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Jump-Filtration Consistent Nonlinear Expectations with $\mathbb{L}^p$ Domains

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Abstract Given $p \in (1, 2]$, the wellposedness of backward stochastic differential equations with jumps (BSDEJs) in $\mathbb{L}^p$ sense gives rise to a so-called g-expectation with $\mathbb{L}^p$ domain under the jump filtration (the one generated by a Brownian motion and a Poisson random measure). In this paper, we extend such a g-expectation to a nonlinear expectation $\mathcal{E}$ with $\mathbb{L}^p$ domain that is consistent with the jump filtration. We study the basic (martingale) properties of the jump-filtration consistent nonlinear expectation $\mathcal{E}$ and show that under certain domination condition, the nonlinear expectation $\mathcal{E}$ can be represented by some g-expectation.

Keywords Backward stochastic differential equations with jumps · $\mathbb{L}^p$ solutions · g-Expectations · Nonlinear expectations consistent with jump filtration · Optional sampling · Doob–Meyer decomposition · Representation theorem

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1 Introduction

The Allais paradox and the Ellsberg paradox prompt people to develop a nonlinear-expectation version of von Neumann–Morgenstern's axiomatic system of expected utilities, a fundamental notion in the modern Economics. Motivated by such a generalization, Peng [28,29] introduced the so-called g-expectations $\mathcal{E}_g$ with $\mathbb{L}^2$ domains

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via backward stochastic differential equations (BSDEs) with generator g. These two seminal works and some following research ([5,6,10,31,36] among others) showed that the g-expectations are closely related to axiom-based coherent and convex risk measures (see [1,17]) in mathematical finance.

Let $\mathbf{F}^B = \{\mathcal{F}_t^B\}_{t\in[0,T]}$ denote the Brownian filtration. Coquet et al. [10] generalized g-expectations with domain $L^2(\mathcal{F}_T^B)$ to Brownian-filtration consistent nonlinear expectations ($\mathbf{F}^B$-expectations for short) with domain $L^2(\mathcal{F}_T^B)$ and systematically analyzed them using the BSDE theory. These authors demonstrated that under the following domination condition

$$\mathcal{E}[\xi] - \mathcal{E}[\eta] \le \mathcal{E}_{g^\mu}[\xi - \eta], \quad \forall\, \xi, \eta \in L^2(\mathcal{F}_T^B) \tag{1.1}$$

with $g^\mu = \mu|z|$ for some $\mu > 0$, a nonlinear expectation $\mathcal{E}$ can be represented by a g-expectation with domain $L^2(\mathcal{F}_T^B)$ or solutions of BSDEs with generator g and square-integrable terminal data.

Let $p \in (1,2]$. Based on our study [41] on $\mathbb{L}^p$ solutions of backward stochastic differential equations with jumps (BSDEJs), we generalized the notion of (conditional) g-expectations to the jump case with $\mathbb{L}^p$ domain and studied their properties in [40]. In the present paper, we further extend these g-expectations to a general class of nonlinear expectations $\mathcal{E}$ with $\mathbb{L}^p$ domains that are consistent with the jump filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the Brownian motion B and the independent Poisson random measure $N_{\mathfrak{p}}$.

1.1 Main Results

An $\mathbf{F}$-consistent nonlinear expectation (or $\mathbf{F}$-expectation for short) is a family of mappings $\mathcal{E} = \big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ satisfying “monotonicity”, “constant preserving” “consistency” “zero-one law”.

When a translation invariant $\mathbf{F}$-expectation $\mathcal{E}$ with domain $L^p(\mathcal{F}_T)$ is dominated by some g^Ξ-expectation (see (2.3)) in sense that

$$\mathcal{E}[\xi] - \mathcal{E}[\eta] \le \mathcal{E}_{g^\Xi}[\xi - \eta], \quad \forall\, \xi, \eta \in L^p(\mathcal{F}_T), \tag{1.2}$$

the corresponding $\mathcal{E}$-martingales still possess many classic properties such as “optional sampling” (Proposition 3.4) and “Doob–Meyer decomposition” (Theorem 3.1). For a translation invariant $\mathbf{F}$-expectation $\mathcal{E}$ with domain $L^p(\mathcal{F}_T)$ under domination (1.2), if the Brownian motion and the Poisson random measure have independent increments under $\mathcal{E}$, the $\mathbf{F}$-expectation $\mathcal{E}$ can be represented by a g-expectation with a deterministic generator g that is independent of y and Lipschitz in (z,u) (see Theorem 4.1 for detailed description).

The significance of such a representation result might be more notable from the following consequence in mathematical finance: In a market with jumps, any *coherent* or *convex* time-consistent risk measure $\rho = \{\rho_t\}_{t\in[0,T]}$ with $\mathbb{L}^p$ domain that satisfies the required domination condition can be represented by the $\mathbb{L}^p$-solution of a BSDEJ with a deterministic Lipschitz generator g. Then one can utilize the BSDEJ theory to analyze the risk measure ρ and employ numerical schemes of BSDEJs to run simulation for financial problems involving ρ.

1.2 Main Contributions

The key to our main representation result (Theorem 4.1) lies in establishing a Doob–Meyer decomposition for **F**-expectations.

Let g^{Ξ_0} be a special Lipschitz generator satisfying (2.3) with $\beta \equiv 0$ and let $\mathcal{E}$ be a translation-invariant **F**-expectation with domain $L^p(\mathcal{F}_T)$ that is dominated by the $g^{\Xi_0}-$expectation in sense of (1.2). Since each $\mathcal{E}$-martingale X is a $\overline{g}^{\Xi_0}-$supermartingale, the upcrossing inequality under the $\overline{g}^{\Xi_0}-$expectation implies that X admits a càdlàg modification. In particular, taking $X_t = \mathcal{E}[\xi|\mathcal{F}_t]$, we can deduce from the a priori $\mathbb{L}^p$-estimate (2.1) of BSDEJs that $\mathcal{E}[\xi|\mathcal{F}_\cdot]$ is continuously dependent on ξ in $\mathbb{L}^p$ sense $\big($see (3.3)$\big)$. Such a continuous dependence is important for the approximation schemes in proving the optional sampling and Doob–Meyer decomposition for the **F**-expectation $\mathcal{E}$ (Proposition 3.4, Theorem 3.1).

Next, we obtain in Proposition 3.3 a semi-martingale decomposition of $\mathcal{E}$-martingales which states that each $\mathcal{E}$-martingale X can be expressed as $X_t = X_0 - \int_0^t \mathfrak{g}_s ds + \int_0^t Z_s dB_s + \int_{(0,t]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), t \in [0, T]$ with $-g^{\Xi_0}(t, -Z_t, -U_t) \le \mathfrak{g}_t \le g^{\Xi_0}(t, Z_t, U_t)$. Also, we show in Proposition 3.6 that the generalized BSDE (3.19) with respect to $\mathcal{E}$ admits a unique p-integrable solution if the driver $f(t, y)$ is continuous in y. Using these two results, we then derive the Doob–Meyer decomposition under the **F**-expectation $\mathcal{E}$ from a priori $\mathbb{L}^p$-estimate for a special BSDEJ (Proposition 2.4) and a monotonic limit theorem of p-integrable jump diffusion processes (Theorem 2.4).

By further assuming that both the Brownian motion and the Poisson random measure have independent increments under $\mathcal{E}$ $\big($see (4.1)$\big)$, we can exploit the Doob–Meyer decomposition and the semi-martingale decomposition of $\mathcal{E}$-martingales to define a deterministic measurable function $g(t, z, u)$ Lipschitz in (z, u) such that $-\int_0^t g(s, z, u) ds + z B_t + \int_{(0,t]} \int_{\mathcal{X}} u(x) \widetilde{N}_{\mathfrak{p}}(ds, dx)$, $t \in [0, T]$ is an $\mathcal{E}$-martingale for any (z, u) $\big($see (4.13)$\big)$. This allows us to eventually represent the **F**-expectation $\mathcal{E}$ by the g-expectation with the deterministic generator $g(t, z, u)$, proving Theorem 4.1.

1.3 Relevant Literature

The BSDEs were introduced by Bismut [4] as adjoint equations for the Pontryagin maximum principle in stochastic control theory. Since Pardoux and Peng [27] commenced a systematical research of BSDEs, the BSDE theory has grown rapidly and has been applied to various areas such as mathematical finance, theoretical economics, stochastic control and optimization, partial differential equations, differential geometry and etc, (see the references in [11,16]).

Among many extensions of BSDEs, Li and Tang [39] introduced into BSDEs a jump term that is driven by a Poisson random measure independent of the Brownian motion; and El Karoui et al. [16] initiated the study of $\mathbb{L}^p$-solutions of BSDEs. We analyzed

$\mathbb{L}^p$ solutions of multi-dimensional BSDEJs in [41] while Kruse and Popier [23,24] studied a similar $\mathbb{L}^p$-solution problem of BSDE under a right-continuous filtration which may be larger than the jump filtration $\mathbf{F}$. For a survey of the recent development of BSDEJs in numerous directions, see e.g. the introduction of Yao [26,41].

Royer [37] extended the g-expectations with $\mathbb{L}^2$ domain to the jump case and showed that under a similar domination condition to (1.2), an $\mathbf{F}$-expectation with domain $L^2(\mathcal{F}_T)$ can be represented by a g-expectation with domain $L^2(\mathcal{F}_T)$. On the other hand, [44] obtained the representation of dominated $\mathbf{F}^B$-expectations with domain $L^p(\mathcal{F}_T^B)$ by g-expectations with domain $L^p(\mathcal{F}_T^B)$. Our paper can be viewed as an extension of Royer [37] to the $\mathbb{L}^p$-domain case and a generalization of Zong and Hu [44] to the jump case, both of which are nontrivial since we have to tackle some technical hurdles arising exclusively in the $\mathbb{L}^p$ jump case (see the "Main contributions" part in the introduction of Yao [41] for details).

Based on the representation of $\mathbf{F}^B$-expectations by Coquet et al. [10], Delbaen et al. [12] derived an integral representation for the minimal penalty term of a dynamic convex risk measure under domination, which further allows [3] to transform an optimal stopping problem under such a risk measure to an equivalent zero-sum game of control and stopping and thus solve the optimal stopping problem. In light of the representation of $\mathbf{F}$-expectations by Royer [37], Tang and Wei [38] obtained an integral representation for the minimal penalty term of a dominated dynamic convex risk measure with jumps while Quenez and Sulem [35] studied the related optimization problem under model ambiguity.

There are many other extensions of Coquet et al. [10]'s representation result: Peng [30] considered an optimal stochastic control problem and showed that any $\mathbf{F}^B$-expectation dominated by the super-evaluation of the control problem is a g-expectation. Peng [33], and later [42,43], studied the representation of $\mathbf{F}^B$-expectations with domain $L^2(\mathcal{F}_T^B)$ (and thus a $\mathbf{F}^B$-dynamic pricing mechanism of square-integrable contingent claims) under a general conditional-expectation version of domination (1.1), which was statistically tested using Chicago Mercantile Exchange's data on options of S&P 500 Futures. In the discrete-time case, Cohen and Elliott [8,9] represented nearly time-consistent nonlinear expectations by solutions to backward stochastic *difference* equations. Bao and Tang [2] showed that under the domination (1.1), an $\mathbf{F}^B$-expection with a floor S can be represented by solutions of a BSDE with reflecting barrier S. Cohen [7] analyzed the representation of filtration-consistent nonlinear expectations in general probability spaces.

As to the quadratic case, Ma and Yao [25] studied the quadratic g-expectations (i.e. g has quadratic growth in z), then Hu et al. [18] represented $\mathbf{F}^B$-consistent quadratic nonlinear expectations (including a large class of convex risk measures that do not satisfying (1.1)) by quadratic g-expectations under a different domination condition. Lately, Kazi-Tani et al. [22] even extended the quadratic g-expectations to the jump case and provided a dual representation for quadratic $\mathbf{F}$-expectations.

Among various other research on nonlinear expectations, Peng [32] used a nonlinear generalization of Kolmogorov's extension theorem to construct a new type of $\mathbf{F}^B$-consistent nonlinear expectations via nonlinear Markov chains, and showed that the corresponding BSDEs under such nonlinear expectations are well-posed.

Recently, Denk et al. [15] discussed a maximal extension for convex nonlinear expectations which admit a representation in terms of countably additive measures. These authors derived a robust Kolmogorov extension theorem and used it to extend nonlinear kernels to an infinite-dimensional path space.

The rest of the paper is organized as follows: We introduce some notations in Sect. 1.4. In Sect. 2, after making basic assumptions on generator g, we review some properties of g-expectations with $\mathbb{L}^p$ domains under jump filtration such as optional sampling, upcrossing inequality and Doob–Meyer decomposition. In Sect. 3, we generalize g-expectations to jump-filtration consistent nonlinear expectations $\mathcal{E}$ with $\mathbb{L}^p$ domains. When a translation-invariant nonlinear expectation $\mathcal{E}$ satisfies certain domination condition, we derive some basic (martingale) properties such as a semi-martingale decomposition of $\mathcal{E}$-martingales, optional sampling and Doob–Meyer decomposition. In Sect. 4, we discuss the representation of translation-invariant nonlinear expectations under domination by g-expectations.

1.4 Notation and Preliminaries

Throughout this paper, we fix a time horizon $T \in (0,\infty)$ and let $(\Omega,\mathcal{F},P)$ be a complete probability space on which a d-dimensional Brownian motion B is defined.

For a generic càdlàg process X, let us denote its corresponding jump process by $\Delta X_t := X_t - X_{t-}$, $t \in [0,T]$ with $X_{0-} := X_0$. Given a measurable space $(\mathcal{X},\mathcal{F}_\mathcal{X})$, let $\mathfrak{p}$ be an $\mathcal{X}$-valued Poisson point process on $(\Omega,\mathcal{F},P)$ that is independent of B. For any scenario $\omega \in \Omega$, let $D_{\mathfrak{p}(\omega)}$ collect all jump times of path $\mathfrak{p}(\omega)$, which is a countable subset of $(0,T]$ (see e.g. [19, Sect. 1.9]). We assume that for some finite measure ν on $\big(\mathcal{X},\mathcal{F}_\mathcal{X}\big)$, the counting measure $N_\mathfrak{p}(dt,dx)$ of $\mathfrak{p}$ on $[0,T]\times\mathcal{X}$ has compensator $E\big[N_\mathfrak{p}(dt,dx)\big] = \nu(dx)dt$. The corresponding compensated Poisson random measure $\widetilde{N}_\mathfrak{p}$ is $\widetilde{N}_\mathfrak{p}(dt,dx) := N_\mathfrak{p}(dt,dx) - \nu(dx)dt$.

For any $t \in [0,T]$, we define sigma-fields

$$\mathcal{F}_t^B := \sigma\big\{B_s; s \le t\big\}, \qquad \mathcal{F}_t^N := \sigma\big\{N_\mathfrak{p}((0,s],A); s \le t, A \in \mathcal{F}_\mathcal{X}\big\},$$
$$\mathcal{F}_t := \sigma\left(\mathcal{F}_t^B \cup \mathcal{F}_t^N\right)$$

and augment them by all P-null sets of $\mathcal{F}$. Clearly, the jump filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the *usual hypotheses* (cf. e.g., [34]). Let $\mathscr{P}$ $\big($resp. $\widehat{\mathscr{P}}\big)$ denote the $\mathbf{F}$-progressively measurable (resp. $\mathbf{F}$-predictable) sigma-field on $[0,T]\times\Omega$, and let $\mathcal{T}$ be the set of all $\mathbf{F}$-stopping times with values in $[0,T]$. For any $\tau \in \mathcal{T}$, we set $\mathcal{T}_\tau := \{\gamma \in \mathcal{T} : \gamma \ge \tau,\ P-\text{a.s.}\}$.

The following spaces of functions will be used in the sequel:

(1) For any $p \in [1,\infty)$, let $L^p_+[0,T]$ be the space of all measurable functions $\psi : [0,T] \mapsto [0,\infty)$ with $\int_0^T \big(\psi(t)\big)^p dt < \infty$.
(2) For $p \in (1,\infty)$, let $L^p_\nu := L^p(\mathcal{X},\mathcal{F}_\mathcal{X},\nu;\mathbb{R})$ be the space of all real-valued, $\mathcal{F}_\mathcal{X}$-measurable functions u with $\|u\|_{L^p_\nu} := \big(\int_\mathcal{X}|u(x)|^p\nu(dx)\big)^{\frac{1}{p}} < \infty$. For any $u_1, u_2 \in L^p_\nu$, we say $u_1 = u_2$ if $u_1(x) = u_2(x)$ for ν-a.s. $x \in \mathcal{X}$.

(3) For any sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$, let $L^0(\mathcal{G})$ be the space of all real-valued, $\mathcal{G}$-measurable random variables and set
- $L^p(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^p(\mathcal{G})} := \big\{E\big[|\xi|^p\big]\big\}^{\frac{1}{p}} < \infty\Big\}$ for any $p \in (1,\infty)$;
- $L^\infty(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^\infty(\mathcal{G})} := \underset{\omega\in\Omega}{\operatorname{esssup}}\, |\xi(\omega)| < \infty\Big\}$.

(4) Let $\mathbb{D}^0$ be the space of all real-valued, $\mathbf{F}$-adapted càdlàg processes, and let $\mathbb{K}^0$ be a subspace of $\mathbb{D}^0$ that includes all $\mathbf{F}$-predictable càdlàg nondecreasing processes X with $X_0 = 0$.

(5) Set $\mathbb{Z}^2_{\text{loc}} := L^2_{\text{loc}}\big([0,T]\times\Omega, \widehat{\mathscr{P}}, dt\times dP; \mathbb{R}^d\big)$, the space of all $\mathbb{R}^d$-valued, $\mathbf{F}$-predictable processes Z with $\int_0^T |Z_t|^2\,dt < \infty$, $P-$a.s.

(6) For any $p \in [1,\infty)$, we let
- $\mathbb{D}^p := \Big\{X \in \mathbb{D}^0 : \|X\|_{\mathbb{D}^p} := \big\{E[X_*^p]\big\}^{\frac{1}{p}} < \infty\Big\}$, where $X_* := \sup\limits_{t\in[0,T]} |X_t| < \infty$.
- $\mathbb{K}^p := \mathbb{K}^0 \cap \mathbb{D}^p = \big\{K \in \mathbb{K}^0 : E[K_T^p] < \infty\big\}$.
- $\mathbb{Z}^{2,p} := \Big\{Z \in \mathbb{Z}^2_{\text{loc}} : \|Z\|_{\mathbb{Z}^{2,p}} := \Big\{E\Big[\big(\int_0^T |Z_t|^2\,dt\big)^{\frac{p}{2}}\Big]\Big\}^{\frac{1}{p}} < \infty\Big\}$.
- $\mathbb{U}^p_{\text{loc}} := L^p_{\text{loc}}\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}, dt\times dP\times\nu(dx); \mathbb{R}\big)$ be the space of all $\widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}$-measurable random fields $U : [0,T]\times\Omega\times\mathcal{X}\to\mathbb{R}$ such that $\int_0^T\int_{\mathcal{X}} |U_t(x)|^p\nu(dx)dt = \int_0^T \|U_t\|^p_{L^p_\nu}dt < \infty$, $P-$a.s.
- $\mathbb{U}^p := \Big\{U \in \mathbb{U}^p_{\text{loc}} : \|U\|_{\mathbb{U}^p} := \big\{E\int_0^T\int_{\mathcal{X}} |U_t(x)|^p\nu(dx)dt\big\}^{\frac{1}{p}} < \infty\Big\} = L^p\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}, dt\times dP\times\nu(dx); \mathbb{R}\big)$.

Given $U \in \mathbb{U}^p_{\text{loc}}$ $\big(\text{resp. } \mathbb{U}^p\big)$, it holds for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$ that $U(t,\omega)\in L^p_\nu$. In virtue of [41, Sect. 1.2], one can define a Poisson stochastic integral of U:

$$M^U_t := \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T],$$

which is a càdlàg local martingale (resp. uniformly integrable martingale) with quadratic variation $[M^U, M^U]_t = \int_{(0,t]}\int_{\mathcal{X}} |U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. The jump process of M^U is $\Delta M^U_t(\omega) = \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}U\big(t,\omega,\mathfrak{p}_t(\omega)\big)$, $t\in(0,T]$. For any $U\in\mathbb{U}^p$, an analogy to (5.1) of Yao [41] shows that

$$E\Bigg[\bigg(\int_{(t,s]}\int_{\mathcal{X}} |U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\bigg)^{\frac{p}{2}}\Bigg] \le E\int_t^s\int_{\mathcal{X}} \big|U_t(x)\big|^p\nu(dx)dt, \quad \forall\, 0\le t<s\le T. \tag{1.3}$$

- Let us simply denote $\mathbb{D}^p\times\mathbb{Z}^{2,p}\times\mathbb{U}^p$ by $\mathbb{S}^p$.

As usual, we set $x^- := (-x)\vee 0$, $x^+ := x\vee 0$ for any $x\in\mathbb{R}$, and use the convention $\inf\emptyset := \infty$. Given $p\in(0,\infty)$, the following inequality will be frequently applied in this paper: For any finite subset $\{a_1,\dots,a_n\}$ of $(0,\infty)$,

$$\left(1 \wedge n^{p-1}\right) \sum_{i=1}^{n} a_i^p \le \left(\sum_{i=1}^{n} a_i\right)^p \le \left(1 \vee n^{p-1}\right) \sum_{i=1}^{n} a_i^p. \tag{1.4}$$

Also, we let c_p denote a generic constant depending only on p (in particular, c_0 stands for a generic constant depending on nothing), whose form may vary from line to line.

2 $\mathbb{L}^p$ Solutions of BSDEs with Jumps and Related g-Expectations

From now on, we fix $p \in (1, 2]$ and set $q := \frac{p}{p-1} \ge 2$.

Based on $\mathbb{L}^p$ solutions of a BSDEJ with generator g, we extended the notion of g-expectations to the jump case with $\mathbb{L}^p$ domains in Yao [40], and analyzed the properties of g-expectations. For purpose of the present paper, we will only review the martingale properties of g-expectations, which are important for our study of jump-filtration consistent nonlinear expectations with $\mathbb{L}^p$ domain in the next two sections.

2.1 $\mathbb{L}^p$ Solutions of BSDEs with Jumps

A mapping $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^p_\nu \to \mathbb{R}$ is called a *p-generator* if it is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})$-measurable.

