On $g-$Evaluations with $\mathbb{L}^p$ Domains under Jump Filtration

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Abstract

Given $p\in(1,2]$, the unique $\mathbb{L}^p$ solutions of backward stochastic differential equations with jumps (BSDEJs) allow us to extend the notion of $g-$evaluations, in particular $g-$expectations, to the jump case with $\mathbb{L}^p$ domains. We explore many important properties of the extended $g-$evaluations including optional sampling, upcrossing inequality, Doob-Meyer decomposition, generator representation and Jensen's inequality. Most of these results are important for the further development of jump-filtration consistent nonlinear expectations with $\mathbb{L}^p$ domains in [95].

Keywords: Backward stochastic differential equation with jumps, $\mathbb{L}^p$ solutions, $g-$evaluation, $g-$expectation, optional sampling, upcrossing inequality, Doob-Meyer decomposition, generator representation, reverse comparison theorem of BSDEJs, Jensen's inequality.

1 Introduction

Let $p\in(1,2]$ and $T\in(0,\infty)$. Given a Lipschitz generator g, [94] showed that for each $p-$integrable terminal data ξ, the real-valued backward stochastic differential equation with jumps (BSDEJ)

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s,U_s)ds-\int_t^T Z_s\,dB_s-\int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T] \tag{1.1}$$

that is driven by a Brownian motion B and an independent $\mathcal{X}-$valued Poisson point process $\mathfrak{p}$ admits a unique $\mathbb{L}^p-$solution (Y^ξ,Z^ξ,U^ξ). In particular, the process Y^ξ can be regarded as the so-called "(conditional) $g-$expectation" of ξ: $\mathcal{E}_g[\xi|\mathcal{F}_t]:=Y^\xi_t$, $t\in[0,T]$. The $g-$expectation $\{\mathcal{E}_g[\xi|\mathcal{F}_t]\}_{t\in[0,T]}$ can be further generalized as $g-$evaluations $\big\{\mathcal{E}^g_{\tau,\gamma}[\xi]\big\}_{\tau<\gamma}$, by considering BSDEJ with random horizon. Such a $g-$evaluations are closely related to a large class of coherent or convex risk measures for $p-$integrable financial positions (which may not be square-integrable) in a market with jumps.

In this paper, we show that as nonlinear expectations with $\mathbb{L}^p$ domains under jump filtration (the filtration generated by B and Poisson random measure $N_{\mathfrak{p}}$), the $g-$evaluations inherit many important (martingale) properties from the classic linear expectations such as optional sampling, upcrossing inequality, Doob-Meyer decomposition, Jensen's inequality and etc. Most of these results will assist us to study finance markets with jumps using nonlinear evaluation criteria or risk measurement.

The well-known Allais paradox suggests people to develop a nonlinear-expectation version of the von Neumann-Morgenstern's axiomatic system of expected utilities, a fundamental notion in the modern economics. Motivated by such a generalization, Peng [77, 80] introduced the concepts of $g-$expectations and $g-$evaluations via backward stochastic differential equations (BSDEs). These two seminal works and some following research ([30, 15, 22, 81, 86] among others) show that the $g-$evaluations are closely related to axiom-based coherent and convex risk measures (see [4, 39]) in mathematical finance: When the generator g is positively homogeneous or convex in (y,z), then $\rho^g_t(\xi):=\mathcal{E}_g[-\xi|\mathcal{F}_t]$ defines a coherent or convex risk measure. Reversely, under certain domination condition (see (4.1) of [30]), a coherent or convex risk measure $\{\rho_t\}_{t\in[0,T]}$ with $\mathbb{L}^2$ domain under Brownian filtration can be represented by some $g-$expectation or the solution of a BSDE with generator g and square-integrable terminal data ξ.

Lin [67] and Royer [87] extended the $g-$expectations to the jump case and obtained a Doob-Meyer decomposition for $g-$expectations with $\mathbb{L}^2$ domains under jump filtration. Under a similar domination condition to (4.1) of [30], [87]

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also showed that a risk measure with $\mathbb{L}^2$ domain is still a $g-$expectation in a financial market with jumps. On the other hand, Ma and Yao [68] generalized the $g-$evaluations to the quadratic case (i.e. the generator g has a quadratic growth in z) while Hu et al. [45] derived a representation of convex risk measures by quadratic $g-$expectations under a different domination condition. Recently, [54] even extended the quadratic $g-$expectations to the jump case and demonstrated that the corresponding martingale properties still hold, such as Doob-Meyer decomposition and downcrossing inequality. Based on these features, they provided a dual representation for dynamic risk measures with jumps.

The present paper starts with a strict comparison theorem for $\mathbb{L}^p-$solutions of BSDEJs (Theorem 2.2) under an additional condition (A3) in u. Theorem 2.2 together with the uniqueness result of BSDEJs in $\mathbb{L}^p$ sense implies that the corresponding $g-$evaluations with $\mathbb{L}^p$ domains under the jump filtration inherits "strict monotonicity ", "constant preserving", "time-consistency", "zero-one law", "translation invariance", "convexity", "positive homogeneity" $\big((\text{g1})-(\text{g7})\big)$ from linear expectations and thus preserves some classic martingale properties such as optional sampling, upcrossing inequality and Doob-Meyer decomposition (Proposition 4.1, Proposition 4.2, Theorem 4.1). In particular, the proof of the Doob-Meyer decomposition for $g-$supermatingales also depends on a *monotonic limit* theorem of $p-$integrable jump diffusion processes with jumps (Theorem A.1) as well as an a priori $\mathbb{L}^p-$estimate of a generalized BSDEJ (Proposition 4.3).

Moreover, we explore other nice properties of $g-$evaluations: Using a result of [94], we can represent a generator g as the limit of the difference quotients of the corresponding $g-$evaluations (see Proposition 5.1), which gives rise to a reverse comparison theorem of BSDEJs (Theorem 5.1). Proposition 5.1 also establishes an equivalence between the convexity (resp. positive homogeneity) of g in (y,z,u) and the convexity (resp. positive homogeneity) of $g-$evaluations, as well as an equivalence between the independence of g on $y-$variable and the translation invariance of $g-$evaluations (Proposition 5.2). When the generator is convex in (z,u), we can use the comparison theorem of BSDEJs again to derive Jensen's inequality of $g-$evaluations (Theorem 5.2).

Main Contributions.

Given $U\in\mathbb{U}^2_{\text{loc}}$, unlike the case of Brownian stochastic integrals, the Burkholder-Davis-Gundy inequality is not applicable for the $p/2-$th power of the Poisson stochastic integral $\int_{(0,t]}\int_{\mathcal{X}} Y_s U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$ (see e.g. Theorem VII.92 of [34]): i.e. $E\Big[\sup\limits_{t\in[0,T]}\Big(\int_{(0,t]}\int_{\mathcal{X}} Y_s U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\Big)^{\frac{p}{2}}\Big]$ cannot be dominated by $E\Big[\big(\int_{(0,T]}\int_{\mathcal{X}}|Y_s|^2|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\big)^{\frac{p}{4}}\Big]$. So to derive an *a priori* $\mathbb{L}^p$ estimate for BSDEJs, we could not follow the classical argument in the proof of [16, Proposition 3.2], neither could we employ the space $\mathbb{U}^{2,p}:=\Big\{U:\,E\Big[\big(\int_0^T\int_{\mathcal{X}}|U_t(x)|^2\nu(dx)dt\big)^{\frac{p}{2}}\Big]<\infty\Big\}$ or the space $\widetilde{\mathbb{U}}^{2,p}:=\Big\{U:\,E\Big[\big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\big)^{\frac{p}{2}}\Big]<\infty\Big\}$ $\Big($Actually one may not be able to compare $E\Big[\big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\big)^{\frac{p}{2}}\Big]$ with $E\Big[\big(\int_0^T\int_{\mathcal{X}}|U_t(x)|^2\nu(dx)dt\big)^{\frac{p}{2}}\Big]\Big)$.

In [94], we started with a generalization of the Poisson stochastic integral for a random field $U\in\mathbb{U}^p$ by constructing a càdlàg uniformly integrable martingale $M^U_t:=\int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$, whose quadratic variation $[M^U,M^U]$ is still $\int_{(0,t]}\int_{\mathcal{X}}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. In deriving the key $\mathbb{L}^p-$type inequality (see Lemma 3.1 of [94]) about the difference $Y=Y^1-Y^2$ of two $p-$integrable solutions to BSDEJs with different parameters, our delicate analysis showed that the variational jump part $\sum\limits_s\Big(|Y_s|^p-|Y_{s-}|^p-p\big\langle|Y_{s-}|^{p-1},\Delta Y_s\big\rangle\Big)$ in the dynamics of $|Y|^p$ will eventually boil down to the term $E\int_0^T\int_{\mathcal{X}}|U^1_t(x)-U^2_t(x)|^p\nu(dx)dt$, which justifies our choice of $\mathbb{U}^p$ over $\mathbb{U}^{2,p}$ or $\widetilde{\mathbb{U}}^{2,p}$ as the space for jump diffusion. The estimation course of the variational jump is full of analytical subtleties, but we managed to overcome them by utilizing some new techniques and special treatments $\big($see (5.11)$-$(5.21) of [94] for details$\big)$.

In the present paper, we developed these techniques to handle similar (bur more complicated) technical hurdles when we are deriving the a priori $\mathbb{L}^p-$estimate for a special BSDEJ in Proposition 4.3 (see (6.56)$-$(6.59)), or when we are measuring the $\mathbb{L}^p-$distance of an increasing sequence of jump diffusion processes Y^n from its limit Y in Theorem A.1 (see (A.40)$-$(A.47) or (A.57)$-$(A.60)). As aforementioned, both Proposition 4.3 and Theorem A.1 are crucial in proving the Doob-Meyer decomposition for $g-$supermatingales (our main Theorem 4.1).

Our analysis in the paper also heavily relies on the follow inequality

$$E\Big[[M^U,M^U]^{\frac{p}{2}}\Big]=E\Big[\Big(\int_0^T\int_{\mathcal{X}}|U_t(x)|^2\nu(dx)dt\Big)^{\frac{p}{2}}\Big]\le E\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^pN_{\mathfrak{p}}(dt,dx)=E\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt.$$

Although many of our results look similar to those with $\mathbb{L}^2$ domains in the non-jump case ([30]) or in jump case ([87]), we have to do more delicate analysis to overcome various technical subtleties raised in the $\mathbb{L}^p-$jump case. For instance, to demonstrate the monotonic limit theorem (Theorem A.1), we nontrivially extend Lemma 2.3 of [78] to the $\mathbb{L}^p-$jump case, see Lemma A.4.

All martingale properties of $g-$evaluations in $\mathbb{L}^p-$jump case, especially the Doob-Meyer decomposition and the monotonic limit theorem, will play important roles in our study of a general class of jump-filtration consistent nonlinear expectations $\mathcal{E}$ with $\mathbb{L}^p-$domains, which encompasses many coherent or convex time-consistent risk measures $\rho=\{\rho_t\}_{t\in[0,T]}$. Under certain domination condition, we show in [95] that the nonlinear expectation $\mathcal{E}$ preserves many important (martingale) properties of linear expectations (including optional sampling and Doob-Meyer decomposition), and thus can be represented by some $g-$expectation. Consequently, one can utilize the BSDEJ theory to systematically analyze the risk measure ρ with $\mathbb{L}^p-$domains and employ numerical schemes of BSDEJs to run simulation for financial problems involving ρ in a market with jumps.

In another of our accompany paper [93], we analyze a BSDEJ with a $p-$integrable reflecting barrier $\mathfrak{L}$ whose generator g is Lipschitz continuous in (y,z,u). We show that such a reflected BSDEJ with $p-$integrable parameters admits a unique $\mathbb{L}^p$ solution, and thus solves the corresponding optimal stopping problem under the $g-$expectation or some dominated risk measure with $\mathbb{L}^p-$domain: the $Y-$component of the unique solution is exactly the *Snell* envelope of process $\mathfrak{L}$ under the $g-$expectation and the first time it meets $\mathfrak{L}$ is an optimal stopping time for maximizing the $g-$expectation of reward $\mathfrak{L}$ or minimizing the risk measure of financial position $\mathfrak{L}$.

Relevant Literature.

The backward stochastic equation (BSDE) was introduced by Bismut [12] as the adjoint equation for the Pontryagin maximum principle in stochastic control theory. Later, Pardoux and Peng [76] commenced a systematical research of BSDEs. Since then, the BSDE theory has grown rapidly and has been applied to various areas such as mathematical finance, theoretical economics, stochastic control and optimization, partial differential equations, differential geometry and etc, (see the references in [38, 31]).

1) Li and Tang [90] introduced into the BSDE a jump term that is driven by a Poisson random measure independent of the Brownian motion. These authors obtained the existence of a unique solution to a BSDEJ with a Lipschitz generator and square-integrable terminal data. Then Barles, Buckdahn and Pardoux [19, 7] showed that the well-posedness of BSDEJs gives rise to a viscosity solution of a semilinear parabolic partial integro-differential equation (PIDE) and thus provides a probabilistic interpretation of such a PIDE. Later, Pardoux [75] relaxed the Lipschitz condition of the generator on variable y by assuming a monotonicity condition on variable y instead. Situ [89] and Mao and Yin [96] even degenerated the monotonicity condition of the generator to a weaker version so as to remove the Lipschitz condition on variable z.

2) During the development of the BSDE theory, some efforts were made in relaxing the square integrability on the terminal data so as to be compatible with the fact that linear BSDEs are well-posed for integrable terminal data or that linear expectations have $\mathbb{L}^1$ domains: El Karoui et al. [38] showed that for any $p-$integrable terminal data, the BSDE with a Lipschitz generator admits a unique $\mathbb{L}^p-$solution. Then Briand and Carmona [14] reduced the Lipschitz condition of the generator on variable y by a strong monotonicity condition as well as a polynomial growth condition on variable y. Later, Briand et al. [16] found that the polynomial growth condition is not necessary if one uses the monotonicity condition similar to that of [75].

We analyzed $\mathbb{L}^p$ solutions of multi-dimensional BSDEJs under a monotone condition in [94], while Kruse and Popier [61, 63] studied a similar $\mathbb{L}^p-$solution problem of BSDE under a right-continuous filtration which may be larger than the jump filtration:

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s,U_s)ds-\int_t^T Z_s\,dB_s-\int_{(t,T]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)-\int_t^T dM_s,\quad t\in[0,T], \tag{1.2}$$

where M is a local martingale orthogonal to the jump filtration. Also, Klimsiak studied $\mathbb{L}^p$ solutions of reflected BSDEs under a general right-continuous filtration in [57].

3) The researches on BSDEs over general filtered probability spaces have recently attracted more and more attention. A series of works [18, 36, 38, 17, 20, 66, 21] are dedicated to the theory of BSDEs (1.2) but driven by a càdlàg martingale under a right-continuous filtration that is also quasi-left continuous. Lately, [13, 74] removed the quasi-left continuity assumption from the filtration so that the quadratic variation of the driving martingale does not need to be absolutely continuous. On the other hand, based on a general martingale representation result due to Davis and Varaiya [32], Cohen and Elliott [25, 26] discussed the case where the driving martingales are not *a priori* chosen but imposed by the filtration; see Hassani and Ouknine [44] for a similar approach on a BSDE in form of a generic map from a space of semimartingales to the spaces of martingales and those of finite-variation processes. Also, Mania and Tevzadze [69] and Jeanblanc et al. [48] studied BSDEs for semimartingales and their applications to mean-variance hedging.

As to BSDEs driven by other discontinuous random sources, Xia [92] and Bandini [6] studied BSDEs driven by a random measure; Confortola et al. [28, 29] considered BSDEs driven by a marked point process; [73, 5, 84, 42] analyzed BSDEs driven by Lévy processes; [2, 88, 55] discussed BSDEs driven by a process with a finite number of marked jumps.

4) There are also plenty of researches on quadratic BSDEJs:

To study the exponential utility maximization problem with an additional liability, Becherer [10] extended Kobylanski [60]'s monotone stability approach to a jump-diffusion model and obtained a unique bounded solution to a related BSDE driven by a random measure whose generator may not be Lipschitz continuous in u. Becherer et al. [11] recently generalized this result for random measures of infinite activity with a non-deterministic compensator. Meanwhile, Morlais [70] utilized a similar monotone stability approach and dynamic programming to show that a special quadratic BSDEJ with bounded terminal data has a unique solution, whose Y component is the value process of an exponential utility maximization problem with jumps. Morlais [71] even obtained an existence result for such quadratic BSDEJs with exponentially integrable terminal data.

For general quadratic BSDEJs with unbounded terminal data, Ngoupeyou [72] and El Karoui et al. [37] extended Barrieu and El Karoui [8]'s quadratic semimartingales approach to the jump case. They managed to obtain an existence result for quadratic-exponential BSDEJs (i.e. quadratic BSDEJs whose generators have a exponential growth in u) with unbounded terminal data. Also, Jeanblanc et al. [49] described the value process of a utility optimization problem under Knightian-uncertainty in a jump setting as a class of quadratic-exponential BSDEJs. When generators of quadratic-exponential BSDEJs are allowed to be locally-Lipschitz, Fujii and Takahashi [40] provided a sufficient condition for the Malliavin's differentiability of such BSDEJs with bounded terminal data while [3] could still employ [60]'s monotone stability approach to show the wellposedness of such BSDEJs.

As to different methods on quadratic BSDEJs, Kazi-Tani et al. [51, 54] exploited the fixed-point approach as in Tevzadze [91] and an exquisite splitting technique to demonstrate the wellposedness of quadratic-exponential BSDEJs with bounded terminal data and applied this result to study the related nonlinear expectations; Laeven and Stadje [64] took a duality approach to characterize the value of an optimal portfolio valuation problem as the unique solution to a BSDEJ with a convex generator which has at most quadratic growth in z.

5) It is worth mentioning that [53, 52] recently made a very interesting development of second-order BSDEs with jumps, and provided a probabilistic interpretation for the related fully-nonlinear PIDEs.

For topics of BSDEJs in other directions, see Cohen and Elliott [23, 24, 27] for BSDEs driven by Markov chains; see Kharroubi et al. [56] for (minimal) solutions to BSDEs with constrained jumps and related quasi-variational inequalities; see Aazizi and Ouknine [1] for a class of constrained BSDEJs and its application in pricing and hedging American options; see Klimsiak and Rozkosz [58, 59] for a general (non-Markovian) BSDE and a related semilinear elliptic equation with measure data whose operator is associated with a regular semi-Dirichlet form; see [62, 43] for BSDEJs with singular terminal data and their applications to optimal position targeting and a non-Markovian liquidation problem respectively; see also [65, 41, 35] for numerical simulation of BSDEJs among other.

The rest of the paper is organized as follows: We introduce some notations in Section 1.1. In Section 2, after making basic assumptions on generator g, we review some properties of $\mathbb{L}^p$ solutions to BSDEJs with generator g (including the wellposedness result, the martingale representation theorem as well as an a priori estimate), and prove a strict comparison theorem for these $\mathbb{L}^p$ solutions. In Section 3, we define the $g-$evaluation with domain $\mathbb{L}^p$ under jump filtration according to the wellposedness of BSDEJs with generator g in $\mathbb{L}^p$ sense. Then we show that the $g-$evaluation preserves many basic properties of linear expectations. In Section 4, we obtained some martingale properties of the $g-$evaluation such as optional sampling, upcrossing inequality and Doob-Meyer decomposition. Section 5 discuss some other fine properties of $g-$evaluations including a generator representation via $g-$evaluations, some of its consequences and Jensen's inequality of the $g-$evaluations. The proofs of our results are relegated to Section 6. We generalize [79]'s monotonic limit theorem for $p-$integrable jump diffusion processes with jumps in the appendix as it is interesting in its own right.

1.1 Notation and Preliminaries

Throughout this paper, we fix a time horizon $T\in(0,\infty)$ and let $(\Omega,\mathcal{F},P)$ be a complete probability space on which a $d-$dimensional Brownian motion B is defined.

For a generic càdlàg process X, we denote its corresponding jump process by $\Delta X_t := X_t - X_{t-}$, $t\in[0,T]$ with $X_{0-} := X_0$. Given a measurable space $(\mathcal{X},\mathcal{F}_\mathcal{X})$, let $\mathfrak{p}$ be an $\mathcal{X}-$valued Poisson point process on $(\Omega,\mathcal{F},P)$ that is independent of B. For any scenario $\omega\in\Omega$, the set $D_{\mathfrak{p}(\omega)}$ of all jump times of path $\mathfrak{p}(\omega)$ is a countable subset of $(0,T]$ (see e.g. Section 1.9 of [46]). We assume that for some finite measure ν on $\big(\mathcal{X},\mathcal{F}_\mathcal{X}\big)$, the counting measure $N_\mathfrak{p}(dt,dx)$ of $\mathfrak{p}$ on $[0,T]\times\mathcal{X}$ has compensator $E\big[N_\mathfrak{p}(dt,dx)\big]=\nu(dx)dt$. The corresponding compensated Poisson random measure $\widetilde{N}_\mathfrak{p}$ will be denoted by $\widetilde{N}_\mathfrak{p}(dt,dx) := N_\mathfrak{p}(dt,dx)-\nu(dx)dt$.

For any $t\in[0,T]$, we define sigma-fields

$$\mathcal{F}^B_t := \sigma\big\{B_s; s\le t\big\}, \qquad \mathcal{F}^N_t := \sigma\big\{N_\mathfrak{p}((0,s],A); s\le t, A\in\mathcal{F}_\mathcal{X}\big\}, \qquad \mathcal{F}_t := \sigma\big(\mathcal{F}^B_t\cup\mathcal{F}^N_t\big)$$

and augment them by all $P-$null sets of $\mathcal{F}$. Clearly, the jump filtration $\mathbf{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ is complete and right-continuous (i.e. satisfies the *usual hypotheses*, see e.g., [82]). Denote by $\mathscr{P}$ $\big($resp. $\widehat{\mathscr{P}}\big)$ the $\mathbf{F}-$progressively measurable (resp. $\mathbf{F}-$predictable) sigma-field on $[0,T]\times\Omega$, and let $\mathcal{T}$ be the collection of all $\mathbf{F}-$stopping times. For any $\tau\in\mathcal{T}$, we set $\mathcal{T}_\tau := \{\gamma\in\mathcal{T}: \gamma\ge\tau,\ P\text{-a.s.}\}$.

Recall that a uniformly integrable càdlàg martingale M is said to be a *BMO* ("Bounded Mean Oscillation") martingale if there exists $C>0$ such that for any $\tau\in\mathcal{T}$

$$E\big[[M,M]_T-[M,M]_\tau|\mathcal{F}_\tau\big]\le C \quad\text{and}\quad |\Delta M_\tau|^2\le C, \quad P\text{-a.s.}$$

The following spaces of functions will be used in the sequel:

1) For any $p\in[1,\infty)$, let $L^p_+[0,T]$ be the space of all measurable functions $\psi:[0,T]\mapsto[0,\infty)$ with $\int_0^T\big(\psi(t)\big)^p dt<\infty$.

2) For $p\in(1,\infty)$, let $L^p_\nu := L^p(\mathcal{X},\mathcal{F}_\mathcal{X},\nu;\mathbb{R})$ be the space of all real-valued, $\mathcal{F}_\mathcal{X}-$measurable functions u with $\|u\|_{L^p_\nu} := \big(\int_\mathcal{X}|u(x)|^p\nu(dx)\big)^{\frac{1}{p}}<\infty$. For any $u_1,u_2\in L^p_\nu$, we say $u_1=u_2$ if $u_1(x)=u_2(x)$ for $\nu-$a.s. $x\in\mathcal{X}$.

3) For any sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$, let $L^0(\mathcal{G})$ be the space of all real-valued, $\mathcal{G}-$measurable random variables and set

- $L^p(\mathcal{G}) := \Big\{\xi\in L^0(\mathcal{G}): \|\xi\|_{L^p(\mathcal{G})} := \Big\{E\big[|\xi|^p\big]\Big\}^{\frac{1}{p}}<\infty\Big\}$ for any $p\in(1,\infty)$;
- $L^\infty(\mathcal{G}) := \Big\{\xi\in L^0(\mathcal{G}): \|\xi\|_{L^\infty(\mathcal{G})} := \underset{\omega\in\Omega}{\text{esssup}}\,|\xi(\omega)|<\infty\Big\}$.

4) Let $\mathbb{D}^0$ be the space of all real-valued, $\mathbf{F}-$adapted càdlàg processes, and let $\mathbb{K}^0$ be a subspace of $\mathbb{D}^0$ that includes all $\mathbf{F}-$predictable càdlàg increasing processes X with $X_0=0$.

5) Set $\mathbb{Z}^2_{\text{loc}} := L^2_{\text{loc}}\big([0,T]\times\Omega,\widehat{\mathscr{P}},dt\times dP;\mathbb{R}^d\big)$, the space of all $\mathbb{R}^d-$valued, $\mathbf{F}-$predictable processes Z with $\int_0^T|Z_t|^2\,dt<\infty$, $P-$a.s.

6) For any $p\in[1,\infty)$, we let

- $\mathbb{D}^p := \Big\{X\in\mathbb{D}^0: \|X\|_{\mathbb{D}^p} := \big\{E[X^p_*]\big\}^{\frac{1}{p}}<\infty\Big\}$, where $X_* := \underset{t\in[0,T]}{\sup}|X_t|<\infty$.
- $\mathbb{K}^p := \mathbb{K}^0\cap\mathbb{D}^p = \big\{K\in\mathbb{K}^0: E[K^p_T]<\infty\big\}$.
- $\mathbb{Z}^{2,p} := \Big\{Z\in\mathbb{Z}^2_{\text{loc}}: \|Z\|_{\mathbb{Z}^{2,p}} := \Big\{E\Big[\big(\int_0^T|Z_t|^2\,dt\big)^{\frac{p}{2}}\Big]\Big\}^{\frac{1}{p}}<\infty\Big\}$. For any $Z\in\mathbb{Z}^{2,p}$, the Burkholder-Davis-Gundy inequality implies that

$$E\bigg[\sup_{t\in[0,T]}\Big|\int_0^t Z_s dB_s\Big|^p\bigg]\le c_p E\bigg[\Big(\int_0^T\big|Z_s\big|^2 ds\Big)^{\frac{p}{2}}\bigg]<\infty \tag{1.3}$$

for some constant $c_p>0$ depending on p. So $\big\{\int_0^t Z_s dB_s\big\}_{t\in[0,T]}$ is a uniformly integrable martingale.

• $\mathbb{U}^p_{\rm loc} := L^p_{\rm loc}\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}, dt\times dP\times\nu(dx);\mathbb{R}\big)$ be the space of all $\widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}-$measurable random fields $U\colon[0,T]\times\Omega\times\mathcal{X}\to\mathbb{R}$ such that $\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt=\int_0^T\|U_t\|^p_{L^p_\nu}dt<\infty$, $P-$a.s.

• $\mathbb{U}^p:=\Big\{U\in\mathbb{U}^p_{\rm loc}\colon \|U\|_{\mathbb{U}^p}:=\big\{E\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt\big\}^{\frac{1}{p}}<\infty\Big\}=L^p\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}, dt\times dP\times\nu(dx);\mathbb{R}\big)$.

For any $U\in\mathbb{U}^p_{\rm loc}$ $\big($resp. $\mathbb{U}^p\big)$, it holds for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$ that $U(t,\omega)\in L^p_\nu$. According to Section 1.2 of [94], we can define a Poisson stochastic integral of U:

$$M^U_t:=\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T], \tag{1.4}$$

which is a càdlàg local martingale (resp. uniformly integrable martingale) with quadratic variation $[M^U,M^U]_t=\int_{(0,t]}\int_{\mathcal{X}}|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. The jump process of M^U is $\Delta M^U_t(\omega)=\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}U\big(t,\omega,\mathfrak{p}_t(\omega)\big)$, $t\in(0,T]$. For any $U\in\mathbb{U}^p$, an analogy to (5.1) of [94] shows that

$$E\Big[\Big(\int_{(t,s]}\int_{\mathcal{X}}|U_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big]\le E\int_t^s\int_{\mathcal{X}}\big|U_t(x)\big|^p\nu(dx)dt, \quad \forall\,0\le t<s\le T. \tag{1.5}$$

• We simply denote $\mathbb{D}^p\times\mathbb{Z}^{2,p}\times\mathbb{U}^p$ by $\mathbb{S}^p$.

As usual, we set $x^-:=(-x)\vee 0$, $x^+:=x\vee 0$ for any $x\in\mathbb{R}$, and use the convention $\inf\emptyset:=\infty$. Given $p\in(0,\infty)$, the following two inequalities will be frequently applied in this paper:

(i) For any $a,b\in\mathbb{R}$, $|a^\pm-b^\pm|\le|a-b|$. (1.6)

(ii) For any finite subset $\{a_1,\cdots,a_n\}$ of $(0,\infty)$, $(1\wedge n^{p-1})\sum_{i=1}^n a_i^p\le\Big(\sum_{i=1}^n a_i\Big)^p\le(1\vee n^{p-1})\sum_{i=1}^n a_i^p$. (1.7)

Also, we let c_p denote a generic constant depending only on p (in particular, c_0 stands for a generic constant depending on nothing), whose form may vary from line to line.

2 $\mathbb{L}^p$ Solutions of BSDEs with Jumps

From now on, we fix $p\in(1,2]$ and set $q:=\frac{p}{p-1}\ge 2$.

A mapping $g:\,[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\to\mathbb{R}$ is called a *$p-$generator* if it is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable. For any $\tau\in\mathcal{T}$,

$$g_\tau(t,\omega,y,z,u):=\mathbf{1}_{\{t<\tau(\omega)\}}\,g(t,\omega,y,z,u), \quad \forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$$

is also $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable.

