_t\big[\,\overline{K}_*\big]\le E_t[K_*]+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\big(E_t\big[K^{i,n}_*(\Pi^t_{t_i})\big]+2\delta_i(x^i_n,y^i_n)\big)=E_t[K_*]+\sum_{i=1}^{\mathcal{I}_o}\sum_{n=1}^{\mathfrak{n}_i}\big(E_{t_i}\big[K^{i,n}_*\big]+2\delta_i(x^i_n,y^i_n)\big)<\infty. \qquad\square$$

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