Definition 2.1 Given $p \in (1, 2]$, let $\xi \in L^0(\mathcal{F}_T)$ and g be a p-generator. A triplet $(Y, Z, U) \in \mathbb{D}^0 \times \mathbb{Z}^2_{\text{loc}} \times \mathbb{U}^p_{\text{loc}}$ is called a solution of a backward stochastic differential equation with jumps that has terminal data ξ and generator g $\big($BSDEJ (ξ, g) for short$\big)$ if $\int_0^T |g(s, Y_s, Z_s, U_s)| ds < \infty$, $P-$a.s. and if it holds $P-$a.s. that

$$\begin{aligned} Y_t &= \xi + \int_t^T g(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s \, dB_s \\ &\quad - \int_{(t,T]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T]. \end{aligned}$$

Let us make the following standard assumptions on p-generators g:

(A1) $\int_0^T |g(t, 0, 0, 0)| dt \in L^p(\mathcal{F}_T)$.

(A2) There exist two $[0, \infty)$-valued, $\mathscr{B}[0, T] \otimes \mathcal{F}_T$-measurable processes β, Λ with $\int_0^T \big(\beta_t^q \vee \Lambda_t^2\big) dt \in L^\infty(\mathcal{F}_T)$ such that for $dt \times dP-$a.s. $(t, \omega) \in [0, T] \times \Omega$

$$\begin{aligned} \big|g(t, \omega, y_1, z_1, u) - g(t, \omega, y_2, z_2, u)\big| &\le \beta(t, \omega)|y_1 - y_2| + \Lambda(t, \omega)|z_1 - z_2|, \\ &\quad \forall\, (y_1, z_1),\ (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d, \\ &\quad \forall\, u \in L^p_\nu. \end{aligned}$$

(A3) There exists a function $\mathfrak{h} : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^p_\nu \times L^p_\nu \to L^q_\nu$ such that
  (i) $\mathfrak{h}$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(L^p_\nu) \otimes \mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable;

(ii) There exist $\kappa_1 \in (-1, 0]$ and $\kappa_2 \geq -\kappa_1$ such that for any $(t, \omega, y, z, u_1, u_2, x) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^p_\nu \times L^p_\nu \times \mathcal{X}$

$$\kappa_1 \leq \big(\mathfrak{h}(t, \omega, y, z, u_1, u_2)\big)(x) \leq \kappa_2;$$

(iii) It holds for $dt \times dP -$ a.s. $(t, \omega) \in [0, T] \times \Omega$ that

$$\begin{aligned} &g(t, \omega, y, z, u_1) - g(t, \omega, y, z, u_2) \\ &\quad \leq \int_{\mathcal{X}} \big(u_1(x) - u_2(x)\big) \cdot \big(\mathfrak{h}(t, \omega, y, z, u_1, u_2)\big)(x)\, \nu(dx), \\ &\forall\, (y, z, u_1, u_2) \in \mathbb{R} \times \mathbb{R}^d \times L^p_\nu \times L^p_\nu. \end{aligned}$$

The parameter quadruplets $\Xi := (\beta, \Lambda, \kappa_1, \kappa_2)$ described in (A2) and (A3) will be referred to as a p-coefficient set. When $\beta \equiv 0$, it will be particularly denoted by Ξ_0.

Remark 2.1 Let $p \in (1, 2]$ and let g be a p-generator.

(1) By (A3) (ii), (iii) and Hölder's inequality, (A2) and (A3) imply
(A2') There exist two $[0, \infty)$-valued, $\mathscr{B}[0, T] \otimes \mathcal{F}_T$-measurable processes β, Λ with $\int_0^T \big(\beta_t^q \vee \Lambda_t^2\big) dt \in L^\infty(\mathcal{F}_T)$ such that for $dt \times dP -$a.s. $(t, \omega) \in [0, T] \times \Omega$

$$\begin{aligned} \big|g(t, \omega, y_1, z_1, u_1) \ &-g(t, \omega, y_2, z_2, u_2)\big| \leq \beta(t, \omega)\big(|y_1 - y_2| + \|u_1 - u_2\|_{L^p_\nu}\big) \\ &+\Lambda(t, \omega)|z_1 - z_2|,\ \forall\, (y_i, z_i, u_i) \in \mathbb{R} \times \mathbb{R}^d \times L^p_\nu,\ i = 1, 2. \end{aligned}$$

(2) If g satisfies (A2') and $\int_0^T |g(t, 0, 0, 0)| dt < \infty$, $P -$ a.s., then Remark 2.1 (2) of Yao [40] shows that for any $(Y, Z, U) \in \mathbb{D}^1 \times \mathbb{Z}^2_{\text{loc}} \times \mathbb{U}^p_{\text{loc}}$, we have $\int_0^T |g(s, Y_s, Z_s, U_s)| ds < \infty$, $P -$ a.s.
(3) If g satisfies (A1), (A2) $\big($resp. (A2')), then $\overline{g}(t, \omega, y, z, u) := -g(t, \omega, -y, -z, -u)$, $(t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^p_\nu$ is also a p-generator satisfying (A1), (A2) $\big($resp. (A2')). If g further satisfies (A3), so does $\overline{g}$.

For simplicity, we set $\widehat{C} := \big\| \int_0^T \big(1 \vee \beta_t^q \vee \Lambda_t^2\big) dt \big\|_{L^\infty(\mathcal{F}_T)}$, and let $\mathcal{C}$ be a generic constant depending on T, $\nu(\mathcal{X})$, p, $\widehat{C}$ (and κ_2 if necessary), whose form may vary from line to line.

For $\mathbb{L}^p$ solutions of BSDEs with jumps, we first quote a wellposedness result and a comparison theorem from of Yao [40, Theorems 2.1, 2.2].

Theorem 2.1 *Given $p \in (1, 2]$, Let g be a p-generator satisfying (A1) and (A2'). For any $\xi \in L^p(\mathcal{F}_T)$, the BSDEJ (ξ, g) admits a unique solution $\big(Y^{\xi,g}, Z^{\xi,g}, U^{\xi,g}\big) \in \mathbb{S}^p$ satisfying*

$$\big\|Y^{\xi,g}\big\|^p_{\mathbb{D}^p} + \big\|Z^{\xi,g}\big\|^p_{\mathbb{Z}^{2,p}} + \big\|U^{\xi,g}\big\|^p_{\mathbb{U}^p} \leq \mathcal{C} E\bigg[\ |\xi|^p + \bigg(\int_0^T |g(t, 0, 0, 0)| dt\bigg)^p\ \bigg]. \tag{2.1}$$

Theorem 2.2 *Let $p \in (1,2]$, $\tau \in \mathcal{T}$ and $\gamma \in \mathcal{T}_\tau$. For $i = 1, 2$, let $\xi_i \in L^0(\mathcal{F}_T)$, let g^i be a p-generator, and let (Y^i, Z^i, U^i) be a solution of BSDEJ (ξ_i, g^i) such that $Y^1 - Y^2 \in \mathbb{D}^p$ and that $Y^1_\gamma \le Y^2_\gamma$, $P-$a.s. For either $i = 1$ or $i = 2$, if g^i satisfies (A2), (A3), and if $g^1(t, Y^{3-i}_t, Z^{3-i}_t, U^{3-i}_t) \le g^2(t, Y^{3-i}_t, Z^{3-i}_t, U^{3-i}_t)$, $dt \times dP -$ a.s. on $]\!]\tau, \gamma[\![$, then it holds $P -$ a.s. that $Y^1_t \le Y^2_t$ for any $t \in [\tau, \gamma]$. If one further has $Y^1_\tau = Y^2_\tau$, $P -$ a.s., then*

(i) *it holds $P -$ a.s. that $Y^1_t = Y^2_t$ for any $t \in [\tau, \gamma]$;*
(ii) *it holds $dt \times dP -$a.s. on $]\!]\tau, \gamma]\!]$ that $(Z^1_t, U^1_t) = (Z^2_t, U^2_t)$ and $g^1(t, Y^i_t, Z^i_t, U^i_t) = g^2(t, Y^i_t, Z^i_t, U^i_t)$, $i = 1, 2$.*

2.2 g-Expectations with Domain $L^p(\mathcal{F}_T)$

The wellposedness result of BSDEs with jumps in $\mathbb{L}^p$ sense (Theorem 2.1) gives rise to a nonlinear expectation, called *g-expectations*, with domain $L^p(\mathcal{F}_T)$, which generalizes the one introduced in Peng [28,29]:

Definition 2.2 Given $p \in (1,2]$, let g be a p-generator satisfying (A1) and (A2'). For any $\xi \in L^p(\mathcal{F}_T)$, define

$$\mathcal{E}_g[\xi|\mathcal{F}_\tau] := Y^{\xi,g}_\tau \in L^p(\mathcal{F}_\tau), \quad \forall\, \tau \in \mathcal{T}$$

as the conditional "g-expectation" of ξ at time τ.

When $g \equiv 0$, the g-expectation is exactly the classic linear expectation: it holds for any $\tau \in \mathcal{T}$ and $\xi \in L^p(\mathcal{F}_T)$ that $\mathcal{E}_g[\xi|\mathcal{F}_\tau] = E[\xi|\mathcal{F}_\tau]$, $P -$ a.s.

Let $p \in (1,2]$ and let g be a p-generator satisfying (A1) and (A2'). We know from Yao [40] that g-expectations with domain $L^p(\mathcal{F}_T)$ inherit the following basic properties from the classic linear expectations: Let $\xi \in L^p(\mathcal{F}_T)$ and $\tau \in \mathcal{T}$.

(g1) "Strict Monotonicity": If g further satisfies (A3), then for any $\eta \in L^p(\mathcal{F}_T)$ with $\xi \le \eta$, $P -$ a.s. one has $\mathcal{E}_g[\xi|\mathcal{F}_\tau] \le \mathcal{E}_g[\eta|\mathcal{F}_\tau]$, $P -$ a.s.; Moreover, if it further holds that $\mathcal{E}_g[\xi|\mathcal{F}_\tau] = \mathcal{E}_g[\eta|\mathcal{F}_\tau]$, $P -$ a.s., then $\xi = \eta$, $P -$ a.s.
(g2) "Constant Preserving": If it holds $dt \times dP -$ a.s. that

$$g(t, y, 0, 0) = 0, \quad \forall\, y \in \mathbb{R}, \tag{2.2}$$

and if ξ is $\mathcal{F}_\tau$-measurable, then $\mathcal{E}_g[\xi|\mathcal{F}_\tau] = \xi$, $P -$ a.s.
(g3) "Time Consistency": Under (2.2), it holds for any $\gamma \in \mathcal{T}_\tau$ that $\mathcal{E}_g\big[\mathcal{E}_g[\xi|\mathcal{F}_\gamma]\big|\mathcal{F}_\tau\big] = \mathcal{E}_g[\xi|\mathcal{F}_\tau]$, $P -$ a.s.
(g4) "Zero-One Law": For any $A \in \mathcal{F}_\tau$, we have $\mathbf{1}_A\mathcal{E}_g[\mathbf{1}_A\xi|\mathcal{F}_\tau] = \mathbf{1}_A\mathcal{E}_g[\xi|\mathcal{F}_\tau]$, $P -$ a.s.; In addition, if $g(t,0,0,0) = 0$, $dt \times dP -$ a.s., then $\mathcal{E}_g[\mathbf{1}_A\xi|\mathcal{F}_\tau] = \mathbf{1}_A\mathcal{E}_g[\xi|\mathcal{F}_\tau]$, $P -$ a.s.
(g5) "Translation Invariance": If g is independent of y, then $\mathcal{E}_g[\xi + \eta|\mathcal{F}_\tau] = \mathcal{E}_g[\xi|\mathcal{F}_\tau] + \eta$, $P -$ a.s. for any $\eta \in L^p(\mathcal{F}_\tau)$.

Now, let us consider two specific p-generators satisfying (A1)–(A3) and their corresponding g-expectations:

Example 2.1 Given $p \in (1,2]$, let Ξ be a p-coefficient set. The functions

$$\begin{aligned} g^{\Xi}(t,\omega,y,z,u) &:= \beta(t,\omega)|y| + \Lambda(t,\omega)|z| - \kappa_1 \int_{\mathcal{X}} u^-(x)\nu(dx) \\ &\quad + \kappa_2 \int_{\mathcal{X}} u^+(x)\nu(dx), \\ \overline{g}^{\Xi}(t,\omega,y,z,u) &:= -g^{\Xi}(t,\omega,-y,-z,-u), \quad \forall\,(t,\omega,y,z,u) \\ &\quad \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu \end{aligned} \tag{2.3}$$

are two p-generators satisfying (A1)–(A3) with respect to the same coefficient set Ξ, where $u^{\pm}(x) := (u(x))^{\pm}$ (See the proof of [40, Example 3.1] for details). It follows that $\mathcal{E}_{\Xi} := \mathcal{E}_{g^{\Xi}}$, $\overline{\mathcal{E}}_{\Xi} := \mathcal{E}_{\overline{g}^{\Xi}}$ are two g-expectations with domain $L^p(\mathcal{F}_T)$.

According to the comparison theorem for BSDEJs (Theorem 2.2), we can bound the variation of a g-expectation by g^{Ξ}-expectation and $\overline{g}^{\Xi}$-expectation as follows.

Proposition 2.1 *Given $p \in (1,2]$, let g be a p-generator satisfying (A1)–(A3) with respect to some p-coefficient set Ξ. For any $\xi,\eta \in L^p(\mathcal{F}_T)$, it holds $P-$a.s. that $\overline{\mathcal{E}}_{\Xi}[\xi-\eta|\mathcal{F}_t] \le \mathcal{E}_g[\xi|\mathcal{F}_t] - \mathcal{E}_g[\eta|\mathcal{F}_t] \le \mathcal{E}_{\Xi}[\xi-\eta|\mathcal{F}_t]$ for any $t \in [0,T]$.*

Proof Fix $\xi,\eta \in L^p(\mathcal{F}_T)$. Set $(\mathcal{Y}^1,\mathcal{Z}^1,\mathcal{U}^1) = \big(Y^{\xi,g}, Z^{\xi,g}, U^{\xi,g}\big)$, $(\mathcal{Y}^2,\mathcal{Z}^2,\mathcal{U}^2) = \big(Y^{\eta,g}, Z^{\eta,g}, U^{\eta,g}\big)$ and $(\mathcal{Y}^3,\mathcal{Z}^3,\mathcal{U}^3) = \big(Y^{\xi-\eta,g^{\Xi}}, Z^{\xi-\eta,g^{\Xi}}, U^{\xi-\eta,g^{\Xi}}\big)$. The $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})$-measurability of g, the $\mathscr{P}$ measurability of process $\mathcal{Y}^2$, the $\widehat{\mathscr{P}}$-measurability of process $\mathcal{Z}^2$ and the $\widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}$-measurability of random field $\mathcal{U}^2$ imply that the mapping

$$\begin{aligned} \overline{g}(t,\omega,y,z,u) := g\big(t,\omega,y+\mathcal{Y}^2(t,\omega), z+\mathcal{Z}^2(t,\omega), u+\mathcal{U}^2(t,\omega)\big) \\ - g\big(t,\omega,\mathcal{Y}^2(t,\omega),\mathcal{Z}^2(t,\omega),\mathcal{U}^2(t,\omega)\big), \end{aligned}$$

$\forall\,(t,\omega,y,z,u) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$ is also $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})$-measurable.

For $\big(\overline{Y},\overline{Z},\overline{U}\big) := (\mathcal{Y}^1-\mathcal{Y}^2, \mathcal{Z}^1-\mathcal{Z}^2, \mathcal{U}^1-\mathcal{U}^2) \in \mathbb{S}^p$, it holds $P-$a.s. that

$$\begin{aligned} \overline{Y}_t = \xi-\eta+\int_t^T \big(g(s,\mathcal{Y}^1_s,\mathcal{Z}^1_s,\mathcal{U}^1_s) - g(s,\mathcal{Y}^2_s,\mathcal{Z}^2_s,\mathcal{U}^2_s)\big)ds \\ - \int_t^T \overline{Z}_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t \in [0,T]. \end{aligned}$$

Namely, $\big(\overline{Y},\overline{Z},\overline{U}\big)$ solves the BSDEJ $(\xi-\eta,\overline{g})$. We can deduce from (A2) and (A3) that $dt\times dP-$a.s.

$$\begin{aligned} &\overline{g}\big(t,\overline{Y}_t,\overline{Z}_t,\overline{U}_t\big) \\ &\quad = g\left(t,\mathcal{Y}^1_t,\mathcal{Z}^1_t,\mathcal{U}^1_t\right) - g\left(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^2_t\right) \\ &\quad = g\left(t,\mathcal{Y}^1_t,\mathcal{Z}^1_t,\mathcal{U}^1_t\right) - g\left(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^1_t\right) + g\left(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^1_t\right) \end{aligned}$$

$$
\begin{aligned}
&-g\left(t,\mathcal{Y}_t^2,\mathcal{Z}_t^2,\mathcal{U}_t^2\right)\\
&\le \beta_t|\overline{Y}_t|+\Lambda_t|\overline{Z}_t|+\int_{\mathcal{X}}\overline{U}_t(x)\cdot\left(\mathfrak{h}\left(t,\mathcal{Y}_t^2,\mathcal{Z}_t^2,\mathcal{U}_t^1,\mathcal{U}_t^2\right)\right)(x)\nu(dx)\\
&\le \beta_t|\overline{Y}_t|+\Lambda_t|\overline{Z}_t|+\kappa_2\int_{\mathcal{X}}\overline{U}_t^+(x)\nu(dx)-\kappa_1\int_{\mathcal{X}}\overline{U}_t^-(x)\nu(dx)\\
&= g^{\Xi}\left(t,\overline{Y}_t,\overline{Z}_t,\overline{U}_t\right).
\end{aligned}
$$

Since g^{Ξ} also satisfies (A2) and (A3) by Example 2.1, applying Theorem 2.2 with $(\tau,\gamma)=(0,T)$, $(g^1,Y^1,Z^1,U^1)=\left(\overline{g},\overline{Y},\overline{Z},\overline{U}\right)$ and $(g^2,Y^2,Z^2,U^2)=(g^{\Xi},\mathcal{Y}^3,\mathcal{Z}^3,\mathcal{U}^3)$ yields that $P-$a.s.

$$
\mathcal{E}^g[\xi|\mathcal{F}_t]-\mathcal{E}^g[\eta|\mathcal{F}_t]=\mathcal{Y}_t^1-\mathcal{Y}_t^2=\overline{Y}_t\le\mathcal{Y}_t^3=\mathcal{E}_{\Xi}[\xi-\eta|\mathcal{F}_t],\qquad \forall t\in[0,T]. \tag{2.4}
$$

Multiplying -1 to BSDEJ$\left(\eta-\xi,g^{\Xi}\right)$ shows that $(-Y^{\eta-\xi,g^{\Xi}},-Z^{\eta-\xi,g^{\Xi}},-U^{\eta-\xi,g^{\Xi}})$ is the unique solution of BSDEJ $\left(\xi-\eta,\overline{g}^{\Xi}\right)$. So $P\left\{-Y_t^{\eta-\xi,g^{\Xi}}=Y_t^{\xi-\eta,\overline{g}^{\Xi}},\ \forall t\in[0,T]\right\}=1$, which together with (2.4) implies that $P-$a.s.

$$
\begin{aligned}
\mathcal{E}^g[\xi|\mathcal{F}_t]-\mathcal{E}^g[\eta|\mathcal{F}_t]&=-\left(\mathcal{E}^g[\eta|\mathcal{F}_t]-\mathcal{E}^g[\xi|\mathcal{F}_t]\right)\ge-\mathcal{E}_{\Xi}[\eta-\xi|\mathcal{F}_t]\\
&=-Y_t^{\eta-\xi,g^{\Xi}}=Y_t^{\xi-\eta,\overline{g}^{\Xi}}=\overline{\mathcal{E}}_{\Xi}[\xi-\eta|\mathcal{F}_t],\quad\forall t\in[0,T].
\end{aligned}
$$

□

2.3 g-Martingales

Let g be a p-generator satisfying (A1) and (A2'). We can define martingales with respect to the g-expectations with domain $L^p(\mathcal{F}_T)$ under jump filtration.

Definition 2.3 Given $p\in(1,2]$, let g be a p-generator satisfying (A1) and (A2'). A real-valued, $\mathbf{F}$-adapted process X is called a g-submartingale (resp. g-supermartingale or g-martingale) if for any $0\le t\le s\le T$, $E[|X_s|^p]<\infty$ and $\mathcal{E}_g[X_s|\mathcal{F}_t]\ge$ (resp. $\le$ or $=$) X_t, $P-$a.s.

The properties of g-martingales, such as "optional sampling theorem", "upcrossing inequality" and "Doob–Meyer decomposition", have been explored inYao [40]. As they will play important roles for developing the martingale properties of filtration-consistent nonlinear expectations in the next section, we cite them completely for ease reference (The following Propositions 2.2, 2.3, 2.4; Theorems 2.3, 2.4 are from Yao [40, Propositions 4.1, 4.2, 4.3, Theorems 4.1, A.1] respectively).

Proposition 2.2 (*Optional Sampling of g-martingales*) *Given* $p\in(1,2]$, *let* g *be a* p-*generator satisfying* (A2), (A3) *and* (2.2). *Let* X *be a* g-*submartingale* (*resp.* g-*supermartingale*) *with* $E[X_*^p]<\infty$ *and let* $\tau\in\mathcal{T}$, $\gamma\in\mathcal{T}_\tau$. *If* X *is right-continuous or if* τ,γ *are finitely valued, then*

$$\mathcal{E}_g[X_\gamma|\mathcal{F}_\tau] \geq (resp. \ \leq)\ X_\tau, \quad P-\text{a.s.}$$

Let us review the notion of *number of upcrossings* for presenting the upcrossing inequality of g-martingales: Given a real-valued process X and two real numbers $a < b$, for any finite subset $\mathcal{D} = \{t_1 < \cdots < t_m\}$ of $[0, T]$, we define the "number of upcrossings" $U_{\mathcal{D}}(a, b; X(\omega))$ of interval $[a, b]$ by the sample path $\{X_t(\omega)\}_{t\in\mathcal{D}}$ as follows: Set $m' := \lceil \frac{m}{2} \rceil$ and $\tau_0 := -1$. For $i = 1, \ldots, m'$, we recursively define

$$\tau_{2i-1}(\omega) := \min\Big\{t \in \mathcal{D} : t > \tau_{2i-2}(\omega),\ X_t(\omega) < a\Big\} \wedge t_m \in \mathcal{T} \quad \text{and}$$
$$\tau_{2i}(\omega) := \min\Big\{t \in \mathcal{D} : t > \tau_{2i-1}(\omega),\ X_t(\omega) > b\Big\} \wedge t_m \in \mathcal{T},$$

with the convention $\min \emptyset = \infty$. Then $U_{\mathcal{D}}(a, b; X(\omega))$ is defined to be the largest integer i such that $\tau_{2i}(\omega) < t_m$. To wit, $U_{\mathcal{D}}\big(a, b; X(\omega)\big) = \sum_{i=1}^{m'} \mathbf{1}_{\{\tau_{2i}(\omega)<t_m\}}$.