We say a $p-$generator g is convex in (y,z,u) if it holds $P-$a.s. that for any $(t,\alpha)\in(0,T)\times[0,1]$ and $(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$, $i=1,2$

$$g\big(t,\alpha y_1+(1-\alpha)y_2,\alpha z_1+(1-\alpha)z_2,\alpha u_1+(1-\alpha)u_2\big)\le\alpha g(t,y_1,z_1,u_1)+(1-\alpha)g(t,y_2,z_2,u_2). \tag{2.1}$$

Also, we say a $p-$generator g is positively homogeneous in (y,z,u) if it holds $P-$a.s. that

$$g(t,\widetilde{\alpha}y,\widetilde{\alpha}z,\widetilde{\alpha}u)=\widetilde{\alpha}g(t,y,z,u), \quad \forall\,(t,\widetilde{\alpha})\in(0,T)\times[0,\infty), \ \forall\,(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu. \tag{2.2}$$

Definition 2.1. *Given $p\in(1,2]$, let $\xi\in L^0(\mathcal{F}_T)$ and g be a $p-$generator. A triplet $(Y,Z,U)\in\mathbb{D}^0\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$ is called a solution of a backward stochastic differential equation with jumps that has terminal data ξ and generator g $\big(BSDEJ(\xi,g)$ for short$\big)$ if $\int_0^T|g(s,Y_s,Z_s,U_s)|ds<\infty$, $P-$a.s. and if (1.1) holds $P-$a.s.*

We shall make the following standard assumptions on $p-$generators g:

(A1) $\int_0^T|g(t,0,0,0)|dt\in L^p(\mathcal{F}_T)$.

(A2) There exist two $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable processes β, Λ with $\int_0^T\big(\beta^q_t\vee\Lambda^2_t\big)dt\in L^\infty(\mathcal{F}_T)$ such that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$

$$\big|g(t,\omega,y_1,z_1,u)-g(t,\omega,y_2,z_2,u)\big|\le\beta(t,\omega)|y_1-y_2|+\Lambda(t,\omega)|z_1-z_2|, \ \forall\,(y_1,z_1), \ (y_2,z_2)\in\mathbb{R}\times\mathbb{R}^d, \ \forall\,u\in L^p_\nu.$$

(A3) There exists a function $\mathfrak{h}\colon [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu \to L^q_\nu$ such that
(i) $\mathfrak{h}$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(L^p_\nu)\otimes\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable;
(ii) There exist $\kappa_1\in(-1,0]$ and $\kappa_2\ge-\kappa_1$ such that for any $(t,\omega,y,z,u_1,u_2,x)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu\times\mathcal{X}$

$$\kappa_1 \le \big(\mathfrak{h}(t,\omega,y,z,u_1,u_2)\big)(x) \le \kappa_2;$$

(iii) It holds for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$ that

$$g(t,\omega,y,z,u_1)-g(t,\omega,y,z,u_2)\le\int_{\mathcal{X}}\big(u_1(x)-u_2(x)\big)\cdot\big(\mathfrak{h}(t,\omega,y,z,u_1,u_2)\big)(x)\,\nu(dx),\ \ \forall\,(y,z,u_1,u_2)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu. \quad (2.3)$$

We refer to $\Xi:=(\beta,\Lambda,\kappa_1,\kappa_2)$ as a $p-$coefficient set.

Remark 2.1. *Let $p\in(1,2]$ and let g be a $p-$generator.*
(1) By (A3) (ii), (iii) and Hölder's inequality, (A2) and (A3) imply
(A2') *There exist two $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable processes β, Λ with $\int_0^T\big(\beta^q_t\vee\Lambda^2_t\big)dt\in L^\infty(\mathcal{F}_T)$ such that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$*

$$\big|g(t,\omega,y_1,z_1,u_1)-g(t,\omega,y_2,z_2,u_2)\big|\le\beta(t,\omega)\big(|y_1-y_2|+\|u_1-u_2\|_{L^p_\nu}\big)+\Lambda(t,\omega)|z_1-z_2|,\ \ \forall\,(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu,\ i=1,2.$$

(2) If g satisfies (A2') and $\int_0^T|g(t,0,0,0)|dt<\infty$, $P-$a.s., then an analogy to Remark 2.1 of [94] shows that for any $(Y,Z,U)\in\mathbb{D}^1\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, one has $\int_0^T|g(s,Y_s,Z_s,U_s)|ds<\infty$, $P-$a.s.
(3) If g satisfies (A1), (A2) (resp. (A2')), then $\overline{g}(t,\omega,y,z,u):=-g(t,\omega,-y,-z,-u)$, $(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$ and g_τ, $\forall\,\tau\in\mathcal{T}$ are also $p-$generators satisfying (A1), (A2) (resp. (A2')). If g further satisfies (A3), so do $\overline{g}$ and g_τ.
(4) We need the assumption "$\kappa_1>-1$" in (A3) (ii) for a strict comparison theorem of BSDEJs (Theorem 2.2) and the upcrossing inequality of $g-$supermartingales (Proposition 4.2). Actually, it is necessary for the Doléans-Dade exponentials $\mathscr{E}.(M)$ in (6.6) and $\mathscr{E}.(M^{\mathcal{D}})$ in (6.40) to be strictly positive martingales (see e.g. [50]), which then allows us to apply Girsanov Theorem to change probabilities in the proofs of Theorem 2.2 and Proposition 4.2.

For simplicity, we set $\widehat{C}:=\big\|\int_0^T\big(1\vee\beta^q_t\vee\Lambda^2_t\big)dt\big\|_{L^\infty(\mathcal{F}_T)}$, and let $\mathcal{C}$ be a generic constant depending on T, $\nu(\mathcal{X})$, p, $\widehat{C}$ (and κ_2 if necessary), whose form may vary from line to line.

For $\mathbb{L}^p$ solutions of BSDEs with jumps, we first quote a wellposedness result, the corresponding martingale representation theorem as well as an a priori estimate from Remark 4.1, Proposition 3.1, Corollary 4.1, Corollary 2.1 and Lemma 3.1 of [94].

Theorem 2.1. *Given $p\in(1,2]$, Let g be a $p-$generator satisfying (A1) and (A2'). For any $\xi\in L^p(\mathcal{F}_T)$, the BSDEJ$(\xi,g)$ admits a unique solution $\big(Y^{\xi,g},Z^{\xi,g},U^{\xi,g}\big)\in\mathbb{S}^p$, which satisfies*

$$\big\|Y^{\xi,g}\big\|^p_{\mathbb{D}^p}+\big\|Z^{\xi,g}\big\|^p_{\mathbb{Z}^{2,p}}+\big\|U^{\xi,g}\big\|^p_{\mathbb{U}^p}\le\mathcal{C}E\bigg[|\xi|^p+\Big(\int_0^T|g(t,0,0,0)|dt\Big)^p\bigg]. \quad (2.4)$$

In particular, for any $\tau\in\mathcal{T}$ and $\xi\in L^p(\mathcal{F}_\tau)$, the unique solution $\big(Y^{\xi,g_\tau},Z^{\xi,g_\tau},U^{\xi,g_\tau}\big)$ of the BSDEJ(ξ,g_τ) in $\mathbb{S}^p$ satisfies that $P\big\{Y^{\xi,g_\tau}_t=Y^{\xi,g_\tau}_{\tau\wedge t},\ t\in[0,T]\big\}=1$ and that $\big(Z^{\xi,g_\tau}_t,U^{\xi,g_\tau}_t\big)=\mathbf{1}_{\{t\le\tau\}}\big(Z^{\xi,g_\tau}_t,U^{\xi,g_\tau}_t\big)$, $dt\times dP-$a.s.

Remark 2.2. *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1) and (A2'). It holds for any $\xi\in L^p(\mathcal{F}_T)$ that $P\big\{Y^{\xi,g}_t=-Y^{-\xi,\overline{g}}_t,\ \forall\,t\in[0,T]\big\}=1$.*

Corollary 2.1. *Let $p\in(1,2]$. For any $\xi\in L^p(\mathcal{F}_T)$, there exists a unique pair $(Z,U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P-$a.s.*

$$E[\xi|\mathcal{F}_t]=E[\xi]+\int_0^t Z_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

Proposition 2.1. *Let $p\in(1,2]$. For $i=1,2$, let $\xi_i\in L^0(\mathcal{F}_T)$, g^i be a $p-$generator, and (Y^i,Z^i,U^i) be a solution of BSDEJ (ξ_i,g^i) such that $Y^1-Y^2\in\mathbb{D}^p$. If g^i satisfies (A2'), then*

$$\big\|Y^1-Y^2\big\|^p_{\mathbb{D}^p}+\big\|Z^1-Z^2\big\|^p_{\mathbb{Z}^{2,p}}+\big\|U^1-U^2\big\|^p_{\mathbb{U}^p}\le\mathcal{C}E\bigg[|\xi_1-\xi_2|^p+\Big(\int_0^T\big|g^1(t,Y^2_t,Z^2_t,U^2_t)-g^2(t,Y^2_t,Z^2_t,U^2_t)\big|dt\Big)^p\bigg]. \quad (2.5)$$

Moreover, we have the following strict comparison theorem for BSDEJs, which will play a key role in the paper.

Theorem 2.2. *Let $p \in (1,2]$, $\tau \in \mathcal{T}$ and $\gamma \in \mathcal{T}_\tau$. For $i=1,2$, let $\xi_i \in L^0(\mathcal{F}_T)$, let g^i be a $p-$generator, and let (Y^i,Z^i,U^i) be a solution of BSDEJ(ξ_i,g^i) such that $Y^1-Y^2 \in \mathbb{D}^p$ and that $Y^1_\gamma \le Y^2_\gamma$, $P-$a.s. For either $i=1$ or $i=2$, if g^i satisfies (A2), (A3), and if $g^1(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t) \le g^2(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)$, $dt\times dP-$a.s. on $]\!]\tau,\gamma[\![$, then it holds $P-$a.s. that $Y^1_t \le Y^2_t$ for any $t\in[\tau,\gamma]$. If one further has $Y^1_\tau = Y^2_\tau$, $P-$a.s., then*
(i) it holds $P-$a.s. that $Y^1_t = Y^2_t$ for any $t\in[\tau,\gamma]$;
(ii) it holds $dt\times dP-$a.s. on $]\!]\tau,\gamma]\!]$ that $(Z^1_t,U^1_t)=(Z^2_t,U^2_t)$ and $g^1(t,Y^i_t,Z^i_t,U^i_t)=g^2(t,Y^i_t,Z^i_t,U^i_t)$, $i=1,2$.

3 $g-$Evaluations with $\mathbb{L}^p$ Domains

The wellposedness result of BSDEs with jumps in $\mathbb{L}^p$ sense (Theorem 2.1) gives rise to a nonlinear expectation, called *$g-$evaluations* with $\mathbb{L}^p$ domains, which generalizes the one introduced in [77] and [80]:

Definition 3.1. *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1), (A2'), and let $\tau\in\mathcal{T}$, $\gamma\in\mathcal{T}_\tau$. Define $g-$evaluation $\mathcal{E}^g_{\tau,\gamma}: L^p(\mathcal{F}_\gamma)\to L^p(\mathcal{F}_\tau)$ by*

$$\mathcal{E}^g_{\tau,\gamma}[\xi] := Y^{\xi,g_\gamma}_\tau, \quad \forall\, \xi\in L^p(\mathcal{F}_\gamma).$$

If $\gamma=T$, we call $\mathcal{E}_g[\xi|\mathcal{F}_\tau] := \mathcal{E}^g_{\tau,T}[\xi]$ the (conditional) $g-$expectation of $\xi\in L^p(\mathcal{F}_T)$ at time τ. By Theorem 2.1, it holds for any $\xi\in L^p(\mathcal{F}_\gamma)$ that

$$\mathbf{1}_{\{\tau=\gamma\}}\mathcal{E}^g_{\tau,\gamma}[\xi] = \mathbf{1}_{\{\tau=\gamma\}} Y^{\xi,g_\gamma}_\tau = \mathbf{1}_{\{\tau=\gamma\}} Y^{\xi,g_\gamma}_\gamma = \mathbf{1}_{\{\tau=\gamma\}} Y^{\xi,g_\tau}_T = \mathbf{1}_{\{\tau=\gamma\}}\xi, \quad P-a.s. \tag{3.1}$$

Lemma 3.1. *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A2') and that $dt\times dP-$a.s.*

$$g(t,y,0,0)=0, \quad \forall\, y\in\mathbb{R}. \tag{3.2}$$

For any $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$, it holds for any $\xi\in L^p(\mathcal{F}_\gamma)$ that $\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}_g[\xi|\mathcal{F}_\tau]$, $P-$a.s. In particular, when $g\equiv 0$, the $g-$evaluation degenerates to the classic linear expectation: for any $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$, it holds for any $\xi\in L^p(\mathcal{F}_\gamma)$ that $\mathcal{E}^g_{\tau,\gamma}[\xi]=E[\xi|\mathcal{F}_\tau]$, $P-$a.s.

Let $p\in(1,2]$ and let g be a $p-$generator satisfying (A1) and (A2'). One can deduce from the uniqueness result and the comparison theorem of $\mathbb{L}^p-$solutions to BSDEJs (Theorem 2.1 and 2.2) as well as Lemma 3.1 that the $g-$evaluations with $\mathbb{L}^p$ domains possess the following basic properties (cf. [81]): Let $\tau\in\mathcal{T}$, $\gamma\in\mathcal{T}_\tau$ and $\xi\in L^p(\mathcal{F}_\gamma)$.
(g1) "Strict Monotonicity": If g further satisfies (A3), then for any $\eta\in L^p(\mathcal{F}_\gamma)$ with $\xi\le\eta$, $P-$a.s. one has $\mathcal{E}^g_{\tau,\gamma}[\xi]\le \mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s.; Moreover, if it further holds that $\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s., then $\xi=\eta$, $P-$a.s.
(g2) "Constant Preserving": Under (3.2), if ξ is $\mathcal{F}_\tau-$measurable, then $\mathcal{E}^g_{\tau,\gamma}[\xi]=\xi$, $P-$a.s.
(g3) "Time Consistency": For any $\zeta\in\mathcal{T}$ with $\tau\le\zeta\le\gamma$, $P-$a.s., it holds $P-$a.s. that $\mathcal{E}^g_{\tau,\zeta}\big[\mathcal{E}^g_{\zeta,\gamma}[\xi]\big]=\mathcal{E}^g_{\tau,\gamma}[\xi]$.
(g4) "Zero$-$One Law": For any $A\in\mathcal{F}_\tau$, we have $\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_A\xi]=\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\xi]$, $P-$a.s.; In addition, if $g(t,0,0,0)=0$, $dt\times dP-$a.s., then $\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_A\xi]=\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\xi]$, $P-$a.s.
(g5) "Translation Invariance": If g is independent of y, then $\mathcal{E}^g_{\tau,\gamma}[\xi+\eta]=\mathcal{E}^g_{\tau,\gamma}[\xi]+\eta$, $P-$a.s. for any $\eta\in L^p(\mathcal{F}_\tau)$.
(g6) "Convexity": If g is convex in (y,z,u), then $\mathcal{E}^g_{\tau,\gamma}[\alpha\xi+(1-\alpha)\eta]\le\alpha\mathcal{E}^g_{\tau,\gamma}[\xi]+(1-\alpha)\mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s. for any $\eta\in L^p(\mathcal{F}_\gamma)$ and $\alpha\in[0,1]$.
(g7) "Positive Homogeneity": If g is positively homogeneous in (y,z,u), then $\mathcal{E}^g_{\tau,\gamma}[\alpha\xi]=\alpha\mathcal{E}^g_{\tau,\gamma}[\xi]$, $P-$a.s. for any $\alpha\in[0,\infty)$.

Now, let us consider two specific $p-$generators satisfying (A1)$-$(A3) and their corresponding $g-$evaluations:

Example 3.1. *Given $p\in(1,2]$, let Ξ be a $p-$coefficient set. The functions*

$$\begin{aligned} g^\Xi(t,\omega,y,z,u) &:= \beta(t,\omega)|y|+\Lambda(t,\omega)|z|-\kappa_1\int_{\mathcal{X}} u^-(x)\nu(dx)+\kappa_2\int_{\mathcal{X}} u^+(x)\nu(dx),\\ \underline{g}^\Xi(t,\omega,y,z,u) &:= -g^\Xi(t,\omega,-y,-z,-u), \quad \forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu \end{aligned}$$

are two $p-$generators satisfying (A1)$-$(A3) with respect to the same coefficient set Ξ, where $u^\pm(x):=(u(x))^\pm$. Then $\mathcal{E}^\Xi_{\tau,\gamma}:=\mathcal{E}^{g^\Xi}_{\tau,\gamma}$ and $\underline{\mathcal{E}}^\Xi_{\tau,\gamma}:=\mathcal{E}^{\underline{g}^\Xi}_{\tau,\gamma}$, $\forall\,\tau\in\mathcal{T}$, $\forall\,\gamma\in\mathcal{T}_\tau$ are two $g-$evaluations with $\mathbb{L}^p$ domains.

In light of the comparison theorem for BSDEJs (Theorem 2.2), we can bound the variation of a $g-$evaluation by $g^{\Xi}-$evaluation and $\overline{g}^{\Xi}-$evaluation as follows.

Proposition 3.1. *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1)$-$(A3) with respect to some $p-$coefficient set Ξ. For any $\tau\in\mathcal{T}$, $\forall\,\gamma\in\mathcal{T}_\tau$ and $\xi,\eta\in L^p(\mathcal{F}_T)$, it holds $P-a.s.$ that $\overline{\mathcal{E}}^{\Xi}_{\tau,\gamma}[\xi-\eta]\le\mathcal{E}^g_{\tau,\gamma}[\xi]-\mathcal{E}^g_{\tau,\gamma}[\eta]\le\mathcal{E}^{\Xi}_{\tau,\gamma}[\xi-\eta]$.*

4 $g-$Martingales

Let g be a $p-$generator satisfying (A1) and (A2'). We can define martingales with respect to the $g-$evaluations that have $\mathbb{L}^p$ domains under jump filtration.

Definition 4.1. *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1) and (A2'). A real-valued, $\mathbf{F}-$adapted process X is called a $g-$submartingale $\big($resp. $g-$supermartingale or $g-$martingale$\big)$ if for any $0\le t\le s\le T$, $E[|X_s|^p]<\infty$ and $\mathcal{E}^g_{t,s}[X_s]\ge$ (resp. $\le$ or $=$) X_t, $P-a.s.$*

The $g-$martingales retain many classic properties such as "optional sampling", "upcrossing inequality" and "Doob-Meyer decomposition".

Let us start with the optional sampling theorem of $g-$martingales, which is important for the Doob-Meyer decomposition of $g-$martingales (Theorem 4.1).

Proposition 4.1. (*Optional Sampling*) *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1)$-$(A3). Let X be a $g-$submartingale (resp. $g-$supermartingale) with $E[X_*^p]<\infty$ and let $\tau\in\mathcal{T}$, $\gamma\in\mathcal{T}_\tau$. If X is right-continuous or if τ, γ are finitely valued, then $\mathcal{E}^g_{\tau,\gamma}[X_\gamma]\ge$ (resp. $\le$) X_τ, $P-a.s.$*

The proof of Proposition 4.1 depends on the following lemma.

Lemma 4.1. *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1) and (A2'). Let $\tau\in\mathcal{T}$ taking values in a finite set $\{0=t_1<\cdots<t_n=T\}$ with $n\ge2$. If $t_i\le t<s\le t_{i+1}$ for some $i\in\{1,\cdots n-1\}$, then for any $\xi\in L^p(\mathcal{F}_{\tau\wedge s})$*

$$\mathcal{E}^g_{\tau\wedge t,\tau\wedge s}[\xi]=\mathbf{1}_{\{\tau\le t_i\}}\xi+\mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathcal{E}^g_{t,s}[\xi],\quad P-a.s. \tag{4.1}$$

To present the upcrossing inequality of $g-$martingales, we recall the notion of *number of upcrossings*: Given a real-valued process X and two real numbers $a<b$, for any finite subset $\mathcal{D}=\{t_1<\cdots<t_m\}$ of $[0,T]$, we define the "number of upcrossings" $U_{\mathcal{D}}(a,b;X(\omega))$ of interval $[a,b]$ by the sample path $\{X_t(\omega)\}_{t\in\mathcal{D}}$ as follows: Set $m':=\lceil\frac{m}{2}\rceil$ and $\tau_0:=-1$. For $i=1,\cdots,m'$, we recursively define

$$\begin{aligned}\tau_{2i-1}(\omega)&:=\min\{t\in\mathcal{D}:\ t>\tau_{2i-2}(\omega),X_t(\omega)<a\}\wedge t_m\in\mathcal{T}\quad\text{and}\\ \tau_{2i}(\omega)&:=\min\{t\in\mathcal{D}:\ t>\tau_{2i-1}(\omega),X_t(\omega)>b\}\wedge t_m\in\mathcal{T},\end{aligned}\tag{4.2}$$

with the convention $\min\emptyset=\infty$. Then $U_{\mathcal{D}}(a,b;X(\omega))$ is set to be the largest integer i such that $\tau_{2i}(\omega)<t_m$. To wit, $U_{\mathcal{D}}\big(a,b;X(\omega)\big)=\sum\limits_{i=1}^{m'}\mathbf{1}_{\{\tau_{2i}(\omega)<t_m\}}$.

Proposition 4.2. (*Upcrossing Inequality*) *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A1)$-$(A3) with respect to some $p-$coefficient set Ξ, and let X be a $g-$supermartingale with $E[X_*^p]<\infty$. For any real numbers $a<b$ and any finite subset $\mathcal{D}=\{t_1<\cdots<t_m\}$ of $[0,T]$, the upcrossing number $U_{\mathcal{D}}(a,b;X)$ of interval $[a,b]$ satisfies*

$$E\Big[\ln\big(1+U_{\mathcal{D}}(a,b;X)\big)\Big]\le\ln\bigg\{\frac{e^{3\widehat{C}}}{b-a}\mathcal{E}^{\Xi}_{0,t_m}\Big[(X_{t_m}-a)^-+\int_0^{t_m}|g(s,0,0,0)|ds\Big]+\frac{|a|e^{3\widehat{C}}}{b-a}+1\bigg\}+\frac{1}{2}\widehat{C}+\big(\kappa_2-\ln(1+\kappa_1)\big)\nu(\mathcal{X})T.$$

The Doob-Meyer decomposition of $g-$martingales will play a crucial role for representing jump-filtration consistent nonlinear expectations with domain $L^p(\mathcal{F}_T)$ by $g-$expectations in our accompanying paper [95].

Theorem 4.1. (*Doob-Meyer Decomposition*) *Given $p\in(1,2]$, let g be a $p-$generator satisfying (A2). Assume that g also satisfies (A3) with $\int_0^T\Lambda_t^{\frac{2p}{2-p}}dt\in L^\infty(\mathcal{F}_T)$ if $p\in(1,2)$, or with $\Lambda\equiv\kappa_\Lambda\in[0,\infty)$ if $p=2$. If $X\in\mathbb{D}^p$ is* *a $g-$supermartingale (resp. $g-$submartingale) and if $E\int_0^T |g(t,0,0,0)|^p dt<\infty$, then there exist unique processes $(Z,U,K)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p\times\mathbb{K}^p$ such that $P-$a.s.*

$$X_t=X_T+\int_t^T g(s,X_s,Z_s,U_s)ds-\int_t^T Z_s dB_s-\int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)+K_T-K_t \ (resp. -K_T+K_t), \quad t\in[0,T]. \quad (4.3)$$

The proof of Theorem 4.1 relies on a *monotonic limit* theorem of jump diffusion processes over $\mathbb{D}^p$ (see Theorem A.1) as well as the following a priori $\mathbb{L}^p-$estimate to a special BSDEJ:

Proposition 4.3. *Given $p\in(1,2]$ and $\xi\in L^p(\mathcal{F}_T)$, let g be a $p-$generator and let X be a real-valued, $\mathbf{F}-$adapted càdlàg process with $X^+\in\mathbb{D}^p$. Let $(Y,Z,U,K)\in\mathbb{D}^p\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}\times\mathbb{K}^p$ satisfies that $P-$a.s.*

$$\begin{cases} Y_t=\xi+\displaystyle\int_t^T g(s,Y_s,Z_s,U_s)ds+K_T-K_t-\int_t^T Z_s\, dB_s-\int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T] \\ \displaystyle\int_0^T \mathbf{1}_{\{Y_{t-}>X_{t-}\}}dK_t=0. \end{cases} \quad (4.4)$$

If there exist three $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable processes $\mathfrak{f}$, β, Λ with $\int_0^T \mathfrak{f}_t dt\in L^p(\mathcal{F}_T)$, $\int_0^T\big(\beta_t^q\vee\Lambda_t^2\big)dt\in L^\infty(\mathcal{F}_T)$ such that

$$\big|g(t,Y_t,Z_t,U_t)\big|\le\mathfrak{f}_t+\beta_t\big(|Y_t|+\|U_t\|_{L^p_\nu}\big)+\Lambda_t|Z_t|, \quad dt\times dP-a.s., \quad (4.5)$$

then $(Z,U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ and

$$\|Y\|^p_{\mathbb{D}^p}+\big\|Z\big\|^p_{\mathbb{Z}^{2,p}}+\big\|U\big\|^p_{\mathbb{U}^p}+E[K_T^p]\le\mathcal{C}E\bigg[|\xi|^p+\Big(\int_0^T\mathfrak{f}_t dt\Big)^p+(X_*^+)^p\bigg]. \quad (4.6)$$

5 Other Fine Properties of $g-$Evaluations

In this section we will extend some fine properties of $g-$evaluations to the jump case with $\mathbb{L}^p$ domains. These properties have been explored for different reasons under Brownian filtration, and thus form an important ingredient of the nonlinear-expectation theory.

In light of Proposition 4.1 of ArXiv version of [94], we can first represent generators g as the limit of the difference quotients of the corresponding $g-$evaluations:

Proposition 5.1. *Given $p\in(1,2]$ and $\kappa_g>0$, let g be a $p-$generator satisfying (A1) and*

(A2") *there exists some $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable process β with $\int_0^T\beta_t^q dt\in L^\infty_+(\mathcal{F}_T)$ such that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$*

$$\big|g(t,\omega,y_1,z_1,u_1)-g(t,\omega,y_2,z_2,u_2)\big|\le\beta(t,\omega)|y_1-y_2|+\kappa_g\big(|z_1-z_2|+\|u_1-u_2\|_{L^p_\nu}\big), \ \ \forall\,(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu, \ i=1,2.$$

Let $(t,y,z,u)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$ such that

$$\lim_{s\to t+}g(s,y,z,u)=g(t,y,z,u), \ P-a.s. \ and \ E\Big[\sup_{s\in[t,t+\delta]}|g(s,y,0,0)|^p\Big]<\infty \ for \ some \ \delta=\delta(t,y)\in(0,T-t]. \quad (5.1)$$

Then it holds $P-$a.s. that $g(t,y,z,u)=\lim\limits_{\varepsilon\to0+}\frac{1}{\varepsilon}\big(\mathcal{E}^g_{t,t+\varepsilon}\big[y+V(t,t+\varepsilon,z,u)\big]-y\big)$, where $V(t,s,z,u):=z(B_s-B_t)+\int_{r\in(t,s]}\int_{\mathcal{X}}u(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)$, $\forall\,s\in(t,T]$.

A simple application of Proposition 5.1 gives rise to the following reverse to Theorem 2.2.

Theorem 5.1. *(A Reverse Comparison Theorem of BSDEJs) Given $p\in(1,2]$, $\kappa_g>0$ and $i=1,2$, let g_i be a $p-$generator satisfying (A1) and (A2"). Let $(t,y,z,u)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$ such that both g_1 and g_2 satisfy (5.1). If there exists $\delta(t,y)\in(0,T-t]$ such that $\mathcal{E}^{g_1}_{t,s}[\xi]\le\mathcal{E}^{g_2}_{t,s}[\xi]$, $P-$a.s. for any $s\in(t,t+\delta(t,y)]$ and $\xi\in L^p(\mathcal{F}_s)$, then it holds $P-$a.s. that $g_1(t,y,z,u)\le g_2(t,y,z,u)$.*

As other consequences of Proposition 5.1, we have the following reverse (g5)−(g7) properties of $g-$evaluations, which show that the convexity (resp. positive homogeneity) of g in (y,z,u) is equivalent to the convexity (resp. positive homogeneity) of $g-$evaluations and that the independence of g on $y-$variable is equivalent to the translation invariance of $g-$evaluations.

Proposition 5.2. *Given $p\in(1,2]$, assume that L^p_ν is a separable space. Let g be a $p-$generator such that for some $\kappa_g>0$, (A2) holds with $\Lambda_t=\kappa_g$, $\forall\, t\in[0,T]$ and that for $P-$a.s. $\omega\in\Omega$*

$$g(t,\omega,y,z,u) \text{ is right continuous in } t\in[0,T) \text{ for any } (y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu. \tag{5.2}$$

(1) If g also satisfies (A1), (A2") and if for any $(t,y)\in[0,T)\times\mathbb{R}$, $E\Big[\sup\limits_{s\in[t,t+\delta]}|g(s,y,0,0)|^p\Big]<\infty$ for certain $\delta=\delta(t,y)\in(0,T-t]$, then g is convex (resp. positively homogeneous) in (y,z,u) if and only if

$$\mathcal{E}^g_{t,s}[\cdot] \text{ is a convex (resp. positively homogeneous) operator on } L^p(\mathcal{F}_s) \text{ for any } 0\le t\le s\le T. \tag{5.3}$$

(2) If g also satisfies (3.2) and (A3), then g is independent of y if and only if

$$\mathcal{E}^g_{0,t}[\xi+c]=\mathcal{E}^g_{0,t}[\xi]+c, \quad \forall\, t\in[0,T],\ \forall\,\xi\in L^p(\mathcal{F}_t),\ \forall\, c\in\mathbb{R}. \tag{5.4}$$

What next is a Jensen's inequality of $g-$evaluations with $\mathbb{L}^p$ domains. Before discussing it, we recall some basic features of convex functions (see [85] for the related notions): Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function and $x\in\mathbb{R}$. One has

$$\begin{cases} f(\lambda x)\le\lambda f(x)+(1-\lambda)f(0), & \text{if } \lambda\in[0,1],\\ f(\lambda x)\ge\lambda f(x)+(1-\lambda)f(0), & \text{if } \lambda\in(0,1)^c=(-\infty,0]\cup[1,\infty). \end{cases} \tag{5.5}$$

Also, the subdifferential $\partial f(x)$ of f at x is the interval $\big[f'_-(x),f'_+(x)\big]$, where $f'_-(x)$ and $f'_+(x)$ are left−derivatives and right−derivatives of f at x respectively.

Theorem 5.2. *(Jensen's Inequality) Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function. Given $p\in(1,2]$, let g be a $p-$generator independent of y such that (A2), (A3) hold and that*

$$g(t,0,0)=0, \quad dt\times dP-a.s. \tag{5.6}$$

Given $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$, let $\xi\in L^p(\mathcal{F}_\gamma)$ such that $E\big[|f(\xi)|^p\big]<\infty$ and that $\partial f\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\cap(0,1)^c\neq\emptyset$, $P-$a.s. If

$$\text{it holds for } dt\times dP-a.s.\ (t,\omega)\in]\!]\tau,\gamma[\![\text{ that } g(t,\omega,z,u) \text{ is convex in } (z,u)\in\mathbb{R}^d\times L^p_\nu, \tag{5.7}$$

then $f\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\le\mathcal{E}^g_{\tau,\gamma}\big[f(\xi)\big]$, $P-$a.s.

6 Proofs

6.1 Proofs of Section 2

Proof of Remark 2.1 (1): We can deduce that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$

$$\begin{aligned} |g(t,\omega,y,z,u_1)-g(t,\omega,y,z,u_2)| &\le \kappa_2\int_{\mathcal{X}}|u_1(x)-u_2(x)|\nu(dx)\\ &\le \kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\|u_1-u_2\|_{L^p_\nu}, \quad \forall\,(y,z,u_1,u_2)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu. \end{aligned} \tag{6.1}$$

So one can take $\widetilde{\beta}_t:=\beta_t\vee\Big(\kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}\Big)$, $\forall\, t\in[0,T]$.
(2) Let $(Y,Z,U)\in\mathbb{D}^1\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$. Fix $n\in\mathbb{N}$. Define

$$\tau_n:=\inf\left\{t\in[0,T]:\int_0^t|g(s,0,0,0)|ds+\int_0^t\big|Z_s\big|^2ds+\int_0^t\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds>n\right\}\wedge T\in\mathcal{T}.$$

Hölder's inequality and (A2') imply that

$$E\Big[\int_0^{\tau_n}\big|g(t,Y_t,Z_t,U_t)\big|dt\Big]\le E\int_0^{\tau_n}\big(|g(t,0,0,0)|+\beta_t|Y_t|+\Lambda_t|Z_t|+\beta_t\|U_t\|_{L^p_\nu}\big)dt$$
$$\le n+E\bigg[Y_*\int_0^T(1\vee\beta^q_t)dt\bigg]+\Big(E\int_0^{\tau_n}\Lambda^2_t dt\Big)^{\frac12}\Big(E\int_0^{\tau_n}|Z_t|^2dt\Big)^{\frac12}+\Big(E\int_0^{\tau_n}\beta^q_t dt\Big)^{\frac1q}\Big(E\int_0^{\tau_n}\|U_t\|^p_{L^p_\nu}dt\Big)^{\frac1p}$$
$$\le n+\widehat{C}\|Y\|_{\mathbb{D}^1}+\big(n\widehat{C}\big)^{\frac12}+n^{\frac1p}\widehat{C}^{\frac1q}<\infty,$$

which shows that $\int_0^{\tau_n}\big|g(t,Y_t,Z_t,U_t)\big|dt<\infty$ except on a $P-$null set $\mathcal{N}_n$. Since $\int_0^T|g(t,0,0,0)|dt<\infty$, $P-$a.s. and since $(Z,U)\in\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, there exists a $P-$null set $\mathcal{N}_0$ such that for any $\omega\in\mathcal{N}^c_0$, $\tau_{\mathfrak{n}}(\omega)=T$ for some $\mathfrak{n}=\mathfrak{n}(\omega)\in\mathbb{N}$. Now, for any $\omega\in\underset{n\in\mathbb{N}\cup\{0\}}{\cap}\mathcal{N}^c_n$, one can deduce that $\int_0^T\big|g(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega))\big|dt=\int_0^{\tau_{\mathfrak{n}}(\omega)}\big|g(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega))\big|dt<\infty$.

(3) Let $\tau\in\mathcal{T}$. If g satisfies (A1), (A2) $\big($resp. (A2')$\big)$, then $\overline{g}$ and g_τ are clearly $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable functions satisfying (A1), (A2) $\big($resp. (A2')$\big)$.

Assume further that g satisfies (A3). Then g_τ satisfies (A3) with $\big(\mathfrak{h}_\tau(t,\omega,y,z,u_1,u_2)\big)(x):=\mathbf{1}_{\{t<\tau(\omega)\}}\big(\mathfrak{h}(t,\omega,y,z,u_1,u_2)\big)(x)\in[\kappa_1,\kappa_2],\ \forall\,(t,\omega,y,z,u_1,u_2,x)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu\times\mathcal{X}$. On the other hand, for any $(t,\omega,y,z,u_1,u_2)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu$, one can deduce that

$$\begin{aligned}\overline{g}(t,\omega,y,z,u_1)-\overline{g}(t,\omega,y,z,u_2)&=g(t,\omega,-y,-z,-u_2)-g(t,\omega,-y,-z,-u_1)\\&\le\int_{\mathcal{X}}\big(-u_2(x)+u_1(x)\big)\cdot\big(\mathfrak{h}(t,\omega,-y,-z,-u_2,-u_1)\big)(x)\nu(dx).\end{aligned}\tag{6.2}$$

So $\overline{g}$ satisfies (2.3) with $\big(\overline{\mathfrak{h}}(t,\omega,y,z,u_1,u_2)\big)(x)=\big(\mathfrak{h}(t,\omega,-y,-z,-u_2,-u_1)\big)(x)\in[\kappa_1,\kappa_2],\ \forall\,x\in\mathcal{X}$. The $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(L^p_\nu)\otimes\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurability of mapping $\mathfrak{h}$ easily implies that of mappings $\mathfrak{h}_\tau$ and $\overline{\mathfrak{h}}$. □

Proof of Theorem 2.1: Let $\xi\in L^p(\mathcal{F}_T)$. Under (A1) and (A2'), the wellposedness of BSDEJ (ξ,g) directly follows from Remark 4.1 of [94]. By (A2'), it holds $dt\times dP-$a.s. that

$$\big|g\big(t,Y^{\xi,g}_t,Z^{\xi,g}_t,U^{\xi,g}_t\big)\big|\le|g(t,0,0,0)|+\beta_t\Big(\big|Y^{\xi,g}_t\big|+\big\|U^{\xi,g}_t\big\|_{L^p(\nu)}\Big)+\Lambda_t\big|Z^{\xi,g}_t\big|.$$

So we see that the condition (3.1) of [94] holds for $(\xi_1,f_1,Y^1,Z^1,U^1)=\big(\xi,g,Y^{\xi,g},Z^{\xi,g},U^{\xi,g}\big)$, $(\xi_2,f_2,Y^2,Z^2,U^2)=(0,0,0,0,0)$ and $(g_t,\Phi_t,\Lambda_t,\Gamma_t,\Upsilon_t)=\big(|g(t,0,0,0)|,\beta_t,\Lambda_t,\beta_t,0\big)$, $t\in[0,T]$. Then Lemma 3.1 and Corollary 4.1 of [94] yields (2.4) and the remaining conclusion. □

Proof of Remark 2.2: Given $\xi\in L^p(\mathcal{F}_T)$, let $(Y,Z,U)\in\mathbb{S}^p$ be the unique solution of BSDEJ (ξ,g). Multiplying -1 to BSDEJ (ξ,g) shows that $(-Y,-Z,-U)\in\mathbb{S}^p$ solves BSDEJ $(-\xi,\overline{g})$. Then we see from Remark 2.1 (3) and Theorem 2.1 that $P\big\{Y^{\xi,g}_t=-Y^{-\xi,\overline{g}}_t,\ \forall\,t\in[0,T]\big\}=1$. □

Proof of Proposition 2.1: By (A2'), it holds $dt\times dP-$a.s. that

$$\big|g^1(t,Y^1_t,Z^1_t,U^1_t)-g^2(t,Y^2_t,Z^2_t,U^2_t)\big|\le\big|g^1(t,Y^2_t,Z^2_t,U^2_t)-g^2(t,Y^2_t,Z^2_t,U^2_t)\big|+\beta_t\big(|Y^1_t-Y^2_t|+\|U^1_t-U^2_t\|_{L^p(\nu)}\big)+\Lambda_t\big|Z^1_t-Z^2_t\big|,$$

which shows that the condition (3.1) of [94] holds for $(\xi_i,f_i,Y^i,Z^i,U^i)=(\xi_i,g^i,Y^i,Z^i,U^i)$, $i=1,2$ and $(g_t,\Phi_t,\Lambda_t,\Gamma_t,\Upsilon_t)=\big(|g^1(t,Y^2_t,Z^2_t,U^2_t)-g^2(t,Y^2_t,Z^2_t,U^2_t)|,\beta_t,\Lambda_t,\beta_t,0\big)$, $t\in[0,T]$. Then Lemma 3.1 of [94] gives rise to (2.5). □

Proof of Theorem 2.2: Without loss of generality, we suppose that g^1 satisfies (A2), (A3) and that

$$g^1(t,Y^2_t,Z^2_t,U^2_t)\le g^2(t,Y^2_t,Z^2_t,U^2_t),\quad dt\times dP-\text{a.s. on }]\!]\tau,\gamma[\![\,.\tag{6.3}$$

1) Set $(Y,Z,U):=(Y^1-Y^2,Z^1-Z^2,U^1-U^2)$ and consider the following $\mathbf{F}-$progressively measurable processes:

$$\begin{aligned}a_t&:=\mathbf{1}_{\{Y_t\ne0\}}\frac{g^1(t,Y^1_t,Z^1_t,U^1_t)-g^1(t,Y^2_t,Z^1_t,U^1_t)}{Y_t},\quad\Theta_t:=e^{\int_0^t a_s ds}>0,\quad\text{and}\\\mathfrak{b}_t&:=\mathbf{1}_{\{Z_t\ne0\}}\frac{g^1(t,Y^2_t,Z^1_t,U^1_t)-g^1(t,Y^2_t,Z^2_t,U^1_t)}{|Z_t|^2}Z_t,\quad\forall\,t\in[0,T].\end{aligned}$$

By (A2), it holds $dt\times dP-$a.s. that

$$|a_t| \le \beta_t \quad \text{and} \quad |\mathfrak{b}_t| \le \Lambda_t. \tag{6.4}$$

Define $\mathfrak{H}_t := \mathfrak{h}(t, Y_t^2, Z_t^2, U_t^1, U_t^2)$ and

$$M_t := \int_0^t \mathfrak{b}_s dB_s + \int_{(0,t]}\int_{\mathcal{X}} \mathfrak{H}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t\in[0,T]. \tag{6.5}$$

Since $E\Big[\big(\int_0^T \Lambda_t^2 dt\big)^{\frac{p}{2}} + \int_0^T \int_{\mathcal{X}} |\mathfrak{H}_t(x)|^p \nu(dx)dt\Big] \le \widehat{C}^{\frac{p}{2}} + \kappa_2^p \nu(\mathcal{X})T < \infty$, we see from (1.3) and (1.4) that M is a uniformly integrable martingale. For any $\zeta\in\mathcal{T}$, (6.4) and (A3) (ii) again yield that $\Delta M(\zeta(\omega),\omega) = \mathbf{1}_{\{\zeta(\omega)\in D_{\mathfrak{p}(\omega)}\}}\mathfrak{H}\big(\zeta(\omega),\omega,\mathfrak{p}(\zeta(\omega),\omega)\big)\in[\kappa_1,\kappa_2]$, $\forall\,\omega\in\Omega$, and that

$$\begin{aligned} E\big[[M,M]_T - [M,M]_\zeta|\mathcal{F}_\zeta\big] &= E\bigg[\int_\zeta^T |\mathfrak{b}_s|^2 ds + \int_{(\zeta,T]}\int_{\mathcal{X}} \big|\mathfrak{H}_s(x)\big|^2 N_{\mathfrak{p}}(ds,dx)\Big|\mathcal{F}_\zeta\bigg] \\ &= E\bigg[\int_\zeta^T |\mathfrak{b}_s|^2 ds + \int_\zeta^T \int_{\mathcal{X}} \big|\mathfrak{H}_s(x)\big|^2 \nu(dx)ds\Big|\mathcal{F}_\zeta\bigg] \le \widehat{C} + \kappa_2^2\nu(\mathcal{X})T < \infty. \end{aligned}$$

Thus, M is a BMO martingale. In virtue of [50], the Doléans-Dade exponential of M

$$\mathscr{E}_t(M) := e^{M_t - \frac{1}{2}\langle M^c\rangle_t} \prod_{0<s\le t}(1+\Delta M_s)e^{-\Delta M_s} > 0, \quad t\in[0,T] \tag{6.6}$$

is a uniformly integrable martingale, where M^c denote the continuous part of M.

Define a probability measure Q by $\frac{dQ}{dP} := \mathscr{E}_T(M)$, which satisfies $\frac{dQ}{dP}\big|_{\mathcal{F}_t} := \mathscr{E}_t(M)$, $\forall\, t\in[0,T]$. The Girsanov's Theorem (e.g. [47, 82]) then shows that $B_t^Q := B_t - \int_0^t \mathfrak{b}_s ds$, $t\in[0,T]$ is a $Q-$Brownian motion and $\widetilde{N}_{\mathfrak{p}}^Q(t,A) := \widetilde{N}_{\mathfrak{p}}(t,A) - \int_{(0,t]}\int_{\mathcal{X}} \mathfrak{H}_s(x)\nu(dx)ds$, $t\in[0,T]$, $A\in\mathcal{F}_{\mathcal{X}}$ is a $Q-$compensated Poisson random measure. By (6.4),

$$\Theta_* \le e^{\int_0^T \beta_t dt} \le e^{\widehat{C}}, \quad P-\text{a.s. and thus } Q-\text{a.s.} \tag{6.7}$$

Now, we fix $0\le t\le s\le T$ and $n\in\mathbb{N}$. Define $\gamma_n := \inf\big\{r\in[\tau,T]: \int_\tau^r |Z_{r'}|^2 dr' + \int_\tau^r \int_{\mathcal{X}} |U_{r'}(x)|^p \nu(dx)dr' > n\big\}\wedge\gamma \in \mathcal{T}_\tau$ and set $\zeta_n := (\tau\vee t)\wedge\gamma_n$, $\varsigma_n := (\tau\vee s)\wedge\gamma_n$. Applying Itô's formula to $\Theta_r Y_r$ on $[\zeta_n,\varsigma_n] = [\tau,\gamma_n]\cap[t,s]$ yields that

$$\Theta_{\zeta_n} Y_{\zeta_n} = \Theta_{\varsigma_n} Y_{\varsigma_n} + \int_{\zeta_n}^{\varsigma_n} \Theta_r\big(\mathfrak{g}_r - a_r Y_r\big)dr - \int_{\zeta_n}^{\varsigma_n} \Theta_r Z_r dB_r - \int_{(\zeta_n,\varsigma_n]}\int_{\mathcal{X}} \Theta_r U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx), \quad P-\text{a.s.} \tag{6.8}$$

where $\mathfrak{g}_r := g^1(r, Y_r^1, Z_r^1, U_r^1) - g^2(r, Y_r^2, Z_r^2, U_r^2)$. By (A3) (iii) and (6.3), it holds $dr\times dP-$a.s. on $]\!]\tau,\gamma[\![$ that

$$\mathfrak{g}_r = a_r Y_r + \mathfrak{b}_r Z_r + g^1(r, Y_r^2, Z_r^2, U_r^1) - g^2(r, Y_r^2, Z_r^2, U_r^2) \le a_r Y_r + \mathfrak{b}_r Z_r + \int_{\mathcal{X}} \mathfrak{H}_r(x)U_r(x)\nu(dx).$$

Plugging this inequality back into (6.8) leads to that

$$\Theta_{\zeta_n} Y_{\zeta_n} \le \Theta_{\varsigma_n} Y_{\varsigma_n} - \big(\mathcal{M}_s^n - \mathcal{M}_t^n + \mathscr{M}_s^n - \mathscr{M}_t^n\big), \quad P-\text{a.s. and thus } Q-\text{a.s.}, \tag{6.9}$$

where $\mathcal{M}_r^n := \int_0^r \mathbf{1}_{\{r'\in(\tau,\gamma_n]\}}\Theta_{r'} Z_{r'} dB_{r'}^Q$ and $\mathscr{M}_r^n := \int_{(0,r]}\int_{\mathcal{X}} \mathbf{1}_{\{r'\in(\tau,\gamma_n]\}}\Theta_{r'} U_{r'}(x)\widetilde{N}_{\mathfrak{p}}^Q(dr',dx)$, $r\in[0,T]$.

We can deduce from the Burkholder-Davis-Gundy inequality, (1.5) and (6.7) that

$$\begin{aligned} E_Q\bigg[\sup_{r\in[0,T]}\big|\mathcal{M}_r^n\big|^p + \sup_{r\in[0,T]}\big|\mathscr{M}_r^n\big|^p\bigg] &\le c_p E_Q\bigg[\Big(\int_\tau^{\gamma_n} |\Theta_r|^2 |Z_r|^2 dr\Big)^{\frac{p}{2}} + \Big(\int_{(\tau,\gamma_n]}\int_{\mathcal{X}} |\Theta_r|^2 |U_r(x)|^2 N_{\mathfrak{p}}(dr,dx)\Big)^{\frac{p}{2}}\bigg] \\ &\le c_p e^{p\widehat{C}} E_Q\bigg[\Big(\int_\tau^{\gamma_n} |Z_r|^2 dr\Big)^{\frac{p}{2}} + \int_\tau^{\gamma_n}\int_{\mathcal{X}} |U_r(x)|^p \nu(dx)dr\bigg] \le c_p e^{p\widehat{C}}\big(n^{\frac{p}{2}} + n\big) < \infty, \end{aligned}$$

thus $\mathcal{M}^n$ and $\mathscr{M}^n$ are two uniformly integrable $Q-$martingales. Taking conditional expectation $E_Q[\ |\mathcal{F}_{\zeta_n}]$ in (6.9) yields that $Q-$a.s.

$$\Theta_{\zeta_n} Y_{\zeta_n} \le E_Q\big[\Theta_{\varsigma_n} Y_{\varsigma_n}|\mathcal{F}_{\zeta_n}\big] = \mathbf{1}_{\{\gamma_n < (\tau\vee t)\wedge\gamma\}} E_Q\big[\Theta_{\varsigma_n} Y_{\varsigma_n}|\mathcal{F}_{\gamma_n}\big] + \mathbf{1}_{\{\gamma_n \ge (\tau\vee t)\wedge\gamma\}} E_Q\big[\Theta_{\varsigma_n} Y_{\varsigma_n}|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big] := \eta_1^n + \eta_2^n. \tag{6.10}$$

As $(Z,U)\in\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, one has $\int_0^T\big(|Z_r|^2+\|U_r\|^p_{L^p_\nu}\big)dr<\infty$, $P-$a.s. and thus $Q-$a.s. So for $Q-$a.s. $\omega\in\Omega$ there exists a $N_\omega\in\mathbb{N}$ such that

$$\text{for any } n\geq N_\omega,\ \gamma_n(\omega)=\gamma(\omega) \text{ and thus } \eta_1^n(\omega)=0. \tag{6.11}$$

It follows that $\lim\limits_{n\to\infty}\eta_1^n=0$, $Q-$a.s. On the other hand, the first equality in (6.11) also shows that $\lim\limits_{n\to\infty}\Theta_{\zeta_n}Y_{\zeta_n}=\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}$ and $\lim\limits_{n\to\infty}\Theta_{\varsigma_n}Y_{\varsigma_n}=\Theta_{(\tau\vee s)\wedge\gamma}Y_{(\tau\vee s)\wedge\gamma}$, $Q-$a.s. even though the process Y may not be left-continuous. For any $n\in\mathbb{N}$, (6.7) shows that $|\Theta_{\varsigma_n}Y_{\varsigma_n}|\leq e^{\widehat{C}}Y_*$, $P-$a.s. Since a slight extension of [83, Proposition A.1 (a)] shows that $E\big[\mathscr{E}^q_T(M)\big]<\infty$, we can deduce from Hölder's inequality that

$$E_Q[Y_*]=E[\mathscr{E}_T(M)Y_*]\leq\|\mathscr{E}_T(M)\|_{L^q(\mathcal{F}_T)}\|Y\|_{\mathbb{D}^p}<\infty. \tag{6.12}$$

As $n\to\infty$ in (6.10), a conditional-expectation version of dominated convergence theorem and (6.11) yield that

$$\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}\leq\lim_{n\to\infty}E_Q\big[\Theta_{\varsigma_n}Y_{\varsigma_n}|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big]=E_Q\big[\Theta_{(\tau\vee s)\wedge\gamma}Y_{(\tau\vee s)\wedge\gamma}|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big],\quad Q-\text{a.s.} \tag{6.13}$$

Taking $s=T$ shows that $\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}\leq E_Q\big[\Theta_\gamma Y_\gamma|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big]\leq 0$, $Q-$a.s. and thus $P-$a.s. It follows that $Y_{(\tau\vee t)\wedge\gamma}\leq 0$, $P-$a.s. By the right continuity of processes Y^1 and Y^2, it holds $P-$a.s. that $Y^1_t\leq Y^2_t$ for any $t\in[\tau,\gamma]$.
2) Suppose further that $Y^1_\tau=Y^2_\tau$, $P-$a.s. For any $t\in[0,T]$, as $\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}\leq 0$, $Q-$a.s., applying (6.13) with $(t,s)=(0,t)$ shows that $0=\Theta_\tau(Y^1_\tau-Y^2_\tau)\leq E_Q\big[\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}|\mathcal{F}_\tau\big]\leq 0$, $Q-$a.s. So $E_Q\big[\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}\big]=E_Q\Big[E_Q\big[\Theta_{(\tau\vee t)\wedge\gamma}Y_{(\tau\vee t)\wedge\gamma}|\mathcal{F}_\tau\big]\Big]=0$, which happens only if $Y_{(\tau\vee t)\wedge\gamma}=0$, $P-$a.s. since $\Theta_{(\tau\vee t)\wedge\gamma}>0$. Using the right continuity of Y^1, Y^2 again shows that $P\big\{Y^1_t=Y^2_t,\ \forall\,t\in[\tau,\gamma]\big\}=1$. It then follows from (1.1) that $P-$a.s.

$$\int_\tau^{(\tau\vee t)\wedge\gamma}\big(g^1(r,Y^1_r,Z^1_r,U^1_r)-g^2(r,Y^2_r,Z^2_r,U^2_r)\big)dr=\int_\tau^{(\tau\vee t)\wedge\gamma}(Z^1_r-Z^2_r)\,dB_r+\int_{(\tau,(\tau\vee t)\wedge\gamma]}\int_{\mathcal{X}}(U^1_r(x)-U^2_r(x))\widetilde{N}_{\mathfrak{p}}(dr,dx),\ t\in[0,T].$$

As continuous finite-variational processes, continuous martingales and discontinuous (jump) martingales are of different natures, any two of them only intersect at 0. So it holds $dt\times dP-$a.s. on $]\!]\tau,\gamma]\!]$ that $Z^1_t=Z^2_t$ and $U^1_t(x)=U^2_t(x)$, $dt\times dP\times\nu(dx)-$a.s. on on $]\!]\tau,\gamma]\!]\times\mathcal{X}$ Since the latter is equivalent to $U^1_t=U^2_t$, $dt\times dP-$a.s. on $]\!]\tau,\gamma]\!]$, we further see that $dt\times dP-$a.s. on $]\!]\tau,\gamma]\!]$

$$g^1(t,Y^1_t,Z^1_t,U^1_t)=g^2(t,Y^2_t,Z^2_t,U^2_t)\ \text{ and thus }\ g^1(t,Y^j_t,Z^j_t,U^j_t)=g^2(t,Y^j_t,Z^j_t,U^j_t),\quad j=1,2.$$ □

6.2 Proofs of Section 3

Proof of Lemma 3.1: Let $\gamma\in\mathcal{T}$. It suffices to show that the unique solution $(Y,Z,U)=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$ of BSDEJ (ξ,g_γ) is also the unique solution of BSDEJ (ξ,g) in $\mathbb{S}^p$: Set $M_t:=\int_0^t Z_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. Since $\big(Z_t,U_t\big)=\mathbf{1}_{\{t\leq\gamma\}}\big(Z_t,U_t\big)$, $dt\times dP-$a.s. by Theorem 2.1, we can deduce from (3.2) that $P-$a.s.

$$\begin{aligned}Y_t\ &=\ \xi+\int_t^T\mathbf{1}_{\{s<\gamma\}}g(s,Y_s,Z_s,U_s)ds-M_T+M_t=\xi+\int_t^T\mathbf{1}_{\{s\leq\gamma\}}g(s,Y_s,Z_s,U_s)ds-M_T+M_t\\ &=\ \xi+\int_t^T g\big(s,Y_s,\mathbf{1}_{\{s\leq\gamma\}}Z_s,\mathbf{1}_{\{s\leq\gamma\}}U_s\big)ds-M_T+M_t=\xi+\int_t^T g\big(s,Y_s,Z_s,U_s\big)ds-M_T+M_t,\quad t\in[0,T],\end{aligned}$$

which shows that (Y,Z,U) is the unique solution of BSDEJ (ξ,g). □

Proof of (g1)−(g7):
1) Let g satisfies (A3) and let $\eta\in L^p(\mathcal{F}_\gamma)$ with $\xi\leq\eta$, $P-$a.s. Applying Theorem 2.2 with $g^1=g^2=g_\gamma$ yields that $P\big\{Y^{\xi,g_\gamma}_t\leq Y^{\eta,g_\gamma}_t,\ \forall\,t\in[\tau,\gamma]\big\}=1$. In particular, $\mathcal{E}^g_{\tau,\gamma}[\xi]=Y^{\xi,g_\gamma}_\tau\leq Y^{\eta,g_\gamma}_\tau=\mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s.

Moreover, if it further holds that $Y^{\xi,g_\gamma}_\tau=\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}^g_{\tau,\gamma}[\eta]=Y^{\eta,g_\gamma}_\tau$, $P-$a.s., Theorem 2.2 again shows that $P\big\{Y^{\xi,g_\gamma}_t=Y^{\eta,g_\gamma}_t,\ \forall\,t\in[\tau,\gamma]\big\}=1$. Then Theorem 2.1 implies that $\xi=Y^{\xi,g_\gamma}_T=Y^{\xi,g_\gamma}_\gamma=Y^{\eta,g_\gamma}_\gamma=Y^{\eta,g_\gamma}_T=\eta$, $P-$a.s., proving (g1).
2) Let g satisfies (3.2). For any $\xi\in L^p(\mathcal{F}_\tau)\subset L^p(\mathcal{F}_\gamma)$, Lemma 3.1 and (3.1) imply that $\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}_g[\xi|\mathcal{F}_\tau]=\mathcal{E}^g_{\tau,\tau}[\xi]=\xi$, $P-$a.s., proving (g2).

3) Set $(Y,Z,U):=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$ and $(\mathcal{Y},\mathcal{Z},\mathcal{U}):=\big(Y^{\eta,g_\zeta},Z^{\eta,g_\zeta},U^{\eta,g_\zeta}\big)$ with $\eta:=\mathcal{E}^g_{\zeta,\gamma}[\xi]\in L^p(\mathcal{F}_\zeta)$. We define $\overline{Y}_t:=\mathbf{1}_{\{t<\zeta\}}\mathcal{Y}_t+\mathbf{1}_{\{t\ge\zeta\}}Y_t$ and $\big(\overline{Z}_t,\overline{U}_t\big):=\mathbf{1}_{\{t\le\zeta\}}(\mathcal{Z}_t,\mathcal{U}_t)+\mathbf{1}_{\{t>\zeta\}}(Z_t,U_t)$, $\forall\, t\in[0,T]$. One can deduce that $\big(\overline{Y},\overline{Z},\overline{U}\big)$ belong to $\mathbb{S}^p$ and that $P-$a.s.

$$\begin{aligned}Y_{\zeta\vee t} &= \xi+\int_{\zeta\vee t}^T g_\gamma(s,Y_s,Z_s,U_s)ds-\int_{\zeta\vee t}^T Z_s dB_s-\int_{(\zeta\vee t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \xi+\int_{\zeta\vee t}^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_{\zeta\vee t}^T \overline{Z}_s dB_s-\int_{(\zeta\vee t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].\end{aligned}\tag{6.14}$$

Let $t\in[0,T]$. Since Theorem 2.1 shows that $(\mathcal{Z}_s,\mathcal{U}_s)=\mathbf{1}_{\{s\le\zeta\}}(\mathcal{Z}_s,\mathcal{U}_s)=\mathbf{1}_{\{s\le\zeta\}}(\overline{Z}_s,\overline{U}_s)$, $ds\times dP-$a.s., taking $t=\zeta$ in (6.14) yields that

$$\begin{aligned}\mathcal{Y}_{\zeta\wedge t} &= \eta+\int_{\zeta\wedge t}^T \mathbf{1}_{\{s<\zeta\}}g(s,\mathcal{Y}_s,\mathcal{Z}_s,\mathcal{U}_s)ds-\int_{\zeta\wedge t}^T \mathcal{Z}_s dB_s-\int_{(\zeta\wedge t,T]}\int_{\mathcal{X}} \mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= Y_\zeta+\int_{\zeta\wedge t}^\zeta \mathbf{1}_{\{s<\gamma\}}g(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_{\zeta\wedge t}^\zeta \overline{Z}_s dB_s-\int_{(\zeta\wedge t,\zeta]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \xi+\int_{\zeta\wedge t}^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_{\zeta\wedge t}^T \overline{Z}_s dB_s-\int_{(\zeta\wedge t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P-\text{a.s.}\end{aligned}\tag{6.15}$$

Multiplying $\mathbf{1}_{\{t\ge\zeta\}}$ to (6.14) and multiplying $\mathbf{1}_{\{t<\zeta\}}$ to (6.15) leads to that

$$\overline{Y}_t=\mathbf{1}_{\{t<\zeta\}}\mathcal{Y}_t+\mathbf{1}_{\{t\ge\zeta\}}Y_t=\xi+\int_t^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P-\text{a.s.}$$

Then we see from the right continuity of process $\overline{Y}$ that $P-$a.s.

$$\overline{Y}_t=\xi+\int_t^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

So $\big(\overline{Y},\overline{Z},\overline{U}\big)$ solves BSDEJ (ξ,g_γ). By uniqueness, one has $P\big\{\overline{Y}_t=Y_t,\ t\in[0,T]\big\}=1$. In particular, applying (3.1) with $(\tau,\gamma,\xi)=\big(\tau,\zeta,\mathcal{E}^g_{\zeta,\gamma}[\xi]\big)$ yields that $\mathcal{E}^g_{\tau,\gamma}[\xi]=Y_\tau=\overline{Y}_\tau=\mathbf{1}_{\{\tau<\zeta\}}\mathcal{Y}_\tau+\mathbf{1}_{\{\tau=\zeta\}}Y_\zeta=\mathbf{1}_{\{\tau<\zeta\}}\mathcal{E}^g_{\tau,\zeta}[\eta]+\mathbf{1}_{\{\tau=\zeta\}}\mathcal{E}^g_{\zeta,\gamma}[\xi]=\mathcal{E}^g_{\tau,\zeta}\big[\mathcal{E}^g_{\zeta,\gamma}[\xi]\big]$, $P-$a.s. Hence, (g3) holds.