Proposition 2.3 (*Upcrossing Inequality of g-martingales*) *Given $p \in (1, 2]$, let g be a p-generator satisfying* (A2), (A3), (2.2) *with respect to some p-coefficient set Ξ, and let X be a g-supermartingale with $E[X_*^p] < \infty$. For any real numbers $a < b$ and any finite subset $\mathcal{D} = \{t_1 < \cdots < t_m\}$ of $[0, T]$, the upcrossing number $U_{\mathcal{D}}(a, b; X)$ of interval $[a, b]$ satisfies*

$$E\Big[\ln\big(1 + U_{\mathcal{D}}(a, b; X)\big)\Big] \leq \ln\left\{\frac{e^{3\widehat{C}}}{b-a}\mathcal{E}_\Xi\big[(X_{t_m} - a)^-\big] + \frac{|a|e^{3\widehat{C}}}{b-a} + 1\right\} + \frac{1}{2}\widehat{C} + \big(\kappa_2 - \ln(1+\kappa_1)\big)\nu(\mathcal{X})T.$$

Theorem 2.3 (*Doob–Meyer Decomposition of g-martingales*) *Given $p \in (1, 2]$, let g be a p-generator satisfying* (2.2) *and* (A2)*. Assume that g also satisfies* (A3) *with $\int_0^T \Lambda_t^{\frac{2p}{2-p}} dt \in L^\infty(\mathcal{F}_T)$ if $p \in (1, 2)$, or with $\Lambda \equiv \kappa_\Lambda \in [0, \infty)$ if $p = 2$. If $X \in \mathbb{D}^p$ is a g-supermartingale* (*resp. g-submartingale*)*, then there exist unique processes $(Z, U, K) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p$ such that $P-$a.s.*

$$X_t = X_T + \int_t^T g\,(s, X_s, Z_s, U_s)\, ds - \int_t^T Z_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx) + K_T - K_t \ (resp. \ - K_T + K_t), \quad t \in [0, T].$$

The Theorem 2.3 relies on the following a priori $\mathbb{L}^p$-estimate to a special BSDEJ and generalized monotonic limit theorem of jump diffusion processes over $\mathbb{D}^p$, both of which are crucial for the proof of the Doob–Meyer decomposition under nonlinear expectation $\mathcal{E}$ (Theorem 3.1).

Proposition 2.4 *Given $p \in (1, 2]$ and $\xi \in L^p(\mathcal{F}_T)$, let g be a p-generator and let X be a real-valued,* **F***-adapted càdlàg process with $X^+ \in \mathbb{D}^p$. Let $(Y, Z, U, K) \in \mathbb{D}^p \times \mathbb{Z}^2_{\text{loc}} \times \mathbb{U}^p_{\text{loc}} \times \mathbb{K}^p$ satisfies that $P-$a.s.*

$$\begin{cases} Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds + K_T - K_t - \int_t^T Z_s\, dB_s \\ \qquad - \int_{(t,T]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T] \\ \int_0^T \mathbf{1}_{\{Y_{t-} > X_{t-}\}} dK_t = 0. \end{cases} \tag{2.5}$$

If there exist three $[0, \infty)$*-valued,* $\mathscr{B}[0, T] \otimes \mathcal{F}_T$*-measurable processes* $\mathfrak{f}$, β, Λ *with* $\int_0^T \mathfrak{f}_t dt \in L^p(\mathcal{F}_T)$, $\int_0^T \big(\beta_t^q \vee \Lambda_t^2\big) dt \in L^\infty(\mathcal{F}_T)$ *such that*

$$\big|g(t, Y_t, Z_t, U_t)\big| \le \mathfrak{f}_t + \beta_t \big(|Y_t| + \|U_t\|_{L^p_\nu}\big) + \Lambda_t |Z_t|, \quad dt \times dP - \text{a.s.}, \tag{2.6}$$

then $(Z, U) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$ *and*

$$\|Y\|^p_{\mathbb{D}^p} + \big\|Z\big\|^p_{\mathbb{Z}^{2,p}} + \big\|U\big\|^p_{\mathbb{U}^p} + E[K_T^p] \le \mathcal{C} E\bigg[|\xi|^p + \Big(\int_0^T \mathfrak{f}_t dt\Big)^p + (X_*^+)^p\bigg].$$

Theorem 2.4 *Given* $p \in (1, 2]$*, let* $\{Y^n\}_{n \in \mathbb{N}}$ *be a series of jump diffusion processes in form of*

$$Y_t^n = Y_0^n - \int_0^t g_s^n ds - K_t^n + \int_0^t Z_s^n dB_s$$
$$+ \int_{(0,t]} \int_{\mathcal{X}} U_s^n(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), \quad \forall\, t \in [0, T],$$

where

(i) $\big\{(g^n, Z^n, U^n)\big\}_{n \in \mathbb{N}}$ *is bounded in* $\mathbb{L}^p\big([0, T] \times \Omega, \mathscr{P}, dt \times dP; \mathbb{R}\big) \times \mathbb{Z}^{2,p} \times \mathbb{U}^p$*, i.e. there exists a* $C > 0$ *such that*

$$\Big(E \int_0^T |g_t^n|^p dt\Big)^{\frac{1}{p}} + \|Z^n\|_{\mathbb{Z}^{2,p}} + \|U^n\|_{\mathbb{U}^p} \le C, \quad \forall\, n \in \mathbb{N};$$

(ii) *For any* $n \in \mathbb{N}$*,* K^n *is an* $\mathbf{F}$*-adapted, continuous increasing process with* $K_0^n = 0$ *and* $K_T^n \in L^p(\mathcal{F}_T)$*;*

(iii) Y^n *is an increasing sequence that is bounded above by some* $X \in \mathbb{D}^p$*, i.e.* $P\{Y_t^n \le Y_t^{n+1} \le X_t,\ \forall\, t \in [0, T]\} = 1$ *for any* $n \in \mathbb{N}$*. Then* $Y_t := \overline{\lim_{n \to \infty}} Y_t^n$*,* $t \in [0, T]$ *is a process of* $\mathbb{D}^p$ *that satisfies*

$$P\Big\{Y_t = \lim_{n \to \infty} \uparrow Y_t^n \le X_t,\ \forall\, t \in [0, T]\Big\} = 1, \tag{2.7}$$

and possesses the following decomposition: There exists $(g, Z, U, K) \in \mathbb{L}^p\big([0, T] \times \Omega, \mathscr{P}, dt \times dP; \mathbb{R}\big) \times \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p$ *such that* $P - \text{a.s.}$

$$
\begin{aligned}
Y_t = Y_0 - \int_0^t g_s ds - K_t + \int_0^t Z_s dB_s \\
+ \int_{(0,t]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), \quad \forall\, t \in [0, T],
\end{aligned} \tag{2.8}
$$

and that for any $\varpi \in (2/p, 2)$

$$
\lim_{n\to\infty} E\bigg[\bigg(\int_0^T |Z_s^n - Z_s|^{\varpi} ds \bigg)^{\frac{p}{2}} + \int_0^T \int_{\mathcal{X}} |U_s^n(x) - U_s(x)|^{\frac{p\varpi}{2}} \nu(dx) ds \bigg] = 0.
$$

Moreover, if Y *has only inaccessible jumps, then* K *is a continuous process and*

$$
\lim_{n\to\infty} E\bigg[\bigg(\int_0^T |Z_s^n - Z_s|^2 ds \bigg)^{\frac{p}{2}} + \int_0^T \int_{\mathcal{X}} |U_s^n(x) - U_s(x)|^p \nu(dx) ds \bigg] = 0. \tag{2.9}
$$

3 Filtration-Consistent Nonlinear Expectations

In this section, we generalize g-expectations to a so-called "filtration-consistent nonlinear expectation" as in Coquet et al. [10]. A large class of nonlinear expectations, such as risk measures or monetary utility functionals, play important role in mathematical finance.

3.1 F-Expectations with Domain $L^p(\mathcal{F}_T)$

Definition 3.1 Let $p \in (1, 2]$.

(1) We call a mapping $\mathcal{E}$: $L^p(\mathcal{F}_T) \to \mathbb{R}$ a "nonlinear expectation" with domain $L^p(\mathcal{F}_T)$ if it satisfies
  - (i) (*Strict Monotonicity*) For any $\xi, \eta \in L^p(\mathcal{F}_T)$ with $\xi \le \eta$, $P-$a.s., $\mathcal{E}[\xi] \le \mathcal{E}[\eta]$; If one further has $\mathcal{E}[\xi] = \mathcal{E}[\eta]$, then $\xi = \eta$, $P-$a.s.
  - (ii) (*Constant Preserving*) $\mathcal{E}[c] = c$, $\forall\, c \in \mathbb{R}$.

(2) A nonlinear expectation $\mathcal{E}$ on $L^p(\mathcal{F}_T)$ is said to be "consistent" with the filtration **F** if for any $\xi \in L^p(\mathcal{F}_T)$ and $t \in [0, T]$, there exists an $\eta = \eta(\xi, t) \in L^p(\mathcal{F}_t)$ such that $\mathcal{E}[\mathbf{1}_A \xi] = \mathcal{E}[\mathbf{1}_A \eta]$ holds for any $A \in \mathcal{F}_t$. By the strict monotonicity, one can check as usual that such a random variable η is unique. We will denote it by $\mathcal{E}[\xi|\mathcal{F}_t]$ and refer to it as the "filtration-consistent conditional nonlinear expectation" (or simply **F**-expectation) of ξ.

(3) Given an **F**-expectation $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ with domain $L^p(\mathcal{F}_T)$, a real-valued, **F**-adapted process X is called an $\mathcal{E}$-submartingale $\big($resp. $\mathcal{E}$-supermartingale or $\mathcal{E}$-martingale$\big)$ if for any $0 \le t \le s \le T$, $E[|X_s|^p] < \infty$ and

$$
\mathcal{E}[X_s|\mathcal{F}_t] \ge (\text{resp.} \ \le \text{ or } =)\ X_t, \quad P-\text{a.s.}
$$

(4) We say an **F**-expectation $\{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}$ with domain $L^p(\mathcal{F}_T)$ to be "translation invariant" if for any $\xi \in L^p(\mathcal{F}_T)$, any $t \in [0,T]$ and any $\eta \in L^p(\mathcal{F}_t)$

$$\mathcal{E}[\xi+\eta|\mathcal{F}_t] = \mathcal{E}[\xi|\mathcal{F}_t] + \eta, \quad P-\text{a.s.}$$

Similar to Peng [31, Proposition 2.2], any **F**-expectation $\{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}$ with domain $L^p(\mathcal{F}_T)$ possesses the following properties: Let $\xi \in L^p(\mathcal{F}_T)$ and $t \in [0,T]$.

(F1) "Monotonicity": For any $\eta \in L^p(\mathcal{F}_T)$ with $\xi \le \eta$, $P-$a.s., $\mathcal{E}[\xi|\mathcal{F}_t] \le \mathcal{E}[\eta|\mathcal{F}_t]$, $P-$a.s.
(F2) "Constant Preserving": If ξ is $\mathcal{F}_t$-measurable, then $\mathcal{E}[\xi|\mathcal{F}_t] = \xi$, $P-$a.s.
(F3) "Time Consistency": For any $s \in [t,T]$, $\mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_s]\big|\mathcal{F}_t\big] = \mathcal{E}[\xi|\mathcal{F}_t]$, $P-$a.s.
(F4) "Zero–One Law": For any $A \in \mathcal{F}_t$, $\mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_t]$, $P-$a.s.

Example 3.1 Let $p \in (1,2]$.

(1) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function with $h(0) = 0$ and satisfying $|h(x)| \le C\big(1+|x|^p\big)$, $\forall\, x \in \mathbb{R}$ for some $C > 0$. Then $\mathcal{E}[\xi|\mathcal{F}_t] := h^{-1}\big(E[h(\xi)|\mathcal{F}_t]\big)$, $\forall\, \xi \in L^p(\mathcal{F}_T)$, $\forall\, t \in [0,T]$ defines an **F**-expectation with domain $L^p(\mathcal{F}_T)$.
(2) Let g be a p-generator satisfying (A2), (A3) and (2.2). One can deduce from (g1)–(g4) and (2.1) that the g-expectation $\{\mathcal{E}_g[\ |\mathcal{F}_t]\}_{t\in[0,T]}$ is an **F**-expectation with domain $L^p(\mathcal{F}_T)$ such that $E\Big[\sup\limits_{t\in[0,T]}\big|\mathcal{E}_g[\xi|\mathcal{F}_t]\big|^p\Big] < CE[|\xi|^p]$ for any $\xi \in L^p(\mathcal{F}_T)$.

Next, let us introduce the notion of the domination of **F**-expectations with domain $L^p(\mathcal{F}_T)$.

Definition 3.2 Given $p \in (1,2]$, let Ξ be a p-coefficient set. We say an **F**-expectation $\{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}$ with domain $L^p(\mathcal{F}_T)$ is "$\mathcal{E}_\Xi$-dominated" if (1.2) holds.

Example 3.2 Given $p \in (1,2]$, let g be a p-generator satisfying (A1)–(A3) with respect to some p-coefficient set Ξ. We see from Proposition 2.1 that the g-expectation $\{\mathcal{E}_g[\ |\mathcal{F}_t]\}_{t\in[0,T]}$ with domain $L^p(\mathcal{F}_T)$ is $\mathcal{E}_\Xi$-dominated in sense of (1.2).

What follows is a conditional expectation version of $\mathcal{E}_{\Xi_0}$-domination.

Proposition 3.1 *Given $p \in (1,2]$, let Ξ be a p-coefficient set with $\beta \equiv 0$, and let $\{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}$ be a translation invariant **F**-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. Then it holds for any $\xi, \eta \in L^p(\mathcal{F}_T)$ and $t \in [0,T]$ that*

$$\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_t] \le \mathcal{E}[\xi|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t] \le \mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t], \quad P-\text{a.s.}, \tag{3.1}$$

and thus that

$$E\Big[\big|\mathcal{E}[\xi|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]\big|^p\Big] \le CE\big[|\xi-\eta|^p\big]. \tag{3.2}$$

Proof Fix $\eta \in L^p(\mathcal{F}_T)$.

(1) By the translation invariance of $\mathcal{E}$, the mapping

$$\mathcal{E}_\eta[\vartheta] := \mathcal{E}[\eta+\vartheta] - \mathcal{E}[\eta], \quad \forall\, \vartheta \in L^p(\mathcal{F}_T)$$

satisfies the "strict monotonicity" and the "constant preserving" in Definition 3.1 (1), and is thus a "nonlinear expectation" with domain $L^p(\mathcal{F}_T)$.
Given $\vartheta \in L^p(\mathcal{F}_T)$ and $t \in [0,T]$, we claim that $\mathcal{E}_\eta[\vartheta|\mathcal{F}_t] = \mathcal{E}[\eta+\vartheta|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]$: To see this, we let $A \in \mathcal{F}_t$. By (F4),

$$\begin{aligned}\mathcal{E}[\eta+\mathbf{1}_A\vartheta|\mathcal{F}_t] &= \mathbf{1}_A\mathcal{E}[\eta+\mathbf{1}_A\vartheta|\mathcal{F}_t] + \mathbf{1}_{A^c}\mathcal{E}[\eta+\mathbf{1}_A\vartheta|\mathcal{F}_t]\\ &= \mathcal{E}[\mathbf{1}_A\eta+\mathbf{1}_A\vartheta|\mathcal{F}_t] + \mathcal{E}[\mathbf{1}_{A^c}\eta|\mathcal{F}_t] = \mathbf{1}_A\mathcal{E}[\eta+\vartheta|\mathcal{F}_t]\\ &\quad +\mathbf{1}_{A^c}\mathcal{E}[\eta|\mathcal{F}_t], \ \ P-\text{a.s.}\end{aligned}$$

Then (F3) and translation invariance of $\mathcal{E}$ imply that

$$\begin{aligned}&\mathcal{E}_\eta\big[\mathbf{1}_A\big(\mathcal{E}[\eta+\vartheta|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]\big)\big] - \mathcal{E}_\eta[\mathbf{1}_A\vartheta]\\ &\quad= \mathcal{E}\big[\eta+\mathbf{1}_A\big(\mathcal{E}[\eta+\vartheta|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]\big)\big] - \mathcal{E}[\eta+\mathbf{1}_A\vartheta]\\ &\quad= \mathcal{E}\Big[\mathcal{E}\big[\eta+\mathbf{1}_A\big(\mathcal{E}[\eta+\vartheta|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]\big)\big|\mathcal{F}_t\big]\Big] - \mathcal{E}[\eta+\mathbf{1}_A\vartheta]\\ &\quad= \mathcal{E}\big[\mathcal{E}[\eta|\mathcal{F}_t] + \mathbf{1}_A\big(\mathcal{E}[\eta+\vartheta|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]\big)\big] - \mathcal{E}[\eta+\mathbf{1}_A\vartheta]\\ &\quad= \mathcal{E}\big[\mathbf{1}_{A^c}\mathcal{E}[\eta|\mathcal{F}_t] + \mathbf{1}_A\mathcal{E}[\eta+\vartheta|\mathcal{F}_t]\big] - \mathcal{E}[\eta+\mathbf{1}_A\vartheta]\\ &\quad= \mathcal{E}\big[\mathcal{E}[\eta+\mathbf{1}_A\vartheta|\mathcal{F}_t]\big] - \mathcal{E}[\eta+\mathbf{1}_A\vartheta] = 0,\end{aligned}$$

proving the claim. Thus $\big\{\mathcal{E}_\eta[\ |\mathcal{F}_t]\big\}_{t\in[0,T]}$ forms a $\mathbf{F}$-expectation. Moreover, the translation invariance and the $\mathcal{E}_{\Xi_0}$-domination of $\mathcal{E}$ lead to those of $\mathcal{E}_\eta$.

(2) Next, let $\xi \in L^p(\mathcal{F}_T)$, $t \in [0,T]$ and set $A := \big\{\mathcal{E}_\eta[\xi-\eta|\mathcal{F}_t] > \mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big\} \in \mathcal{F}_t$. Since $\mathcal{E}_\eta[\vartheta] \le \mathcal{E}_{\Xi_0}[\vartheta]$ for any $\vartheta \in L^p(\mathcal{F}_T)$ and since $\mathbf{1}_A\big(\mathcal{E}_\eta[\xi-\eta|\mathcal{F}_t] - \mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big) \ge 0$, $P-$ a.s., we can deduce from (F1)–(F4), the translation invariance of $\mathcal{E}_\eta$ and (g2), (g3), (g5) of g^{Ξ_0}-expectations that

$$\begin{aligned}0 = \mathcal{E}_\eta[0] &\le \mathcal{E}_\eta\big[\mathbf{1}_A\big(\mathcal{E}_\eta[\xi-\eta|\mathcal{F}_t] - \mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big)\big]\\ &= \mathcal{E}_\eta\big[\mathcal{E}_\eta[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t] - \mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t]\big]\\ &= \mathcal{E}_\eta\Big[\mathcal{E}_\eta\big[\mathbf{1}_A(\xi-\eta) - \mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t]\big|\mathcal{F}_t\big]\Big]\\ &= \mathcal{E}_\eta\big[\mathbf{1}_A(\xi-\eta) - \mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t]\big]\\ &\le \mathcal{E}_{\Xi_0}\big[\mathbf{1}_A(\xi-\eta) - \mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t]\big]\\ &= \mathcal{E}_{\Xi_0}\Big[\mathcal{E}_{\Xi_0}\big[\mathbf{1}_A(\xi-\eta) - \mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t]\big|\mathcal{F}_t\big]\Big]\\ &= \mathcal{E}_{\Xi_0}\big[\mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)\big|\mathcal{F}_t] - \mathcal{E}_{\Xi_0}[\mathbf{1}_A(\xi-\eta)|\mathcal{F}_t]\big]\\ &= \mathcal{E}_{\Xi_0}[0] = 0.\end{aligned}$$

Then the strict monotonicity of $\mathcal{E}_\eta[\ ]$ implies that $\mathbf{1}_A\big(\mathcal{E}_\eta[\xi-\eta|\mathcal{F}_t]-\mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big)=0$, $P-$a.s. It follows that $P(A)=0$ or equivalently, $\mathcal{E}[\xi|\mathcal{F}_t]-\mathcal{E}[\eta|\mathcal{F}_t]=\mathcal{E}_\eta[\xi-\eta|\mathcal{F}_t]\le\mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]$, $P-$ a.s.

(3) Let $\xi,\eta\in L^p(\mathcal{F}_T)$. If (Y,Z,U) is the unique solution of BSDEJ $\big(\eta-\xi,g^{\Xi_0}\big)$, multiplying -1 shows that $(-Y,-Z,-U)$ is the unique solution of BSDEJ $\big(\xi-\eta,\overline{g}^{\Xi_0}\big)$. Namely, $P\big\{-\mathcal{E}_{\Xi_0}[\eta-\xi|\mathcal{F}_t]=\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_t],\ \ \forall\, t\in[0,T]\big\}=1$. It follows from the $\mathcal{E}_{\Xi_0}$-domination of $\mathcal{E}$ that

$$\mathcal{E}_\eta[\xi-\eta]=\mathcal{E}[\xi]-\mathcal{E}[\eta]=-\big(\mathcal{E}[\eta]-\mathcal{E}[\xi]\big)\ge-\mathcal{E}_{\Xi_0}[\eta-\xi]=\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta].$$

For any $t\in[0,T]$, using similar arguments to those in part 2) yields that $\mathcal{E}[\xi|\mathcal{F}_t]-\mathcal{E}[\eta|\mathcal{F}_t]=\mathcal{E}_\eta[\xi-\eta|\mathcal{F}_t]\ge\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]$, $P-$ a.s., which proves (3.1). Then one can deduce from (2.1) and Example 2.1 that

$$\begin{aligned}
&E\Big[\big|\mathcal{E}[\xi|\mathcal{F}_t]-\mathcal{E}[\eta|\mathcal{F}_t]\big|^p\Big]\\
&\le E\Big[\big|\mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big|^p\vee\big|\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big|^p\Big]\\
&\le E\Big[\sup_{s\in[0,T]}\big|\mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_s]\big|^p+\sup_{s\in[0,T]}\big|\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_s]\big|^p\Big]\\
&\le\mathcal{C}E\big[|\xi-\eta|^p\big].
\end{aligned}$$

□

3.2 $\mathcal{E}$-Martingales

In this subsection, we will study basic properties of $\mathcal{E}$-martingales such as optional sampling theorem and Doob–Meyer decomposition. All of them rely on a path regular result of $\mathcal{E}$-martingales as follows:

Proposition 3.2 *Given $p\in(1,2]$, let Ξ be a p-coefficient set with $\beta\equiv0$, and let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant $\mathbf{F}$-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. Then each $\mathcal{E}$-martingale admits a càdlàg modification. Consequently, we are able to upgrade the inequality (3.2):*

$$E\Big[\sup_{t\in[0,T]}\big|\mathcal{E}[\xi|\mathcal{F}_t]-\mathcal{E}[\eta|\mathcal{F}_t]\big|^p\Big]\le\mathcal{C}E\big[|\xi-\eta|^p\big],\quad\forall\,\xi,\eta\in L^p(\mathcal{F}_T).\qquad(3.3)$$

Proof Set $\mathcal{D}_T:=\{k2^{-n}\in(0,T):k,n\in\mathbb{N}\}\cup\{0,T\}$. Clearly, $q_n^+(t):=\frac{\lceil 2^n t\rceil}{2^n}\wedge T\in\mathcal{D}_T$, $\forall\,t\in[0,T]$, $\forall\,n\in\mathbb{N}$. Let X be an $\mathcal{E}$-martingale with $E[X_*^p]<\infty$ and define the right-limit process of X by

$$\widehat{X}_t:=\varliminf_{n\to\infty}X_{q_n^+(t)}\in[-\infty,\infty],\qquad\forall\,t\in[0,T].$$

The right-continuity of filtration $\mathbf{F}$ implies that the process $\widehat{X}$ is $\mathbf{F}$-adapted.