4a) Fix $A\in\mathcal{F}_\tau$. Set $(Y^1,Z^1,U^1):=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$ and $(Y^2,Z^2,U^2):=\big(Y^{\mathbf{1}_A\xi,g_\gamma},Z^{\mathbf{1}_A\xi,g_\gamma},U^{\mathbf{1}_A\xi,g_\gamma}\big)$. Given $i=1,2$, applying Corollary 2.1 with $\xi=\mathbf{1}_A Y^i_\tau\in L^p(\mathcal{F}_\tau)$ shows that there exists a unique pair $\big(\mathcal{Z}^i,\mathcal{U}^i\big)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P-$a.s., $\mathcal{Y}^i_t:=E\big[\mathbf{1}_A Y^i_\tau|\mathcal{F}_t\big]=E\big[\mathbf{1}_A Y^i_\tau\big]+\int_0^t\mathcal{Z}^i_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}\mathcal{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. We define $\overline{Y}^i_t:=\mathbf{1}_{\{t<\tau\}}\mathcal{Y}^i_t+\mathbf{1}_{\{t\ge\tau\}}\mathbf{1}_A Y^i_t$, $\big(\overline{Z}^i_t,\overline{U}^i_t\big):=\mathbf{1}_{\{t\le\tau\}}(\mathcal{Z}^i_t,\mathcal{U}^i_t)+\mathbf{1}_{\{t>\tau\}}\mathbf{1}_A(Z^i_t,U^i_t)$, $\forall\, t\in[0,T]$, and can deduce that $\big(\overline{Y}^i,\overline{Z}^i,\overline{U}^i\big)$ belong to $\mathbb{S}^p$.

For any $t\in[0,T]$, since $\{\tau\le t\}\in\mathcal{F}_\tau$, we see that $A\cap\{\tau\le t\}\in\mathcal{F}_t$ and thus $A\cap\{\tau\le t<\gamma\}\in\mathcal{F}_t$. Then $\big\{\mathbf{1}_A\mathbf{1}_{\{\tau\le t<\gamma\}}\big\}_{t\in[0,T]}$ is an $\mathbf{F}-$adapted càdlàg process. It follows that

$$g_A(t,\omega,y,z,u):=\mathbf{1}_{\{\omega\in A\}}\mathbf{1}_{\{\tau(\omega)\le t<\gamma(\omega)\}}g(t,\omega,y,z,u),\quad \forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$$

is a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable mapping that satisfies (A1) and (A2').

Given $t\in[0,T]$, multiplying $\mathbf{1}_A$ to the BSDEJ (ξ,g_γ) over period $[\tau\vee t,T]$ yields that

$$\begin{aligned}\mathbf{1}_A Y^1_{\tau\vee t} &= \mathbf{1}_A\xi+\int_{\tau\vee t}^T \mathbf{1}_A\mathbf{1}_{\{s<\gamma\}}g(s,Y^1_s,Z^1_s,U^1_s)ds-\int_{\tau\vee t}^T \mathbf{1}_A Z^1_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\mathbf{1}_A U^1_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A\xi+\int_{\tau\vee t}^T \mathbf{1}_A\mathbf{1}_{\{\tau\le s<\gamma\}}g(s,\mathbf{1}_A Y^1_s,\mathbf{1}_A Z^1_s,\mathbf{1}_A U^1_s)ds-\int_{\tau\vee t}^T \overline{Z}^1_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\overline{U}^1_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A\xi+\int_{\tau\vee t}^T g_A\big(s,\overline{Y}^1_s,\overline{Z}^1_s,\overline{U}^1_s\big)ds-\int_{\tau\vee t}^T \overline{Z}^1_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\overline{U}^1_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P-\text{a.s.}\end{aligned}\tag{6.16}$$

Similarly, multiplying $\mathbf{1}_A$ to the BSDEJ $(\mathbf{1}_A\xi, g_\gamma)$ over period $[\tau\vee t, T]$ yields that

$$\mathbf{1}_A Y^2_{\tau\vee t} = \mathbf{1}_A\xi + \int_{\tau\vee t}^T g_A\big(s,\overline{Y}^2_s,\overline{Z}^2_s,\overline{U}^2_s\big)ds - \int_{\tau\vee t}^T \overline{Z}^2_s dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \overline{U}^2_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P\text{-a.s.} \tag{6.17}$$

Fix $i=1,2$. The right continuity of process Y^i, (6.16) and (6.17) shows that P-a.s.

$$\mathbf{1}_A Y^i_{\tau\vee t} = \mathbf{1}_A\xi + \int_{\tau\vee t}^T g_A\big(s,\overline{Y}^i_s,\overline{Z}^i_s,\overline{U}^i_s\big)ds - \int_{\tau\vee t}^T \overline{Z}^i_s dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \overline{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T]. \tag{6.18}$$

Let $t\in[0,T]$. Since $\mathcal{Y}^i_\tau = E\big[\mathbf{1}_A Y^i_\tau|\mathcal{F}_\tau\big] = \mathbf{1}_A Y^i_\tau$, P-a.s. taking $t=\tau$ in (6.18) yields that

$$\begin{aligned}\mathcal{Y}^i_{\tau\wedge t} &= \mathcal{Y}^i_\tau - \int_{\tau\wedge t}^\tau \mathcal{Z}^i_s dB_s - \int_{(\tau\wedge t,\tau]}\int_{\mathcal{X}} \mathcal{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) = \mathbf{1}_A Y^i_\tau - \int_{\tau\wedge t}^\tau \overline{Z}^i_s dB_s - \int_{(\tau\wedge t,\tau]}\int_{\mathcal{X}} \overline{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A\xi + \int_{\tau\wedge t}^T g_A(s,\overline{Y}^i_s,\overline{Z}^i_s,\overline{U}^i_s)ds - \int_{\tau\wedge t}^T \overline{Z}^i_s dB_s - \int_{(\tau\wedge t,T]}\int_{\mathcal{X}} \overline{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P\text{-a.s.}\end{aligned} \tag{6.19}$$

Multiplying $\mathbf{1}_{\{t\geq\tau\}}$ to (6.18) and multiplying $\mathbf{1}_{\{t<\tau\}}$ to (6.19) leads to that

$$\overline{Y}^i_t = \mathbf{1}_{\{t<\tau\}}\mathcal{Y}^i_t + \mathbf{1}_{\{t\geq\tau\}}\mathbf{1}_A Y^i_t = \mathbf{1}_A\xi + \int_t^T g_A(s,\overline{Y}^i_s,\overline{Z}^i_s,\overline{U}^i_s)ds - \int_t^T \overline{Z}^i_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} \overline{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P\text{-a.s.}$$

The right continuity of process $\overline{Y}^i$ then implies that P-a.s.

$$\overline{Y}^i_t = \mathbf{1}_A\xi + \int_t^T g_A(s,\overline{Y}^i_s,\overline{Z}^i_s,\overline{U}^i_s)ds - \int_t^T \overline{Z}^i_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} \overline{U}^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T].$$

Thus both $\big(\overline{Y}^1,\overline{Z}^1,\overline{U}^1\big)$ and $\big(\overline{Y}^2,\overline{Z}^2,\overline{U}^2\big)$ solve BSDEJ $(\mathbf{1}_A\xi, g_A)$. By uniqueness, one has $P\big\{\overline{Y}^1_t = \overline{Y}^2_t,\ t\in[0,T]\big\}=1$. It follows that $\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\xi] = \mathbf{1}_A Y^1_\tau = \overline{Y}^1_\tau = \overline{Y}^2_\tau = \mathbf{1}_A Y^2_\tau = \mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_A\xi]$, P-a.s.

4b) Next, suppose that $g(t,0,0,0)=0$, $dt\times dP$-a.s. Set $(Y,Z,U) := \big(Y^{\xi,g_\gamma}, Z^{\xi,g_\gamma}, U^{\xi,g_\gamma}\big)$. Since $\eta := \mathbf{1}_A Y_\tau \in L^p(\mathcal{F}_\tau)$, Theorem 2.1 shows that the BSDEJ (η, g_τ) admits a unique solution $(\mathcal{Y},\mathcal{Z},\mathcal{U})\in\mathbb{S}^p$. We define $\overline{Y}_t := \mathbf{1}_{\{t<\tau\}}\mathcal{Y}_t + \mathbf{1}_{\{t\geq\tau\}}\mathbf{1}_A Y_t$, $\big(\overline{Z}_t,\overline{U}_t\big) := \mathbf{1}_{\{t\leq\tau\}}(\mathcal{Z}_t,\mathcal{U}_t) + \mathbf{1}_{\{t>\tau\}}\mathbf{1}_A(Z_t,U_t)$, $\forall\, t\in[0,T]$. Like $\big(\overline{Y}^i,\overline{Z}^i,\overline{U}^i\big)$, the processes $\big(\overline{Y},\overline{Z},\overline{U}\big)$ also belong to $\mathbb{S}^p$.

Given $t\in[0,T]$, similar to (6.16), multiplying $\mathbf{1}_A$ to the BSDEJ (ξ, g_γ) over period $[\tau\vee t, T]$ again yields that

$$\begin{aligned}\mathbf{1}_A Y_{\tau\vee t} &= \mathbf{1}_A\xi + \int_{\tau\vee t}^T \mathbf{1}_A g_\gamma(s,Y_s,Z_s,U_s)ds - \int_{\tau\vee t}^T \mathbf{1}_A Z_s dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \mathbf{1}_A U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A\xi + \int_{\tau\vee t}^T g_\gamma(s,\mathbf{1}_A Y_s,\mathbf{1}_A Z_s,\mathbf{1}_A U_s)ds - \int_{\tau\vee t}^T \overline{Z}_s dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A\xi + \int_{\tau\vee t}^T g_\gamma\big(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s\big)ds - \int_{\tau\vee t}^T \overline{Z}_s dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P\text{-a.s.}\end{aligned}$$

By the right continuity of process Y, it holds P-a.s. that

$$\mathbf{1}_A Y_{\tau\vee t} = \mathbf{1}_A\xi + \int_{\tau\vee t}^T g_\gamma\big(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s\big)ds - \int_{\tau\vee t}^T \overline{Z}_s dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T]. \tag{6.20}$$

Let $t\in[0,T]$. Taking $t=\tau$ in (6.20) and using an analogy to (6.15) yield that

$$\begin{aligned}\mathcal{Y}_{\tau\wedge t} &= \eta + \int_{\tau\wedge t}^T \mathbf{1}_{\{s<\tau\}} g(s,\mathcal{Y}_s,\mathcal{Z}_s,\mathcal{U}_s)ds - \int_{\tau\wedge t}^T \mathcal{Z}_s dB_s - \int_{(\tau\wedge t,T]}\int_{\mathcal{X}} \mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A Y_\tau + \int_{\tau\wedge t}^\tau \mathbf{1}_{\{s<\gamma\}} g(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds - \int_{\tau\wedge t}^\tau \overline{Z}_s dB_s - \int_{(\tau\wedge t,\tau]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \mathbf{1}_A\xi + \int_{\tau\wedge t}^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds - \int_{\tau\wedge t}^T \overline{Z}_s dB_s - \int_{(\tau\wedge t,T]}\int_{\mathcal{X}} \overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P\text{-a.s.}\end{aligned} \tag{6.21}$$

Multiplying $\mathbf{1}_{\{t\ge\tau\}}$ to (6.20) and multiplying $\mathbf{1}_{\{t<\tau\}}$ to (6.21) leads to that

$$\overline{Y}_t = \mathbf{1}_{\{t<\tau\}}\mathcal{Y}_t+\mathbf{1}_{\{t\ge\tau\}}\mathbf{1}_A Y_t = \mathbf{1}_A\xi+\int_t^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P\text{−a.s.}$$

The right continuity of process $\overline{Y}$ then shows that P−a.s.

$$\overline{Y}_t=\mathbf{1}_A\xi+\int_t^T g_\gamma(s,\overline{Y}_s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

So $\big(\overline{Y},\overline{Z},\overline{U}\big)$ solves BSDEJ $(\mathbf{1}_A\xi,g_\gamma)$. By uniqueness, one has $P\big\{\overline{Y}_t = Y_t^{\mathbf{1}_A\xi,g_\gamma},\ t\in[0,T]\big\}=1$. It follows that $\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_A\xi]=Y_\tau^{\mathbf{1}_A\xi,g_\gamma}=\overline{Y}_\tau=\mathbf{1}_A Y_\tau=\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\xi]$, P−a.s., proving (g4).

5) Assume that g is independent of y. Set $(Y,Z,U):=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$ and let $\eta\in L^p(\mathcal{F}_\tau)$. In light of Theorem 2.1, the BSDEJ $(Y_\tau+\eta,g_\tau)$ admits a unique solution $(\mathcal{Y},\mathcal{Z},\mathcal{U})\in\mathbb{S}^p$. We define $\overline{Y}_t:=\mathbf{1}_{\{t<\tau\}}\mathcal{Y}_t+\mathbf{1}_{\{t\ge\tau\}}(Y_t+\eta)$ and $\big(\overline{Z}_t,\overline{U}_t\big):=\mathbf{1}_{\{t\le\tau\}}(\mathcal{Z}_t,\mathcal{U}_t)+\mathbf{1}_{\{t>\tau\}}(Z_t,U_t)$, $\forall\, t\in[0,T]$. One can deduce that $\big(\overline{Y},\overline{Z},\overline{U}\big)$ belong to $\mathbb{S}^p$.

Given $t\in[0,T]$, adding η to the BSDEJ (ξ,g_γ) over period $[\tau\vee t,T]$ again yields that

$$\begin{aligned}Y_{\tau\vee t}+\eta \;&=\; \xi+\eta+\int_{\tau\vee t}^T g_\gamma(s,Z_s,U_s)ds-\int_{\tau\vee t}^T Z_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &=\; \xi+\eta+\int_{\tau\vee t}^T g_\gamma\big(s,\overline{Z}_s,\overline{U}_s\big)ds-\int_{\tau\vee t}^T \overline{Z}_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P\text{−a.s.}\end{aligned}$$

By the right continuity of process Y, it holds P−a.s. that

$$Y_{\tau\vee t}+\eta=\xi+\eta+\int_{\tau\vee t}^T g_\gamma\big(s,\overline{Z}_s,\overline{U}_s\big)ds-\int_{\tau\vee t}^T \overline{Z}_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T]. \tag{6.22}$$

Let $t\in[0,T]$. Since Theorem 2.1 shows that $(\mathcal{Z}_s,\mathcal{U}_s)=\mathbf{1}_{\{s\le\tau\}}(\mathcal{Z}_s,\mathcal{U}_s)=\mathbf{1}_{\{s\le\tau\}}(\overline{Z}_s,\overline{U}_s)$, $ds\times dP$−a.s., taking $t=\tau$ in (6.22) yields that

$$\begin{aligned}\mathcal{Y}_{\tau\wedge t} \;&=\; Y_\tau+\eta+\int_{\tau\wedge t}^T \mathbf{1}_{\{s<\tau\}}g(s,\mathcal{Z}_s,\mathcal{U}_s)ds-\int_{\tau\wedge t}^T \mathcal{Z}_s dB_s-\int_{(\tau\wedge t,T]}\int_{\mathcal{X}}\mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= Y_\tau+\eta+\int_{\tau\wedge t}^\tau \mathbf{1}_{\{s<\gamma\}}g(s,\overline{Z}_s,\overline{U}_s)ds-\int_{\tau\wedge t}^\tau \overline{Z}_s dB_s-\int_{(\tau\wedge t,\tau]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \xi+\eta+\int_{\tau\wedge t}^T g_\gamma(s,\overline{Z}_s,\overline{U}_s)ds-\int_{\tau\wedge t}^T \overline{Z}_s dB_s-\int_{(\tau\wedge t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P\text{−a.s.}\end{aligned} \tag{6.23}$$

Multiplying $\mathbf{1}_{\{t\ge\tau\}}$ to (6.22) and multiplying $\mathbf{1}_{\{t<\tau\}}$ to (6.23) leads to that

$$\overline{Y}_t = \mathbf{1}_{\{t<\tau\}}\mathcal{Y}_t+\mathbf{1}_{\{t\ge\tau\}}(Y_t+\eta) = \xi+\eta+\int_t^T g_\gamma(s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P\text{−a.s.}$$

The right continuity of process $\overline{Y}$ then shows that P−a.s.

$$\overline{Y}_t=\xi+\eta+\int_t^T g_\gamma(s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

So $\big(\overline{Y},\overline{Z},\overline{U}\big)$ solves BSDEJ $(\xi+\eta,g_\gamma)$. By uniqueness, one has $P\big\{\overline{Y}_t = Y_t^{\xi+\eta,g_\gamma},\ t\in[0,T]\big\}=1$. It follows that $\mathcal{E}^g_{\tau,\gamma}[\xi+\eta]=Y_\tau^{\xi+\eta,g_\gamma}=\overline{Y}_\tau=Y_\tau+\eta=\mathcal{E}^g_{\tau,\gamma}[\xi]+\eta$, P−a.s. Therefore, (g5) holds.

6) Assume that g is convex in (y,z,u) and let $\eta\in L^p(\mathcal{F}_\gamma)$, $\alpha\in[0,1]$. We set $(Y^1,Z^1,U^1):=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$, $(Y^2,Z^2,U^2):=\big(Y^{\eta,g_\gamma},Z^{\eta,g_\gamma},U^{\eta,g_\gamma}\big)$ and $\big(\overline{Y},\overline{Z},\overline{U}\big):=\big(\alpha Y^1+(1-\alpha)Y^2,\alpha Z^1+(1-\alpha)Z^2,\alpha U^1+(1-\alpha)U^2\big)$. As $\mathfrak{g}_t:=\alpha g_\gamma(t,Y^1_t,Z^1_t,U^1_t)+(1-\alpha)g_\gamma(t,Y^2_t,Z^2_t,U^2_t)$, $t\in[0,T]$ is an $\mathbf{F}$−progressively measurable process, one can regard it as a special $p-$generator. It holds $P-$a.s. that

$$\begin{aligned}\overline{Y}_t &= \alpha Y^1_t+(1-\alpha)Y^2_t = \alpha\xi+(1-\alpha)\eta+\int_t^T \big(\alpha g_\gamma(s,Y^1_s,Z^1_s,U^1_s)+(1-\alpha)g_\gamma(s,Y^2_s,Z^2_s,U^2_s)\big)ds\\ &\quad -\int_t^T\big(\alpha Z^1_s+(1-\alpha)Z^2_s\big)dB_s-\int_{(t,T]}\int_{\mathcal{X}}\big(\alpha U^1_s(x)+(1-\alpha)U^2_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \alpha\xi+(1-\alpha)\eta+\int_t^T \mathfrak{g}_s ds-\int_t^T \overline{Z}_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T].\end{aligned}$$

Since the convexity of g in (y,z,u) shows that $P\big\{g_\gamma\big(t,\overline{Y}_t,\overline{Z}_t,\overline{U}_t\big)\le \mathfrak{g}_t,\ \forall\, t\in(0,T)\big\}=1$, an application of Theorem 2.2 with $(g^1,Y^1,Z^1,U^1)=\big(g_\gamma,Y^{\alpha\xi+(1-\alpha)\eta,g_\gamma},Z^{\alpha\xi+(1-\alpha)\eta,g_\gamma},U^{\alpha\xi+(1-\alpha)\eta,g_\gamma}\big)$ and $(g^2,Y^2,Z^2,U^2)=\big(\mathfrak{g},\overline{Y},\overline{Z},\overline{U}\big)$ yields that $P\big\{Y^{\alpha\xi+(1-\alpha)\eta,g_\gamma}_t\le\overline{Y}_t,\ \forall\, t\in[\tau,\gamma]\big\}=1$. Hence, we obtain $\mathcal{E}^g_{\tau,\gamma}\big[\alpha\xi+(1-\alpha)\eta\big]=Y^{\alpha\xi+(1-\alpha)\eta,g_\gamma}_\tau\le\overline{Y}_\tau=\alpha\mathcal{E}^g_{\tau,\gamma}[\xi]+(1-\alpha)\mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s.

7) Next, assume that g is positively homogeneous in (y,z,u). Let $\widetilde{\alpha}\in[0,\infty)$ and set $(Y,Z,U):=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$. It holds $P-$a.s. that

$$\begin{aligned}\widetilde{\alpha}Y_t &= \widetilde{\alpha}\xi+\int_t^T\widetilde{\alpha}g_\gamma(s,Y_s,Z_s,U_s)ds-\int_t^T\widetilde{\alpha}Z_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\widetilde{\alpha}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \widetilde{\alpha}\xi+\int_t^T g_\gamma(s,\widetilde{\alpha}Y_s,\widetilde{\alpha}Z_s,\widetilde{\alpha}U_s)ds-\int_t^T\widetilde{\alpha}Z_s dB_s-\int_{(t,T]}\int_{\mathcal{X}}\widetilde{\alpha}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T],\end{aligned}$$

which shows that $(\widetilde{\alpha}Y,\widetilde{\alpha}Z,\widetilde{\alpha}U)\in\mathbb{S}^p$ solves BSDEJ $(\widetilde{\alpha}\xi,g_\gamma)$. Thus, $P\big\{Y^{\widetilde{\alpha}\xi,g_\gamma}_t=\widetilde{\alpha}Y_t,\ t\in[0,T]\big\}=1$. In particular, $\mathcal{E}^g_{\tau,\gamma}[\widetilde{\alpha}\xi]=Y^{\widetilde{\alpha}\xi,g_\gamma}_\tau=\widetilde{\alpha}Y_\tau=\widetilde{\alpha}\mathcal{E}^g_{\tau,\gamma}[\xi]$, $P-$a.s. □

Proof of Example 3.1: 1) Since (1.6) and Hölder's inequality imply that

$$\int_{\mathcal{X}}\big|u^\pm_1(x)-u^\pm_2(x)\big|\nu(dx)\le\int_{\mathcal{X}}\big|u_1(x)-u_2(x)\big|\nu(dx)\le\big(\nu(\mathcal{X})\big)^{\frac1q}\|u_1-u_2\|_{L^p_\nu}, \quad \forall\, u_1,u_2\in L^p_\nu,$$

we see that $u\to\int_{\mathcal{X}}u^\pm(x)\nu(dx)$ is a continuous function on L^p_ν. It follows that g^Ξ and $\overline{g}^\Xi$ are two $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable mappings. Clearly, g^Ξ and $\overline{g}^\Xi$ satisfy (A2) with coefficients (β,Λ), and $\overline{g}^\Xi(\cdot,0,0,0)\equiv g^\Xi(\cdot,0,0,0)\equiv0$.

2) To verify (A3) for g^Ξ, we let $(t,\omega,y,z,u_1,u_2)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu$. As

$$\begin{aligned}g^\Xi(t,\omega,y,z,u_1)-g^\Xi(t,\omega,y,z,u_2) &= -\kappa_1\int_{\mathcal{X}}\big(u^-_1(x)-u^-_2(x)\big)\nu(dx)+\kappa_2\int_{\mathcal{X}}\big(u^+_1(x)-u^+_2(x)\big)\nu(dx)\\ &= \int_{\mathcal{X}}\big[\kappa_1\big(u_1(x)-u_2(x)\big)+(\kappa_2-\kappa_1)\big(u^+_1(x)-u^+_2(x)\big)\big]\nu(dx), \qquad (6.24)\end{aligned}$$

g^Ξ satisfies (2.3) with $\big(\mathfrak{h}_\kappa(t,\omega,y,z,u_1,u_2)\big)(x):=\kappa_1+\mathbf{1}_{\{u_1(x)\neq u_2(x)\}}(\kappa_2-\kappa_1)\frac{u^+_1(x)-u^+_2(x)}{u_1(x)-u_2(x)}$, $\forall\, x\in\mathcal{X}$. Clearly, $\mathfrak{h}_\kappa(t,\omega,y,z,u_1,u_2)$ is a real-valued, $\mathcal{F}_{\mathcal{X}}-$measurable function. Since

$$a^+\le b^+ \quad \text{for any } a,b\in\mathbb{R} \text{ with } a\le b, \qquad (6.25)$$

we can deduce from (1.6) that $\kappa_1\le\big(\mathfrak{h}_\kappa(t,\omega,y,z,u_1,u_2)\big)(x)=\kappa_1+\mathbf{1}_{\{u_1(x)\neq u_2(x)\}}(\kappa_2-\kappa_1)\frac{|u^+_1(x)-u^+_2(x)|}{|u_1(x)-u_2(x)|}\le\kappa_2$, which implies that $\mathfrak{h}_\kappa(t,\omega,y,z,u_1,u_2)\in L^q_\nu$.

It remains to show that the mapping $\mathfrak{h}_\kappa$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(L^p_\nu)\otimes\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable: let $(t,\omega,y,z,u_1,u_2)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu$, let $\lambda>0$ and define $f_\lambda(\alpha):=\frac{\alpha^+}{\lambda}\wedge1$, $\alpha\in\mathbb{R}$. For any $u\in L^p_\nu$, (6.25) and (1.6) show that the function $\big(\phi^{u_1}_\lambda(u)\big)(x):=f_\lambda\big(u_1(x)-u(x)\big)\frac{u^+_1(x)-u^+(x)}{u_1(x)-u(x)}$, $\forall\, x\in\mathcal{X}$ takes values in $[0,1]$, so $\phi^{u_1}_\lambda(u)\in L^q_\nu$.

We first show that $\phi^{u_1}_\lambda$ is a continuous mapping from L^p_ν to L^q_ν. Fix $\varepsilon>0$ and set $\delta=\delta(\lambda,\varepsilon):=\frac{\lambda\varepsilon}{3}(2\nu(\mathcal{X}))^{-\frac1q}$. Let $u,\widetilde{u}\in L^p_\nu$ with $\|\widetilde{u}-u\|_{L^p_\nu}<\delta(1/2)^{\frac1p}(\varepsilon/2)^{\frac{1}{p-1}}$. Since

$$\begin{aligned}\Big|\frac{u^+_1(x)-u^+(x)}{u_1(x)-u(x)}-\frac{u^+_1(x)-\widetilde{u}^+(x)}{u_1(x)-\widetilde{u}(x)}\Big| &= \Big|\frac{(\widetilde{u}^+(x)-u^+(x))(u_1(x)-\widetilde{u}(x))+(u^+_1(x)-\widetilde{u}^+(x))(u(x)-\widetilde{u}(x))}{(u_1(x)-u(x))(u_1(x)-\widetilde{u}(x))}\Big|\\ &\le \frac{|\widetilde{u}(x)-u(x)||u_1(x)-\widetilde{u}(x)|+|u_1(x)-\widetilde{u}(x)||u(x)-\widetilde{u}(x)|}{|u_1(x)-u(x)||u_1(x)-\widetilde{u}(x)|}=2\frac{|u(x)-\widetilde{u}(x)|}{|u_1(x)-u(x)|}, \quad \forall\, x\in\mathcal{X}\end{aligned}$$

and since the Lipschitz coefficient of f_λ is no larger than $1/\lambda$, one can deduce from (1.6) that for any $x\in\{|\widetilde{u}-u|<\delta\}$

$$\big|\big(\phi^{u_1}_\lambda(u)\big)(x)-\big(\phi^{u_1}_\lambda(\widetilde{u})\big)(x)\big|=\bigg|f_\lambda\big((u_1-u)(x)\big)\Big(\frac{u_1^+(x)-u^+(x)}{u_1(x)-u(x)}-\frac{u_1^+(x)-\widetilde{u}^+(x)}{u_1(x)-\widetilde{u}(x)}\Big)+\big(f_\lambda\big((u_1-u)(x)\big)-f_\lambda\big((u_1-\widetilde{u})(x)\big)\big)\frac{u_1^+(x)-\widetilde{u}^+(x)}{u_1(x)-\widetilde{u}(x)}\bigg|$$
$$\le 2\frac{(u_1(x)-u(x))^+}{\lambda}\frac{|u(x)-\widetilde{u}(x)|}{|u_1(x)-u(x)|}+\frac{1}{\lambda}|u(x)-\widetilde{u}(x)|\le\frac{3}{\lambda}\big|u(x)-\widetilde{u}(x)\big|<\frac{3\delta}{\lambda}.$$

It follows that

$$\int_{\mathcal{X}}\Big|\big(\phi^{u_1}_\lambda(u)\big)(x)-\big(\phi^{u_1}_\lambda(\widetilde{u})\big)(x)\Big|^q\nu(dx)\le\int_{\{|\widetilde{u}-u|<\delta\}}\Big(\frac{3\delta}{\lambda}\Big)^q\nu(dx)+\int_{\{|\widetilde{u}-u|\ge\delta\}}2^q\,\nu(dx)\le\Big(\frac{3\delta}{\lambda}\Big)^q\nu(\mathcal{X})+\frac{2^q}{\delta^p}\|\widetilde{u}-u\|^p_{L^p_\nu}<\varepsilon^q,$$

or $\big\|(\phi^{u_1}_\lambda(u))-(\phi^{u_1}_\lambda(\widetilde{u}))\big\|_{L^q_\nu}\le\varepsilon$. This shows that

the mapping $\phi^{u_1}_\lambda$ is uniformly continuous from L^p_ν to L^q_ν and thus $\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable. (6.26)

For any $u\in L^p_\nu$, we define a function $\phi^{u_1}(u)\in L^q_\nu$ by

$$\big(\phi^{u_1}(u)\big)(x):=\lim_{\lambda\to0}\big(\phi^{u_1}_\lambda(u)\big)(x)=\mathbf{1}_{\{u_1(x)-u(x)>0\}}\frac{u_1^+(x)-u^+(x)}{u_1(x)-u(x)}\in[-1,1],\quad\forall\,x\in\mathcal{X}.$$

In light of the bounded convergence theorem, $\lim\limits_{\lambda\to0}\|\phi^{u_1}_\lambda(u)-\phi^{u_1}(u)\|^q_{L^q_\nu}=\lim\limits_{\lambda\to0}\int_{\mathcal{X}}\big|(\phi^{u_1}_\lambda(u))(x)-(\phi^{u_1}(u))(x)\big|^q\nu(dx)=0$. Namely, $\phi^{u_1}(u)$ is the limit of $\big\{\phi^{u_1}_\lambda(u)\big\}_{\lambda>0}$ in L^q_ν. It then follows from (6.26) that the mapping ϕ^{u_1} is $\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable.