(1) For any $0 \le t < s \le T$, applying (3.1) with $(\xi, \eta) = (X_s, 0)$ and using (F2) yield that $X_t = \mathcal{E}[X_s|\mathcal{F}_t] \ge \overline{\mathcal{E}}_{\Xi_0}[X_s|\mathcal{F}_t]$, $P-$a.s. So X is also an $\overline{g}^{\Xi_0}$-supermartingale. We claim that

$$P\Big\{\widehat{X}_t = \lim_{n\to\infty} X_{q_n^+(t)} \in \mathbb{R}, \ \ \forall\, t \in [0, T]\Big\} = 1. \tag{3.4}$$

To see this, we let $-\infty < a < b < \infty$ and let $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of $\mathcal{D}_T$ such that $\underset{n}{\cup}\mathcal{D}_n = \mathcal{D}_T$ and that $T \in \mathcal{D}_n$ for any $n \in \mathbb{N}$. Since Example 2.1 shows that $\overline{g}^{\Xi_0}$ also satisfies (A2), (A3) and (2.2) with the same coefficient set Ξ_0, applying Proposition 2.3 with $g = \overline{g}^{\Xi_0}$, we obtain that for any $n \in \mathbb{N}$

$$\begin{aligned} E\Big[\ln\big(1 + U_{\mathcal{D}_n}(a, b; X)\big)\Big] &\le \ln\bigg\{\frac{e^{3\widehat{C}}}{b-a}\mathcal{E}_{\Xi_0}\big[a^+ + |X_T|\big] + \frac{|a|e^{3\widehat{C}}}{b-a} + 1\bigg\} \\ &\quad + \frac{1}{2}\widehat{C} + \big(\kappa_2 - \ln(1+\kappa_1)\big)\nu(\mathcal{X})T < \infty. \end{aligned}$$

As $\lim\limits_{n\to\infty} \uparrow U_{\mathcal{D}_n}(a, b; X) = U_{\mathcal{D}_T}(a, b; X)$, the monotone convergence theorem implies that

$$\begin{aligned} E\Big[\ln\big(1 + U_{\mathcal{D}_T}(a, b; X)\big)\Big] &= \lim_{n\to\infty} \uparrow E\Big[\ln\big(1 + U_{\mathcal{D}_n}(a, b; X)\big)\Big] \\ &\le \ln\bigg\{\frac{e^{3\widehat{C}}}{b-a}\Big(\mathcal{E}_{\Xi_0}\big[a^+ + |X_T|\big] + |a|\Big) + 1\bigg\} \\ &\quad + \frac{1}{2}\widehat{C} + \big(\kappa_2 - \ln(1+\kappa_1)\big)\nu(\mathcal{X})T. \end{aligned}$$

So $U_{\mathcal{D}_T}(a, b; X) < \infty$, $P-$a.s. Using a classical argument (see e.g. [21, Proposition 1.3.14]) leads to that

$$\lim_{s\nearrow t, s\in\mathcal{D}_T} X_s \text{ and } \lim_{s\searrow t, s\in\mathcal{D}_T} X_s \text{ exist and are finite for any } t \in [0, T].$$

Then (3.4) follows, and thus one can regard $\widehat{X}$ as a real-valued, $\mathbf{F}$-adapted càdlàg process.

(2) Next, fix $t \in [0, T]$. For any $n \in \mathbb{N}$, applying (3.1) with $(\xi, \eta, t) = (X_T, 0, q_n^+(t))$ and using (F2) yield that $P-$a.s.

$$\begin{aligned} \big|X_{q_n^+(t)}\big|^p &= \big|\mathcal{E}\big[X_T\big|\mathcal{F}_{q_n^+(t)}\big]\big|^p \le \big|\mathcal{E}_{\Xi_0}\big[X_T\big|\mathcal{F}_{q_n^+(t)}\big]\big|^p \vee \big|\overline{\mathcal{E}}_{\Xi_0}\big[X_T\big|\mathcal{F}_{q_n^+(t)}\big]\big|^p \\ &\le \sup_{s\in[0,T]}\big|\mathcal{E}_{\Xi_0}\big[X_T\big|\mathcal{F}_s\big]\big|^p + \sup_{s\in[0,T]}\big|\overline{\mathcal{E}}_{\Xi_0}\big[X_T\big|\mathcal{F}_s\big]\big|^p := \xi_{\Xi_0}. \end{aligned} \tag{3.5}$$

It follows from (3.4) that

$$\big|\widehat{X}_t\big|^p = \lim_{n\to\infty}\big|X_{q_n^+(t)}\big|^p \le \xi_{\Xi_0}, \quad P-\text{a.s.} \tag{3.6}$$

As (2.1) shows that

$$E\big[\xi_{\Xi_0}\big] \leq C E[|X_T|^p], \tag{3.7}$$

we see that $\widehat{X}_t \in L^p(\mathcal{F}_T)$.
For any $n \in \mathbb{N}$, (F2) and (3.2) imply that

$$\begin{aligned} E\Big[\big|\mathcal{E}[X_{q_n^+(t)}|\mathcal{F}_t] - \widehat{X}_t\big|^p\Big] &= E\Big[\big|\mathcal{E}[X_{q_n^+(t)}|\mathcal{F}_t] - \mathcal{E}\big[\widehat{X}_t|\mathcal{F}_t\big]\big|^p\Big] \\ &\leq C E\Big[\big|X_{q_n^+(t)} - \widehat{X}_t\big|^p\Big]. \end{aligned} \tag{3.8}$$

Since (1.4), (3.5) and (3.6) show that $\big|X_{q_n^+(t)} - \widehat{X}_t\big|^p \leq 2^{p-1}\big(\big|X_{q_n^+(t)}\big|^p + \big|\widehat{X}_t\big|^p\big) \leq 2^p \xi_{\Xi_0}$, one can deduce from (3.4), (3.7) and the dominated convergence theorem that $\lim_{n\to\infty} E\big[\big|X_{q_n^+(t)} - \widehat{X}_t\big|^p\big] = 0$. It then follows from (3.8) that $\lim_{n\to\infty} E\Big[\big|\mathcal{E}\big[X_{q_n^+(t)}|\mathcal{F}_t\big] - \widehat{X}_t\big|^p\Big] = 0$. So there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\widehat{X}_t = \lim_{i\to\infty} \mathcal{E}\big[X_{q_{n_i}^+(t)}|\mathcal{F}_t\big] = X_t, \quad P-\text{a.s.}$$

Therefore, $\widehat{X}$ is a càdlàg modification of X.

(3) Let $\xi, \eta \in L^p(\mathcal{F}_T)$. We can deduce from (F2), (3.1), (2.1) and Example 2.1 that $E\Big[\sup_{t\in[0,T]} \big|\mathcal{E}[\xi|\mathcal{F}_t]\big|^p\Big] \leq E\Big[\sup_{t\in[0,T]} \Big(\big|\mathcal{E}_{\Xi_0}[\xi|\mathcal{F}_t]\big|^p \vee \big|\overline{\mathcal{E}}_{\Xi_0}[\xi|\mathcal{F}_t]\big|^p\Big)\Big] \leq E\Big[\sup_{t\in[0,T]} \big|\mathcal{E}_{\Xi_0}[\xi|\mathcal{F}_t]\big|^p + \sup_{t\in[0,T]} \big|\overline{\mathcal{E}}_{\Xi_0}[\xi|\mathcal{F}_t]\big|^p\Big] \leq C E\big[|\xi|^p\big] < \infty$, and similarly that $E\Big[\sup_{t\in[0,T]} \big|\mathcal{E}[\eta|\mathcal{F}_t]\big|^p\Big] \leq C E\big[|\eta|^p\big] < \infty$.
According to Part (2), the $\mathcal{E}$-martingales $\{\mathcal{E}[\xi|\mathcal{F}_t]\}_{t\in[0,T]}$ and $\{\mathcal{E}[\eta|\mathcal{F}_t]\}_{t\in[0,T]}$ have càdlàg modifications, we still denote them by $\{\mathcal{E}[\xi|\mathcal{F}_t]\}_{t\in[0,T]}$ and $\{\mathcal{E}[\eta|\mathcal{F}_t]\}_{t\in[0,T]}$ respectively. Using (3.1), (2.1) and Example 2.1 again yields

$$\begin{aligned} &E\Big[\sup_{t\in[0,T]} \big|\mathcal{E}[\xi|\mathcal{F}_t] - \mathcal{E}[\eta|\mathcal{F}_t]\big|^p\Big] \\ &\leq E\Big[\sup_{t\in[0,T]} \Big(\big|\mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big|^p \vee \big|\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big|^p\Big)\Big] \\ &\leq E\Big[\sup_{t\in[0,T]} \big|\mathcal{E}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big|^p + \sup_{t\in[0,T]} \big|\overline{\mathcal{E}}_{\Xi_0}[\xi-\eta|\mathcal{F}_t]\big|^p\Big] \\ &\leq C E\big[|\xi-\eta|^p\big]. \end{aligned}$$

□

From now on, for any translation invariant, $\mathcal{E}_{\Xi_0}$-dominated $\mathbf{F}$-expectation $\mathcal{E}$, we will just consider the càdlàg modification of any $\mathcal{E}$-martingale, which turns out to be a semi-martingale under P in the following form:

Proposition 3.3 *Given $p \in (1,2]$, let Ξ be a p-coefficient set with $\beta \equiv 0$ such that $\int_0^T \Lambda_t^{\frac{2p}{2-p}} dt \in L^\infty(\mathcal{F}_T)$ if $p \in (1,2)$ or $\Lambda \equiv \kappa_\Lambda \in [0,\infty)$ if $p = 2$. Also, let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant $\mathbf{F}$-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. For $i = 1, 2$, let $X^i \in \mathbb{D}^p$ be an $\mathcal{E}$-martingale, then there exist a real-valued, $\mathbf{F}$-progressively measurable process $\mathfrak{g}^i$ and $(Z^i, U^i) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$ such that*

(1) $P\big\{X_t^i = X_T^i + \int_t^T \mathfrak{g}_s^i ds - \int_t^T Z_s^i dB_s - \int_{(t,T]} \int_{\mathcal{X}} U_s^i(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), \quad \forall t \in [0,T]\big\} = 1.$
(2) $\underline{g}^{\Xi_0}(t, Z_t^i, U_t^i) \le \mathfrak{g}_t^i \le g^{\Xi_0}(t, Z_t^i, U_t^i)$, $dt \times dP-$a.s.
(3) $\underline{g}^{\Xi_0}(t, Z_t^1 - Z_t^2, U_t^1 - U_t^2) \le \mathfrak{g}_t^1 - \mathfrak{g}_t^2 \le g^{\Xi_0}(t, Z_t^1 - Z_t^2, U_t^1 - U_t^2)$, $dt \times dP-$a.s.

Proof (1) Let $i = 1, 2$. For $0 \le t \le s \le T$, (F2) and (3.1) show that

$$\underline{\mathcal{E}}_{\Xi_0}[X_s^i|\mathcal{F}_t] \le X_t^i = \mathcal{E}[X_s^i|\mathcal{F}_t] - \mathcal{E}[0|\mathcal{F}_t] \le \mathcal{E}_{\Xi_0}[X_s^i|\mathcal{F}_t], \quad P-\text{a.s.}$$

Thus X^i is a g^{Ξ_0}-submartingale and a $\underline{g}^{\Xi_0}$-supermartingale. By Example 2.1, g^{Ξ_0}, $\underline{g}^{\Xi_0}$ satisfies (A2), (A3) and (2.2) with the same coefficient set Ξ_0. In light of Theorem 2.3, there exist unique processes $(Z^i, U^i, K^i) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p$ such that $P-$a.s.

$$\begin{aligned} X_t^i &= X_T^i + \int_t^T g^{\Xi_0}(s, Z_s^i, U_s^i) ds - \int_t^T Z_s^i dB_s \\ &\quad - \int_{(t,T]} \int_{\mathcal{X}} U_s^i(x) \widetilde{N}_{\mathfrak{p}}(ds, dx) - K_T^i + K_t^i, \quad t \in [0,T], \end{aligned} \tag{3.9}$$

and there exist unique processes $(\overline{Z}^i, \overline{U}^i, \overline{K}^i) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p$ such that $P-$a.s.

$$\begin{aligned} X_t^i &= X_T^i + \int_t^T \underline{g}^{\Xi_0}(s, \overline{Z}_s^i, \overline{U}_s^i) ds - \int_t^T \overline{Z}_s^i dB_s \\ &\quad - \int_{(t,T]} \int_{\mathcal{X}} \overline{U}_s^i(x) \widetilde{N}_{\mathfrak{p}}(ds, dx) + \overline{K}_T^i - \overline{K}_t^i, \quad t \in [0,T]. \end{aligned} \tag{3.10}$$

Subtracting (3.10) from (3.9) yields that $P-$a.s.

$$\begin{aligned} &\int_0^t \Big[g^{\Xi_0}(s, Z_s^i, U_s^i) - \underline{g}^{\Xi_0}(s, \overline{Z}_s^i, \overline{U}_s^i) \Big] ds \\ &= \int_0^t (Z_s^i - \overline{Z}_s^i) dB_s + \int_{(0,t]} \int_{\mathcal{X}} \big(U_s^i(x) - \overline{U}_s^i(x)\big) \widetilde{N}_{\mathfrak{p}}(ds, dx) \\ &\quad + K_t^i + \overline{K}_t^i, t \in [0,T]. \end{aligned} \tag{3.11}$$

The jump times of the stochastic integral $\left\{\int_{(0,t]}\int_{\mathcal{X}}\left(U^i_s(x) - \overline{U}^i_s(x)\right)\widetilde{N}_{\mathfrak{p}}(ds,dx)\right\}_{t\in[0,T]}$ are totally inaccessible, while the jumps of the $\mathbf{F}$-predictable càdlàg increasing process $K^i + \overline{K}^i$ are exhausted by a sequence $\{\zeta^i_n\}_{n\in\mathbb{N}}$ of $\mathbf{F}$-predictable stopping times (i.e. $\{(t,\omega)\in[0,T]\times\Omega : \Delta K^i_t(\omega)+\Delta\overline{K}^i_t(\omega)>0\}$ is a union of graphs $[\![\zeta^i_n]\!]$ and these graphs are disjoint on $(0,T)$, see e.g. "Complements to Chapter IV" of Dellacherie and Meyer [14] or [20, Proposition I.2.24] for details). So we see from (3.11) that for $P-$a.s. $\omega\in\Omega$

$$\begin{aligned}\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\left(\Delta K^i_t(\omega)+\Delta\overline{K}^i_t(\omega)\right) &= 0 \text{ and } 0 = \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\left(U^i-\overline{U}^i\right)(t,\omega,\mathfrak{p}_t(\omega)) \\ &\quad + \mathbf{1}_{\{t\notin D_{\mathfrak{p}(\omega)}\}}\left(\Delta K^i_t(\omega)+\Delta\overline{K}^i_t(\omega)\right),\ \ \forall t\in[0,T].\end{aligned}$$

It follows that $P-$a.s.

$$\Delta K^i_t + \Delta\overline{K}^i_t = 0, \quad \forall t\in[0,T] \ \text{ and } \ U^i_t(x)=\overline{U}^i_t(x), \quad \forall (t,x)\in[0,T]\times\mathcal{X}. \tag{3.12}$$

The former shows that the increasing processes K^i and $\overline{K}^i$ have $P-$a.s. continuous paths, which together with the latter and (3.11) implies that $Z^i_t = \overline{Z}^i_t$, $dt\times dP-$a.s. and that $P-$a.s.

$$K^i_t + \overline{K}^i_t = \int_0^t \left[g^{\Xi_0}(s,Z^i_s,U^i_s) - \overline{g}^{\Xi_0}(s,Z^i_s,U^i_s)\right]ds, \quad t\in[0,T]. \tag{3.13}$$

Hence, both K^i and $\overline{K}^i$ are absolutely continuous processes: there exist nonnegative $\mathbf{F}$-progressively measurable processes $a^i,\overline{a}^i$ such that $P-$a.s.

$$K^i_t = \int_0^t a^i_s ds, \quad \text{and} \quad \overline{K}^i_t = \int_0^t \overline{a}^i_s ds, \quad t\in[0,T].$$

Then we see from (3.9) that the real-valued, $\mathbf{F}$-progessively measurable process $\mathfrak{g}^i_t := g^{\Xi_0}(t,Z^i_t,U^i_t) - a^i_t$, $t\in[0,T]$ together with (Z^i,U^i) leads to that $P-$a.s.

$$X^i_t = X^i_T + \int_t^T \mathfrak{g}^i_s ds - \int_t^T Z^i_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} U^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T]. \tag{3.14}$$

By (3.13), it holds $dt\times dP-$a.s. that

$$\begin{aligned}g^{\Xi_0}(t,Z^i_t,U^i_t) \ge \mathfrak{g}^i_t = g^{\Xi_0}(t,Z^i_t,U^i_t) - a^i_t = \overline{g}^{\Xi_0}(t,Z^i_t,U^i_t) + \overline{a}^i_t \\ \ge \overline{g}^{\Xi_0}(t,Z^i_t,U^i_t).\end{aligned}$$

(2) We know from the proof of Proposition 3.1 that

$$\widetilde{\mathcal{E}}[\vartheta|\mathcal{F}_t] := \mathcal{E}[X_T^2 + \vartheta|\mathcal{F}_t] - \mathcal{E}[X_T^2|\mathcal{F}_t], \quad \forall\, t \in [0, T], \ \forall\, \vartheta \in L^p(\mathcal{F}_T)$$

is also a translation invariant **F**-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. Since $X_t^1 - X_t^2 = \mathcal{E}[X_T^1|\mathcal{F}_t] - \mathcal{E}[X_T^2|\mathcal{F}_t] = \widetilde{\mathcal{E}}[X_T^1 - X_T^2|\mathcal{F}_t], t \in [0, T]$ is an $\widetilde{\mathcal{E}}$-martingale, an application of (3.14) yields that for some real-valued, **F**-progressively measurable process $\widetilde{\mathfrak{g}}$ and $\big(\widetilde{Z}, \widetilde{U}\big) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$, it holds $P-$a.s. that

$$\begin{aligned} X_t^1 - X_t^2 = X_T^1 - X_T^2 &+ \int_t^T \widetilde{\mathfrak{g}}_s ds - \int_t^T \widetilde{Z}_s dB_s \\ &- \int_{(t,T]}\int_{\mathcal{X}} \widetilde{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T], \end{aligned} \tag{3.15}$$

and that

$$\underline{g}^{\Xi_0}(t, \widetilde{Z}_t, \widetilde{U}_t) \le \widetilde{\mathfrak{g}}_t \le g^{\Xi_0}(t, \widetilde{Z}_t, \widetilde{U}_t), \quad dt \times dP - \text{a.s.} \tag{3.16}$$

As $X^1 - X^2$ also satisfies that $P-$ a.s.

$$\begin{aligned} X_t^1 - X_t^2 = X_T^1 - X_T^2 &+ \int_t^T (\mathfrak{g}_s^1 - \mathfrak{g}_s^2) ds - \int_t^T (Z_s^1 - Z_s^2) dB_s \\ &- \int_{(t,T]}\int_{\mathcal{X}} \big(U_s^1(x) - U_s^2(x)\big)\widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T], \end{aligned}$$

an comparison with (3.15) shows that $\big(\widetilde{\mathfrak{g}}_t, \widetilde{Z}_t\big) = \big(\mathfrak{g}_t^1 - \mathfrak{g}_t^2, Z_t^1 - Z_t^2\big), dt \times dP -$ a.s. and that $\widetilde{U}_t(x) = U_t^1(x) - U_t^2(x)$, $dt \times dP \times \nu(dx)$-a.s. Plugging these equalities into (3.16) yields that

$$\begin{aligned} &\underline{g}^{\Xi_0}\left(t, Z_t^1 - Z_t^2, U_t^1 - U_t^2\right) \\ &\quad \le \mathfrak{g}_t^1 - \mathfrak{g}_t^2 \le g^{\Xi_0}\left(t, Z_t^1 - Z_t^2, U_t^1 - U_t^2\right), dt \times dP - \text{a.s.} \end{aligned}$$

□

Let Ξ be a p-coefficient set with $\beta \equiv 0$, and let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant **F**-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. For any $\xi \in L^p(\mathcal{F}_T)$ and $\tau \in \mathcal{T}$, define $\mathcal{E}[\xi|\mathcal{F}_\tau] = \Upsilon_\tau^\xi$, where Υ^ξ denotes the càdlàg modification of the $\mathcal{E}$-martingale $\big\{\mathcal{E}[\xi|\mathcal{F}_t]\big\}_{t\in[0,T]}$. Analogous to Proposition 2.2, one has an optional sampling theorem for $\mathcal{E}$-martingales.

Proposition 3.4 (*Optional Sampling of $\mathcal{E}$-martingales*) *Given $p \in (1, 2]$, let Ξ be a p-coefficient set with $\beta \equiv 0$, and let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant* **F***-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. For any right-continuous $\mathcal{E}$-submartingale (resp. $\mathcal{E}$-supermartingale) X with $E[X_*^p] < \infty$, it holds for any $\tau, \gamma \in \mathcal{T}$ that*

$$X_{\tau\wedge\gamma} \le (resp.\ \ge)\ \mathcal{E}[X_\gamma|\mathcal{F}_\tau], \quad P-\text{a.s.}$$

Proof Let us only consider the $\mathcal{E}$-submartingale case, as the other cases can be derived similarly.

Fix $\tau, \gamma \in \mathcal{T}$ and let $t \in [0, T], n \in \mathbb{N}$. We set $t_i^n := t + \frac{i}{2^n}(T-t)$ for $i = 0, \cdots, 2^n$, $A_1^n := \{t \le \gamma \vee t \le t_1^n\} \in \mathcal{F}_{t_1^n}$, $A_i^n := \{t_{i-1}^n < \gamma \vee t \le t_i^n\} \in \mathcal{F}_{t_i^n}$ for $i = 2, \cdots, 2^n$, and define $\gamma_n := \sum_{i=1}^{2^n} \mathbf{1}_{A_i^n} t_i^n \in \mathcal{T}$.

For any $i = 1, \cdots, 2^n - 1$, set $\xi_i^n := \mathbf{1}_{\{\gamma_n \le t_i^n\}} X_{\gamma_n} + \mathbf{1}_{\{\gamma_n \ge t_{i+1}^n\}} X_{t_{i+1}^n}$. Since $\{\gamma_n \le t_i^n\} = \{\gamma_n \ge t_{i+1}^n\}^c \in \mathcal{F}_{t_i^n}$ and since $\mathbf{1}_{\{\gamma_n \le t_i^n\}} X_{\gamma_n} \in \mathcal{F}_{t_i^n}$, (F2), (F4) and the $\mathcal{E}$-submartingality of X imply that $P-$ a.s.

$$\begin{aligned}
\mathcal{E}\big[\xi_i^n|\mathcal{F}_{t_i^n}\big] &= \mathbf{1}_{\{\gamma_n \le t_i^n\}}\mathcal{E}\big[\xi_i^n|\mathcal{F}_{t_i^n}\big] + \mathbf{1}_{\{\gamma_n \ge t_{i+1}^n\}}\mathcal{E}\big[\xi_i^n|\mathcal{F}_{t_i^n}\big]\\
&= \mathcal{E}\big[\mathbf{1}_{\{\gamma_n \le t_i^n\}} X_{\gamma_n}|\mathcal{F}_{t_i^n}\big] + \mathcal{E}\big[\mathbf{1}_{\{\gamma_n \ge t_{i+1}^n\}} X_{t_{i+1}^n}|\mathcal{F}_{t_i^n}\big]\\
&= \mathbf{1}_{\{\gamma_n \le t_i^n\}} X_{\gamma_n} + \mathbf{1}_{\{\gamma_n \ge t_{i+1}^n\}}\mathcal{E}\big[X_{t_{i+1}^n}|\mathcal{F}_{t_i^n}\big]\\
&\ge \mathbf{1}_{\{\gamma_n \le t_i^n\}} X_{\gamma_n} + \mathbf{1}_{\{\gamma_n \ge t_{i+1}^n\}} X_{t_i^n}\\
&= \mathbf{1}_{\{\gamma_n \le t_{i-1}^n\}} X_{\gamma_n} + \mathbf{1}_{\{\gamma_n \ge t_i^n\}} X_{t_i^n} = \xi_{i-1}^n.
\end{aligned}$$

Taking $\mathcal{E}[\ |\mathcal{F}_t]$, we see from (F1) and (F3) that $\mathcal{E}\big[\xi_i^n|\mathcal{F}_t\big] = \mathcal{E}\big[\mathcal{E}[\xi_i^n|\mathcal{F}_{t_i^n}]\big|\mathcal{F}_t\big] \ge \mathcal{E}\big[\xi_{i-1}^n|\mathcal{F}_t\big]$, $P-$ a.s. It then follows that

$$\begin{aligned}
\mathcal{E}[X_{\gamma_n}|\mathcal{F}_t] = \mathcal{E}[\xi_{2^n-1}^n|\mathcal{F}_t] \ge \mathcal{E}[\xi_{2^n-2}^n|\mathcal{F}_t] \ge \cdots&\\
\ge \mathcal{E}[\xi_0^n|\mathcal{F}_t] = \mathcal{E}[X_{t_1^n}|\mathcal{F}_t] \ge X_t, \quad P-\text{a.s.}&
\end{aligned}$$

By (3.2), $E\Big[\big|\mathcal{E}[X_{\gamma_n}|\mathcal{F}_t] - \mathcal{E}[X_{\gamma\vee t}|\mathcal{F}_t]\big|^p\Big] \le CE\big[|X_{\gamma_n} - X_{\gamma\vee t}|^p\big]$.

Since $\lim\limits_{n\to\infty} \downarrow \gamma_n = \gamma \vee t$ and since $E[X_*^p] < \infty$, the right-continuity of X and the dominated convergence theorem imply that $\lim\limits_{n\to\infty} E\big[|X_{\gamma_n} - X_{\gamma\vee t}|^p\big] = 0$ and thus $\lim\limits_{n\to\infty} E\big[\big|\mathcal{E}[X_{\gamma_n}|\mathcal{F}_t] - \mathcal{E}[X_{\gamma\vee t}|\mathcal{F}_t]\big|^p\big] = 0$. So there exists a subsequence $\{n_j\}_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\mathcal{E}[X_{\gamma\vee t}|\mathcal{F}_t] = \lim_{j\to\infty} \mathcal{E}[X_{\gamma_{n_j}}|\mathcal{F}_t] \ge X_t, \quad P-\text{a.s.}$$

Since $\{\gamma \le t\} \in \mathcal{F}_t$, (F4) and translation invariance of $\mathcal{E}$ show that $P-$ a.s.

$$\begin{aligned}
\mathcal{E}[X_\gamma|\mathcal{F}_t] &= \mathcal{E}\big[\mathbf{1}_{\{\gamma>t\}} X_{\gamma\vee t} + \mathbf{1}_{\{\gamma\le t\}} X_{\gamma\wedge t}\big|\mathcal{F}_t\big]\\
&= \mathbf{1}_{\{\gamma>t\}}\mathcal{E}\big[X_{\gamma\vee t}\big|\mathcal{F}_t\big] + \mathbf{1}_{\{\gamma\le t\}} X_{\gamma\wedge t} \ge \mathbf{1}_{\{\gamma>t\}} X_t + \mathbf{1}_{\{\gamma\le t\}} X_{\gamma\wedge t} = X_{\gamma\wedge t}.
\end{aligned}$$

Hence, it holds except on a P-null set $\mathcal{N}$ that $\mathcal{E}[X_\gamma|\mathcal{F}_t] \ge X_{\gamma\wedge t}$, $\forall\, t \in \big([0, T)\cap\mathbb{Q}\big)\cup\{T\}$ and that the paths $\mathcal{E}[X_\gamma|\mathcal{F}_\cdot]$ and $X_{\gamma\wedge\cdot}$ are both càdlàg, thanks to Proposition 3.2. Eventually, it holds on $\mathcal{N}^c$ that

$$\mathcal{E}[X_\gamma|\mathcal{F}_t] \ge X_{\gamma\wedge t}, \quad \forall\, t \in [0, T], \quad \text{and thus} \quad \mathcal{E}[X_\gamma|\mathcal{F}_\tau] \ge X_{\tau\wedge\gamma}.$$

$\square$

In light of Propositions 3.2 and 3.4, we can extend the properties (F1)–(F4) of **F**-expectations.

Proposition 3.5 *Given $p \in (1,2]$, let Ξ be a p-coefficient set with $\beta \equiv 0$, and let $\{\mathcal{E}[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}$ be a translation invariant $\mathbf{F}$-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. Then for any $\xi \in L^p(\mathcal{F}_T)$ and $\tau \in \mathcal{T}$, the following statements hold:*

- (F1*) *"Monotonicity": For any $\eta \in L^p(\mathcal{F}_T)$ with $\xi \le \eta$, $P-$a.s., $\mathcal{E}[\xi|\mathcal{F}_\tau] \le \mathcal{E}[\eta|\mathcal{F}_\tau]$, $P-$a.s.*
- (F2*) *"Constant Preserving": If ξ is $\mathcal{F}_\tau$-measurable, then $\mathcal{E}[\xi|\mathcal{F}_\tau] = \xi$, $P-$a.s.*
- (F3*) *"Time Consistency": For any $\gamma \in \mathcal{T}_\tau$, $\mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_\gamma]\big|\mathcal{F}_\tau\big] = \mathcal{E}[\xi|\mathcal{F}_\tau]$, $P-$a.s.*
- (F4*) *"Zero–One Law": For any $A \in \mathcal{F}_\tau$, $\mathcal{E}[\mathbf{1}_A\xi|\mathcal{F}_\tau] = \mathbf{1}_A\mathcal{E}[\xi|\mathcal{F}_\tau]$, $P-$a.s.*
- (F5*) *"Tr0anslation Invariance": For any $\eta \in L^p(\mathcal{F}_\tau)$, $\mathcal{E}[\xi+\eta|\mathcal{F}_\tau] = \mathcal{E}[\xi|\mathcal{F}_\tau]+\eta$, $P-$a.s.*

Proof Let $\xi \in L^p(\mathcal{F}_T)$, $\tau \in \mathcal{T}$ and $\gamma \in \mathcal{T}_\tau$. Applying (3.3) with $\eta = 0$ and using (F2) yield that $E\Big[\big(\Upsilon^\xi_*\big)^p\Big] = E\Big[\sup\limits_{t\in[0,T]}\big|\mathcal{E}[\xi|\mathcal{F}_t]\big|^p\Big] \le \mathcal{C}E\big[|\xi|^p\big] < \infty$. Then Proposition 3.4 shows that $\mathcal{E}[\xi|\mathcal{F}_\tau] = \Upsilon^\xi_\tau = \mathcal{E}[\Upsilon^\xi_\gamma|\mathcal{F}_\tau] = \mathcal{E}\big[\mathcal{E}[\xi|\mathcal{F}_\gamma]|\mathcal{F}_\tau\big]$, $P-$a.s., so (F3*) holds.

Next, let $\eta \in L^p(\mathcal{F}_T)$ with $\xi \le \eta$, $P-$a.s., let $\zeta \in L^p(\mathcal{F}_\tau)$ and let $A \in \mathcal{F}_\tau$. Given $n \in \mathbb{N}$, we set $t^n_i := \frac{i}{2^n}T$ for $i = 0, \cdots, 2^n$, $\mathcal{A}^n_1 := \{0 \le \tau \le t^n_1\} \in \mathcal{F}_{t^n_1}$, $\mathcal{A}^n_i := \{t^n_{i-1} < \tau \le t^n_i\} \in \mathcal{F}_{t^n_i}$ for $i = 2, \cdots, 2^n$, and define $\tau_n := \sum_{i=1}^{2^n}\mathbf{1}_{\mathcal{A}^n_i}t^n_i \in \mathcal{T}$.

(i) By (F1), $\mathcal{E}[\xi|\mathcal{F}_{\tau_n}] = \sum_{i=1}^{2^n}\mathbf{1}_{\mathcal{A}^n_i}\mathcal{E}\big[\xi|\mathcal{F}_{t^n_i}\big] \le \sum_{i=1}^{2^n}\mathbf{1}_{\mathcal{A}^n_i}\mathcal{E}\big[\eta|\mathcal{F}_{t^n_i}\big] = \mathcal{E}[\eta|\mathcal{F}_{\tau_n}]$, $P-$a.s.;

(ii) Since $\mathbf{1}_{\mathcal{A}^n_i}\zeta \in \mathcal{F}_{t^n_i}$ for any $i = 1, \cdots, 2^n$, one can deduce from (F2), (F4) and the translation invariance of $\mathcal{E}$ that

$$\begin{aligned}\mathcal{E}[\zeta|\mathcal{F}_{\tau_n}] &= \sum_{i=1}^{2^n}\mathbf{1}_{\mathcal{A}^n_i}\mathcal{E}\big[\zeta|\mathcal{F}_{t^n_i}\big] = \sum_{i=1}^{2^n}\mathcal{E}\big[\mathbf{1}_{\mathcal{A}^n_i}\zeta|\mathcal{F}_{t^n_i}\big]\\ &= \sum_{i=1}^{2^n}\mathbf{1}_{\mathcal{A}^n_i}\zeta = \zeta, \quad P-\text{a.s.},\end{aligned} \tag{3.17}$$

and that

$$\begin{aligned}\mathcal{E}[\xi+\zeta|\mathcal{F}_{\tau_n}] &= \sum_{i=1}^{2^n}\mathbf{1}_{\mathcal{A}^n_i}\mathcal{E}\big[\xi+\zeta|\mathcal{F}_{t^n_i}\big] = \sum_{i=1}^{2^n}\mathcal{E}\big[\mathbf{1}_{\mathcal{A}^n_i}\xi+\mathbf{1}_{\mathcal{A}^n_i}\zeta|\mathcal{F}_{t^n_i}\big]\\ &= \sum_{i=1}^{2^n}\big(\mathbf{1}_{\mathcal{A}^n_i}\mathcal{E}\big[\xi|\mathcal{F}_{t^n_i}\big]+\mathbf{1}_{\mathcal{A}^n_i}\zeta\big) = \mathcal{E}[\xi|\mathcal{F}_{\tau_n}]+\zeta, \ \ P-\text{a.s.}\end{aligned} \tag{3.18}$$

(iii) As $A \cap \mathcal{A}_i^n \in \mathcal{F}_{t_i^n}$ for any $i = 1, \ldots, 2^n$, (F2) implies that

$$\begin{aligned} \mathbf{1}_A \mathcal{E}[\xi|\mathcal{F}_{\tau_n}] &= \sum_{i=1}^{2^n} \mathbf{1}_{A\cap\mathcal{A}_i^n} \mathcal{E}\big[\xi|\mathcal{F}_{t_i^n}\big] = \sum_{i=1}^{2^n} \mathcal{E}\big[\mathbf{1}_{A\cap\mathcal{A}_i^n}\xi|\mathcal{F}_{t_i^n}\big] \\ &= \sum_{i=1}^{2^n} \mathbf{1}_{\mathcal{A}_i^n} \mathcal{E}\big[\mathbf{1}_A\xi|\mathcal{F}_{t_i^n}\big] = \mathcal{E}\big[\mathbf{1}_A\xi|\mathcal{F}_{\tau_n}\big], \quad P-\text{a.s.} \end{aligned}$$

Clearly, $\lim\limits_{n\to\infty} \downarrow \tau_n = \tau$. Thus, letting $n \to \infty$ in (i), (3.17), (3.18) and (iii) leads to (F1*), (F2*), (F4*), (F5*). □

3.3 Doob–Meyer Decomposition of $\mathcal{E}$-Supermartingales

Let $\mathcal{E}$ be an **F**-expectation. The Doob–Meyer decomposition of $\mathcal{E}$-supermartingales requires the study of the following (generalized) BSDE with respect to $\mathcal{E}$

$$Y_t + \int_0^t f(s, Y_s)ds = \mathcal{E}\bigg[\xi + \int_0^T f(s, Y_s)ds\bigg|\mathcal{F}_t\bigg], \quad t \in [0, T], \tag{3.19}$$

whose well-posedness is based on the fixed-point argument and (3.3):

Proposition 3.6 *Given $p \in (1, 2]$, let Ξ be a p-coefficient set with $\beta \equiv 0$, and let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant* **F***-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. Also, let $f : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable function such that $\int_0^T |f(t, 0)|dt \in L^p(\mathcal{F}_T)$ and that for some $C_f > 0$, it holds $dt \times dP -$ a.s. that $\big|f(t, y_1) - f(t, y_2)\big| \le C_f|y_1 - y_2|$ for any $y_1, y_2 \in \mathbb{R}$.*

Then for any $\xi \in L^p(\mathcal{F}_T)$, the BSDE (3.19) admits a unique solution $Y \in \mathbb{D}^p$, i.e. the unique process in $\mathbb{D}^p$ that satisfies (3.19) $P -$ a.s.

Proof In the inequality (3.3), the constant coefficient C does not depend on the choice of ξ and η. To temporarily freeze the form of such a constant, we rephrase it by $\widehat{C}$ in this proof.

Set $\mathfrak{a} := 2\widehat{C}T^{p-1}C_f^p$. Let $\mathbb{L}^p_{\mathfrak{a}}$ collect all real-valued, **F**-progressively measurable processes Y with $\|Y\|_{\mathbb{L}^p_{\mathfrak{a}}} := \big\{E\int_0^T e^{\mathfrak{a}t}|Y_t|^p dt\big\}^{\frac{1}{p}} < \infty$. Clearly, $\mathbb{L}^p_{\mathfrak{a}}$ is a complete space under norm $\| \ \|_{\mathbb{L}^p_{\mathfrak{a}}}$.

Fix $\xi \in L^p(\mathcal{F}_T)$ and let $Y \in \mathbb{L}^p_{\mathfrak{a}}$. Since (1.4) shows that

$$\begin{aligned} E\bigg[\Big(\int_0^T |f(t, Y_t)|dt\Big)^p\bigg] &\le E\bigg[\Big(\int_0^T (|f(t, 0)| + C_f|Y_t|)dt\Big)^p\bigg] \\ &\le 2^{p-1}E\bigg[\Big(\int_0^T |f(t, 0)|dt\Big)^p + C_f^p T^{p-1}\int_0^T |Y_t|^p dt\bigg] \\ &\le 2^{p-1}E\bigg[\Big(\int_0^T |f(t, 0)|dt\Big)^p\bigg] + 2^{p-1}C_f^p T^{p-1}\|Y\|^p_{\mathbb{L}^p_{\mathfrak{a}}} < \infty, \end{aligned} \tag{3.20}$$

Proposition 3.2 shows that $\Psi_t(Y) := \mathcal{E}\big[\xi + \int_0^T f(s, Y_s)ds|\mathcal{F}_t\big] - \int_0^t f(s, Y_s)ds$, $t \in [0, T]$ is a real-valued, $\mathbf{F}$-adapted càdlàg process. Applying (3.3) with $(\xi, \eta) = \big(\xi + \int_0^T f(s, Y_s)ds, 0\big)$, we see from (F2), (1.4) and (3.20) that

$$
\begin{aligned}
&E\Big[\big(\Psi_*(Y)\big)^p\Big] \\
&\le 2^{p-1}E\bigg[\sup_{t\in[0,T]}\bigg|\mathcal{E}\bigg[\xi + \int_0^T f(s, Y_s)ds\Big|\mathcal{F}_t\bigg]\bigg|^p + \sup_{t\in[0,T]}\bigg|\int_0^t f(s, Y_s)ds\bigg|^p\bigg] \\
&\le 2^{p-1}E\bigg[\widehat{\mathcal{C}}\bigg|\xi + \int_0^T f(s, Y_s)ds\bigg|^p + \bigg(\int_0^T \big|f(s, Y_s)\big|ds\bigg)^p\bigg] \\
&\le 2^{p-1}E\bigg[2^{p-1}\widehat{\mathcal{C}}|\xi|^p + (1 + 2^{p-1}\widehat{\mathcal{C}})\bigg(\int_0^T \big|f(s, Y_s)\big|ds\bigg)^p\bigg] < \infty,
\end{aligned}
$$

so $\Psi(Y) \in \mathbb{D}^p \subset \mathbb{L}^p_{\mathfrak{a}}$.

To see that Ψ is a contraction map on $\mathbb{L}^p_{\mathfrak{a}}$, we let $\mathcal{Y}$ be another process in $\mathbb{L}^p_{\mathfrak{a}}$. Given $t \in [0, T]$, since the translation invariance of $\mathcal{E}$ shows that $\Psi_t(Y) - \Psi_t(\mathcal{Y}) = \mathcal{E}\big[\xi + \int_t^T f(s, Y_s)ds\big|\mathcal{F}_t\big] - \mathcal{E}\big[\xi + \int_t^T f(s, \mathcal{Y}_s)ds\big|\mathcal{F}_t\big]$, $P-$a.s., one can deduce from (3.3) and Hölder's inequality that

$$
\begin{aligned}
&E\Big[\big|\Psi_t(Y) - \Psi_t(\mathcal{Y})\big|^p\Big] \\
&\le E\bigg[\sup_{t'\in[0,T]}\bigg|\mathcal{E}\bigg[\xi + \int_t^T f(s, Y_s)ds\Big|\mathcal{F}_{t'}\bigg] - \mathcal{E}\bigg[\xi + \int_t^T f(s, \mathcal{Y}_s)ds\Big|\mathcal{F}_{t'}\bigg]\bigg|^p\bigg] \\
&\le \widehat{\mathcal{C}}E\bigg[\bigg|\int_t^T \big(f(s, Y_s) - f(s, \mathcal{Y}_s)\big)ds\bigg|^p\bigg] \\
&\le \widehat{\mathcal{C}}T^{p-1}E\int_t^T \big|f(s, Y_s) - f(s, \mathcal{Y}_s)\big|^p ds \le \widehat{\mathcal{C}}T^{p-1}C_f^p E\int_t^T |Y_s - \mathcal{Y}_s|^p ds.
\end{aligned}
$$

It follows from Fubini's Theorem that

$$
\begin{aligned}
\big\|\Psi(Y) - \Psi(\mathcal{Y})\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} &= E\int_0^T e^{\mathfrak{a}t}|\Psi_t(Y) - \Psi_t(\mathcal{Y})|^p dt \\
&= \int_0^T e^{\mathfrak{a}t}E\Big[\big|\Psi_t(Y) - \Psi_t(\mathcal{Y})\big|^p\Big]dt \\
&\le \frac{\mathfrak{a}}{2}\int_0^T e^{\mathfrak{a}t}E\int_t^T |Y_s - \mathcal{Y}_s|^p dsdt \\
&= \frac{\mathfrak{a}}{2}E\int_0^T \bigg(|Y_s - \mathcal{Y}_s|^p\int_0^s e^{\mathfrak{a}t}dt\bigg)ds \\
&= \frac{1}{2}E\int_0^T (e^{\mathfrak{a}s} - 1)|Y_s - \mathcal{Y}_s|^p ds \le \frac{1}{2}\|Y - \mathcal{Y}\|^p_{\mathbb{L}^p_{\mathfrak{a}}}. \qquad (3.21)
\end{aligned}
$$

Hence, Ψ is a contraction map on $\mathbb{L}^p_{\mathfrak{a}}$ and thus admits a unique fixed point $\widehat{Y} \in \mathbb{L}^p_{\mathfrak{a}}$.