Define $\big(\widehat{\phi}^{u_1}(u)\big)(x):=\mathbf{1}_{\{u_1(x)-u(x)<0\}}\frac{u_1^+(x)-u^+(x)}{u_1(x)-u(x)}\in[-1,1]$, $\forall\,u\in L^p_\nu$, $\forall\,x\in\mathcal{X}$. One can similarly show that $\widehat{\phi}^{u_1}$ is also a $\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable mapping. Consequently, the mapping $u\to\mathfrak{h}_\kappa(t,\omega,y,z,u_1,u)$ is again $\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable. Symmetrically, the mapping $u\to\mathfrak{h}_\kappa(t,\omega,y,z,u,u_2)$ is $\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable. Putting them together yields the expected measurability of $\mathfrak{h}_\kappa$.

3) Similar to (6.2), we see from (6.24) that for any $(t,\omega,y,z,u_1,u_2)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu$

$$\overline{g}^{\Xi}(t,\omega,y,z,u_1)-\overline{g}^{\Xi}(t,\omega,y,z,u_2)=\int_{\mathcal{X}}\big(-u_2(x)+u_1(x)\big)\cdot\big(\mathfrak{h}_\kappa(t,\omega,-y,-z,-u_2,-u_1)\big)(x)\nu(dx).$$

So $\overline{g}^{\Xi}$ satisfies (2.3) with $\big(\overline{\mathfrak{h}}_\kappa(t,\omega,y,z,u_1,u_2)\big)(x)=\big(\mathfrak{h}_\kappa(t,\omega,-y,-z,-u_2,-u_1)\big)(x)\in[\kappa_1,\kappa_2]$, $\forall\,x\in\mathcal{X}$. Clearly, the mapping $\overline{\mathfrak{h}}_\kappa$ is also $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(L^p_\nu)\otimes\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable. Therefore, $\overline{g}^{\Xi}$ also satisfies (A3). □

Proof of Proposition 3.1: Fix $\tau\in\mathcal{T}$, $\forall\,\gamma\in\mathcal{T}_\tau$ and $\xi,\eta\in L^p(\mathcal{F}_T)$. Set $(\mathcal{Y}^1,\mathcal{Z}^1,\mathcal{U}^1)=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$, $(\mathcal{Y}^2,\mathcal{Z}^2,\mathcal{U}^2)=\big(Y^{\eta,g_\gamma},Z^{\eta,g_\gamma},U^{\eta,g_\gamma}\big)$ and $(\mathcal{Y}^3,\mathcal{Z}^3,\mathcal{U}^3)=\big(Y^{\xi-\eta,\overline{g}^{\Xi}_\gamma},Z^{\xi-\eta,\overline{g}^{\Xi}_\gamma},U^{\xi-\eta,\overline{g}^{\Xi}_\gamma}\big)$. The $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurability of g, the $\mathscr{P}-$measurability of process $\mathcal{Y}^2$, the $\widehat{\mathscr{P}}-$measurability of process $\mathcal{Z}^2$ and the $\widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}-$measurability of random field $\mathcal{U}^2$ imply that the mapping

$$\overline{g}(t,\omega,y,z,u):=g\big(t,\omega,y+\mathcal{Y}^2(t,\omega),z+\mathcal{Z}^2(t,\omega),u+\mathcal{U}^2(t,\omega)\big)-g\big(t,\omega,\mathcal{Y}^2(t,\omega),\mathcal{Z}^2(t,\omega),\mathcal{U}^2(t,\omega)\big),$$

$\forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$ is also $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable.

For $\big(\overline{Y},\overline{Z},\overline{U}\big):=(\mathcal{Y}^1-\mathcal{Y}^2,\mathcal{Z}^1-\mathcal{Z}^2,\mathcal{U}^1-\mathcal{U}^2)\in\mathbb{S}^p$, it holds $P-$a.s. that

$$\overline{Y}_t=\xi-\eta+\int_t^T\mathbf{1}_{\{t<\gamma\}}\big(g(s,\mathcal{Y}^1_s,\mathcal{Z}^1_s,\mathcal{U}^1_s)-g(s,\mathcal{Y}^2_s,\mathcal{Z}^2_s,\mathcal{U}^2_s)\big)ds-\int_t^T\overline{Z}_sdB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

Namely, $\big(\overline{Y},\overline{Z},\overline{U}\big)$ solves the BSDEJ $(\xi-\eta,\overline{g}_\gamma)$. We can deduce from (A2) and (A3) that $dt\times dP-$a.s.

$$\overline{g}\big(t,\overline{Y}_t,\overline{Z}_t,\overline{U}_t\big)=g(t,\mathcal{Y}^1_t,\mathcal{Z}^1_t,\mathcal{U}^1_t)-g(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^2_t)=g(t,\mathcal{Y}^1_t,\mathcal{Z}^1_t,\mathcal{U}^1_t)-g(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^1_t)+g(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^1_t)-g(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^2_t)$$
$$\le\beta_t|\overline{Y}_t|+\Lambda_t|\overline{Z}_t|+\int_{\mathcal{X}}\overline{U}_t(x)\cdot\big(\mathfrak{h}(t,\mathcal{Y}^2_t,\mathcal{Z}^2_t,\mathcal{U}^1_t,\mathcal{U}^2_t)\big)(x)\nu(dx)$$
$$\le\beta_t|\overline{Y}_t|+\Lambda_t|\overline{Z}_t|+\kappa_2\int_{\mathcal{X}}\overline{U}^+_t(x)\nu(dx)-\kappa_1\int_{\mathcal{X}}\overline{U}^-_t(x)\nu(dx)=g^{\Xi}\big(t,\overline{Y}_t,\overline{Z}_t,\overline{U}_t\big).$$

Since g^{Ξ} also satisfies (A2) and (A3) by Example 3.1, applying Theorem 2.2 with $(\tau,\gamma)=(0,T)$, $(Y^1,Z^1,U^1)=\big(\overline{Y},\overline{Z},\overline{U}\big)$ and $(Y^2,Z^2,U^2)=(\mathcal{Y}^3,\mathcal{Z}^3,\mathcal{U}^3)$ yields that $P\big\{\mathcal{Y}^1_t-\mathcal{Y}^2_t=\overline{Y}_t\le\mathcal{Y}^3_t,\ \forall\, t\in[0,T]\big\}=1$. In particular,

$$\mathcal{E}^g_{\tau,\gamma}[\xi]-\mathcal{E}^g_{\tau,\gamma}[\eta]=\mathcal{Y}^1_\tau-\mathcal{Y}^2_\tau\le\mathcal{Y}^3_\tau=\mathcal{E}^{\Xi}_{\tau,\gamma}[\xi-\eta],\quad P-\text{a.s.} \tag{6.27}$$

Multiplying -1 to BSDEJ $\big(\eta-\xi,g^{\Xi}_\gamma\big)$ shows that $(-Y^{\eta-\xi,g^{\Xi}_\gamma},-Z^{\eta-\xi,g^{\Xi}_\gamma},-U^{\eta-\xi,g^{\Xi}_\gamma})$ is the unique solution of BSDEJ $\big(\xi-\eta,\overline{g}^{\Xi}_\gamma\big)$. So $P\Big\{-Y^{\eta-\xi,g^{\Xi}_\gamma}_t=Y^{\xi-\eta,\overline{g}^{\Xi}_\gamma}_t,\ \forall\, t\in[0,T]\Big\}=1$, which together with (6.27) implies that

$$\mathcal{E}^g_{\tau,\gamma}[\xi]-\mathcal{E}^g_{\tau,\gamma}[\eta]=-\big(\mathcal{E}^g_{\tau,\gamma}[\eta]-\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\ge-\mathcal{E}^{\Xi}_{\tau,\gamma}[\eta-\xi]=-Y^{\eta-\xi,g^{\Xi}_\gamma}_\tau=Y^{\xi-\eta,\overline{g}^{\Xi}_\gamma}_\tau=\overline{\mathcal{E}}^{\Xi}_{\tau,\gamma}[\xi-\eta],\quad P-\text{a.s.}\qquad\square$$

6.3 Proofs of Section 4

Proof of Lemma 4.1: Let $t_i\le t<s\le t_{i+1}$ for some $i\in\{1,\cdots n-1\}$ and let $\xi\in L^p(\mathcal{F}_{\tau\wedge s})$. Set $(Y,Z,U):=\big(Y^{\xi,g_{\tau\wedge s}},Z^{\xi,g_{\tau\wedge s}},U^{\xi,g_{\tau\wedge s}}\big)$ and $\big(\widetilde{Y},\widetilde{Z},\widetilde{U}\big):=\big(Y^{\xi,g_s},Z^{\xi,g_s},U^{\xi,g_s}\big)$.

Let $t'\in[t,s]$. Since $\{\tau\le t_i\}=\{\tau\ge t_{i+1}\}^c\in\mathcal{F}_{t_i}\subset\mathcal{F}_{t'}$, and since $(Z_r,U_r)=\mathbf{1}_{\{r\le\tau\wedge s\}}(Z_r,U_r)$, $dr\times dP-$a.s. by Theorem 2.1, multiplying $\mathbf{1}_{\{\tau\le t_i\}}$ and $\mathbf{1}_{\{\tau\ge t_{i+1}\}}$ to BSDEJ $\big(\xi,g_{\tau\wedge s}\big)$ over period $[t',T]$ respectively yields that $P-$a.s.

$$\begin{aligned}\mathbf{1}_{\{\tau\le t_i\}}Y_{t'}&=\mathbf{1}_{\{\tau\le t_i\}}\xi+\int_{t'}^T\mathbf{1}_{\{\tau\le t_i\}}\mathbf{1}_{\{r<\tau\wedge s\}}g(r,Y_r,Z_r,U_r)dr-\int_{t'}^T\mathbf{1}_{\{\tau\le t_i\}}\mathbf{1}_{\{r\le\tau\wedge s\}}Z_rdB_r\\&\quad-\int_{(t',T]}\int_{\mathcal{X}}\mathbf{1}_{\{\tau\le t_i\}}\mathbf{1}_{\{r\le\tau\wedge s\}}U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)=\mathbf{1}_{\{\tau\le t_i\}}\xi,\end{aligned}\tag{6.28}$$

and that $P-$a.s.

$$\begin{aligned}\mathbf{1}_{\{\tau\ge t_{i+1}\}}Y_{t'}&=\mathbf{1}_{\{\tau\ge t_{i+1}\}}\xi+\int_{t'}^T\mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathbf{1}_{\{r<\tau\wedge s\}}g(r,Y_r,Z_r,U_r)dr-\int_{t'}^T\mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathbf{1}_{\{r\le\tau\wedge s\}}Z_rdB_r\\&\quad-\int_{(t',T]}\int_{\mathcal{X}}\mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathbf{1}_{\{r\le\tau\wedge s\}}U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)\\&=\mathbf{1}_{\{\tau\ge t_{i+1}\}}\xi+\int_{t'}^s\mathbf{1}_{\{\tau\ge t_{i+1}\}}g(r,Y_r,Z_r,U_r)dr-\int_{t'}^s\mathbf{1}_{\{\tau\ge t_{i+1}\}}Z_rdB_r-\int_{(t',s]}\int_{\mathcal{X}}\mathbf{1}_{\{\tau\ge t_{i+1}\}}U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx).\end{aligned}\tag{6.29}$$

Also, an analogy to (6.28) shows that $P-$a.s.

$$\mathbf{1}_{\{\tau\le t_i\}}\widetilde{Y}_{t'}=\mathbf{1}_{\{\tau\le t_i\}}\xi+\int_{t'}^s\mathbf{1}_{\{\tau\le t_i\}}g\big(r,\widetilde{Y}_r,\widetilde{Z}_r,\widetilde{U}_r\big)dr-\int_{t'}^s\mathbf{1}_{\{\tau\le t_i\}}\widetilde{Z}_rdB_r-\int_{(t',s]}\int_{\mathcal{X}}\mathbf{1}_{\{\tau\le t_i\}}\widetilde{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx).\tag{6.30}$$

Next, set $(\mathcal{Y}_r,\mathcal{Z}_r,\mathcal{U}_r):=\mathbf{1}_{\{\tau\le t_i\}}\big(\widetilde{Y}_r,\widetilde{Z}_r,\widetilde{U}_r\big)+\mathbf{1}_{\{\tau\ge t_{i+1}\}}(Y_r,Z_r,U_r)$, $\forall\, r\in[t,s]$. As $\mathcal{Y}_t\in L^p(\mathcal{F}_t)$, Theorem 2.1 shows that the BSDEJ $(\mathcal{Y}_t,g_t)$ admits a unique solution $(\mathscr{Y},\mathscr{Z},\mathscr{U})\in\mathbb{S}^p$. Define $\overline{Y}_r:=\mathbf{1}_{\{r<t\}}\mathscr{Y}_r+\mathbf{1}_{\{r\ge t\}}\mathcal{Y}_{r\wedge s}$ and $\big(\overline{Z}_r,\overline{U}_r\big):=\mathbf{1}_{\{r\le t\}}(\mathscr{Z}_r,\mathscr{U}_r)+\mathbf{1}_{\{t<r\le s\}}(\mathcal{Z}_r,\mathcal{U}_r)$, $\forall\, r\in[0,T]$. One can deduce that $\big(\overline{Y},\overline{Z},\overline{U}\big)$ belong to $\mathbb{S}^p$.

For any $t'\in[t,T]$, adding (6.29) to (6.30) yields that

$$\begin{aligned}\overline{Y}_{t'}&=\mathcal{Y}_{t'\wedge s}=\mathbf{1}_{\{\tau\le t_i\}}\widetilde{Y}_{t'\wedge s}+\mathbf{1}_{\{\tau\ge t_{i+1}\}}Y_{t'\wedge s}=\xi+\int_{t'\wedge s}^s g(r,\mathcal{Y}_r,\mathcal{Z}_r,\mathcal{U}_r)dr-\int_{t'\wedge s}^s\mathcal{Z}_rdB_r-\int_{(t'\wedge s,s]}\int_{\mathcal{X}}\mathcal{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)\\&=\xi+\int_{t'}^T g_s(r,\overline{Y}_r,\overline{Z}_r,\overline{U}_r)dr-\int_{t'}^T\overline{Z}_rdB_r-\int_{(t',T]}\int_{\mathcal{X}}\overline{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx),\quad P-\text{a.s.}\end{aligned}\tag{6.31}$$

On the other hand, for any $\widehat{t}\in[0,t)$, as Theorem 2.1 shows that $\overline{Y}_t=\mathcal{Y}_t=\mathscr{Y}_T=\mathscr{Y}_t$, $P-$a.s., we have

$$\begin{aligned}\overline{Y}_{\widehat{t}}-\overline{Y}_t&=\mathscr{Y}_{\widehat{t}}-\mathscr{Y}_t=\int_{\widehat{t}}^t g_t(r,\mathscr{Y}_r,\mathscr{Z}_r,\mathscr{U}_r)dr-\int_{\widehat{t}}^t\mathscr{Z}_rdB_r-\int_{(\widehat{t},t]}\int_{\mathcal{X}}\mathscr{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)\\&=\int_{\widehat{t}}^t g\big(r,\overline{Y}_r,\overline{Z}_r,\overline{U}_r\big)dr-\int_{\widehat{t}}^t\overline{Z}_rdB_r-\int_{(\widehat{t},t]}\int_{\mathcal{X}}\overline{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx),\quad P-\text{a.s.}\end{aligned}$$

Taking $t'=t$ in (6.31) yields that

$$\overline{Y}_{\widehat{t}}=\xi+\int_{\widehat{t}}^{T} g_s\big(r,\overline{Y}_r,\overline{Z}_r,\overline{U}_r\big)dr-\int_{\widehat{t}}^{T}\overline{Z}_r dB_r-\int_{(\widehat{t},T]}\int_{\mathcal{X}}\overline{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx),\quad P\text{-a.s.} \tag{6.32}$$

By the right-continuity of $\overline{Y}$, we see from (6.31) and (6.32) that P−a.s.

$$\overline{Y}_{t'}=\xi+\int_{t'}^{T} g_s(r,\overline{Y}_r,\overline{Z}_r,\overline{U}_r)dr-\int_{t'}^{T}\overline{Z}_r dB_r-\int_{(t',T]}\int_{\mathcal{X}}\overline{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx),\quad t'\in[0,T],$$

which shows that $\big(\overline{Y},\overline{Z},\overline{U}\big)$ solves BSDEJ (ξ,g_s). It follows that $\mathcal{E}^g_{t,s}[\xi]=Y^{\xi,g_s}_t=\overline{Y}_t=\mathcal{Y}_t$, P−a.s. Then applying (6.28) with $t'=t$, we see from Theorem 2.1 again that P−a.s.

$$\mathcal{E}^g_{\tau\wedge t,\tau\wedge s}[\xi]=Y_{\tau\wedge t}=Y_t=\mathbf{1}_{\{\tau\le t_i\}}Y_t+\mathbf{1}_{\{\tau\ge t_{i+1}\}}Y_t=\mathbf{1}_{\{\tau\le t_i\}}\xi+\mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathcal{Y}_t=\mathbf{1}_{\{\tau\le t_i\}}\xi+\mathbf{1}_{\{\tau\ge t_{i+1}\}}\mathcal{E}^g_{t,s}[\xi]. \qquad \square$$

Proof of Proposition 4.1: Let us only consider the g−submartingale case, as the other cases can be derived similarly.

1) Assume first that γ takes values in a finite set $\{0=t_1<\cdots<t_n=T\}$.

If $t\in[t_n,T]$, (3.1) shows that $\mathcal{E}^g_{\gamma\wedge t,\gamma}[X_\gamma]=Y^{X_\gamma,g_\gamma}_{\gamma\wedge t}=Y^{X_\gamma,g_\gamma}_{\gamma}=\mathcal{E}^g_{\gamma,\gamma}[X_\gamma]=X_\gamma=X_{\gamma\wedge t}$, P−a.s. Then let us inductively argue that for any $t\in[0,T]$,

$$\mathcal{E}^g_{\gamma\wedge t,\gamma}[X_\gamma]\ge X_{\gamma\wedge t},\ \ P\text{-a.s.} \tag{6.33}$$

Suppose that for some $i\in\{2,\cdots,n\}$, (6.33) holds for each $t\in[t_i,T]$. Given $t\in[t_{i-1},t_i)$, the (g1), (g3) properties of g−evaluations and (4.1) imply that

$$\mathcal{E}^g_{\gamma\wedge t,\gamma}[X_\gamma]=\mathcal{E}^g_{\gamma\wedge t,\gamma\wedge t_i}\big[\mathcal{E}^g_{\gamma\wedge t_i,\gamma}[X_\gamma]\big]\ge\mathcal{E}^g_{\gamma\wedge t,\gamma\wedge t_i}[X_{\gamma\wedge t_i}]=\mathbf{1}_{\{\gamma\le t_{i-1}\}}X_{\gamma\wedge t_i}+\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{E}^g_{t,t_i}[X_{\gamma\wedge t_i}],\quad P\text{-a.s.} \tag{6.34}$$

Since $\{\gamma\ge t_i\}=\{\gamma\le t_{i-1}\}^c\in\mathcal{F}_{t_{i-1}}\subset\mathcal{F}_t$, the (g4) of g−evaluations and the g−submartingality of X show that P−a.s.

$$\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{E}^g_{t,t_i}[X_{\gamma\wedge t_i}]=\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{E}^g_{t,t_i}[\mathbf{1}_{\{\gamma\ge t_i\}}X_{\gamma\wedge t_i}]=\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{E}^g_{t,t_i}[\mathbf{1}_{\{\gamma\ge t_i\}}X_{t_i}]=\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{E}^g_{t,t_i}[X_{t_i}]\ge\mathbf{1}_{\{\gamma\ge t_i\}}X_t=\mathbf{1}_{\{\gamma\ge t_i\}}X_{\gamma\wedge t}.$$

Putting it back to (6.34) proves (6.33) for any $t\in[t_{i-1},T]$. This completes the inductive step. Hence, (6.33) holds for any $t\in[0,T]$.

If τ is also finitely valued, for example in $\{0=s_1<\cdots<s_m=T\}$, then we see from (6.33) that P−a.s.

$$\mathcal{E}^g_{\tau,\gamma}[X_\gamma]=Y^{X_\gamma,g_\gamma}_\tau=Y^{X_\gamma,g_\gamma}_{\gamma\wedge\tau}=\sum_{j=1}^m\mathbf{1}_{\{\tau=s_j\}}Y^{X_\gamma,g_\gamma}_{\gamma\wedge s_j}=\sum_{j=1}^m\mathbf{1}_{\{\tau=s_j\}}\mathcal{E}^g_{\gamma\wedge s_j,\gamma}[X_\gamma]\ge\sum_{j=1}^m\mathbf{1}_{\{\tau=s_j\}}X_{\gamma\wedge s_j}=X_{\gamma\wedge\tau}=X_\tau.$$

2) Next, assume that X is right-continuous but τ,γ are general stopping times. Set $(Y,Z,Y):=\big(Y^{X_\gamma,g_\gamma},Z^{X_\gamma,g_\gamma},U^{X_\gamma,g_\gamma}\big)$. For any $n\in\mathbb{N}$, we set $t^n_i:=\frac{i}{2^n}T$, $i=0,\cdots,2^n$ and define $\tau_n:=\sum_{i=1}^{2^n}\mathbf{1}_{\{t^n_{i-1}<\tau\le t^n_i\}}t^n_i$ and $\gamma_n:=\sum_{i=1}^{2^n}\mathbf{1}_{\{t^n_{i-1}<\gamma\le t^n_i\}}t^n_i\in\mathcal{T}$.

Let $m,n\in\mathbb{N}$ with $m>n$ and set $(Y^n,Z^n,Y^n):=\big(Y^{X_{\gamma_n},g_{\gamma_n}},Z^{X_{\gamma_n},g_{\gamma_n}},U^{X_{\gamma_n},g_{\gamma_n}}\big)$. Since $\tau_m\le\tau_n\le\gamma_n$, Part 1 shows that $Y^n_{\tau_m}=\mathcal{E}^g_{\tau_m,\gamma_n}[X_{\gamma_n}]\ge X_{\tau_m}$, P−a.s. As $\lim\limits_{m\to\infty}\downarrow\tau_m=\tau$, the right continuity of processes Y^n and X implies that

$$Y^n_\tau=\lim_{m\to\infty}Y^n_{\tau_m}\ge\lim_{m\to\infty}X_{\tau_m}=X_\tau,\quad P\text{-a.s.} \tag{6.35}$$

By Proposition 2.1,

$$E[|Y^n_\tau-Y_\tau|^p]\le\big\|Y^n-Y\big\|^p_{\mathbb{D}^p}\le\mathcal{C}E\bigg[|X_{\gamma_n}-X_\gamma|^p+\Big(\int_\gamma^{\gamma_n}\big|g(t,Y_t,Z_t,U_t)\big|dt\Big)^p\bigg]. \tag{6.36}$$

Also, (A1)−(A3), (6.1), (1.7) and Hölder's inequality implies that

$$\begin{aligned}E\bigg[\Big(\int_0^T\big|g(t,Y_t,Z_t,U_t)\big|dt\Big)^p\bigg]&\le E\bigg[\Big(\int_0^T\big(|g(t,0,0,0)|+\beta_t|Y_t|+\Lambda_t|Z_t|+\kappa_2(\nu(\mathcal{X}))^{\frac1q}\|U_t\|_{L^p_\nu}\big)dt\Big)^p\bigg]\\&\le4^{p-1}E\bigg[\Big(\int_0^T|g(t,0,0,0)|dt\Big)^p+\widehat{C}^{\frac{p}{q}}TY^p_*+\widehat{C}^{\frac{p}{2}}\Big(\int_0^T|Z_t|^2dt\Big)^{\frac{p}{2}}+\kappa_2^p(\nu(\mathcal{X})T)^{\frac{p}{q}}\int_0^T\|U_t\|^p_{L^p_\nu}dt\bigg]<\infty.\end{aligned}$$

Since $E[X_*^p]<\infty$ and since $\lim\limits_{n\to\infty}\downarrow\gamma_n=\gamma$, letting $n\to\infty$ in (6.36), we can deduce from the right-continuity of X and the dominated convergence theorem that $\lim\limits_{n\to\infty}E[|Y_\tau^n-Y_\tau|^p]=0$. So there exists a subsequence $\{n_i\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that $Y_\tau=\lim\limits_{i\to\infty}Y_\tau^{n_i}$, $P-$a.s. It then follows from (6.35) that $Y_\tau=\lim\limits_{i\to\infty}Y_\tau^{n_i}\ge X_\tau$, $P-$a.s. or $\mathcal{E}^g_{\tau,\gamma}[X_\gamma]\ge X_\tau$, $P-$a.s. □

Proof of Proposition 4.2: We simply denote $g_t^0:=g(t,0,0,0)$, $t\in[0,T]$.

1) *Like Part 1 in the proof of Theorem 2.2, we first construct an equivalent probability $Q^{\mathcal{D}}$ to P.*

Let $i\in\{1,\cdots,2m'\}$ and let τ_i be the finitely valued $\mathbf{F}-$stopping time as defined in (4.2). We set $(Y^i,Z^i,U^i):=\big(Y^{X_{\tau_i},g_{\tau_i}},Z^{X_{\tau_i},g_{\tau_i}},U^{X_{\tau_i},g_{\tau_i}}\big)$. Since $(Z_t^i,U_t^i)=\mathbf{1}_{\{t\le\tau_i\}}(Z_t^i,U_t^i)$, $dt\times dP-$a.s. by Theorem 2.1, it holds $P-$a.s. that

$$Y^i_{\tau_i\wedge t}=X_{\tau_i}+\int_{\tau_i\wedge t}^{\tau_i}g(s,Y_s^i,Z_s^i,U_s^i)ds-\int_{\tau_i\wedge t}^{\tau_i}Z_s^idB_s-\int_{(\tau_i\wedge t,\tau_i]}\int_{\mathcal{X}}U_s^i(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad\forall\,t\in[0,T].\tag{6.37}$$

Clearly, $\mathfrak{a}_t^i:=\mathbf{1}_{\{Y_t^i\ne0\}}\dfrac{g(t,Y_t^i,Z_t^i,U_t^i)-g(t,0,Z_t^i,U_t^i)}{Y_t^i}$, $\mathfrak{b}_t^i:=\mathbf{1}_{\{Z_t^i\ne0\}}\dfrac{g(t,0,Z_t^i,U_t^i)-g(t,0,0,U_t^i)}{|Z_t^i|^2}Z_t^i$, $\forall\,t\in[0,T]$ are two $\mathbf{F}-$progressively measurable processes. By (A2), it holds $dt\times dP-$a.s. that

$$|\mathfrak{a}_t^i|\le\beta_t\quad\text{and}\quad|\mathfrak{b}_t^i|\le\Lambda_t.\tag{6.38}$$

Also, setting $\mathfrak{H}_t^i:=\mathfrak{h}(t,0,0,0,U_t^i)$, $t\in[0,T]$, we can deduce from (A2), (A3) (iii) that $dt\times dP-$a.s.

$$g(t,Y_t^i,Z_t^i,U_t^i)-g_t^0=\mathfrak{a}_t^iY_t^i+\mathfrak{b}_t^iZ_t^i+g(t,0,0,U_t^i)-g(t,0,0,0)\ge\mathfrak{a}_t^iY_t^i+\mathfrak{b}_t^iZ_t^i+\int_{\mathcal{X}}\mathfrak{H}_t^i(x)U_t^i(x)\nu(dx).\tag{6.39}$$

Similar to (6.5), $M_t^{\mathcal{D}}:=\int_0^T\big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{b}_s^i\big)dB_s+\int_{(0,T]}\int_{\mathcal{X}}\big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{H}_s^i(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$ is a uniformly integrable martingale. For any $\zeta\in\mathcal{T}$, we see from (6.38) and (A3) (ii) that

$$\Big|\Delta M^{\mathcal{D}}(\zeta(\omega),\omega)\Big|=\mathbf{1}_{\{\zeta(\omega)\in D_{\mathfrak{p}(\omega)}\}}\bigg|\sum_{i=1}^{2m'}\mathbf{1}_{\{\zeta(\omega)\in(\tau_{i-1}(\omega),\tau_i(\omega)]\}}\big(\mathfrak{H}^i\big(\zeta(\omega),\omega,\mathfrak{p}(\zeta(\omega),\omega)\big)\bigg|\le\kappa_2,\quad\forall\,\omega\in\Omega,$$

and that

$$E\Big[\big[M^{\mathcal{D}},M^{\mathcal{D}}\big]_T-\big[M^{\mathcal{D}},M^{\mathcal{D}}\big]_\tau\Big|\mathcal{F}_\tau\Big]=E\bigg[\int_\tau^T\bigg(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\big|\mathfrak{b}_s^i\big|^2\bigg)ds+\int_{(\tau,T]}\int_{\mathcal{X}}\bigg(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\big|\mathfrak{H}_s^i(x)\big|^2\bigg)N_{\mathfrak{p}}(ds,dx)\bigg|\mathcal{F}_\tau\bigg]$$
$$=E\bigg[\int_\tau^T\bigg(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\big|\mathfrak{b}_s^i\big|^2\bigg)ds+\int_\tau^T\int_{\mathcal{X}}\bigg(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\big|\mathfrak{H}_s^i(x)\big|^2\bigg)\nu(dx)ds\bigg|\mathcal{F}_\tau\bigg]\le\widehat{C}+\kappa_2^2\nu(\mathcal{X})T<\infty.$$

Thus, $M^{\mathcal{D}}$ is a BMO martingale. In virtue of [50], the Doléans-Dade exponential of $M^{\mathcal{D}}$

$$\mathscr{E}_t(M^{\mathcal{D}}):=e^{M_t^{\mathcal{D}}-\frac12\langle M^{\mathcal{D},c}\rangle_t}\prod_{0<s\le t}(1+\Delta M_s^{\mathcal{D}})e^{-\Delta M_s^{\mathcal{D}}}>0,\quad t\in[0,T]\tag{6.40}$$

is a uniformly integrable martingale, where $M^{\mathcal{D},c}$ denote the continuous part of $M^{\mathcal{D}}$.