Set $\mathscr{Y} := \Psi\big(\widehat{Y}\big) \in \mathbb{D}^p$. By (3.21), $\big\|\Psi(\mathscr{Y}) - \mathscr{Y}\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} = \big\|\Psi(\mathscr{Y}) - \Psi\big(\widehat{Y}\big)\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} \le \frac{1}{2}\big\|\mathscr{Y} - \widehat{Y}\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} = \frac{1}{2}\big\|\Psi\big(\widehat{Y}\big) - \widehat{Y}\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} = 0$, which implies that $\Psi_t(\mathscr{Y}) = \mathscr{Y}_t$, $dt \times dP -$ a.s. Then one can deduce from the right-continuity of $\mathscr{Y}$ and $\Psi_t(\mathscr{Y})$ that $P\{\Psi_t(\mathscr{Y}) = \mathscr{Y}_t,\ \forall\, t \in [0,T]\} = 1$, namely, $\mathscr{Y}$ is a solution of BSDE (3.19).

Let $\mathscr{Y}' \in \mathbb{D}^p$ be another solution of (3.19). Clearly, $P\{\Psi_t(\mathscr{Y}') = \mathscr{Y}'_t,\ \forall\, t \in [0,T]\} = 1$ implies that $\Psi_t(\mathscr{Y}') = \mathscr{Y}'_t$, $dt \times dP -$ a.s. Using (3.21) again shows that $\big\|\mathscr{Y}' - \mathscr{Y}\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} = \big\|\Psi(\mathscr{Y}') - \Psi(\mathscr{Y})\big\|^p_{\mathbb{L}^p_{\mathfrak{a}}} \le \frac{1}{2}\|\mathscr{Y}' - \mathscr{Y}\|^p_{\mathbb{L}^p_{\mathfrak{a}}}$. So $\|\mathscr{Y}' - \mathscr{Y}\|^p_{\mathbb{L}^p_{\mathfrak{a}}} = 0$ or $\mathscr{Y}'_t = \mathscr{Y}_t$, $dt \times dP -$ a.s. It follows from the right-continuity of $\mathscr{Y}$ and $\mathscr{Y}'$ that $P\{\mathscr{Y}'_t = \mathscr{Y}_t,\ \forall\, t \in [0,T]\} = 1$. Therefore, $\mathscr{Y} \in \mathbb{D}^p$ is a unique solution of BSDE (3.19). □

Given a $\mathcal{E}$-supermartingale X with only inaccessible jumps, by analyzing penalized BSDEs with respect to $\mathcal{E}$

$$Y^n_t + n\int_0^t (X_s - Y^n_s)ds = \mathcal{E}\bigg[X_T + n\int_0^T (X_s - Y^n_s)ds\bigg|\mathcal{F}_t\bigg], \quad t \in [0,T] \tag{3.22}$$

and utilizing Theorem 2.4, we can derive a Doob–Meyer decomposition of X.

Theorem 3.1 (*Doob–Meyer Decomposition of $\mathcal{E}$-martingales*) *Given $p \in (1,2]$, let Ξ be a p-coefficient set with $\beta \equiv 0$ such that $\int_0^T \Lambda_t^{\frac{2p}{2-p}} dt \in L^\infty(\mathcal{F}_T)$ if $p \in (1,2)$ or $\Lambda \equiv \kappa_\Lambda \in [0,\infty)$ if $p = 2$. Also, let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant* $\mathbf{F}$*-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated. If $X \in \mathbb{D}^p$ is an $\mathcal{E}$-supermartingale with only inaccessible jumps, then there exists a continuous process $K \in \mathbb{K}^p$ such that $X + K$ is an $\mathcal{E}$-martingale.*

Proof Let $X \in \mathbb{D}^p$ be an $\mathcal{E}$-supermartingale with only inaccessible jumps.

Fix $n \in \mathbb{N}$. Since $E\big[(\int_0^T |X_t|dt)^p\big] \le T^p E[X_*^p] < \infty$, Proposition 3.6 shows that the penalized BSDE (3.22) admits a unique solution $Y^n \in \mathbb{D}^p$.

(1) *We first show that*

$$P\big\{Y^n_t \le X_t,\ \forall\, t \in [0,T]\big\} = 1. \tag{3.23}$$

Let $i \in \mathbb{N}$. In light of the Debut Theorem (see e.g. Theorem IV.50 of [13]), $\tau^n_i := \inf\{t \in [0,T] : Y^n_t \ge X_t + 1/i\} \wedge T$ defines an $\mathbf{F}$-stopping time. As

$$Y^n_T = \mathcal{E}\bigg[X_T + n\int_0^T (X_s - Y^n_s)ds\bigg|\mathcal{F}_T\bigg] - n\int_0^T (X_s - Y^n_s)ds = X_T, \quad P - \text{a.s.} \tag{3.24}$$

by (F2), the $\mathbf{F}$-stopping time $\gamma^n_i := \inf\{t \in [\tau^n_i, T] : Y^n_t \le X_t\}$ satisfies $\tau^n_i \le \gamma^n_i \le T$, $P -$ a.s. And the right continuity of process $Y^n - X$ implies that

$$Y^n_{\gamma^n_i} \le X_{\gamma^n_i}, \quad P - \text{a.s.} \tag{3.25}$$

Since $Y^n_s > X_s$ over period $[\tau^n_i, \gamma^n_i)$, we can deduce from (F1*), (F3*), (F5*), (3.25), the $\mathcal{E}$-supermartingality of X as well as Proposition 3.4 that

$$
\begin{aligned}
Y^n_{\tau^n_i} &= \mathcal{E}\Big[X_T + n\int_0^T (X_s - Y^n_s)ds\Big|\mathcal{F}_{\tau^n_i}\Big] - n\int_0^{\tau^n_i}(X_s - Y^n_s)ds \\
&= \mathcal{E}\Big[\mathcal{E}\Big[X_T + n\int_0^T (X_s - Y^n_s)ds\Big|\mathcal{F}_{\gamma^n_i}\Big]\Big|\mathcal{F}_{\tau^n_i}\Big] - n\int_0^{\tau^n_i}(X_s - Y^n_s)ds \\
&= \mathcal{E}\Big[Y^n_{\gamma^n_i} + n\int_0^{\gamma^n_i} (X_s - Y^n_s)ds\Big|\mathcal{F}_{\tau^n_i}\Big] - n\int_0^{\tau^n_i}(X_s - Y^n_s)ds \\
&= \mathcal{E}\Big[Y^n_{\gamma^n_i} + n\int_{\tau^n_i}^{\gamma^n_i} (X_s - Y^n_s)ds\Big|\mathcal{F}_{\tau^n_i}\Big] \\
&\le \mathcal{E}\big[Y^n_{\gamma^n_i}\big|\mathcal{F}_{\tau^n_i}\big] \le \mathcal{E}\big[X_{\gamma^n_i}\big|\mathcal{F}_{\tau^n_i}\big] \le X_{\tau^n_i}
\end{aligned}
\tag{3.26}
$$

holds except on a P-null set $\mathcal{N}^n_i$. For all $\omega \in \Omega$ except on a P-null set $\widetilde{\mathcal{N}}_n$, the paths $Y^n_\cdot(\omega) - X_\cdot(\omega)$ is right-continuous. Given $\omega \in \{\tau^n_i < T\} \cap \widetilde{\mathcal{N}}^c_n$, the definition of τ^n_i and the right-continuity of the paths $Y^n_\cdot(\omega) - X_\cdot(\omega)$ imply that $Y^n\big(\tau^n_i(\omega), \omega\big) \ge X\big(\tau^n_i(\omega), \omega\big) + 1/i$. Comparing this inequality with (3.26) shows that $\{\tau^n_i < T\} \cap \widetilde{\mathcal{N}}^c_n \subset \mathcal{N}^n_i$, and it follows that $\{\tau^n_i < T\} \subset \widetilde{\mathcal{N}}_n \cup \mathcal{N}^n_i$. Taking union over $i \in \mathbb{N}$ yields that

$$
\begin{aligned}
\{Y^n_t > X_t,\ \text{for some } t \in [0,T)\} &= \underset{i\in\mathbb{N}}{\cup}\{Y^n_t \ge X_t + 1/i, \\
\text{for some } t \in [0,T)\} &\subset \underset{i\in\mathbb{N}}{\cup}\{\tau^n_i < T\} \subset \widetilde{\mathcal{N}}_n \cup \Big(\underset{i\in\mathbb{N}}{\cup}\mathcal{N}^n_i\Big).
\end{aligned}
$$

So $P\{Y^n_t \le X_t,\ \forall\, t \in [0,T)\} = 1$, which together with (3.24) proves (3.23).

(2) It follows from (3.23) that $K^n_t := n\int_0^t (X_s - Y^n_s)ds,\ t \in [0,T]$ is an $\mathbf{F}$-adapted, continuous increasing process with $K^n_0 = 0$. By (1.4), $E\big[(K^n_T)^p\big] \le n^p T^p 2^{p-1} E\big[X^p_* + (Y^n_*)^p\big] < \infty$, so $K^n \in \mathbb{K}^p$.

As $Y^n + K^n \in \mathbb{D}^p$ is an $\mathcal{E}$-martingale, Proposition 3.3 shows that for some real-valued, $\mathbf{F}$-progressively measurable process $\mathfrak{g}^n$ and some $(Z^n, U^n) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$, it holds $P-$a.s. that

$$
\begin{aligned}
Y^n_t + K^n_t = Y^n_T + K^n_T &+ \int_t^T \mathfrak{g}^n_s ds - \int_t^T Z^n_s dB_s \\
&- \int_{(t,T]}\int_{\mathcal{X}} U^n_s(x)\widetilde{\mathfrak{N}}_{\mathfrak{p}}(ds,dx), \quad t \in [0,T]
\end{aligned}
\tag{3.27}
$$

and that $\underline{g}^{\Xi_0}(t, Z^n_t, U^n_t) \le \mathfrak{g}^n_t \le g^{\Xi_0}(t, Z^n_t, U^n_t)$, $dt \times dP-$a.s. Using Hölder's inequality, we see from the latter that

$$
\begin{aligned}
|\mathfrak{g}^n_t| &\le \Lambda_t |Z^n_t| + \kappa_2 \int_{\mathcal{X}} \big|U^n_t(x)\big|\nu(dx) \\
&\le \Lambda_t |Z^n_t| + \kappa_2 \big(\nu(\mathcal{X})\big)^{\frac{1}{q}} \|U^n_t\|_{L^p_\nu}, \quad dt \times dP-\text{a.s.}
\end{aligned}
$$

Thus, as a p-generator independent of (y, z, u)-variables, $\mathfrak{g}^n$ satisfies (2.6) with $\mathfrak{f}_t = 0$.
By (3.23), one has $P\{Y^n_{t-} \le X_{t-},\ \ \forall t \in [0,T]\} = 1$, or $P\{\mathbf{1}_{\{Y^n_{t-} > X_{t-}\}} = 0,\ \ \forall t \in [0,T]\} = 1$, which together with (3.27) and (3.24) shows that (2.5) holds with $g = \mathfrak{g}^n$ and $(Y, Z, U, K, \xi) = (Y^n, Z^n, U^n, K^n, X_T) \in \mathbb{D}^p \times \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p \times L^p(\mathcal{F}_T)$. Applying Proposition 2.4 yields that

$$\begin{aligned} \|Y^n\|^p_{\mathbb{D}^p} + \big\|Z^n\big\|^p_{\mathbb{Z}^{2,p}} + \big\|U^n\big\|^p_{\mathbb{U}^p} + E\big[(K^n_T)^p\big] &\le \mathcal{C}E\big[|X_T|^p + (X^+_*)^p\big] \\ &\le \mathcal{C}E\big[X^p_*\big], \end{aligned} \tag{3.28}$$

where the constant $\mathcal{C}$ does not depend on n.

(3) *Next, we show that* $P\{Y^n_t \le Y^{n+1}_t,\ t \in [0,T]\} = 1$.
Let $j \in \mathbb{N}$. By the Debut Theorem again, $\zeta^n_j := \inf\{t \in [0,T] : Y^n_t \ge Y^{n+1}_t + 1/j\} \wedge T$ defines an $\mathbf{F}$-stopping time. As $Y^n_T = Y^{n+1}_T = X_T$, $P-$a.s. by (F2), the $\mathbf{F}$-stopping time $\sigma^n_j := \inf\{t \in [\zeta^n_j, T] : Y^n_t \le Y^{n+1}_t\}$ satisfies $\zeta^n_j \le \sigma^n_j \le T$, $P-$a.s. And the right continuity of process $Y^n - Y^{n+1}$ implies that

$$Y^n_{\sigma^n_j} \le Y^{n+1}_{\sigma^n_j}, \quad P-\text{a.s.} \tag{3.29}$$

Since $Y^n_s > Y^{n+1}_s$ over period $[\zeta^n_j, \sigma^n_j)$, we see from part (1) that $n\int_{\zeta^n_j}^{\sigma^n_j}(X_s - Y^n_s)ds \le n\int_{\zeta^n_j}^{\sigma^n_j}(X_s - Y^{n+1}_s)ds \le (n+1)\int_{\zeta^n_j}^{\sigma^n_j}(X_s - Y^{n+1}_s)ds$, $P-$a.s. Then (F1*), (F3*), (F5*) and (3.29) yield that

$$\begin{aligned} Y^n_{\zeta^n_j} &= \mathcal{E}\bigg[X_T + n\int_0^T (X_s - Y^n_s)ds\Big|\mathcal{F}_{\zeta^n_j}\bigg] - n\int_0^{\zeta^n_j}(X_s - Y^n_s)ds \\ &= \mathcal{E}\bigg[\mathcal{E}\bigg[X_T + n\int_0^T (X_s - Y^n_s)ds\Big|\mathcal{F}_{\sigma^n_j}\bigg]\bigg|\mathcal{F}_{\zeta^n_j}\bigg] - n\int_0^{\zeta^n_j}(X_s - Y^n_s)ds \\ &= \mathcal{E}\bigg[Y^n_{\sigma^n_j} + n\int_{\zeta^n_j}^{\sigma^n_j}(X_s - Y^n_s)ds\Big|\mathcal{F}_{\zeta^n_j}\bigg] \\ &\le \mathcal{E}\bigg[Y^{n+1}_{\sigma^n_j} + (n+1)\int_{\zeta^n_j}^{\sigma^n_j}(X_s - Y^{n+1}_s)ds\Big|\mathcal{F}_{\zeta^n_j}\bigg] \\ &= \mathcal{E}\bigg[\mathcal{E}\bigg[X_T + (n+1)\int_0^T (X_s - Y^{n+1}_s)ds\Big|\mathcal{F}_{\sigma^n_j}\bigg]\bigg|\mathcal{F}_{\zeta^n_j}\bigg] \\ &\quad -(n+1)\int_0^{\zeta^n_j}(X_s - Y^{n+1}_s)ds \\ &= \mathcal{E}\bigg[X_T + (n+1)\int_0^T (X_s - Y^{n+1}_s)ds\Big|\mathcal{F}_{\zeta^n_j}\bigg] \\ &\quad -(n+1)\int_0^{\zeta^n_j}(X_s - Y^{n+1}_s)ds = Y^{n+1}_{\zeta^n_j}, \quad P-\text{a.s.} \end{aligned}$$

Using similar arguments to those below (3.26), we obtain that $P\{Y_t^n \le Y_t^{n+1},\ t \in [0,T]\} = 1$.

In light of Theorem 2.4, $Y_t := \varlimsup_{n\to\infty} Y_t^n, t \in [0,T]$ defines a process of $\mathbb{D}^p$ satisfying (2.7), and there exists $(g, Z, U, K) \in \mathbb{L}^p\big([0,T]\times\Omega, \mathscr{P}, dt\times dP; \mathbb{R}\big) \times \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p$ such that (2.8) holds. According to the proof of Theorem 2.4, the process g is the weak limit of $\{\mathfrak{g}^n\}_{n\in\mathbb{N}}$ in $\mathbb{L}^p\big([0,T]\times\Omega, \mathscr{P}, dt\times dP; \mathbb{R}\big)$.

Given $n \in \mathbb{N}$, (3.23), Hölder's inequality and (3.28) show that

$$0 \le E\int_0^T (X_t - Y_t^n)dt = \frac{1}{n}E[K_T^n] \le \frac{1}{n}\big\{E[(K_T^n)^p]\big\}^{\frac{1}{p}} \le \frac{1}{n}\mathcal{C}\big\{E\big[X_*^p\big]\big\}^{\frac{1}{p}}. \tag{3.30}$$

Since it holds $P-$a.s. that $X_t - Y_t^n \le X_t - Y_t^1,\ \forall\, t \in [0,T]$ by the monotonicity of $\{Y^n\}_{n\in\mathbb{N}}$ and since $E\int_0^T (X_t - Y_t^1)dt \le TE[X_*^+ + Y_*^1] \le TE\big[1+(X_*^+ + Y_*^1)^p\big] < \infty$, letting $n \to \infty$ in (3.30), we know from the dominated convergence theorem that $E\int_0^T (X_t - Y_t)dt = \lim_{n\to\infty} E\int_0^T (X_t - Y_t^n)dt = 0$. This equality and (2.7) imply that $X_t - Y_t = 0$, $dt \times dP-$a.s., which together with the right-continuity of processes $X - Y$ yields

$$P\{X_t = Y_t,\ \ \forall\, t \in [0,T]\} = 1. \tag{3.31}$$

So the process Y also has only inaccessible jumps. Then Theorem 2.4 further yields that K is a continuous process and (2.9) holds.

Let $m, n \in \mathbb{N}$. Since Proposition 3.3 also shows that

$$\underline{g}^{\Xi_0}(t, Z_t^m - Z_t^n, U_t^m - U_t^n) \le \mathfrak{g}_t^m - \mathfrak{g}_t^n \le g^{\Xi_0}(t, Z_t^m - Z_t^n, U_t^m - U_t^n), dt \times dP - \text{a.s.}$$

Hölder's inequality implies that

$$\begin{aligned}
&E\int_0^T \big|\mathfrak{g}_t^m - \mathfrak{g}_t^n\big|^p dt \\
&\le E\int_0^T \Big(\Lambda_t|Z_t^m - Z_t^n| + \kappa_2\int_{\mathcal{X}}\big|U_t^m(x) - U_t^n(x)\big|\nu(dx)\Big)^p dt \\
&\le 2^{p-1}\bigg(\Big\|\int_0^T \Lambda_t^{\frac{2p}{2-p}}dt\Big\|_{L^\infty(\mathcal{F}_T)}\bigg)^{\frac{2-p}{2}} E\bigg[\Big(\int_0^T |Z_t^m - Z_t^n|^2 dt\Big)^{\frac{p}{2}}\bigg] \\
&\quad + 2^{p-1}\kappa_2^p\big(\nu(\mathcal{X})\big)^{p-1}E\int_0^T\int_{\mathcal{X}}\big|U_t^m(x) - U_t^n(x)\big|^p\nu(dx)dt \quad \text{for } p \in (1,2),
\end{aligned}$$

and similarly that

$$\begin{aligned}
E\int_0^T \big|\mathfrak{g}_t^m - \mathfrak{g}_t^n\big|^2 dt &\le E\int_0^T \Big(\kappa_\Lambda|Z_t^m - Z_t^n| + \kappa_2\int_{\mathcal{X}}\big|U_t^m(x) - U_t^n(x)\big|\nu(dx)\Big)^2 dt \\
&\le 2\kappa_\Lambda^2 E\int_0^T |Z_t^m - Z_t^n|^2 dt + 2\kappa_2^2\nu(\mathcal{X}) \\
&\quad \times E\int_0^T\int_{\mathcal{X}}\big|U_t^m(x) - U_t^n(x)\big|^2\nu(dx)dt.
\end{aligned}$$

So we see from (2.9) that $\{\mathfrak{g}^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$, let $\widetilde{\mathfrak{g}}$ be its strong limit (and thus weak limit). By the uniqueness of the weak limit of $\{\mathfrak{g}^n\}_{n\in\mathbb{N}}$ in $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$, we have $\widetilde{\mathfrak{g}}_t=g_t$, $dt\times dP-$a.s., and it follows that

$$\lim_{n\to\infty}E\int_0^T\big|\mathfrak{g}_t^n-g_t\big|^p dt=0. \tag{3.32}$$

For any $n\in\mathbb{N}$, since $E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t^n(x)-U_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big]\le E\int_0^T\int_{\mathcal{X}}\big|U_t^n(x)-U_t(x)\big|^p\nu(dx)dt$ by (1.3), we can deduce from (3.3), Hölder's inequality and the Burkholder–Davis–Gundy inequality that

$$\begin{aligned}
&E\bigg[\sup_{t\in[0,T]}\big|\mathcal{E}[Y_T^n+K_T^n|\mathcal{F}_t]-\mathcal{E}[Y_T+K_T|\mathcal{F}_t]\big|^p+\sup_{t\in[0,T]}\big|Y_t^n+K_t^n-Y_t-K_t\big|^p\bigg]\\
&\le CE\bigg[\sup_{t\in[0,T]}\big|Y_t^n+K_t^n-Y_t-K_t\big|^p\bigg]\\
&\le CE\bigg[|Y_0^n-Y_0|^p+\Big(\int_0^T|\mathfrak{g}_t^n-g_t|dt\Big)^p+\sup_{t\in[0,T]}\Big|\int_0^t(Z_s^n-Z_s)dB_s\Big|^p\\
&\quad+\sup_{t\in[0,T]}\Big|\int_{(0,t]}\int_{\mathcal{X}}\big(U_s^n(x)-U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\Big|^p\bigg]\\
&\le CE\bigg[|Y_0^n-Y_0|^p+T^{p-1}\int_0^T|\mathfrak{g}_t^n-g_t|^p dt+\Big(\int_0^T|Z_t^n-Z_t|^2dt\Big)^{\frac{p}{2}}\\
&\quad+\Big(\int_{(0,T]}\int_{\mathcal{X}}\big|U_t^n(x)-U_t(x)\big|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\bigg]\\
&\le CE\bigg[|Y_0^n-Y_0|^p+T^{p-1}\int_0^T|\mathfrak{g}_t^n-g_t|^p dt\\
&\quad+\Big(\int_0^T|Z_t^n-Z_t|^2dt\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}\big|U_t^n(x)-U_t(x)\big|^p\nu(dx)dt\bigg].
\end{aligned}$$

As $n\to\infty$, (2.7), (3.32) and (2.9) show that $\lim_{n\to\infty}E\bigg[\sup_{t\in[0,T]}\big|\mathcal{E}[Y_T^n+K_T^n|\mathcal{F}_t]-\mathcal{E}[Y_T+K_T|\mathcal{F}_t]\big|^p+\sup_{t\in[0,T]}\big|Y_t^n+K_t^n-Y_t-K_t\big|^p\bigg]=0$. Hence, there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\begin{aligned}
\lim_{i\to\infty}\sup_{t\in[0,T]}\Big|\mathcal{E}[Y_T^{n_i}+K_T^{n_i}|\mathcal{F}_t]-\mathcal{E}[Y_T+K_T|\mathcal{F}_t]\Big|&=\lim_{i\to\infty}\sup_{t\in[0,T]}\big|Y_t^{n_i}+K_t^{n_i}-Y_t-K_t\big|\\
&=0,\quad P-\text{a.s.},
\end{aligned}$$

which further implies that $P\Big\{\lim_{i\to\infty}\mathcal{E}[Y_T^{n_i}+K_T^{n_i}|\mathcal{F}_t]=\mathcal{E}[Y_T+K_T|\mathcal{F}_t],\ \lim_{i\to\infty}\big(Y_t^{n_i}+K_t^{n_i}\big)=Y_t+K_t,\ \ \forall\, t\in[0,T]\Big\}=1$. It then follows from (3.31) and the $\mathcal{E}$-martingality of $Y^{n_i}+K^{n_i}$'s that $P-$a.s.

$$\mathcal{E}[X_T + K_T|\mathcal{F}_t] = \mathcal{E}[Y_T + K_T|\mathcal{F}_t] = \lim_{i\to\infty}\mathcal{E}[Y_T^{n_i} + K_T^{n_i}|\mathcal{F}_t] = \lim_{i\to\infty}\big(Y_t^{n_i} + K_t^{n_i}\big)$$
$$= Y_t + K_t = X_t + K_t, \quad \forall\, t \in [0, T].$$

Therefore, $X + K$ is an $\mathcal{E}$-martingale. □

4 Representation of an F-Expectation by a g-Expectation

On domain $L^p(\mathcal{F}_T)$, We have seen in Example 3.1 (2) that a g-expectation is particular case of filtration-consistent nonlinear expectations. Inversely, we will show in this section that a translation-invariant **F**-expectation under domination (1.2) can be identified as a g-expectation and thus expressed as the $\mathbb{L}^p$ solution of a BSDE with jump

$$\mathcal{E}[\xi|\mathcal{F}_t] = \xi + \int_t^T g(s, Z_s, U_s)ds - \int_t^T Z_s\, dB_s$$
$$- \int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T].$$

Consequently, one can use the techniques and analytic tools in the BSDE theory to study filtration-consistent nonlinear expectations which include a large class of risk measures and monetary utility functionals in mathematical finance.