Define a probability measure $Q^{\mathcal{D}}$ by $\frac{dQ^{\mathcal{D}}}{dP}:=\mathscr{E}_T(M^{\mathcal{D}})$, which satisfies $\frac{dQ^{\mathcal{D}}}{dP}\big|_{\mathcal{F}_t}:=\mathscr{E}_t(M^{\mathcal{D}})$, $\forall\,t\in[0,T]$. The Girsanov's Theorem shows that $B_t^{\mathcal{D}}:=B_t-\int_0^t\big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{b}_s^i\big)ds$, $t\in[0,T]$ is a $Q^{\mathcal{D}}-$Brownian motion and $\widetilde{N}_{\mathfrak{p}}^{\mathcal{D}}(t,A):=\widetilde{N}_{\mathfrak{p}}(t,A)-\int_{(0,t]}\int_{\mathcal{X}}\big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{H}_s^i(x)\big)\nu(dx)ds$, $t\in[0,T]$, $A\in\mathcal{F}_{\mathcal{X}}$ is a $Q^{\mathcal{D}}-$compensated Poisson random measure.

2) *Next, we show that* $(b-a)E_{Q^{\mathcal{D}}}\big[U_{\mathcal{D}}(a,b;X)\big]\le e^{2\widehat{C}}E_{Q^{\mathcal{D}}}\bigg[|a|\widehat{C}+(X_{t_m}-a)^-+\int_0^{t_m}|g_s^0|ds\bigg].$ (6.41)

By (6.38), the $\mathbf{F}-$adapted continuous process $\Theta_t^{\mathcal{D}}:=\exp\Big\{\int_0^t\big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{a}_s^i\big)ds\Big\}$, $t\in[0,T]$ satisfies that

$$e^{-\widehat{C}}\le e^{-\int_0^T\beta_sds}\le\inf_{t\in[0,T]}\Theta_t^{\mathcal{D}}\le\sup_{t\in[0,T]}\Theta_t^{\mathcal{D}}\le e^{\int_0^T\beta_sds}\le e^{\widehat{C}},\quad P\text{-a.s. and thus }Q^{\mathcal{D}}\text{-a.s.}\tag{6.42}$$

Let $i\in\{1,\cdots,2m'\}$ and $n\in\mathbb{N}$. We define $\gamma^i_n:=\inf\big\{t\in[0,T]: \int_0^t|Z^i_s|^2ds+\int_0^t\int_{\mathcal{X}}|U^i_s(x)|^p\nu(dx)ds>n\big\}\wedge T\in\mathcal{T}$. Applying Itô's formula to $\Theta^{\mathcal{D}}_t Y^i_t$ over period $[\tau_{i-1}\wedge\gamma^i_n,\tau_i\wedge\gamma^i_n]$, we can deduce from (6.37) and (6.39) that

$$\begin{aligned}\Theta^{\mathcal{D}}_{\tau_{i-1}\wedge\gamma^i_n}Y^i_{\tau_{i-1}\wedge\gamma^i_n}&=\Theta^{\mathcal{D}}_{\tau_i\wedge\gamma^i_n}Y^i_{\tau_i\wedge\gamma^i_n}+\int_{\tau_{i-1}\wedge\gamma^i_n}^{\tau_i\wedge\gamma^i_n}\Theta^{\mathcal{D}}_s\big(g(s,Y^i_s,Z^i_s,U^i_s)-\mathfrak{a}^i_sY^i_s\big)ds-\int_{\tau_{i-1}\wedge\gamma^i_n}^{\tau_i\wedge\gamma^i_n}\Theta^{\mathcal{D}}_sZ^i_sdB_s\\&\quad-\int_{(\tau_{i-1}\wedge\gamma^i_n,\tau_i\wedge\gamma^i_n]}\int_{\mathcal{X}}\Theta^{\mathcal{D}}_sU^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\&\ge\Theta^{\mathcal{D}}_{\tau_i\wedge\gamma^i_n}Y^i_{\tau_i\wedge\gamma^i_n}+\int_{\tau_{i-1}\wedge\gamma^i_n}^{\tau_i\wedge\gamma^i_n}\Theta^{\mathcal{D}}_sg^0_sds-\big(\mathcal{M}^{i,n}_{\tau_i}-\mathcal{M}^{i,n}_{\tau_{i-1}}+\mathscr{M}^{i,n}_{\tau_i}-\mathscr{M}^{i,n}_{\tau_{i-1}}\big),\quad P-\text{a.s. and thus } Q^{\mathcal{D}}-a.s.,\end{aligned}\tag{6.43}$$

where $\mathcal{M}^{i,n}_t:=\int_0^t\mathbf{1}_{\{s\le\gamma^i_n\}}\Theta^{\mathcal{D}}_sZ^i_sdB^{\mathcal{D}}_s$ and $\mathscr{M}^{i,n}_t:=\int_{(0,t]}\int_{\mathcal{X}}\mathbf{1}_{\{s\le\gamma^i_n\}}\Theta^{\mathcal{D}}_sU^i_s(x)\widetilde{N}^{\mathcal{D}}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. Then the Burkholder-Davis-Gundy inequality, (1.5) and (6.42) imply that

$$E_{Q^{\mathcal{D}}}\Big[\sup_{t\in[0,T]}\big|\mathcal{M}^{i,n}_t\big|^p+\sup_{t\in[0,T]}\big|\mathscr{M}^{i,n}_t\big|^p\Big]\le c_pE_{Q^{\mathcal{D}}}\Big[\Big(\int_0^{\gamma^i_n}(\Theta^{\mathcal{D}}_s)^2|Z^i_s|^2ds\Big)^{\frac{p}{2}}+\int_0^{\gamma^i_n}\int_{\mathcal{X}}(\Theta^{\mathcal{D}}_s)^p|U^i_s(x)|^p\nu(dx)ds\Big]\le c_pe^{p\widehat{C}}\big(n^{\frac{p}{2}}+n\big)<\infty.$$

So both $\mathcal{M}^{i,n}$ and $\mathscr{M}^{i,n}$ are two uniformly integrable $Q^{\mathcal{D}}-$martingales. Taking conditional expectation $E_{Q^{\mathcal{D}}}[\,|\mathcal{F}_{\tau_{i-1}\wedge\gamma^i_n}]$ in (6.43) yields that

$$\Theta^{\mathcal{D}}_{\tau_{i-1}\wedge\gamma^i_n}Y^i_{\tau_{i-1}\wedge\gamma^i_n}+\int_0^{\tau_{i-1}\wedge\gamma^i_n}\Theta^{\mathcal{D}}_sg^0_sds\ge E_{Q^{\mathcal{D}}}\Big[\Theta^{\mathcal{D}}_{\tau_i\wedge\gamma^i_n}Y^i_{\tau_i\wedge\gamma^i_n}+\int_0^{\tau_i\wedge\gamma^i_n}\Theta^{\mathcal{D}}_sg^0_sds\Big|\mathcal{F}_{\tau_{i-1}\wedge\gamma^i_n}\Big],\quad Q^{\mathcal{D}}-a.s.\tag{6.44}$$

Set $\eta^{\mathcal{D}}_{i,n}:=\Theta^{\mathcal{D}}_{\tau_i\wedge\gamma^i_n}Y^i_{\tau_i\wedge\gamma^i_n}-\Theta^{\mathcal{D}}_{\tau_i}X_{\tau_i}-\int_{\tau_i\wedge\gamma^i_n}^{\tau_i}\Theta^{\mathcal{D}}_sg^0_sds$. Doob's martingale inequality shows that

$$\varepsilon\,Q^{\mathcal{D}}\Big\{\sup_{t\in[0,T]}\big|E_{Q^{\mathcal{D}}}[\eta^{\mathcal{D}}_{i,n}|\mathcal{F}_t]\big|\ge\varepsilon\Big\}\le E_{Q^{\mathcal{D}}}\Big[\big|\eta^{\mathcal{D}}_{i,n}\big|\Big],\quad\forall\,\varepsilon>0.\tag{6.45}$$

As $(Z^i,U^i)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$, we have $\int_0^T\big(|Z^i_s|^2+\|U^i_s\|^p_{L^p_\nu}\big)ds<\infty$, $P-$a.s. and thus $Q^{\mathcal{D}}-$a.s. So for $Q^{\mathcal{D}}-$a.s. $\omega\in\Omega$ there exists a $N^i_\omega=N^{\mathcal{D},i}_\omega\in\mathbb{N}$ such that

$$\gamma^i_n(\omega)=T \text{ for any } n\ge N^i_\omega.\tag{6.46}$$

It follows that $\lim_{n\to\infty}\Theta^{\mathcal{D}}_{\tau_{i-1}\wedge\gamma^i_n}Y^i_{\tau_{i-1}\wedge\gamma^i_n}=\Theta^{\mathcal{D}}_{\tau_{i-1}}Y^i_{\tau_{i-1}}$ and $\lim_{n\to\infty}\Theta^{\mathcal{D}}_{\tau_i\wedge\gamma^i_n}Y^i_{\tau_i\wedge\gamma^i_n}=\Theta^{\mathcal{D}}_{\tau_i}Y^i_{\tau_i}=\Theta^{\mathcal{D}}_{\tau_i}X_{\tau_i}$, $Q-$a.s. even though Y^i may not be left-continuous. In particular, the second limit together with (6.46) further shows that $\lim_{n\to\infty}\eta^{\mathcal{D}}_{i,n}=0$. Since $|\eta^{\mathcal{D}}_{i,n}|\le e^{\widehat{C}}\big(Y^i_*+X_*+\int_0^T|g^0_s|ds\big)$, $\forall\,n\in\mathbb{N}$ by (6.42), an analogy to (6.12), (1.7) and (A1) show that

$$E_{Q^{\mathcal{D}}}\Big[Y^i_*+X_*+\int_0^T|g^0_s|ds\Big]\le3^{\frac1q}\big\|\mathscr{E}_T(M^{\mathcal{D}})\big\|_{L^q(\mathcal{F}_T)}\Big\{E\Big[(Y^i_*)^p+X^p_*+\Big(\int_0^T|g^0_s|ds\Big)^p\Big]\Big\}^{\frac1p}<\infty.$$

Letting $n\to\infty$ in (6.45), we can deduce from the dominated convergence theorem that $\lim_{n\to\infty}Q^{\mathcal{D}}\Big\{\sup_{t\in[0,T]}\big|E_{Q^{\mathcal{D}}}[\eta^{\mathcal{D}}_{i,n}|\mathcal{F}_t]\big|\ge\varepsilon\Big\}=0$, $\forall\,\varepsilon>0$ or $\big\{\sup_{t\in[0,T]}\big|E_{Q^{\mathcal{D}}}[\eta^{\mathcal{D}}_{i,n}|\mathcal{F}_t]\big|\big\}_{n\in\mathbb{N}}$ converges to 0 in probability $Q^{\mathcal{D}}$. Hence, there exists a sequence $\{n_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}\sup_{t\in[0,T]}\big|E_{Q^{\mathcal{D}}}[\eta^{\mathcal{D}}_{i,n_j}|\mathcal{F}_t]\big|=0$, $Q^{\mathcal{D}}-$a.s. By (6.44), it holds $Q^{\mathcal{D}}-$a.s. that

$$\begin{aligned}E_{Q^{\mathcal{D}}}\Big[\Theta^{\mathcal{D}}_{\tau_i}X_{\tau_i}+\int_0^{\tau_i}\Theta^{\mathcal{D}}_sg^0_sds\Big|\mathcal{F}_{\tau_{i-1}\wedge\gamma^i_{n_j}}\Big]&\le\Theta^{\mathcal{D}}_{\tau_{i-1}\wedge\gamma^i_{n_j}}Y^i_{\tau_{i-1}\wedge\gamma^i_{n_j}}+\int_0^{\tau_{i-1}\wedge\gamma^i_{n_j}}\Theta^{\mathcal{D}}_sg^0_sds-E_{Q^{\mathcal{D}}}\big[\eta^{\mathcal{D}}_{i,n_j}\big|\mathcal{F}_{\tau_{i-1}\wedge\gamma^i_{n_j}}\big]\\&\le\Theta^{\mathcal{D}}_{\tau_{i-1}\wedge\gamma^i_{n_j}}Y^i_{\tau_{i-1}\wedge\gamma^i_{n_j}}+\int_0^{\tau_{i-1}\wedge\gamma^i_{n_j}}\Theta^{\mathcal{D}}_sg^0_sds+\sup_{t\in[0,T]}\big|E_{Q^{\mathcal{D}}}[\eta^{\mathcal{D}}_{i,n_j}|\mathcal{F}_t]\big|,\quad\forall\,j\in\mathbb{N}.\end{aligned}$$

As $j\to\infty$, we see from (6.46) and the right continuity of process $E_{Q^{\mathcal{D}}}\big[\Theta^{\mathcal{D}}_{\tau_i}X_{\tau_i}+\int_0^{\tau_i}\Theta^{\mathcal{D}}_sg^0_sds|\mathcal{F}_t\big]$, $t\in[0,T]$ that

$$E_{Q^{\mathcal{D}}}\Big[\Theta^{\mathcal{D}}_{\tau_i}X_{\tau_i}+\int_0^{\tau_i}\Theta^{\mathcal{D}}_sg^0_sds\Big|\mathcal{F}_{\tau_{i-1}}\Big]\le\Theta^{\mathcal{D}}_{\tau_{i-1}}Y^i_{\tau_{i-1}}+\int_0^{\tau_{i-1}}\Theta^{\mathcal{D}}_sg^0_sds\le\Theta^{\mathcal{D}}_{\tau_{i-1}}X_{\tau_{i-1}}+\int_0^{\tau_{i-1}}\Theta^{\mathcal{D}}_sg^0_sds,\quad Q^{\mathcal{D}}-a.s.\tag{6.47}$$

In the second inequality above, we used the $g-$supermartingality of X: $X_{\tau_{i-1}} \ge \mathcal{E}^g_{\tau_{i-1},\tau_i}[X_{\tau_i}] = Y^i_{\tau_{i-1}}$, $P-$a.s., and thus $Q^{\mathcal{D}}-$a.s.

Let $i=1,\cdots,m'$. As $X_{\tau_{2i}}>b$ on $\{\tau_{2i}<t_m\}$,

$$\mathbf{1}_{\{\tau_{2i-1}<t_m\}}(X_{\tau_{2i}}-a)=\mathbf{1}_{\{\tau_{2i}<t_m\}}(X_{\tau_{2i}}-a)+\mathbf{1}_{\{\tau_{2i-1}<t_m=\tau_{2i}\}}(X_{t_m}-a)\ge \mathbf{1}_{\{\tau_{2i}<t_m\}}(b-a)-\mathbf{1}_{\{\tau_{2i-1}<t_m=\tau_{2i}\}}(X_{t_m}-a)^-.$$

Also, since $X_{\tau_{2i-1}}<a$ on $\{\tau_{2i-1}<t_m\}$, we can deduce from (6.47) that $Q^{\mathcal{D}}-$a.s.

$$\begin{aligned}\mathbf{1}_{\{\tau_{2i-1}<t_m\}}\Theta^{\mathcal{D}}_{\tau_{2i-1}}a &\ge \mathbf{1}_{\{\tau_{2i-1}<t_m\}}\Theta^{\mathcal{D}}_{\tau_{2i-1}}X_{\tau_{2i-1}} \ge E_{Q^{\mathcal{D}}}\bigg[\mathbf{1}_{\{\tau_{2i-1}<t_m\}}\Big(\Theta^{\mathcal{D}}_{\tau_{2i}}X_{\tau_{2i}}+\int_{\tau_{2i-1}}^{\tau_{2i}}\Theta^{\mathcal{D}}_s g^0_s ds\Big)\Big|\mathcal{F}_{\tau_{2i-1}}\bigg]\\ &\ge E_{Q^{\mathcal{D}}}\bigg[\Theta^{\mathcal{D}}_{\tau_{2i}}\Big(\mathbf{1}_{\{\tau_{2i-1}<t_m\}}a+\mathbf{1}_{\{\tau_{2i}<t_m\}}(b-a)-\mathbf{1}_{\{\tau_{2i-1}<t_m=\tau_{2i}\}}(X_{t_m}-a)^-\Big)+\mathbf{1}_{\{\tau_{2i-1}<t_m\}}\int_{\tau_{2i-1}}^{\tau_{2i}}\Theta^{\mathcal{D}}_s g^0_s ds\Big|\mathcal{F}_{\tau_{2i-1}}\bigg].\end{aligned}$$

Taking $E_{Q^{\mathcal{D}}}[\,]$ then yields that

$$(b-a)E_{Q^{\mathcal{D}}}\big[\mathbf{1}_{\{\tau_{2i}<t_m\}}\Theta^{\mathcal{D}}_{\tau_{2i}}\big]\le E_{Q^{\mathcal{D}}}\bigg[\mathbf{1}_{\{\tau_{2i-1}<t_m\}}a\big(\Theta^{\mathcal{D}}_{\tau_{2i-1}}-\Theta^{\mathcal{D}}_{\tau_{2i}}\big)+\mathbf{1}_{\{\tau_{2i-1}<t_m=\tau_{2i}\}}\Theta^{\mathcal{D}}_{\tau_{2i}}(X_{t_m}-a)^-+e^{\widehat{C}}\int_{\tau_{2i-1}}^{\tau_{2i}}|g^0_s|ds\bigg].$$

Since $\Theta^{\mathcal{D}}_{\tau_{2i}}-\Theta^{\mathcal{D}}_{\tau_{2i-1}}=\int_{\tau_{2i-1}}^{\tau_{2i}}\Theta^{\mathcal{D}}_s\mathfrak{a}^{2i}_s ds$, (6.42) and (6.38) implies that

$$\begin{aligned}(b-a)e^{-\widehat{C}}E_{Q^{\mathcal{D}}}\big[\mathbf{1}_{\{\tau_{2i}<t_m\}}\big]&\le(b-a)E_{Q^{\mathcal{D}}}\big[\mathbf{1}_{\{\tau_{2i}<t_m\}}\Theta^{\mathcal{D}}_{\tau_{2i}}\big]\\ &\le e^{\widehat{C}}E_{Q^{\mathcal{D}}}\bigg[\mathbf{1}_{\{\tau_{2i-1}<t_m\}}|a|\int_{\tau_{2i-1}}^{\tau_{2i}}\beta_s ds+\mathbf{1}_{\{\tau_{2i-1}<t_m=\tau_{2i}\}}(X_{t_m}-a)^-+\int_{\tau_{2i-1}}^{\tau_{2i}}|g^0_s|ds\bigg].\end{aligned}$$

Summing up over $i\in\{1,\cdots,m'\}$, we obtain (6.41).

3) *In this step, we show that* $E_{Q^{\mathcal{D}}}[\xi]\le e^{\widehat{C}}\mathcal{E}^{\Xi}_{0,t_m}[\xi], \quad \forall\,\xi\in L^p(\mathcal{F}_{t_m}).$ (6.48)

To see this, we let $\xi\in L^p(\mathcal{F}_{t_m})$ and $(Y,Z,U):=\big(Y^{\xi,g^{\Xi}_{t_m}},Z^{\xi,g^{\Xi}_{t_m}},U^{\xi,g^{\Xi}_{t_m}}\big)$. As $\tau_{2m'}=t_m$, (6.38) and (A3) (ii) show that

$$\begin{aligned}Z_s\Big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{b}^i_s\Big)+\int_{\mathcal{X}}U_s(x)\Big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{H}^i_s(x)\Big)\nu(dx)&\le\Lambda_s|Z_s|+\int_{\mathcal{X}}\Big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\big(\kappa_2U^+_s(x)-\kappa_1U^-_s(x)\big)\Big)\nu(dx)\\ &\le\Lambda_s|Z_s|+\kappa_2\int_{\mathcal{X}}U^+_s(x)\nu(dx)-\kappa_1\int_{\mathcal{X}}U^-_s(x)\nu(dx)\ \text{ holds } ds\times dP\text{-a.s. on } [0,t_m]\times\Omega.\end{aligned} \tag{6.49}$$

Let $k\in\mathbb{N}$ and define $\gamma_k:=\inf\big\{t\in[0,T]:\int_0^t|Z_s|^2ds+\int_0^t\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds>k\big\}\wedge T\in\mathcal{T}$. Set $\phi_t:=e^{\int_0^{t\wedge t_m}sgn(Y_s)\beta_s ds}\le e^{\widehat{C}}$, $t\in[0,T]$. Applying Itô's formula to ϕ_tY_t over period $[0,\gamma_k]$, we can deduce from (6.49) that

$$\begin{aligned}Y_0&=\phi_{\gamma_k}Y_{\gamma_k}+\int_0^{\gamma_k}\phi_s\Lambda_s|Z_s|ds-\kappa_1\int_0^{\gamma_k}\int_{\mathcal{X}}\phi_sU^-_s(x)\nu(dx)ds+\kappa_2\int_0^{\gamma_k}\int_{\mathcal{X}}\phi_sU^+_s(x)\nu(dx)ds\\ &\quad-\int_0^{\gamma_k}\phi_sZ_sdB_s-\int_{(0,\gamma_k]}\int_{\mathcal{X}}\phi_sU_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &\ge\phi_{\gamma_k}Y_{\gamma_k}+\int_0^{\gamma_k}\phi_sZ_s\Big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{b}^i_s\Big)ds+\int_0^{\gamma_k}\int_{\mathcal{X}}\phi_sU_s(x)\Big(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\mathfrak{H}^i_s(x)\Big)\nu(dx)ds\\ &\quad-\int_0^{\gamma_k}\phi_sZ_sdB_s-\int_{(0,\gamma_k]}\int_{\mathcal{X}}\phi_sU_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)=\phi_{\gamma_k}Y_{\gamma_k}-\mathcal{M}^k_T-\mathscr{M}^k_T,\quad P\text{-a.s. or } Q^{\mathcal{D}}-a.s.,\end{aligned} \tag{6.50}$$

where $\mathcal{M}^k_t:=\int_0^t\mathbf{1}_{\{s\le\gamma_k\}}\phi_sZ_sdB^{\mathcal{D}}_s$ and $\mathscr{M}^k_t:=\int_{(0,t]}\int_{\mathcal{X}}\mathbf{1}_{\{s\le\gamma_k\}}\phi_sU_s(x)\widetilde{N}^{\mathcal{D}}_{\mathfrak{p}}(ds,dx)$. The Burkholder-Davis-Gundy inequality and (1.5) imply that

$$E_{Q^{\mathcal{D}}}\Big[\sup_{r\in[0,T]}\big|\mathcal{M}^k_r\big|^p+\sup_{r\in[0,T]}\big|\mathscr{M}^k_r\big|^p\Big]\le c_pE_{Q^{\mathcal{D}}}\bigg[\Big(\int_0^{\gamma_k}\phi^2_r|Z_r|^2dr\Big)^{\frac{p}{2}}+\int_0^{\gamma_k}\int_{\mathcal{X}}\phi^p_r|U_r(x)|^p\nu(dx)dr\bigg]\le c_pe^{p\widehat{C}}\big(k^{\frac{p}{2}}+k\big)<\infty,$$

thus $\mathcal{M}$ and $\mathscr{M}$ are two uniformly integrable $Q^{\mathcal{D}}-$martingales. Taking expectation $E_{Q^{\mathcal{D}}}[\,]$ in (6.50) yields that

$$\mathcal{E}^{\Xi}_{0,t_m}[\xi]=Y_0\ge E_{Q^{\mathcal{D}}}\big[\phi_{\gamma_k}Y_{\gamma_k}\big]\ge e^{-\widehat{C}}E_{Q^{\mathcal{D}}}\big[Y_{\gamma_k}\big]. \tag{6.51}$$

As $(Z,U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$, one has $\int_0^T\big(|Z_s|^2+\|U_s\|^p_{L^p_\nu}\big)ds<\infty$, $P-$a.s. and thus $Q^{\mathcal{D}}-$a.s. So for $Q^{\mathcal{D}}-$a.s. $\omega\in\Omega$ there exists a $K_\omega\in\mathbb{N}$ such that $\gamma_k(\omega)=T$ for any $k\ge K_\omega$. It follows that $\lim\limits_{k\to\infty}Y_{\gamma_k}=Y_T=\xi$, $Q^{\mathcal{D}}-$a.s. even though the process Y may not be left-continuous. An analogy to (6.12) yields that $E_{Q^{\mathcal{D}}}[Y_*]=E\big[\mathscr{E}_T(M^{\mathcal{D}})Y_*\big]\le\|\mathscr{E}_T(M^{\mathcal{D}})\|_{L^q(\mathcal{F}_T)}\|Y\|_{\mathbb{D}^p}<\infty$. Then letting $k\to\infty$ in (6.51), we obtain (6.48) from the dominated convergence theorem.

Now, taking $\xi=(X_{t_m}-a)^-+\int_0^{t_m}|g^0_s|ds$ in (6.48) and setting $\eta:=1+U_{\mathcal{D}}(a,b;X)$, one can deduce from (6.41), Jensen's inequality, (6.38) and (A3) (ii) that

$$\begin{aligned}
1+\frac{e^{3\widehat{C}}}{b-a}&\bigg(|a|+\mathcal{E}^{\Xi}_{0,t_m}\Big[(X_{t_m}-a)^-+\int_0^{t_m}|g^0_s|ds\Big]\bigg)\ge 1+\frac{e^{2\widehat{C}}}{b-a}\bigg(|a|\widehat{C}+E_{Q^{\mathcal{D}}}\Big[(X_{t_m}-a)^-+\int_0^{t_m}|g^0_s|ds\Big]\bigg)\\
&\ge E_{Q^{\mathcal{D}}}[\eta]=E\big[\eta\mathscr{E}_T(M^{\mathcal{D}})\big]\ge\exp\bigg\{E\bigg[\ln\eta+M^{\mathcal{D}}_T-\frac12\langle M^{\mathcal{D},c}\rangle_T+\sum_{0<s\le T}\big(\ln(1+\Delta M^{\mathcal{D}}_s)-\Delta M^{\mathcal{D}}_s\big)\bigg]\bigg\}\\
&=\exp\bigg\{E\bigg[\ln\eta-\frac12\int_0^T\bigg(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}|\mathfrak{b}^i_s|^2\bigg)ds+\int_{(0,T]}\int_{\mathcal{X}}\bigg(\sum_{i=1}^{2m'}\mathbf{1}_{\{s\in(\tau_{i-1},\tau_i]\}}\Big(\ln\big(1+\mathfrak{H}^i_s(x)\big)-\mathfrak{H}^i_s(x)\Big)\bigg)N_{\mathfrak{p}}(ds,dx)\bigg]\bigg\}\\
&\ge\exp\bigg\{E\bigg[\ln\eta-\frac12\int_0^{t_m}\Lambda_s^2ds+\big(\ln(1+\kappa_1)-\kappa_2\big)N_{\mathfrak{p}}\big((0,t_m],\mathcal{X}\big)\bigg]\bigg\}\ge\exp\Big\{E\big[\ln\eta\big]-\frac12\widehat{C}+\big(\ln(1+\kappa_1)-\kappa_2\big)\nu(\mathcal{X})T\Big\}.
\end{aligned}$$

Then the conclusion follows. □

Proof of Proposition 4.3: We set $\wp:=\big(2^{p-4}\,p(p-1)\big)^{\frac1p}$ and define processes

$$a_t:=\beta_t+\frac{\Lambda_t^2}{p-1}+\frac{p-1}{p}\wp^{-q}\beta_t^q+\frac1p\wp^p\nu(\mathcal{X})\quad\text{and}\quad A_t:=p\int_0^t a_sds,\quad t\in[0,T].$$

Then $C_A:=\|A_T\|_{L^\infty(\mathcal{F}_T)}\le\big(p+q+(p-1)\wp^{-q}\big)\widehat{C}+\wp^p\nu(\mathcal{X})T$.

The process Y has two jumps sources: the jump times of the stochastic integral M^U are totally inaccessible, while the jumps of the $\mathbf{F}-$predictable càdlàg increasing process K are exhausted by a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of $\mathbf{F}-$predictable stopping times $\big($i.e. $\{(t,\omega)\in[0,T]\times\Omega:\Delta K_t(\omega)>0\}$ is a union of graphs $[\![\zeta_n]\!]$ and these graphs are disjoint on $(0,T)$, see e.g. "Complements to Chapter IV" of [34] or Proposition I.2.24 of [47] for details$\big)$. In particular, one can deduce that for $P-$a.s. $\omega\in\Omega$

$$\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\Delta K_t(\omega)=0\quad\text{and}\quad\Delta Y_t(\omega)=\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}U\big(t,\omega,\mathfrak{p}_t(\omega)\big)-\mathbf{1}_{\{t\notin D_{\mathfrak{p}(\omega)}\}}\Delta K_t(\omega),\quad\forall\,t\in[0,T]. \tag{6.52}$$

Since the càdlàg increasing process K and the Poisson stochastic integral M^U jump countably many times along their $P-$a.s. paths, so does process Y: i.e.

$$\big\{t\in[0,T]:Y_{t-}(\omega)\ne Y_t(\omega)\big\}\text{ is a countable subset of }[0,T]\text{ for }P-\text{a.s. }\omega\in\Omega. \tag{6.53}$$

1) Fix $n\in\mathbb{N}$ and define $\tau_n:=\inf\Big\{t\in[0,T]:\int_0^t\big(|Z_s|^2+\|U_s\|^p_{L^p_\nu}\big)ds>n\Big\}\wedge T\in\mathcal{T}$. For any $\varepsilon\in(0,1]$, the function $\varphi_\varepsilon(x):=\big(|x|^2+\varepsilon\big)^{\frac12}$, $x\in\mathbb{R}$ has the following derivatives of its $p-$th power:

$$D\,\varphi^p_\varepsilon(x)=p\,\varphi^{p-2}_\varepsilon(x)\,x\quad\text{and}\quad D^2\,\varphi^p_\varepsilon(x)=p\,\varphi^{p-2}_\varepsilon(x)+p(p-2)\varphi^{p-4}_\varepsilon(x)\,x^2\ge p(p-1)\varphi^{p-2}_\varepsilon(x). \tag{6.54}$$

Now, let us fix $(t,\varepsilon)\in[0,T]\times(0,1]$. Applying Itô's formula (see e.g. Theorem VIII.27 of [34] or Theorem II.32 of [82]) to process $e^{A_s}\varphi^p_\varepsilon(Y_s)$ over the interval $[\tau_n\wedge t,\tau_n]$ yields that

$$\begin{aligned}
&e^{A_{\tau_n\wedge t}}\varphi^p_\varepsilon(Y_{\tau_n\wedge t})+\frac12\int_{\tau_n\wedge t}^{\tau_n}e^{A_s}D^2\varphi^p_\varepsilon(Y_s)|Z_s|^2ds+\sum_{s\in(\tau_n\wedge t,\tau_n]}e^{A_s}\Big(\varphi^p_\varepsilon(Y_s)-\varphi^p_\varepsilon(Y_{s-})-D\varphi^p_\varepsilon(Y_{s-})\Delta Y_s\Big)\\
&\quad=e^{A_{\tau_n}}\varphi^p_\varepsilon(Y_{\tau_n})+p\int_{\tau_n\wedge t}^{\tau_n}e^{A_s}\big[\varphi^{p-2}_\varepsilon(Y_s)\,Y_s\,g(s,Y_s,Z_s,U_s)-a_s\varphi^p_\varepsilon(Y_s)\big]ds\\
&\qquad+p\int_{\tau_n\wedge t}^{\tau_n}e^{A_s}\varphi^{p-2}_\varepsilon(Y_{s-})Y_{s-}dK_s-p\big(M^n_T-M^n_t+\mathcal{M}^n_T-\mathcal{M}^n_t\big),\quad P-\text{a.s.},
\end{aligned} \tag{6.55}$$

where $M^n_s := M^{n,\varepsilon}_s = \int_0^{\tau_n\wedge s} e^{A_r}\varphi_\varepsilon^{p-2}(Y_{r-})Y_{r-}Z_r dB_r$ and $\mathcal{M}^n_s := \mathcal{M}^{n,\varepsilon}_s = \int_{(0,\tau_n\wedge s]}\int_{\mathcal{X}} e^{A_r}\varphi_\varepsilon^{p-2}(Y_{r-})Y_{r-}U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)$, $\forall\, s\in[0,T]$. Similar to (5.10) of [94], we can deduce from Taylor's Expansion Theorem and (6.54) that $P-$a.s.

$$\varphi_\varepsilon^p\big(Y_s\big)-\varphi_\varepsilon^p\big(s,Y_{s-}\big)-D\varphi_\varepsilon^p\big(s,Y_{s-}\big)\Delta Y_s \ge p(p-1)\big|\Delta Y_s\big|^2\int_0^1(1-\alpha)\varphi_\varepsilon^{p-2}\big(Y_{s-}+\alpha\Delta Y_s\big)d\alpha. \tag{6.56}$$

When $|Y_{s-}|\le|\Delta Y_s|$, one has $\varphi_\varepsilon^{p-2}(Y_{s-}+\alpha\Delta Y_s)\ge\big((|Y_{s-}|+\alpha|\Delta Y_s|)^2+\varepsilon\big)^{\frac{p}{2}-1}\ge\big(4|\Delta Y_s|^2+\varepsilon\big)^{\frac{p}{2}-1}\ge 2^{p-2}\big(|\Delta Y_s|^2+\varepsilon\big)^{\frac{p}{2}-1}$, $\forall\,\alpha\in[0,1]$. So it follows from (6.56) and (6.52) that for $P-$a.s. $\omega\in\Omega$

$$\begin{aligned}
&\sum_{s\in(\tau_n(\omega)\wedge t,\tau_n(\omega)]} e^{A_s(\omega)}\Big(\varphi_\varepsilon^p\big(Y_s(\omega)\big)-\varphi_\varepsilon^p\big(Y_{s-}(\omega)\big)-D\varphi_\varepsilon^p\big(Y_{s-}(\omega)\big)\Delta Y_s(\omega)\Big)\\
&\quad\ge 2^{p-3}p(p-1)\sum_{s\in(\tau_n(\omega)\wedge t,\tau_n(\omega)]}\mathbf{1}_{\{|Y_{s-}(\omega)|\le|\Delta Y_s(\omega)|\}}e^{A_s(\omega)}\big|\Delta Y_s(\omega)\big|^2\big(|\Delta Y_s(\omega)|^2+\varepsilon\big)^{\frac{p}{2}-1}\\
&\quad\ge 2^{p-3}p(p-1)\sum_{s\in D_{\mathfrak{p}(\omega)}\cap(\tau_n(\omega)\wedge t,\tau_n(\omega)]}\mathbf{1}_{\{|Y_{s-}(\omega)|\le|U(s,\omega,\mathfrak{p}_s(\omega))|\}}e^{A_s(\omega)}\big|U(s,\omega,\mathfrak{p}_s(\omega))\big|^2\big(|U(s,\omega,\mathfrak{p}_s(\omega))|^2+\varepsilon\big)^{\frac{p}{2}-1}\\
&\quad= 2^{p-3}p(p-1)\bigg(\int_{(\tau_n\wedge t,\tau_n]}\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}e^{A_s}\big|U_s(x)\big|^2\big(|U_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}N_{\mathfrak{p}}(ds,dx)\bigg)(\omega).
\end{aligned}\tag{6.57}$$

Also, (4.5) and Young's inequality imply that $P-$a.s.

$$\begin{aligned}
\varphi_\varepsilon^{p-2}(Y_s)\,Y_s\,g(s,Y_s,Z_s,U_s)&\le\varphi_\varepsilon^{p-2}(Y_s)|Y_s|\big(\mathfrak{f}_s+\beta_t\big(|Y_s|+\|U_s\|_{L^p_\nu}\big)+\Lambda_s|Z_s|\big)\\
&\le\mathfrak{f}_s\varphi_\varepsilon^{p-1}(Y_s)+\beta_s\varphi_\varepsilon^p(Y_s)+\Lambda_s\varphi_\varepsilon^{p-2}(Y_s)|Y_s||Z_s|+\beta_s\varphi_\varepsilon^{p-1}(Y_s)\|U_s\|_{L^p_\nu}\\
&\le\mathfrak{f}_s\varphi_\varepsilon^{p-1}(Y_s)+\bigg(\beta_s+\frac{\Lambda_s^2}{p-1}+\frac{1}{q}\wp^{-q}\beta_s^q\bigg)\varphi_\varepsilon^p(Y_s)+\frac{p-1}{4}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2+\frac{1}{p}\wp^p\|U_s\|^p_{L^p_\nu}\ \text{ for a.e. } s\in[0,T].
\end{aligned}$$

Since an analogy to (5.12) of [94] shows that $\|U_s\|^p_{L^p_\nu}\le\varphi_\varepsilon^p(Y_{s-})\nu(\mathcal{X})+\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}|U_s(x)|^p\nu(dx)$ for any $s\in[0,T]$, we see from (6.53) that $P-$a.s.

$$\begin{aligned}
\varphi_\varepsilon^{p-2}(Y_s)\,Y_s\,g(s,Y_s,Z_s,U_s)\ &\le\ \mathfrak{f}_s\varphi_\varepsilon^{p-1}(Y_s)+a_s\varphi_\varepsilon^p(Y_s)+\frac{p-1}{4}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2\\
&\qquad+\frac{1}{p}\wp^p\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}|U_s(x)|^p\nu(dx)\ \text{ for a.e. } s\in[0,T].
\end{aligned}\tag{6.58}$$

The function $\psi(x):=x\varphi_\varepsilon^{p-2}(x)=x(x^2+\varepsilon)^{\frac{p}{2}-1}$, $x\in\mathbb{R}$ has strictly positive derivative $\frac{d}{dx}\psi(x)=(x^2+\varepsilon)^{\frac{p}{2}-2}\big((p-1)x^2+\varepsilon\big)>0$, so it satisfies $\psi(x)\le\psi(x^+)\le(x^+)^{p-1}$, $\forall\,x\in\mathbb{R}$. Then one can deduce from the flat-off condition in (4.4) that $P-$a.s.

$$\int_{\tau_n\wedge t}^{\tau_n}e^{A_s}\varphi_\varepsilon^{p-2}(Y_{s-})Y_{s-}dK_s=\int_{\tau_n\wedge t}^{\tau_n}\mathbf{1}_{\{Y_{s-}\le X_{s-}\}}e^{A_s}\psi(Y_{s-})dK_s\le\int_{\tau_n\wedge t}^{\tau_n}\mathbf{1}_{\{Y_{s-}\le X_{s-}\}}e^{A_s}\psi(X_{s-})dK_s\le e^{C_A}\int_0^{\tau_n}\big(X^+_{s-}\big)^{p-1}dK_s.$$

Plugging this inequality together with (6.54), (6.57), (6.58) back into (6.55) yield that

$$\begin{aligned}
&e^{A_{\tau_n\wedge t}}\varphi_\varepsilon^p(Y_{\tau_n\wedge t})+\frac{p}{4}(p-1)\int_{\tau_n\wedge t}^{\tau_n}e^{A_s}\,\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2ds+2\wp^p\int_{(\tau_n\wedge t,\tau_n]}\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}e^{A_s}\big|U_s(x)\big|^2\big(|U_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}N_{\mathfrak{p}}(ds,dx)\\
&\qquad\le\eta^\varepsilon_t+\wp^p\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}e^{A_s}|U_s(x)|^p\nu(dx)ds-p\,(M_T-M_t+\mathcal{M}_T-\mathcal{M}_t),\quad P-\text{a.s.},
\end{aligned}$$

where $\eta^\varepsilon_t=\eta^{n,\varepsilon}_t:=e^{C_A}\big(\varphi_\varepsilon^p(Y_{\tau_n})+p\int_{\tau_n\wedge t}^{\tau_n}\varphi_\varepsilon^{p-1}(Y_s)\mathfrak{f}_s ds+p\int_0^{\tau_n}(X^+_{s-})^{p-1}dK_s\big)$. Since $E\Big[\sup_{s\in[0,T]}\varphi_\varepsilon^p(Y_s)\Big]\le E\Big[\sup_{s\in[0,T]}|Y_s|^p\Big]+\varepsilon^{\frac{p}{2}}=\|Y\|^p_{\mathbb{D}^p}+\varepsilon^{\frac{p}{2}}<\infty$ by (1.7), Young's inequality implies that

$$\begin{aligned}
E[\eta^\varepsilon_t]\ &\le\ e^{C_A}E\bigg[\sup_{s\in[0,\tau_n]}\varphi_\varepsilon^p(Y_s)+p\sup_{s\in[0,\tau_n]}\varphi_\varepsilon^{p-1}(Y_s)\int_0^{\tau_n}\mathfrak{f}_s ds+pK_{\tau_n}\sup_{s\in[0,\tau_n]}(X^+_s)^{p-1}\bigg]\\
&\le\ e^{C_A}E\bigg[p\sup_{s\in[0,T]}\varphi_\varepsilon^p(Y_s)+\Big(\int_0^T\mathfrak{f}_s ds\Big)^p+(p-1)\sup_{s\in[0,T]}(X^+_s)^p+K_T^p\bigg]<\infty.
\end{aligned}$$

Then using similar arguments to those that lead to (5.24) of [94], we can obtain that

$$E\bigg[\sup_{s\in[0,\tau_n]}|Y_s|^p+\Big(\int_0^{\tau_n}|Z_s|^2ds\Big)^{\frac{p}{2}}+\int_0^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds\bigg]\le\mathcal{C}\mathscr{J}_n, \tag{6.59}$$

where $\mathscr{J}_n:=E\big[|Y_{\tau_n}|^p+\big(\int_0^T\mathfrak{f}_sds\big)^p+\int_0^{\tau_n}(X^+_{s-})^{p-1}dK_s\big]$.

2) Since it holds $P-$a.s. that $Y_0=Y_t+\int_0^t g(s,Y_s,Z_s,U_s)ds+K_t-\int_0^t Z_sdB_s-\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ \forall\,t\in[0,T]$, (4.5) and Hölder's inequality imply that $P-$a.s.

$$\begin{aligned}K_{\tau_n}&\le 2\sup_{t\in[0,\tau_n]}|Y_t|+\int_0^{\tau_n}\big(\mathfrak{f}_s+\beta_s|Y_s|+\Lambda_s|Z_s|+\beta_s\|U_s\|_{L^p_\nu}\big)ds+\bigg|\int_0^{\tau_n}Z_sdB_s\bigg|+\bigg|\int_{(0,\tau_n]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\bigg|\\&\le\int_0^T\mathfrak{f}_sds+(2+\widehat{C})\sup_{t\in[0,\tau_n]}|Y_t|+\widehat{C}^{\frac12}\bigg(\int_0^{\tau_n}|Z_s|^2ds\bigg)^{\frac12}+\bigg(\int_0^T\beta_s^qds\bigg)^{\frac1q}\bigg(\int_0^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds\bigg)^{\frac1p}\\&\quad+\sup_{t\in[0,T]}\bigg|\int_0^{\tau_n\wedge t}Z_sdB_s\bigg|+\sup_{t\in[0,T]}\bigg|\int_{(0,\tau_n\wedge t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\bigg|.\end{aligned} \tag{6.60}$$

Taking $p-$th power and then taking expectation in (6.60), one can deduce from (1.7), the Burkholder-Davis-Gundy inequality, (6.59) and (1.5) that

$$\begin{aligned}6^{1-p}E\big[K^p_{\tau_n}\big]&\le E\bigg[\Big(\int_0^T\mathfrak{f}_sds\Big)^p\bigg]+(2+\widehat{C})^pE\bigg[\sup_{t\in[0,\tau_n]}|Y_t|^p\bigg]+\Big(\widehat{C}^{\frac p2}+c_p\Big)E\bigg[\bigg(\int_0^{\tau_n}|Z_s|^2ds\bigg)^{\frac p2}\bigg]\\&\quad+\widehat{C}^{\frac pq}E\int_0^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds+c_pE\bigg[\bigg(\int_{(0,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)\bigg)^{\frac p2}\bigg]\\&\le\mathcal{C}E\bigg[\Big(\int_0^T\mathfrak{f}_sds\Big)^p+\sup_{t\in[0,\tau_n]}|Y_t|^p+\bigg(\int_0^{\tau_n}|Z_s|^2ds\bigg)^{\frac p2}+\int_0^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds\bigg]\\&\le\mathcal{C}\mathscr{J}_n\le\frac{6^{1-p}}{2}E\big[K^p_{\tau_n}\big]+\mathcal{C}\lambda_n,\end{aligned}$$

with $\lambda_n:=E\big[|Y_{\tau_n}|^p+\big(\int_0^T\mathfrak{f}_sds\big)^p+(X^+_*)^p\big]$. It follows that $E\big[K^p_{\tau_n}\big]\le\mathcal{C}\lambda_n$ and thus that $\mathscr{J}_n\le\mathcal{C}E\big[K^p_{\tau_n}\big]+\mathcal{C}\lambda_n\le\mathcal{C}\lambda_n$. Then we see from (6.59) that

$$E\bigg[\sup_{s\in[0,\tau_n]}|Y_s|^p+\Big(\int_0^{\tau_n}|Z_s|^2ds\Big)^{\frac p2}+\int_0^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds+K^p_{\tau_n}\bigg]\le\mathcal{C}\lambda_n. \tag{6.61}$$

As $(Z,U)\in\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, it holds for all $\omega\in\Omega$ except on a $P-$null set $\mathcal{N}$ that $\tau_{\mathfrak{n}}(\omega)=T$ for some $\mathfrak{n}=\mathfrak{n}(\omega)\in\mathbb{N}$. For any $\omega\in\mathcal{N}^c$, $\lim_{n\to\infty}Y(\tau_n(\omega),\omega)=Y(T,\omega)=\xi(\omega)$ and $\lim_{n\to\infty}K(\tau_n(\omega),\omega)=K(T,\omega)$ although the paths $Y_\cdot(\omega)$, $K_\cdot(\omega)$ may not be left-continuous. Therefore, letting $n\to\infty$ in (6.61), we can deduce (4.6) from the monotone convergence theorem and the dominated convergence theorem. □

Proof of Theorem 4.1: 1) Let X first be a $g-$supermartingale. Fix $n\in\mathbb{N}$. Clearly,

$$g^n(t,\omega,y,z,u):=g(t,\omega,y,z,u)+n\big(X(t,\omega)-y\big),\quad\forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$$

defines a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable mapping that satisfies (A3) automatically. By (1.7) and Hölder's inequality , $E\Big[\big(\int_0^T|g^n(t,0,0,0)|dt\big)^p\Big]\le 2^{p-1}E\big[T\int_0^T|g(t,0,0,0)|^pdt+(nT)^pX^p_*\big]<\infty$, so (A1) holds for g^n. Also, (A2) implies that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$

$$\begin{aligned}\big|g^n(t,\omega,y_1,z_1,u)-g^n(t,\omega,y_2,z_2,u)\big|&\le|g(t,\omega,y_1,z_1,u)-g(t,\omega,y_2,z_2,u)|+n|y_1-y_2|\\&\le\big(\beta(t,\omega)+n\big)|y_1-y_2|+\Lambda(t,\omega)|z_1-z_2|,\quad\forall\,(y_1,z_1),\ (y_2,z_2)\in\mathbb{R}\times\mathbb{R}^d,\ \forall\,u\in L^p_\nu.\end{aligned}$$

Hence, g^n satisfies (A2) with $\beta^n_t:=\beta_t+n$, $t\in[0,T]$, which is a $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable process with

$\big\| \int_0^T (\beta_t^n)^q dt \big\|_{L^\infty(\mathcal{F}_T)} \le 2^{q-1}\Big(\big\| \int_0^T \beta_t^q dt \big\|_{L^\infty(\mathcal{F}_T)} + n^q T \Big) < \infty$. Then we know from Theorem 2.1 that the BSDEJ (X_T, g^n) admits a unique solution $(Y^n, Z^n, U^n) \in \mathbb{S}^p$.

1a) *We first show that* $P\big\{Y_t^n \le X_t,\ \forall\, t \in [0,T]\big\} = 1.$ (6.62)

Let $i \in \mathbb{N}$. In light of the Debut Theorem (see e.g. Theorem IV.50 of [33]), $\tau_i^n := \inf\{t \in [0,T] : Y_t^n \ge X_t + 1/i\} \wedge T$ defines an $\mathbf{F}-$stopping time. As $Y_T^n = X_T$, $P-$a.s., the $\mathbf{F}-$stopping time $\gamma_i^n := \inf\{t \in [\tau_i^n, T] : Y_t^n \le X_t\}$ satisfies $\tau_i^n \le \gamma_i^n \le T$, $P-$a.s. And the right continuity of process $Y^n - X$ implies that

$$Y_{\gamma_i^n}^n \le X_{\gamma_i^n}, \quad P-\text{a.s.} \tag{6.63}$$

In light of Theorem 2.1, the unique solution $(\mathcal{Y}^n, \mathcal{Z}^n, \mathcal{U}^n) \in \mathbb{S}^p$ of BSDEJ $\big(Y_{\gamma_i^n}^n, g_{\gamma_i^n}\big)$ satisfies $\mathcal{Y}_{\gamma_i^n}^n = \mathcal{Y}_T^n = Y_{\gamma_i^n}^n$, $P-$a.s. Since $Y_s^n > X_s$ over period $[\tau_i^n, \gamma_i^n)$, it holds $P-$a.s. that

$$g^n\big(s, Y_s^n, Z_s^n, U_s^n\big) = g\big(s, Y_s^n, Z_s^n, U_s^n\big) + n(X_s - Y_s^n) \le g\big(s, Y_s^n, Z_s^n, U_s^n\big) = g_{\gamma_i^n}\big(s, Y_s^n, Z_s^n, U_s^n\big), \quad \forall\, s \in (\tau_i^n, \gamma_i^n).$$

Since $g_{\gamma_i^n}$ also satisfies (A2) and (A3) by Remark 2.1 (3), applying Theorem 2.2 with $(\tau, \gamma) = (\tau_i^n, \gamma_i^n)$, $(Y^1, Z^1, U^1) = (Y^n, Z^n, U^n)$ and $(Y^2, Z^2, U^2) = (\mathcal{Y}^n, \mathcal{Z}^n, \mathcal{U}^n)$, we can deduce from (6.63), the monotonicity (g1) of $g-$evaluations, the $g-$supermartingality of X as well as Proposition 4.1 that

$$Y_{\tau_i^n}^n \le \mathcal{Y}_{\tau_i^n}^n = \mathcal{E}^g_{\tau_i^n, \gamma_i^n}\big[Y_{\gamma_i^n}^n\big] \le \mathcal{E}^g_{\tau_i^n, \gamma_i^n}\big[X_{\gamma_i^n}\big] \le X_{\tau_i^n} \tag{6.64}$$

holds except on a $P-$null set $\mathcal{N}_i^n$. For all $\omega \in \Omega$ except on a $P-$null set $\widetilde{\mathcal{N}}_n$, the paths $Y_\cdot^n(\omega) - X_\cdot(\omega)$ is right-continuous. Given $\omega \in \{\tau_i^n < T\} \cap \widetilde{\mathcal{N}}_n^c$, the definition of τ_i^n and the right-continuity of the paths $Y_\cdot^n(\omega) - X_\cdot(\omega)$ imply that $Y^n\big(\tau_i^n(\omega), \omega\big) \ge X\big(\tau_i^n(\omega), \omega\big) + 1/i$. Comparing this inequality with (6.64) shows that $\{\tau_i^n < T\} \cap \widetilde{\mathcal{N}}_n^c \subset \mathcal{N}_i^n$, and it follows that $\{\tau_i^n < T\} \subset \widetilde{\mathcal{N}}_n \cup \mathcal{N}_i^n$.

Taking union over $i \in \mathbb{N}$ yields that

$$\{Y_t^n > X_t, \text{ for some } t \in [0,T)\} = \underset{i \in \mathbb{N}}{\cup} \{Y_t^n \ge X_t + 1/i, \text{ for some } t \in [0,T)\} \subset \underset{i \in \mathbb{N}}{\cup} \{\tau_i^n < T\} \subset \widetilde{\mathcal{N}}_n \cup \Big(\underset{i \in \mathbb{N}}{\cup} \mathcal{N}_i^n \Big).$$

So $P\{Y_t^n \le X_t, \forall\, t \in [0,T)\} = 1$, which together with $P\{Y_T^n = X_T\} = 1$ proves (6.62).

1b) Then $K_t^n := n\int_0^t (X_s - Y_s^n) ds$, $t \in [0,T]$ is an $\mathbf{F}-$adapted, continuous increasing process with $K_0^n = 0$. By (1.7), $E\big[(K_T^n)^p\big] \le n^p E\big[(\int_0^T (X_s - Y_s^n) ds)^p\big] \le 2^{p-1}(nT)^p E\big[X_*^p + (Y_*^n)^p\big] < \infty$. So $K^n \in \mathbb{K}^p$. We also see from (6.62) that $P\{Y_{t-}^n \le X_{t-}, \ \forall\, t \in [0,T]\} = 1$ or $P\{\mathbf{1}_{\{Y_{t-}^n > X_{t-}\}} = 0, \ \forall\, t \in [0,T]\} = 1$, which shows that (4.4) holds with $g = g$ and $(Y, Z, U, K, \xi) = (Y^n, Z^n, U^n, K^n, X_T) \in \mathbb{D}^p \times \mathbb{Z}^{2,p} \times \mathbb{U}^p \times \mathbb{K}^p \times L^p(\mathcal{F}_T)$. Since Remark 2.1 (1) implies that g satisfies (4.5) with $\mathfrak{f}_t = |g(t,0,0,0)|$, an application of Proposition 4.3 yields that

$$\|Y^n\|_{\mathbb{D}^p}^p + \big\|Z^n\big\|_{\mathbb{Z}^{2,p}}^p + \big\|U^n\big\|_{\mathbb{U}^p}^p + E\big[(K_T^n)^p\big] \le \mathcal{C} E\bigg[|X_T|^p + \Big(\int_0^T |g(t,0,0,0)| dt \Big)^p + (X_*^+)^p \bigg] \le \mathcal{C}\alpha_X, \tag{6.65}$$

where the constant $\mathcal{C}$ does not depend on n and $\alpha_X := E\big[X_*^p + \int_0^T |g(t,0,0,0)|^p dt\big]$. Since g satisfies (A2), (A3), we can deduce from Hölder's inequality, (1.7), (6.1) and (6.65) that

$$\begin{aligned}
E\int_0^T \big|g^n(t, Y_t^n, Z_t^n, U_t^n)\big|^p dt &= E\int_0^T \big|g(t, Y_t^n, Z_t^n, U_t^n) + n(X_t - Y_t^n)\big|^p dt \\
&\le 6^{p-1}\Big\{ E\int_0^T \big[(\beta_t^p + n^p)|Y_t^n|^p + \Lambda_t^p |Z_t^n|^p + \kappa_2^p (\nu(\mathcal{X}))^{\frac{p}{q}} \|U_t^n\|_{L_\nu^p}^p + |g(t,0,0,0)|^p \big] dt + n^p T E\big[X_*^p\big] \Big\} \\
&\le 6^{p-1}\bigg\{ (\widehat{C} + n^p T)\|Y^n\|_{\mathbb{D}^p}^p + \bigg\| \int_0^T \Lambda_t^{\frac{2p}{2-p}} dt \bigg\|_{L^\infty(\mathcal{F}_T)}^{\frac{2-p}{2}} E\bigg[\Big(\int_0^T |Z_t^n|^2 dt \Big)^{\frac{p}{2}} \bigg] + \kappa_2^p (\nu(\mathcal{X}))^{\frac{p}{q}} E\int_0^T \|U_t^n\|_{L_\nu^p}^p dt + (1 + n^p T)\alpha_X \bigg\} \\
&\le \mathcal{C}\bigg(1 + n^p + \bigg\| \int_0^T \Lambda_t^{\frac{2p}{2-p}} dt \bigg\|_{L^\infty(\mathcal{F}_T)}^{\frac{2-p}{2}} + \kappa_2^p \bigg)\alpha_X < \infty \quad \text{for } p \in (1,2),
\end{aligned}$$

and similarly that

$$E\int_0^T \big|g^n(t,Y_t^n,Z_t^n,U_t^n)\big|^2 dt \le 6\Big\{E\int_0^T \big[(\beta_t^2+n^2)|Y_t^n|^2+\kappa_\Lambda^2|Z_t^n|^2+\kappa_2^2\nu(\mathcal{X})\|U_t^n\|_{L^2_\nu}^2+|g(t,0,0,0)|^2\big]dt+n^2TE\big[X_*^2\big]\Big\}$$
$$\le 6\bigg\{(\widehat{C}+n^2T)\|Y^n\|_{\mathbb{D}^2}^2+\kappa_\Lambda^2E\int_0^T|Z_t^n|^2dt+\kappa_2^2\,\nu(\mathcal{X})E\int_0^T\|U_t^n\|_{L^2_\nu}^2dt+(1+n^2T)\alpha_X\bigg\}\le \mathcal{C}\big(1+n^2+\kappa_\Lambda^2+\kappa_2^2\big)\alpha_X<\infty.$$

By (6.62) again, it holds $P-$a.s. that $g^n(t,Y_t^n,Z_t^n,U_t^n)\le g^{n+1}(t,Y_t^n,Z_t^n,U_t^n)$ for any $t\in[0,T]$. Since g^{n+1} satisfies (A2) and (A3), Theorem 2.2 implies that $P\{Y_t^n\le Y_t^{n+1},\ \forall\, t\in[0,T]\}=1$. In light of Theorem A.1, $Y_t:=\overline{\lim_{n\to\infty}}Y_t^n$, $t\in[0,T]$ defines a process of $\mathbb{D}^p$ satisfies (A.4), and there exist $(g,Z,U,K)\in\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)\times\mathbb{Z}^{2,p}\times\mathbb{U}^p\times\mathbb{K}^p$ such that (A.5) holds $P-$a.s. and that (A.6) holds for any $\varpi\in(2/p,2)$. According to the proof of Theorem A.1, the process g is the weak limit of processes $\big\{g(t,Y_t^n,Z_t^n,U_t^n)\big\}_{t\in[0,T]}$, $n\in\mathbb{N}$ in $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$.