For the representation of a translation-invariant **F**-expectation, we additional require that both Brownian motion B and Poisson random measure $N_{\mathfrak{p}}$ have independent increments under $\mathcal{E}$, i.e., it holds for any $t \in [0, T)$, $\delta \in (0, T-t]$ and $(z, u) \in \mathbb{R}^d \times L^p_\nu$ that

$$\mathcal{E}\Big[z(B_{t+\delta} - B_t) + \int_{(t,t+\delta]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds, dx)\Big|\mathcal{F}_t\Big]$$
$$= \mathcal{E}\Big[z(B_{t+\delta} - B_t) + \int_{(t,t+\delta]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds, dx)\Big], \quad P-\text{a.s.} \tag{4.1}$$

This assumption can be verified by g-expectations with deterministic generators.

Example 4.1 Given $p \in (1, 2]$, let $g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^p_\nu \to \mathbb{R}$ be a $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})$-measurable function satisfies

(i) It holds for a.e. $t \in [0, T]$ that $g(t, y, 0, 0) = 0, \ \forall\, y \in \mathbb{R}$;
(ii) For some $c(\cdot) \in L^q_+[0, T]$, it holds for a.e. $t \in [0, T]$ that

$$\big|g(t, y_1, z_1, u) - g(t, y_2, z_2, u)\big| \le c(t)\big(|y_1 - y_2| + |z_1 - z_2| + \|u_1 - u_2\|_{L^p_\nu}\big),$$
$$\forall\, (y_1, z_1, u_1),\ (y_2, z_2, u_2) \in \mathbb{R} \times \mathbb{R}^d \times L^p_\nu.$$

Then (4.1) holds for the g-expectation $\mathcal{E}_g$.

Proof Fix $t \in [0, T)$, $\delta \in (0, T-t]$ and $(z, u) \in \mathbb{R}^d \times L^p_\nu$.

Clearly, $B^t_s := B_s - B_t$, $s \in [t, T]$ is a d-dimensional Brownian motion over period $[t, T]$ with $B^t_t = 0$ while $N^t_{\mathfrak{p}}\big((t, s], A\big) := N_{\mathfrak{p}}\big((0, s], A\big) - N_{\mathfrak{p}}\big((0, t], A\big)$, $s \in [t, T]$, $A \in \mathscr{F}_{\mathcal{X}}$ is the counting measure of $\mathfrak{p}$ on $(t, T] \times \mathcal{X}$ with compensator $E\big[N^t_{\mathfrak{p}}(ds, dx)\big] = \nu(dx)ds$. And $\widetilde{N}^t_{\mathfrak{p}}\big((t, s], A\big) := \widetilde{N}_{\mathfrak{p}}\big((0, s], A\big) - \widetilde{N}_{\mathfrak{p}}\big((0, t], A\big)$, $s \in [t, T]$, $A \in \mathscr{F}_{\mathcal{X}}$ is the corresponding compensated Poisson random measure over period $[t, T]$. For any $s \in [t, T]$, we define sigma-fields

$$
\begin{aligned}
\mathcal{F}^{B^t}_s &:= \sigma\big\{B^t_r; r \in [t, s]\big\} \subset \mathcal{F}^B_s, \quad \mathcal{F}^{N^t}_s \\
&:= \sigma\big\{N^t_{\mathfrak{p}}((0, r], A); r \in [t, s], A \in \mathscr{F}_{\mathcal{X}}\big\} \subset \mathcal{F}^N_s, \;\; \mathcal{F}^t_s := \sigma\Big(\mathcal{F}^{B^t}_s \cup \mathcal{F}^{N^t}_s\Big) \subset \mathcal{F}_s
\end{aligned}
$$

and augment them by all P-null sets of $\mathcal{F}$ (In fact, $\mathcal{F}^t_t$ is the collection of all $\mathcal{F}$-measurable sets with P-measure 0 or 1). The jump filtration $\mathbf{F}^t = \{\mathcal{F}^t_s\}_{s \in [t, T]}$ over period $[t, T]$ still satisfies the usual hypotheses. Let $\widehat{\mathscr{P}}^t$ be the $\mathbf{F}^t$-predictable sigma-field on $[t, T] \times \Omega$.

Set $\xi := z(B_{t+\delta} - B_t) + \int_{(t,t+\delta]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds, dx) = zB^t_{t+\delta} + \int_{(t,t+\delta]}\int_{\mathcal{X}} u(x)$ $\widetilde{N}^t_{\mathfrak{p}}(ds, dx) \in \mathcal{F}^t_{t+\delta}$. We can deduce from (1.4), the Burkholder-Davis-Gundy inequality and (1.3) that

$$
\begin{aligned}
E\big[|\xi|^p\big] &\le 2^{p-1}E\bigg[|z|^p \sup_{s \in [t,t+\delta]} |B^t_s|^p + \sup_{s \in [t,t+\delta]}\bigg|\int_{(t,s]}\int_{\mathcal{X}} u(x)\widetilde{N}^t_{\mathfrak{p}}(ds, dx)\bigg|^p\bigg] \\
&\le c_p E\bigg[|z|^p \delta^{\frac{p}{2}} + \bigg(\int_{(t,t+\delta]}\int_{\mathcal{X}} |u(x)|^2 N^t_{\mathfrak{p}}(ds, dx)\bigg)^{\frac{p}{2}}\bigg] \\
&= c_p|z|^p\delta^{\frac{p}{2}} + c_p E\bigg[\bigg(\int_{(t,t+\delta]}\int_{\mathcal{X}} |u(x)|^2 N_{\mathfrak{p}}(ds, dx)\bigg)^{\frac{p}{2}}\bigg] \\
&\le c_p|z|^p\delta^{\frac{p}{2}} + c_p E\int_t^{t+\delta}\int_{\mathcal{X}} |u(x)|^p \nu(dx)dt \\
&= c_p\big(|z|^p\delta^{\frac{p}{2}} + \|u\|^p_{L^p_\nu}\delta\big) < \infty.
\end{aligned} \tag{4.2}
$$

As $g|_{[t,T]}$ is a deterministic p-generator satisfying (2.2) and (A2'), Theorem 2.1 shows that the following BSDE with jumps over period $[t, T]$

$$
\begin{aligned}
\mathcal{Y}_s = \xi + \int_s^T g(r, \mathcal{Y}_r, \mathcal{Z}_r, \mathcal{U}_r)ds - \int_t^T \mathcal{Z}_r\, dB^t_r \\
- \int_{(s,T]}\int_{\mathcal{X}} \mathcal{U}_r(x)\widetilde{N}^t_{\mathfrak{p}}(dr, dx), \quad s \in [t, T]
\end{aligned}
$$

admits a unique solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{U}) \in \mathbb{D}^p_{\mathbf{F}^t}\big([t, T]\big) \times \mathbb{Z}^{2,p}_{\mathbf{F}^t}\big([t, T]\big) \times \mathbb{U}^p_{\mathbf{F}^t}\big([t, T]\big)$. Here $\mathbb{D}^p_{\mathbf{F}^t}\big([t, T]\big)$ denotes the space of all real-valued, $\mathbf{F}^t$-adapted càdlàg processes $\{Y_s\}_{s \in [t,T]}$ satisfying $E\Big[\sup_{s \in [t,T]} |Y_s|^p\Big] < \infty$, $\mathbb{Z}^{2,p}_{\mathbf{F}^t}\big([t, T]\big)$ denotes the space of all

$\mathbb{R}^d$-valued, $\mathbf{F}^t$-predictable processes $\{Z_s\}_{s\in[t,T]}$ satisfying $E\Big[\big(\int_t^T |Z_s|^2 ds\big)^{\frac{p}{2}}\Big] < \infty$, and $\mathbb{U}^p_{\mathbf{F}^t}\big([t,T]\big)$ denotes the space of all $\widehat{\mathscr{P}^t} \otimes \mathcal{F}_{\mathcal{X}}$-measurable random fields U : $[t,T]\times\Omega\times\mathcal{X}\to\mathbb{R}$ satisfying $E\int_t^T\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds<\infty$. In particular, $\mathcal{Y}_t$ is a real number.

Since $\mathcal{F}^t_s \subset \mathcal{F}_s$ for any $s\in[t,T]$, we see that $\big(\widetilde{\mathcal{Y}}_s,\widetilde{\mathcal{Z}}_s,\widetilde{\mathcal{U}}_s\big) := \big(\mathcal{Y}_{t\vee s},\mathbf{1}_{\{s>t\}}\mathcal{Z}_s,\mathbf{1}_{\{s>t\}}\mathcal{U}_s\big)$, $s\in[0,T]$ belongs to $\mathbb{S}^p$ and satisfies

$$\begin{aligned}
\widetilde{\mathcal{Y}}_s &= \mathcal{Y}_{t\vee s}\\
&= \xi + \int_{t\vee s}^T g(r,\mathcal{Y}_r,\mathcal{Z}_r,\mathcal{U}_r)ds - \int_{t\vee s}^T \mathcal{Z}_r\,dB^t_r - \int_{(t\vee s,T]}\int_{\mathcal{X}} \mathcal{U}_r(x)\widetilde{N}^t_{\mathfrak{p}}(dr,dx)\\
&= \xi + \int_{t\vee s}^T g(r,\mathcal{Y}_r,\mathcal{Z}_r,\mathcal{U}_r)dr - \int_{t\vee s}^T \mathcal{Z}_r\,dB_r - \int_{(t\vee s,T]}\int_{\mathcal{X}} \mathcal{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)\\
&= \xi + \int_s^T \mathbf{1}_{\{r>t\}}g(r,\mathcal{Y}_r,\mathcal{Z}_r,\mathcal{U}_r)dr - \int_s^T \mathbf{1}_{\{r>t\}}\mathcal{Z}_r\,dB_r\\
&\quad - \int_{(s,T]}\int_{\mathcal{X}} \mathbf{1}_{\{r>t\}}\mathcal{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)\\
&= \xi + \int_s^T g\big(r,\widetilde{\mathcal{Y}}_r,\widetilde{\mathcal{Z}}_r,\widetilde{\mathcal{U}}_r\big)dr - \int_s^T \mathbf{1}_{\{r>t\}}\widetilde{\mathcal{Z}}_r\,dB_r\\
&\quad - \int_{(s,T]}\int_{\mathcal{X}} \widetilde{\mathcal{U}}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx), \quad s\in[0,T],
\end{aligned}$$

which shows that $\big(\widetilde{\mathcal{Y}},\widetilde{\mathcal{Z}},\widetilde{\mathcal{U}}\big)$ solves BSDEJ$\,(\xi,g)$. It follows that $\mathcal{E}_g[\xi|\mathcal{F}_s] = \widetilde{\mathcal{Y}}_s$, $\forall\, s\in[0,T]$. Taking $s=t$ yields that

$$\begin{aligned}
&\mathcal{E}_g\bigg[z(B_{t+\delta}-B_t) + \int_{(t,t+\delta]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\Big|\mathcal{F}_t\bigg]\\
&= \mathcal{E}_g[\xi|\mathcal{F}_t] = \widetilde{\mathcal{Y}}_t = \mathcal{Y}_t = \widetilde{\mathcal{Y}}_0 = \mathcal{E}_g[\xi|\mathcal{F}_0] = \mathcal{E}_g[\xi]\\
&= \mathcal{E}_g\bigg[z(B_{t+\delta}-B_t) + \int_{(t,t+\delta]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\bigg].
\end{aligned}$$

□

Let $\mathscr{L}([0,T])$ denote the Lebesgue sigma-field on $[0,T]$. We are ready to state one of the main results of the paper: Under the domination (1.2), a translation invariant $\mathbf{F}$-expectation with domain $L^p(\mathcal{F}_T)$ that satisfies (4.1) can be represented by a g-expectation with deterministic generator g:

Theorem 4.1 *Given $p\in(1,2]$, let Ξ be a p-coefficient set in which $\beta\equiv 0$ and $\Lambda\equiv\kappa_\Lambda\in[0,\infty)$. Also, let $\big\{\mathcal{E}[\cdot|\mathcal{F}_t]\big\}_{t\in[0,T]}$ be a translation invariant $\mathbf{F}$-expectation with domain $L^p(\mathcal{F}_T)$ that is $\mathcal{E}_{\Xi_0}$-dominated and satisfies (4.1). Then there exists a deterministic function $g:[0,T]\times\mathbb{R}^d\times L^p_\nu\to\mathbb{R}$ that is $\mathscr{L}([0,T])\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})$- measurable and satisfies*

(i) *For a.e.* $t \in [0,T]$, $g(t,0,0)=0$;
(ii) *For any* $t \in [0,T]$ *and* $(z_1,u_1),(z_2,u_2) \in \mathbb{R}^d \times L^p_\nu$, $\big|g(t,z_1,u_1) - g(t,z_2,u_2)\big| \le \kappa_\Lambda|z_1-z_2| + \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\|u_1-u_2\|_{L^p_\nu}$;
(iii) *For any* $\xi \in L^p(\mathcal{F}_T)$, $P\big\{\mathcal{E}[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi|\mathcal{F}_t],\ t \in [0,T]\big\} = 1$.

Proof (1) Fix $\theta := (z,u) \in \mathbb{R}^d \times L^p_\nu$. We define

$$Y^\theta_t := -\int_0^t g^{\Xi_0}(z,u)ds + zB_t + \int_{(0,t]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ \overline{Y}^\theta_t := -\int_0^t \overline{g}^{\Xi_0}(z,u)ds + zB_t + \int_{(0,t]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ t \in [0,T],$$

which is a real-valued, $\mathbf{F}$-adapted càdlàg process with only inaccessible jumps. Since Hölder's inequality shows that

$$\big|g^{\Xi_0}(z,u)\big| \vee \big|\overline{g}^{\Xi_0}(z,u)\big| \le \kappa_\Lambda|z| + \kappa_2\int_{\mathcal{X}}|u(x)|\nu(dx) \le \kappa_\Lambda|z| + \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\|u\|_{L^p_\nu} := C_\theta, \quad \forall t \in [0,T], \tag{4.3}$$

an analogy to (4.2) implies that

$$\begin{aligned} E\Big[\big(Y^\theta_*\big)^p \vee \big(\overline{Y}^\theta_*\big)^p\Big] &\le 4^{p-1}E\Bigg[(\kappa_\Lambda|z|T)^p + \kappa_2^pT^p\big(\nu(\mathcal{X})\big)^{p-1}\|u\|^p_{L^p_\nu} + z^pB^p_* \\ &\quad + \sup_{t\in[0,T]}\Big|\int_{(0,t]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(dt,dx)\Big|^p\Bigg] \\ &\le c_p\Big\{(\kappa_\Lambda|z|T)^p + \kappa_2^pT^p\big(\nu(\mathcal{X})\big)^{p-1}\|u\|^p_{L^p_\nu} \\ &\quad + |z|^pT^{\frac{p}{2}} + \|u\|^p_{L^p_\nu}T\Big\} < \infty. \end{aligned}$$

Clearly, $\big(Y^\theta, z, u\big)$ is the unique solution of BSDEJ $\big(Y^\theta_T, g^{\Xi_0}\big)$ and $\Big(\overline{Y}^\theta, z, u\Big)$ is the unique solution of BSDEJ $\Big(\overline{Y}^\theta_T, \overline{g}^{\Xi_0}\Big)$. To wit, Y^θ is a g^{Ξ_0}-martingale and $\overline{Y}^\theta$ is a $\overline{g}^{\Xi_0}$-martingale.

For any $t \in [0,T]$, applying (3.1) with $(\xi,\eta) = \big(Y^\theta_T, 0\big)$ yields that $Y^\theta_t = \mathcal{E}_{g^{\Xi_0}}\big[Y^\theta_T\big|\mathcal{F}_t\big] \ge \mathcal{E}[Y^\theta_T|\mathcal{F}_t]$, $P-$a.s. So Y^θ is an $\mathcal{E}$-supermartingale with only inaccessible jumps. In light of Theorem 3.1, we can find a continuous process $K^\theta \in \mathbb{K}^p$ such that $Y^\theta + K^\theta \in \mathbb{D}^p$ is an $\mathcal{E}$-martingale. Then Proposition 3.3 shows that there exist a real-valued, $\mathbf{F}$-progressively measurable process $\mathfrak{g}^\theta$ and $(Z^\theta, U^\theta) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$ such that $P-$a.s.

$$\begin{aligned} &Y^\theta_T + \int_t^T \mathfrak{g}^\theta_s ds - \int_t^T Z^\theta_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} U^\theta_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) + K^\theta_T - K^\theta_t = Y^\theta_t \\ &= Y^\theta_T + \int_t^T g^{\Xi_0}(z,u)ds - \int_t^T zdB_s - \int_{(t,T]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ \forall t \in [0,T]. \end{aligned} \tag{4.4}$$

So it holds $P-$ a.s. that

$$\int_0^t \big(g^{\Xi_0}(z,u)-\mathfrak{g}_s^\theta\big)ds$$
$$=\int_0^t (z-Z_s^\theta)dB_s+\int_{(0,t]}\int_{\mathcal{X}}\big(u(x)-U_s^\theta(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)+K_t^\theta,\quad \forall\, t\in[0,T]$$

Using similar arguments to those lead to (3.12) and (3.13), we obtain that

$$P\big\{U_t^\theta(x)=u(x),\ \forall\,(t,x)\in[0,T]\times\mathcal{X}\big\}=1 \quad\text{and}\quad Z_t^\theta=z,\quad dt\times dP-\text{a.s.} \tag{4.5}$$

Proposition 3.3 also shows that $dt\times dP-$ a.s.

$$\underline{g}^{\Xi_0}(z,u)=\underline{g}^{\Xi_0}(Z_t^\theta,U_t^\theta)\le\mathfrak{g}_t^\theta\le g^{\Xi_0}(Z_t^\theta,U_t^\theta)=g^{\Xi_0}(z,u). \tag{4.6}$$

(2) *In this step, we define* $g(t,z,u)$, $t\in[0,T]$ *and show that*

$$\mathfrak{g}_t^\theta=g(t,z,u),\quad dt\times dP-\text{a.s. on } [0,T)\times\Omega. \tag{4.7}$$

Set $g(T,z,u):=0$. Let $t\in[0,T)$. We set $\delta_n(t):=\frac{1}{n}\wedge(T-t)$, $\mathfrak{D}_n(t,z,u):=z(B_{t+\delta_n(t)}-B_t)+\int_{(t,t+\delta_n(t)]}\int_{\mathcal{X}}u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\in L^p\big(\mathcal{F}_{t+\delta_n(t)}\big)$, $\alpha_n(t,z,u):=\frac{1}{\delta_n(t)}\,\mathcal{E}\big[\mathfrak{D}_n(t,z,u)\big]$ for any $n\in\mathbb{N}$ and define $g(t,z,u):=\overline{\lim_{n\to\infty}}\,\alpha_n(t,z,u)$.

For any $n\in\mathbb{N}$, since $\int_t^{t+\delta_n(t)}g^{\Xi_0}(z,u)ds$ is a real number, applying (3.1) with $(\xi,\eta)=\big(\mathfrak{D}_n(t,z,u),0\big)$, one can deduce from (4.1), (F2), (g5) of g^{Ξ_0}-expectations and the g^{Ξ_0}-martingality of Y^θ that

$$\begin{aligned}\mathcal{E}\big[\mathfrak{D}_n(t,z,u)\big]&=\mathcal{E}\big[\mathfrak{D}_n(t,z,u)|\mathcal{F}_t\big]\le\mathcal{E}_{\Xi_0}\big[\mathfrak{D}_n(t,z,u)|\mathcal{F}_t\big]\\&=\mathcal{E}_{\Xi_0}\bigg[Y_{t+\delta_n(t)}^\theta-Y_t^\theta+\int_t^{t+\delta_n(t)}g^{\Xi_0}(z,u)ds\bigg|\mathcal{F}_t\bigg]\\&=\mathcal{E}_{\Xi_0}\big[Y_{t+\delta_n(t)}^\theta|\mathcal{F}_t\big]-Y_t^\theta+\int_t^{t+\delta_n(t)}g^{\Xi_0}(z,u)ds\\&=\int_t^{t+\delta_n(t)}g^{\Xi_0}(z,u)ds,\quad P-\text{a.s.}\end{aligned} \tag{4.8}$$

Using a similar argument on $\underline{g}^{\Xi_0}$-martingality of $\overline{Y}^\theta$ yields that $P-$ a.s.

$$\begin{aligned}\mathcal{E}\big[\mathfrak{D}_n(t,z,u)\big]&\ge\overline{\mathcal{E}}_{\Xi_0}\bigg[\overline{Y}_{t+\delta_n(t)}^\theta-\overline{Y}_t^\theta+\int_t^{t+\delta_n(t)}\underline{g}^{\Xi_0}(z,u)ds\bigg|\mathcal{F}_t\bigg]\\&=\int_t^{t+\delta_n(t)}\underline{g}^{\Xi_0}(z,u)ds.\end{aligned}$$

By (4.3), $\big|\mathcal{E}[\mathfrak{D}_n(t,z,u)]\big| \le \int_t^{t+\delta_n(t)} \big|g^{\Xi_0}(z,u)\big| \vee \big|\overline{g}^{\Xi_0}(z,u)\big|ds \le C_\theta \delta_n(t)$. Letting $n \to \infty$ yields that

$$-C_\theta \le g(t,z,u) = \overline{\lim_{n\to\infty}} \frac{1}{\delta_n(t)} \mathcal{E}\big[\mathfrak{D}_n(t,z,u)\big] \le C_\theta, \quad \forall\, t \in [0,T).$$

To see (4.7), we set $\xi_n^t := \frac{1}{\delta_n(t)} \int_t^{t+\delta_n(t)} \big(\mathfrak{g}_s^\theta - \mathfrak{g}_t^\theta\big)ds$, for any $t \in [0,T)$ and $n \in \mathbb{N}$. The $\mathbf{F}$-progressively measurability of process $\mathfrak{g}^\theta$ implies that the mapping $\mathfrak{g}^\theta(t,\omega) : [0,T]\times\Omega \to \mathbb{R}$ is $\mathscr{B}([0,T])\otimes\mathcal{F}_T$-measurable. So for $P-$a.s. $\omega\in\Omega$, (4.6) and (4.3) show that

$$\text{the function } t \to \mathfrak{g}_t^\theta(\omega) \text{ is } \mathscr{B}([0,T])/\mathscr{B}(\mathbb{R})\text{-measurable,} \tag{4.9}$$

and that

$$|\mathfrak{g}_t^\theta(\omega)| \le \big|g^{\Xi_0}(z,u)\big| \vee \big|\overline{g}^{\Xi_0}(z,u)\big| \le C_\theta \quad \text{for a.e. } t \in [0,T]. \tag{4.10}$$

Lebesgue differentiation theorem then yields that for $P-$a.s. $\omega\in\Omega$, $\lim_{n\to\infty} \xi_n^t(\omega) = 0$ for a.e. $t\in[0,T)$. By (4.10) and Fubini Theorem, there exists a Lebesgue-null set $\mathfrak{E} = \mathfrak{E}(z,u)$ of $[0,T)$ such that for any $t \in [0,T)\backslash\mathfrak{E}$,

$$\lim_{n\to\infty} \xi_n^t = 0 \quad \text{and} \quad |\mathfrak{g}_t^\theta| \le C_\theta, \quad P-\text{a.s.} \tag{4.11}$$

Let $t\in[0,T)\backslash\mathfrak{E}$. For any $n\in\mathbb{N}$, the translation invariance of $\mathcal{E}$, (4.1), (4.4) and the $\mathcal{E}$-martingality of $Y^\theta + K^\theta$ implies that that $P-$a.s.

$$\begin{aligned}
&\mathcal{E}\big[\mathfrak{D}_n(t,z,u)\big] - \delta_n(t)\mathfrak{g}_t^\theta \\
&\quad = \mathcal{E}\Big[Y^\theta_{t+\delta_n(t)} + K^\theta_{t+\delta_n(t)} - Y_t^\theta - K_t^\theta + \int_t^{t+\delta_n(t)} \mathfrak{g}_s^\theta ds\Big|\mathcal{F}_t\Big] - \delta_n(t)\mathfrak{g}_t^\theta \\
&\quad = \mathcal{E}\Big[Y^\theta_{t+\delta_n(t)} + K^\theta_{t+\delta_n(t)} + \int_t^{t+\delta_n(t)} (\mathfrak{g}_s^\theta - \mathfrak{g}_t^\theta) ds\Big|\mathcal{F}_t\Big] - Y_t^\theta - K_t^\theta \\
&\quad = \mathcal{E}\Big[Y^\theta_{t+\delta_n(t)} + K^\theta_{t+\delta_n(t)} + \int_t^{t+\delta_n(t)} (\mathfrak{g}_s^\theta - \mathfrak{g}_t^\theta) ds\Big|\mathcal{F}_t\Big] \\
&\qquad - \mathcal{E}\big[Y^\theta_{t+\delta_n(t)} + K^\theta_{t+\delta_n(t)}\big|\mathcal{F}_t\big].
\end{aligned}$$

In light of (3.2), $E\big[|\alpha_n(t,z,u) - \mathfrak{g}_t^\theta|^p\big] \le \mathscr{C} E\big[|\xi_n^t|^p\big]$, where $\mathscr{C}$ is a constant only depending on T, $\nu(\mathcal{X})$, p, κ_2 and κ_Λ. Since (4.10) and (4.11) show that $P\{|\xi_n^t| \le 2C_\theta, \forall n \in \mathbb{N}\} = 1$, the bounded convergence theorem yields that $\lim_{n\to\infty} E[|\xi_n^t|^p] = 0$ and thus that $\lim_{n\to\infty} E\big[|\alpha_n(t,z,u) - \mathfrak{g}_t^\theta|^p\big] = 0$. Let $\{n_j\}_{j\in\mathbb{N}}$ be an arbitrary subsequence of $\mathbb{N}$. As $\lim_{j\to\infty} E\big[|\alpha_{n_j}(t,z,u) - \mathfrak{g}_t^\theta|^p\big] = 0$, one can find a subsequence $\{n'_j = n'_j(t,z,u)\}_{j\in\mathbb{N}}$ of $\{n_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \alpha_{n'_j}(t,z,u) = \mathfrak{g}_t^\theta, \quad P-\text{a.s.} \tag{4.12}$$

In particular, this shows the sequence $\{\alpha_{n_j}(t,z,u)\}_{j\in\mathbb{N}}$ of real numbers has a convergent subsequence $\{\alpha_{n'_j}(t,z,u)\}_{j\in\mathbb{N}}$. In turn, the sequence $\{\alpha_n(t,z,u)\}_{n\in\mathbb{N}}$ is convergent itself, so $g(t,z,u) = \lim\limits_{n\to\infty}\alpha_n(t,z,u)$. Putting it back into (4.12) yields that $\mathfrak{g}^\theta_t = g(t,z,u)$, $P-$a.s. Then an application of Fubini's Theorem gives rise to (4.7).

(3) *We first show that item (i) and (ii) hold.*

One can deduce from (4.7) and (4.6) that $\overline{g}^{\Xi_0}(z,u) \le g(t,z,u) \le g^{\Xi_0}(z,u)$ for a.e. $t\in[0,T]$. In particular, taking $\theta=(z,u)=(0,0)$ proves item (i).

Given $i=1,2$, we let $\theta_i := (z_i,u_i) \in \mathbb{R}^d\times L^p_\nu$ and set $\mathfrak{D}_n(t,i) := z_i(B_{t+\delta_n(t)}-B_t)+\int_{(t,t+\delta_n(t)]}\int_{\mathcal{X}} u_i(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \in L^p\big(\mathcal{F}_{t+\delta_n(t)}\big)$ for any $t\in[0,T)$ and $n\in\mathbb{N}$.

We also set $\overline{\theta} := (\overline{z},\overline{u}) = (z_1-z_2,u_1-u_2)$. Let $t\in[0,T)$ and $n\in\mathbb{N}$. Similar to (4.8), we can deduce from (4.1), (3.1), (g5) of g^{Ξ_0}-expectations and the g^{Ξ_0}-martingality of $Y^{\overline{\theta}}$ that

$$
\begin{aligned}
\mathcal{E}\big[\mathfrak{D}_n(t,1)\big]-\mathcal{E}\big[\mathfrak{D}_n(t,2)\big] &= \mathcal{E}\big[\mathfrak{D}_n(t,1)\big|\mathcal{F}_t\big]-\mathcal{E}\big[\mathfrak{D}_n(t,2)\big|\mathcal{F}_t\big]\\
&\le \mathcal{E}_{\Xi_0}\big[\mathfrak{D}_n(t,1)-\mathfrak{D}_n(t,2)\big|\mathcal{F}_t\big]\\
&= \mathcal{E}_{\Xi_0}\left[Y^{\overline{\theta}}_{t+\delta_n(t)}-Y^{\overline{\theta}}_t+\int_t^{t+\delta_n(t)} g^{\Xi_0}(\overline{z},\overline{u})ds\middle|\mathcal{F}_t\right]\\
&= \int_t^{t+\delta_n(t)} g^{\Xi_0}(\overline{z},\overline{u})ds \le C_{\overline{\theta}}\delta_n(t), \quad P-\text{a.s.}
\end{aligned}
$$

Letting $n\to\infty$, we obtain that $g(t,z_1,u_1) = \varlimsup\limits_{n\to\infty}\frac{1}{\delta_n(t)}\mathcal{E}\big[\mathfrak{D}_n(t,1)\big] \le \varlimsup\limits_{n\to\infty}\frac{1}{\delta_n(t)}\mathcal{E}\big[\mathfrak{D}_n(t,2)\big]+C_{\overline{\theta}} = g(t,z_2,u_2)+\kappa_\Lambda|\overline{z}|+\kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\|\overline{u}\|_{L^p_\nu}$. Reversing the roles of $\theta_1=(z_1,u_1)$ and $\theta_2=(z_2,u_2)$ yields that $g(t,z_2,u_2)\le g(t,z_1,u_1)+\kappa_\Lambda|\overline{z}|+\kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\|\overline{u}\|_{L^p_\nu}$. So item (ii) holds.

(4) *Next, we show that the function $g(t,z,u):[0,T]\times\mathbb{R}^d\times L^p_\nu\to\mathbb{R}$ is $\mathscr{L}([0,T])\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})$-measurable.*

Fix $\lambda\in\mathbb{R}$, it suffice to show that $A_\lambda := \big\{(t,z,u)\in[0,T]\times\mathbb{R}^d\times L^p_\nu : g(t,z,u)<\lambda\big\}\in\mathscr{L}([0,T])\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)$.

We also fix $n\in\mathbb{N}$ and set $\mathfrak{r}_n := \frac{1}{n}\Big(\kappa_\Lambda+\kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\Big)^{-1}$. For any $(z,u)\in\mathbb{R}^d\times L^p_\nu$, define $O_n(z,u) := \{(z',u')\in\mathbb{R}^d\times L^p_\nu : |z-z'|^2+\|u-u'\|^2_{L^p_\nu}<\mathfrak{r}_n^2\}$, which is the open ball centered at (z,u) with radius $\mathfrak{r}_n$ in $\mathbb{R}^d\times L^p_\nu$. Since the space $\mathbb{R}^d\times L^p_\nu$ is separable and thus Lindelöf, there exists a sequence $\big\{\big(z^n_j,u^n_j\big)\big\}_{j\in\mathbb{N}}$ of $\mathbb{R}^d\times L^p_\nu$ such that $\bigcup\limits_{j\in\mathbb{N}} O_n\big(z^n_j,u^n_j\big)=\mathbb{R}^d\times L^p_\nu$.

Let $j\in\mathbb{N}$. By (4.7), it holds for $P-$a.s. $\omega\in\Omega$ that $\mathfrak{g}^\theta_t(\omega)=g(t,z,u)$ for a.e. $t\in[0,T]$, which together with (4.9) implies that the function $t\to g(t,z,u)$ on $[0,T]$ is $\mathscr{L}([0,T])/\mathscr{B}(\mathbb{R})$-measurable. So $D^n_j := \{t\in[0,T] : g(t,z^n_j,u^n_j) < \lambda-1/n\}$ belongs to $\mathscr{L}([0,T])$. For any $(t,z,u)\in D^n_j\times O_n(z^n_j,u^n_j)$, item (ii) shows that

$$g(t,z,u) \le g\left(t, z_j^n, u_j^n\right) + \kappa_\Lambda \left|z - z_j^n\right| + \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}} \|u - u_j^n\|_{L_\nu^p} < \lambda - 1/n \\ + \left(\kappa_\Lambda + \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\right)\mathfrak{r}_n = \lambda,$$

which implies that $D_j^n \times O_n(z_j^n, u_j^n) \subset A_\lambda$.

On the other hand, for any $(t,z,u) \in \mathcal{A}_n^\lambda := \big\{(t,z,u) \in [0,T] \times \mathbb{R}^d \times L_\nu^p : g(t,z,u) < \lambda - 2/n\big\}$, since $(z,u) \in O_n\big(z_\ell^n, u_\ell^n\big)$ for some $\ell \in \mathbb{N}$, one can deduce from item (ii) again that

$$g(t, z_\ell^n, u_\ell^n) \le g(t,z,u) + \kappa_\Lambda \left|z - z_\ell^n\right| + \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}} \|u - u_\ell^n\|_{L_\nu^p} < \lambda - 2/n \\ + \left(\kappa_\Lambda + \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\right)\mathfrak{r}_n = \lambda - 1/n.$$

So $t \in D_\ell^n$ and it follows that $\mathcal{A}_n^\lambda \subset \underset{j\in\mathbb{N}}{\cup}\Big(D_j^n \times O_n(z_j^n, u_j^n)\Big) \subset A_\lambda$. As $A_\lambda = \underset{n\in\mathbb{N}}{\cup}\mathcal{A}_n^\lambda$, taking union over $n \in \mathbb{N}$ yields that $A_\lambda = \underset{n\in\mathbb{N}}{\cup}\ \underset{j\in\mathbb{N}}{\cup}\Big(D_j^n \times O_n(z_j^n, u_j^n)\Big) \in \mathscr{L}([0,T]) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}\big(L_\nu^p\big)$.

(5) *Finally, let us verify item (iii).*
We have seen from part (3) and (4) that $g : [0,T] \times \mathbb{R}^d \times L_\nu^p \to \mathbb{R}$ is a $\mathscr{L}([0,T]) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}\big(L_\nu^p\big)/\mathscr{B}(\mathbb{R})$-measurable function satisfying (i) and (ii). Namely, g is a deterministic p-generator satisfying (2.2) and (A2').

Let $(z,u) \in \mathbb{R}^d \times L_\nu^p$. We see from (4.4), (4.7) and (4.5) that $P-$a.s.

$$Y_t^\theta = Y_T^\theta + \int_t^T g(s,z,u)ds - z(B_T - B_t) \\ - \int_{(t,T]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) + K_T^\theta - K_t^\theta, \quad t \in [0,T]. \tag{4.13}$$

The translation invariance of $\mathcal{E}$ and the $\mathcal{E}$-martingality of $Y^\theta + K^\theta$ then imply that for any $0 \le t < s \le T$

$$\mathcal{E}\big[\eta(t,s,z,u)|\mathcal{F}_t\big] = \mathcal{E}\big[Y_s^\theta + K_s^\theta - Y_t^\theta - K_t^\theta\big|\mathcal{F}_t\big] \\ = \mathcal{E}\big[Y_s^\theta + K_s^\theta\big|\mathcal{F}_t\big] - Y_t^\theta - K_t^\theta = 0, \quad P-\text{a.s.}, \tag{4.14}$$

with $\eta(t,s,z,u) := z(B_s - B_t) + \int_{(t,s]}\int_{\mathcal{X}} u(x)\widetilde{N}_{\mathfrak{p}}(dr,dx) - \int_t^s g(r,z,u)dr$.

Let $(Z,U) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$ be in form of simple processes, i.e.

$$\big(Z_t(\omega), U_t(\omega)\big) = \sum_{i=1}^N \mathbf{1}_{\{t\in(t_i,t_{i+1}]\}}\left(\sum_{j=1}^{\ell_i}\mathbf{1}_{\{\omega\in A_j^i\}}\big(z_j^i, u_j^i\big)\right), \quad (t,\omega) \in [0,T] \times \Omega, \tag{4.15}$$

where $0 = t_1 < \cdots < t_{N+1} = T$; and for $i = 1, \cdots, N$, $\{z^i_j\}_{j=1}^{\ell_i} \subset \mathbb{R}^d$, $\{u^i_j\}_{j=1}^{\ell_i} \subset L^p_\nu$ and $\{A^i_j\}_{j=1}^{\ell_i} \subset \mathcal{F}_{t_i}$ is a partition of Ω. (By refining, we can let Z and U have the same time partition and the same $\mathcal{F}_{t_i}$-measurable partition of Ω for each $i = 1, \cdots, N$.) For any $\forall t \in [0, T]$, we set $\mathfrak{Y}_t := \int_t^T Z_s dB_s + \int_{(t,T]} \int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx) - \int_t^T g(s, Z_s, U_s)ds$ and claim that

$$\mathcal{E}[\mathfrak{Y}_t|\mathcal{F}_t] = 0, \quad P-\text{a.s.} \tag{4.16}$$

Clearly, $\mathcal{E}[\mathfrak{Y}_T|\mathcal{F}_T] = 0$, $P-$a.s. Assume next that for some $i = 1, \cdots, N$, (4.16) holds for any $t \in [t_{i+1}, T]$. Given $t \in [t_i, t_{i+1})$, (F3), (F4), translation invariance of $\mathcal{E}$ and (4.14) imply that $P-$a.s.

$$\begin{aligned}
\mathcal{E}[\mathfrak{Y}_t|\mathcal{F}_t] &= \mathcal{E}\Big[\mathcal{E}\Big[\mathfrak{Y}_{t_{i+1}} + \sum_{j=1}^{\ell_i} \mathbf{1}_{A^i_j}\eta\big(t, t_{i+1}, z^i_j, u^i_j\big)\Big|\mathcal{F}_{t_{i+1}}\Big]\Big|\mathcal{F}_t\Big] \\
&= \mathcal{E}\Big[\mathcal{E}\big[\mathfrak{Y}_{t_{i+1}}|\mathcal{F}_{t_{i+1}}\big] + \sum_{j=1}^{\ell_i} \mathbf{1}_{A^i_j}\eta\big(t, t_{i+1}, z^i_j, u^i_j\big)\Big|\mathcal{F}_t\Big] \\
&= \sum_{j=1}^{\ell_i} \mathbf{1}_{A^i_j}\mathcal{E}\Big[\sum_{j=1}^{\ell_i} \mathbf{1}_{A^i_j}\eta\big(t, t_{i+1}, z^i_j, u^i_j\big)\Big|\mathcal{F}_t\Big] \\
&= \sum_{j=1}^{\ell_i} \mathcal{E}\Big[\mathbf{1}_{A^i_j}\eta\big(t, t_{i+1}, z^i_j, u^i_j\big)|\mathcal{F}_t\Big] = \sum_{j=1}^{\ell_i} \mathbf{1}_{A^i_j}\mathcal{E}\Big[\eta\big(t, t_{i+1}, z^i_j, u^i_j\big)|\mathcal{F}_t\Big] = 0.
\end{aligned}$$

Using mathematical induction shows that (4.16) holds for any $t \in [0, T]$.

Now, let $\xi \in L^p(\mathcal{F}_T)$. Theorem 2.1 shows that the BSDEJ (ξ, g) admits a unique solution $(Y, Z, U) \in \mathbb{S}^p$. In light of Theorem IV.67 of [13], we can approximate (Z, U) in $\mathbb{Z}^{2,p} \times \mathbb{U}^p$ by a sequence of simple processes $\{(Z^n, U^n)\}_{n\in\mathbb{N}}$ in form of (4.15): $\lim_{n\to\infty} \|Z^n - Z\|_{\mathbb{Z}^{2,p}} = \lim_{n\to\infty} \|U^n - U\|_{\mathbb{U}^p} = 0$. Given $n \in \mathbb{N}$, since the translation invariance of $\mathcal{E}$ and (4.16) imply that

$$\begin{aligned}
\mathcal{E}[\xi|\mathcal{F}_t] - \mathcal{E}_g[\xi|\mathcal{F}_t] &= \mathcal{E}[\xi|\mathcal{F}_t] - Y_t = \mathcal{E}\big[\xi - Y_t|\mathcal{F}_t\big] \\
&= \mathcal{E}\Big[\int_t^T Z_s dB_s + \int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx) \\
&\quad - \int_t^T g(s, Z_s, U_s)ds\Big|\mathcal{F}_t\Big] \\
&\quad - \mathcal{E}\Big[\int_t^T Z^n_s dB_s + \int_{(t,T]}\int_{\mathcal{X}} U^n_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx) \\
&\quad - \int_t^T g(s, Z^n_s, U^n_s)ds\Big|\mathcal{F}_t\Big], \quad P-\text{a.s.},
\end{aligned}$$

one can deduce from (3.2), (1.4), Burkholder–Davis–Gundy inequality, item (ii), (1.3) and Hölder's inequality that

$$
\begin{aligned}
&E\Big[\big|\mathcal{E}[\xi|\mathcal{F}_t]-\mathcal{E}_g[\xi|\mathcal{F}_t]\big|^p\Big]\\
&\le \mathscr{C}E\Bigg[\bigg|\int_t^T (Z_s-Z_s^n)dB_s+\int_{(t,T]}\int_{\mathcal{X}}\big(U_s(x)-U_s^n(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\
&\quad -\int_t^T \big(g(s,Z_s,U_s)-g(s,Z_s^n,U_s^n)\big)ds\bigg|^p\Bigg]\\
&\le \mathscr{C}E\Bigg[\bigg(\int_0^T |Z_s-Z_s^n|^2ds\bigg)^{\frac{p}{2}}+\bigg(\int_{(0,T]}\int_{\mathcal{X}}\big|U_s(x)-U_s^n(x)\big|^2 N_{\mathfrak{p}}(ds,dx)\bigg)^{\frac{p}{2}}\\
&\quad +\bigg(\int_0^T |Z_s-Z_s^n|ds\bigg)^p+\bigg(\int_0^T \|U_s-U_s^n\|_{L_\nu^p}ds\bigg)^p\Bigg]\\
&\le \mathscr{C}E\Bigg[\bigg(\int_0^T |Z_s-Z_s^n|^2ds\bigg)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}\big|U_s(x)-U_s^n(x)\big|^p\,\nu(dx)ds\Bigg]\\
&=\mathscr{C}\Big(\|Z^n-Z\|_{\mathbb{Z}^{2,p}}^p+\|U^n-U\|_{\mathbb{U}^p}^p\Big).
\end{aligned}
$$

Here $\mathscr{C}$ stands for a generic constant depending on T, $\nu(\mathcal{X})$, p, κ_2 and κ_Λ, whose form changes from line to line. Letting $n\to\infty$ yields that $\mathcal{E}[\xi|\mathcal{F}_t]=\mathcal{E}_g[\xi|\mathcal{F}_t]$, $P-$a.s., which together with the right-continuity of $\mathcal{E}$-martingale $\big\{\mathcal{E}[\xi|\mathcal{F}_t]\big\}_{t\in[0,T]}$ and g-martingale $\big\{\mathcal{E}_g[\xi|\mathcal{F}_t]\big\}_{t\in[0,T]}$ leads to item (iii). □

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