Let $\varpi\in(2/p,2)$ and set $\varrho:=\frac{p\varpi}{2}\in(1,p)$. Hölder's inequality, (A2), (A3), (6.1) and (1.7) imply that

$$E\int_0^T\big|g(t,Y_t^n,Z_t^n,U_t^n)-g(t,Y_t,Z_t,U_t)\big|^\varrho dt\le 3^{\varrho-1}E\int_0^T\bigg\{\beta_t^\varrho(Y_t-Y_t^n)^\varrho+\Lambda_t^\varrho|Z_t^n-Z_t|^\varrho+\kappa_2^\varrho\Big(\int_{\mathcal{X}}|U_t^n(x)-U_t(x)|\nu(dx)\Big)^\varrho\bigg\}dt$$
$$\le 3^{\varrho-1}E\int_0^T\beta_t^\varrho(Y_t-Y_t^n)^\varrho dt+3^{\varrho-1}\Big\|\int_0^T\Lambda_t^{\frac{2\varrho}{2-p}}dt\Big\|_{L^\infty(\mathcal{F}_T)}^{\frac{2-p}{2}}E\bigg[\Big(\int_0^T|Z_t^n-Z_t|^\varpi dt\Big)^{\frac{p}{2}}\bigg]$$
$$+3^{\varrho-1}\kappa_2^\varrho(\nu(\mathcal{X}))^{\varrho-1}E\int_0^T\int_{\mathcal{X}}|U_t^n(x)-U_t(x)|^\varrho\nu(dx)dt\quad\text{for } p\in(1,2),\tag{6.66}$$

and similarly that

$$E\int_0^T\big|g(t,Y_t^n,Z_t^n,U_t^n)-g(t,Y_t,Z_t,U_t)\big|^\varrho dt\le 3^{\varrho-1}E\int_0^T\beta_t^\varrho(Y_t-Y_t^n)^\varrho dt+3^{\varrho-1}\kappa_\Lambda^\varrho E\int_0^T|Z_t^n-Z_t|^\varpi dt$$
$$+3^{\varrho-1}\kappa_2^\varrho(\nu(\mathcal{X}))^{\varrho-1}E\int_0^T\int_{\mathcal{X}}|U_t^n(x)-U_t(x)|^\varrho\nu(dx)dt.\tag{6.67}$$

Since $Y_t-Y_t^n\le X_t-Y_t^1$, $dt\times dP-$a.s. by the monotonicity of $\{Y^n\}_{n\in\mathbb{N}}$ and since $E\int_0^T\beta_t^\varrho(X_t-Y_t^1)^\varrho dt\le\big\|\int_0^T\beta_t^\varrho dt\big\|_{L^\infty(\mathcal{F}_T)}\cdot E\big[(X_*^++Y_*^1)^\varrho\big]\le\big\|\int_0^T 1\vee\beta_t^q dt\big\|_{L^\infty(\mathcal{F}_T)}\cdot E\big[1+(X_*^++Y_*^1)^p\big]<\infty$, letting $n\to\infty$ in (6.66) and (6.67), we can deduce from the dominated convergence theorem and (A.6) that $\lim_{n\to\infty}E\int_0^T\big|g(t,Y_t^n,Z_t^n,U_t^n)-g(t,Y_t,Z_t,U_t)\big|^\varrho dt=0$. Then processes $\big\{g(t,Y_t^n,Z_t^n,U_t^n)\big\}_{t\in[0,T]}$, $n\in\mathbb{N}$ strongly converge and thus weakly converge to process $\big\{g(t,Y_t,Z_t,U_t)\big\}_{t\in[0,T]}$ in $\mathbb{L}^\varrho\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$. However, the weak convergence of $\big\{g(t,Y_t^n,Z_t^n,U_t^n)\big\}_{t\in[0,T]}$'s to g in $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$ implies that $\big\{g(t,Y_t^n,Z_t^n,U_t^n)\big\}_{t\in[0,T]}$'s also weakly converge to g in $\mathbb{L}^\varrho\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$. So by the uniqueness of the weak limit of processes $\big\{g(t,Y_t^n,Z_t^n,U_t^n)\big\}_{t\in[0,T]}$, $n\in\mathbb{N}$ in $\mathbb{L}^\varrho\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$, we obtain

$$g_t=g(t,Y_t,Z_t,U_t),\quad dt\times dP-\text{a.s.}\tag{6.68}$$

Given $n\in\mathbb{N}$, (6.62), Hölder's inequality and (6.65) show that

$$0\le E\int_0^T(X_t-Y_t^n)dt=\frac{1}{n}E[K_T^n]\le\frac{1}{n}\big\{E[(K_T^n)^p]\big\}^{\frac{1}{p}}\le\frac{1}{n}\mathcal{C}\big\{E\big[X_*^p\big]\big\}^{\frac{1}{p}}.\tag{6.69}$$

Since it holds $P-$a.s. that $X_t-Y_t^n\le X_t-Y_t^1$, $\forall\, t\in[0,T]$ by the monotonicity of $\{Y^n\}_{n\in\mathbb{N}}$ and since $E\int_0^T(X_t-Y_t^1)dt\le TE[X_*^++Y_*^1]\le TE\big[1+(X_*^++Y_*^1)^p\big]<\infty$, letting $n\to\infty$ in (6.69), we know from the dominated convergence theorem that $E\int_0^T(X_t-Y_t)dt=\lim_{n\to\infty}E\int_0^T(X_t-Y_t^n)dt=0$. This equality and (A.4) imply that $X_t-Y_t=0$, $dt\times dP-$a.s., which together with the right-continuity of processes $X-Y$ yields $P\{X_t=Y_t,\ \forall\, t\in[0,T]\}=1$. Putting it and (6.68) back to (A.5) leads to (4.3) for the case of $g-$supermartingale.

1c) Let $\big(\widetilde{Z},\widetilde{U},\widetilde{K}\big)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p\times\mathbb{K}^p$ be another triplet of processes such that $P-$a.s.

$$X_t=X_T+\int_t^T g(s,X_s,\widetilde{Z}_s,\widetilde{U}_s)ds-\int_t^T\widetilde{Z}_sdB_s-\int_{(t,T]}\int_{\mathcal{X}}\widetilde{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)+\widetilde{K}_T-\widetilde{K}_t,\quad t\in[0,T].$$

Subtracting it from (4.3) yields that $P-$a.s.

$$\int_0^t \big[g(s,X_s,Z_s,U_s)-g(s,X_s,\widetilde{Z}_s,\widetilde{U}_s)\big]ds+K_t-\widetilde{K}_t=\int_0^t \big(Z_s-\widetilde{Z}_s\big)dB_s+\int_{(0,t]}\int_{\mathcal{X}}\big(U_s(x)-\widetilde{U}_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ t\in[0,T]. \quad (6.70)$$

An analogy to (6.52) shows that for $P-$a.s. $\omega\in\Omega$

$$\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\big(\Delta K_t(\omega)-\Delta\widetilde{K}_t(\omega)\big)=0 \text{ and } 0=\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\big(U(t,\omega,\mathfrak{p}_t(\omega))-\widetilde{U}(t,\omega,\mathfrak{p}_t(\omega))\big)-\mathbf{1}_{\{t\notin D_{\mathfrak{p}(\omega)}\}}\big(\Delta K_t(\omega)-\Delta\widetilde{K}_t(\omega)\big),\ \ \forall\, t\in[0,T],$$

which implies that $P-$a.s., $\Delta K_t=\Delta\widetilde{K}_t$, $\forall\, t\in[0,T]$ and $U_t(x)=\widetilde{U}_t(x)$, $\forall\,(t,x)\in[0,T]\times\mathcal{X}$. It then follows from (6.70) that $P-$a.s., $\int_0^t\big[g(s,X_s,Z_s,U_s)-g(s,X_s,\widetilde{Z}_s,U_s)\big]ds+K^c_t-\widetilde{K}^c_t=\int_0^t\big(Z_s-\widetilde{Z}_s\big)dB_s$, $\forall\, t\in[0,T]$, where K^c (resp. $\widetilde{K}^c$) denotes the continuous part of K (resp. $\widetilde{K}$). Since the set of continuous martingales and that of continuous finite-variation processes only intersect at constants, one can deduce that $Z_t=\widetilde{Z}_t$, $dt\times dP-$a.s. and thus that $P\{K^c_t=\widetilde{K}^c_t,\ \forall\, t\in[0,T]\}=1$.

2) Next, let X be a $g-$submartingale. For any $0\le t\le s\le T$, Remark 2.2 shows that $-X_t\ge-\mathcal{E}^g_{t,s}[X_s]=-Y^{X_s,g_s}_t=Y^{-X_s,\overline{g}_s}_t=\mathcal{E}^{\overline{g}}_{t,s}[-X_s]$, $P-$a.s. So $-X$ is a $\overline{g}-$supermartingale. By part 1, there exist unique processes $(\overline{Z},\overline{U},\overline{K})\in\mathbb{Z}^{2,p}\times\mathbb{U}^p\times\mathbb{K}^p$ such that $P-$a.s.

$$\begin{aligned}-X_t &= -X_T+\int_t^T\overline{g}(s,-X_s,\overline{Z}_s,\overline{U}_s)ds-\int_t^T\overline{Z}_sdB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)+\overline{K}_T-\overline{K}_t\\ &= -X_T-\int_t^T g(s,X_s,-\overline{Z}_s,-\overline{U}_s)ds-\int_t^T\overline{Z}_sdB_s-\int_{(t,T]}\int_{\mathcal{X}}\overline{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)+\overline{K}_T-\overline{K}_t,\quad t\in[0,T].\end{aligned}$$

Then $(Z,U,K):=(-\overline{Z},-\overline{U},\overline{K})\in\mathbb{Z}^{2,p}\times\mathbb{U}^p\times\mathbb{K}^p$ are the unique processes satisfying that $P-$a.s.

$$X_t=X_T+\int_t^T g(s,X_s,Z_s,U_s)ds-\int_t^T Z_sdB_s-\int_{(t,T]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)-K_T+K_t,\quad t\in[0,T]. \qquad \square$$

6.4 Proofs of Section 5

Proof of Proposition 5.2: 1) Assume that g also satisfies (A1), (A2") and that for any $(t,y)\in[0,T)\times\mathbb{R}$, $E\Big[\sup\limits_{s\in[t,t+\delta]}|g(s,y,0,0)|^p\Big]<\infty$ for certain $\delta=\delta(t,y)\in(0,T-t]$. The necessity of (5.3) directly follows from (g6)$-$(g7) of $g-$evaluations with $\mathbb{L}^p$ domains.

To show the sufficiency of (5.3), we let $(t,\alpha,\widetilde{\alpha})\in(0,T)\times[0,1]\times[0,\infty)$ and $(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$, $i=1,2$. Proposition 5.1 and (5.3) show that

$$\begin{aligned}&g\big(t,\alpha y_1+(1-\alpha)y_2,\alpha z_1+(1-\alpha)z_2,\alpha u_1+(1-\alpha)u_2\big)\\ &\quad=\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\Big(\mathcal{E}^g_{t,t+\varepsilon}\big[\alpha(y_1+V(t,t+\varepsilon,z_1,u_1))+(1-\alpha)(y_2+V(t,t+\varepsilon,z_2,u_2))\big]-\big(\alpha y_1+(1-\alpha)y_2\big)\Big)\\ &\quad\le\alpha\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\big(\mathcal{E}^g_{t,t+\varepsilon}\big[y_1+V(t,t+\varepsilon,z_1,u_1)\big]-y_1\big)+(1-\alpha)\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\big(\mathcal{E}^g_{t,t+\varepsilon}\big[y_2+V(t,t+\varepsilon,z_2,u_2)\big]-y_2\big)\\ &\quad=\alpha g(t,y_1,z_1,u_1)+(1-\alpha)g(t,y_2,z_2,u_2),\quad P-\text{a.s.},\end{aligned}$$

and that

$$\begin{aligned}g\big(t,\widetilde{\alpha}y_1,\widetilde{\alpha}z_1,\widetilde{\alpha}u_1\big)&=\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\big(\mathcal{E}^g_{t,t+\varepsilon}\big[\widetilde{\alpha}(y_1+V(t,t+\varepsilon,z_1,u_1))\big]-\widetilde{\alpha}y_1\big)\\ &=\widetilde{\alpha}\lim_{\varepsilon\to0+}\frac{1}{\varepsilon}\big(\mathcal{E}^g_{t,t+\varepsilon}\big[y_1+V(t,t+\varepsilon,z_1,u_1)\big]-y_1\big)=\widetilde{\alpha}g(t,y_1,z_1,u_1),\quad P-\text{a.s.}\end{aligned}$$

Then (A2") and the separability of L^p_ν imply that for any $t\in(0,T)$, it holds $P-$a.s. that (2.1) holds for any $\alpha\in[0,1]$ and $(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$, $i=1,2$ and it holds $P-$a.s. that (2.2) holds for any $\widetilde{\alpha}\in[0,\infty)$ and $(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$. Moreover, we see from (5.2) that $P-$a.s., (2.1) holds for any $(t,\alpha)\in(0,T)\times[0,1]$ and $(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$, $i=1,2$, and that $P-$a.s., (2.2) holds for any $(t,\widetilde{\alpha})\in(0,T)\times[0,\infty)$ and $(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$.

2) Next, assume that g also satisfies (3.2) and (A3). The necessity of (5.4) directly follows from (g5) of $g-$evaluations with $\mathbb{L}^p$ domains.

To see the sufficiency of (5.4), we fix $c\in\mathbb{R}$. By (6.1), g is also Lipschitz in u with coefficient $\kappa_2\big(\nu(\mathcal{X})\big)^{\frac{1}{q}}$. Clearly, $g^c(t,\omega,y,z,u):=g(t,\omega,y-c,z,u)$, $\forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$ is still a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable mapping that satisfies (A2), (3.2) and that is Lipschitz in u.

Let $t\in[0,T]$, $\xi\in L^p(\mathcal{F}_t)$ and set $(Y,Z,U):=\big(Y^{\xi-c,g_t},Z^{\xi-c,g_t},U^{\xi-c,g_t}\big)$. Adding c to BSDEJ $(\xi-c,g_t)$ shows that $\mathcal{Y}_s:=Y_s+c$, $s\in[0,T]$ satisfies that

$$\mathcal{Y}_s=\xi+\int_s^T\mathbf{1}_{\{r<t\}}g^c(r,\mathcal{Y}_r,Z_r,U_r)dr-\int_s^T Z_r dB_r-\int_{(s,T]}\int_{\mathcal{X}}U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx),\quad\forall\,s\in[0,T].$$

Namely, $(\mathcal{Y},Z,U)\in\mathbb{S}^p$ satisfies BSDEJ (ξ,g^c_t). By uniqueness, it holds $P-$a.s. that $\mathcal{E}^{g^c}_{s,t}[\xi]=\mathcal{Y}_s=Y_s+c=\mathcal{E}^g_{s,t}[\xi-c]+c$, $\forall\,s\in[0,t]$. In particular, taking $s=0$, we see from (5.4) that

$$\mathcal{E}^{g^c}_{0,t}[\xi]=\mathcal{E}^g_{0,t}[\xi-c]+c=\mathcal{E}^g_{0,t}[\xi],\quad\forall\,t\in[0,T],\ \forall\,\xi\in L^p(\mathcal{F}_t).\tag{6.71}$$

Next, let $t\in[0,T)$, $s\in(t,T]$ and $\xi\in L^p(\mathcal{F}_s)$. We set $A:=\big\{\mathcal{E}^g_{t,s}[\xi]<\mathcal{E}^{g^c}_{t,s}[\xi]\big\}\in\mathcal{F}_t$. Using (6.71) with $(t,\xi)=(s,\mathbf{1}_A\xi)$ and $(t,\xi)=\big(t,\mathbf{1}_A\mathcal{E}^{g^c}_{t,s}[\xi]\big)$ respectively, we can deduce from (g3) and (g4) that

$$\mathcal{E}^g_{0,t}\big[\mathbf{1}_A\mathcal{E}^g_{t,s}[\xi]\big]=\mathcal{E}^g_{0,t}\big[\mathcal{E}^g_{t,s}[\mathbf{1}_A\xi]\big]=\mathcal{E}^g_{0,s}[\mathbf{1}_A\xi]=\mathcal{E}^{g^c}_{0,s}[\mathbf{1}_A\xi]=\mathcal{E}^{g^c}_{0,t}\big[\mathbf{1}_A\mathcal{E}^{g^c}_{t,s}[\xi]\big]=\mathcal{E}^g_{0,t}\big[\mathbf{1}_A\mathcal{E}^{g^c}_{t,s}[\xi]\big],\quad P-\text{a.s.}$$

As $\mathbf{1}_A\mathcal{E}^g_{t,s}[\xi]\le\mathbf{1}_A\mathcal{E}^{g^c}_{t,s}[\xi]$, $P-$a.s., the strict monotonicity (g1) of $g-$evaluation implies that $\mathbf{1}_A\mathcal{E}^g_{t,s}[\xi]=\mathbf{1}_A\mathcal{E}^{g^c}_{t,s}[\xi]$, $P-$a.s. It follows that $P\big\{\mathcal{E}^g_{t,s}[\xi]<\mathcal{E}^{g^c}_{t,s}[\xi]\big\}=0$. Similarly, one can get $P\big\{\mathcal{E}^g_{t,s}[\xi]>\mathcal{E}^{g^c}_{t,s}[\xi]\big\}=0$. So $\mathcal{E}^{g^c}_{t,s}[\xi]=\mathcal{E}^g_{t,s}[\xi]$, $P-$a.s. In light of Proposition 5.1, it holds for any $(t,z,u)\in[0,T)\times\mathbb{R}^d\times L^p_\nu$ that $g(t,c,z,u)=g^c(t,c,z,u)=g(t,0,z,u)$, $P-$a.s. So one can deduce from (A2), (6.1) and the separability of L^p_ν that for any $t\in[0,T)$, it holds $P-$a.s. that $g(t,y,z,u)=g(t,0,z,u)$, $\forall\,(y,z,u)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$. Then (5.2) implies that $P-$a.s. $g(t,y,z,u)=g(t,0,z,u)$, $\forall\,(t,y,z,u)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$. □

Proof of Theorem 5.2: Since both f'_- and f'_+ are non-decreasing functions by the convexity of f, $\psi(x):=\mathbf{1}_{\{f'_-(x)\le 0\}}f'_-(x)+\mathbf{1}_{\{f'_-(x)>0\}}f'_+(x)$, $x\in\mathbb{R}$ is also a non-decreasing function and thus Borel-measurable on $\mathbb{R}$. Then $\eta:=\psi\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big)$ defines a $\mathcal{F}_\tau-$measurable random variable that satisfies

$$\eta\big(\xi-\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\le f(\xi)-f\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big).\tag{6.72}$$

Fix $n\in\mathbb{N}$ and set $A_n:=\big\{\big|\mathcal{E}^g_{\tau,\gamma}[\xi]\big|+|\eta|\le n\big\}\in\mathcal{F}_\tau$. As $\mathbf{1}_{A_n}\eta\,\xi,\mathbf{1}_{A_n}f(\xi)\in L^p(\mathcal{F}_\gamma)$ and $\mathbf{1}_{A_n}\eta\,\mathcal{E}^g_{\tau,\gamma}[\xi],\mathbf{1}_{A_n}f\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\in L^\infty(\mathcal{F}_\gamma)$, (6.72), (g1) and (g5) imply that $P-$a.s.

$$\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_{A_n}\eta\,\xi]-\mathbf{1}_{A_n}\eta\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}^g_{\tau,\gamma}\big[\mathbf{1}_{A_n}\eta\big(\xi-\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\big]\le\mathcal{E}^g_{\tau,\gamma}\big[\mathbf{1}_{A_n}f(\xi)-\mathbf{1}_{A_n}f\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big)\big]=\mathcal{E}^g_{\tau,\gamma}\big[\mathbf{1}_{A_n}f(\xi)\big]-\mathbf{1}_{A_n}f\big(\mathcal{E}^g_{\tau,\gamma}[\xi]\big).\tag{6.73}$$

Set $(Y,Z,U)=\big(Y^{\xi,g_\gamma},Z^{\xi,g_\gamma},U^{\xi,g_\gamma}\big)$. Applying Corollary 2.1 with $\xi=\mathbf{1}_{A_n}\eta\,Y_\tau\in L^p(\mathcal{F}_\tau)$ shows that there exists a unique pair $\big(Z^n,U^n\big)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P-$a.s.

$$Y^n_t:=E\big[\mathbf{1}_{A_n}\eta\,Y_\tau|\mathcal{F}_t\big]=E\big[\mathbf{1}_{A_n}\eta\,Y_\tau\big]+\int_0^t Z^n_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U^n_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

We define $\overline{Y}^n_t:=\mathbf{1}_{\{t<\tau\}}Y^n_t+\mathbf{1}_{\{t\ge\tau\}}\mathbf{1}_{A_n}\eta Y_t$, $\big(\overline{Z}^n_t,\overline{U}^n_t\big):=\mathbf{1}_{\{t\le\tau\}}(Z^n_t,U^n_t)+\mathbf{1}_{\{t>\tau\}}\mathbf{1}_{A_n}\eta\,(Z_t,U_t)$, $\forall\,t\in[0,T]$, and can deduce that $\big(\overline{Y}^n,\overline{Z}^n,\overline{U}^n\big)$ belong to $\mathbb{S}^p$.

For any $t\in[0,T]$, since $\mathbf{1}_{A_n}\eta$ is $\mathcal{F}_\tau-$measurable, we see that $\eta\mathbf{1}_{A_n}\mathbf{1}_{\{\tau\le t\}}$ is $\mathcal{F}_t-$measurable. It follows that $\big\{\eta\mathbf{1}_{A_n}\mathbf{1}_{\{\tau\le t<\gamma\}}\big\}_{t\in[0,T]}$ is an $\mathbf{F}-$adapted càdlàg process and $g^n_t:=\eta\mathbf{1}_{A_n}\mathbf{1}_{\{\tau\le t<\gamma\}}g(t,Z_t,U_t)$, $t\in[0,T]$ is a $\mathbf{F}-$progressively measurable process. In particular, we can regard g^n as a special $p-$generator.

Given $t\in[0,T]$, multiplying $\mathbf{1}_{A_n}\eta\in\mathcal{F}_\tau$ to the BSDEJ (ξ,g_γ) over period $[\tau\vee t,T]$ yields that

$$\begin{aligned}\mathbf{1}_{A_n}\eta Y_{\tau\vee t}&=\mathbf{1}_{A_n}\eta\,\xi+\int_{\tau\vee t}^T\mathbf{1}_{A_n}\eta g_\gamma(s,Z_s,U_s)ds-\int_{\tau\vee t}^T\mathbf{1}_{A_n}\eta Z_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\mathbf{1}_{A_n}\eta U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\&=\mathbf{1}_{A_n}\eta\,\xi+\int_{\tau\vee t}^T g^n_s ds-\int_{\tau\vee t}^T\overline{Z}^n_s dB_s-\int_{(\tau\vee t,T]}\int_{\mathcal{X}}\overline{U}^n_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad P-\text{a.s.}\end{aligned}$$

By the right continuity of process Y, it holds $P-$a.s. that

$$\mathbf{1}_{A_n}\eta Y_{\tau\vee t} = \mathbf{1}_{A_n}\eta\,\xi + \int_{\tau\vee t}^T g_s^n ds - \int_{\tau\vee t}^T \overline{Z}_s^n dB_s - \int_{(\tau\vee t,T]}\int_{\mathcal{X}} \overline{U}_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T]. \tag{6.74}$$

Let $t\in[0,T]$. Since $Y_\tau^n = E\big[\mathbf{1}_{A_n}\eta\,Y_\tau|\mathcal{F}_\tau\big] = \mathbf{1}_{A_n}\eta\,Y_\tau$, $P-$a.s. taking $t=\tau$ in (6.74) yields that

$$\begin{aligned} Y_{\tau\wedge t}^n &= Y_\tau^n - \int_{\tau\wedge t}^\tau Z_s^n dB_s - \int_{(\tau\wedge t,\tau]}\int_{\mathcal{X}} U_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) = \mathbf{1}_{A_n}\eta Y_\tau - \int_{\tau\wedge t}^\tau \overline{Z}_s^n dB_s - \int_{(\tau\wedge t,\tau]}\int_{\mathcal{X}} \overline{U}_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \\ &= \mathbf{1}_{A_n}\eta\,\xi + \int_{\tau\wedge t}^T g_s^n ds - \int_{\tau\wedge t}^T \overline{Z}_s^n dB_s - \int_{(\tau\wedge t,T]}\int_{\mathcal{X}} \overline{U}_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P-\text{a.s.} \end{aligned} \tag{6.75}$$

Multiplying $\mathbf{1}_{\{t\ge\tau\}}$ to (6.74) and multiplying $\mathbf{1}_{\{t<\tau\}}$ to (6.75) leads to that

$$\overline{Y}_t^n = \mathbf{1}_{\{t<\tau\}}Y_t^n + \mathbf{1}_{\{t\ge\tau\}}\mathbf{1}_{A_n}\eta Y_t = \mathbf{1}_{A_n}\eta\,\xi + \int_t^T g_s^n ds - \int_t^T \overline{Z}_s^n dB_s - \int_{(t,T]}\int_{\mathcal{X}} \overline{U}_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P-\text{a.s.}$$

The right continuity of process $\overline{Y}^n$ then implies that $P-$a.s.

$$\overline{Y}_t^n = \mathbf{1}_{A_n}\eta\,\xi + \int_t^T g_s^n ds - \int_t^T \overline{Z}_s^n dB_s - \int_{(t,T]}\int_{\mathcal{X}} \overline{U}_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T].$$

Hence $\big(\overline{Y}^n, \overline{Z}^n, \overline{U}^n\big)$ solves BSDEJ $(\mathbf{1}_{A_n}\eta\,\xi, g^n)$.

As $\psi(x)\in(0,1)^c$ for any $x\in\mathbb{R}$ with $\partial f(x)\cap(0,1)^c\neq\emptyset$, one can deduce that $\eta\in(0,1)^c$, $P-$a.s. Then (5.7) and an analogy to (5.5) imply that $\eta g(t,Z_t,U_t)\le g(t,\eta Z_t,\eta U_t)$ $dt\times dP-$a.s. on $]\!]\tau,\gamma[\![$. And we further see from (5.6) that

$$g_t^n = \mathbf{1}_{A_n}\eta\,g(t,Z_t,U_t) \le \mathbf{1}_{A_n}g(t,\eta Z_t,\eta U_t) = g(t,\mathbf{1}_{A_n}\eta Z_t,\mathbf{1}_{A_n}\eta U_t) = g_\gamma\big(t,\overline{Z}_t^n,\overline{U}_t^n\big)$$

holds $dt\times dP-$a.s. on $]\!]\tau,\gamma[\![$. Applying Theorem 2.2 with $(g^1,g^2)=(g^n,g_\gamma)$ and $i=2$ yields that $P\big\{\mathbf{1}_{A_n}\eta\,Y_t = \overline{Y}_t^n \le Y_t^{\mathbf{1}_{A_n}\eta\,\xi,g_\gamma},\ \forall\,t\in[\tau,\gamma]\big\}=1$. In particular, we have $\mathbf{1}_{A_n}\eta\,\mathcal{E}_{\tau,\gamma}^g[\xi] = \mathbf{1}_{A_n}\eta\,Y_\tau \le Y_\tau^{\mathbf{1}_{A_n}\eta\,\xi,g_\gamma} = \mathcal{E}_{\tau,\gamma}^g[\mathbf{1}_{A_n}\eta\,\xi]$, $P-$a.s., which together with (6.73) shows that

$$\mathbf{1}_{A_n}f\big(\mathcal{E}_{\tau,\gamma}^g[\xi]\big) \le \mathcal{E}_{\tau,\gamma}^g\big[\mathbf{1}_{A_n}f(\xi)\big] = Y_\tau^{\mathbf{1}_{A_n}f(\xi),g_\gamma}, \quad P-\text{a.s.} \tag{6.76}$$

In light of Proposition 2.1,

$$E\Big[\big|Y_\tau^{\mathbf{1}_{A_n}f(\xi),g_\gamma} - Y_\tau^{f(\xi),g_\gamma}\big|^p\Big] \le \big\|Y^{\mathbf{1}_{A_n}f(\xi),g_\gamma} - Y^{f(\xi),g_\gamma}\big\|_{\mathbb{D}^p}^p \le \mathcal{C}E\big[|\mathbf{1}_{A_n}f(\xi)-f(\xi)|^p\big], \tag{6.77}$$

where the constant $\mathcal{C}$ does not depend on n. Since $\lim\limits_{n\to\infty}\uparrow \mathbf{1}_{A_n}=1$, $P-$a.s. and since $E\big[|f(\xi)|^p\big]<\infty$, letting $n\to\infty$ in (6.77), we can deduce from the dominated convergence theorem that $\lim\limits_{n\to\infty}E\Big[\big|Y_\tau^{\mathbf{1}_{A_n}f(\xi),g_\gamma} - Y_\tau^{f(\xi),g_\gamma}\big|^p\Big]=0$. Then we can find a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim\limits_{i\to\infty}Y_\tau^{\mathbf{1}_{A_{n_i}}f(\xi),g_\gamma} = Y_\tau^{f(\xi),g_\gamma}$, $P-$a.s. Eventually, letting $i\to\infty$ in (6.76), we obtain $f\big(\mathcal{E}_{\tau,\gamma}^g[\xi]\big) = \lim\limits_{i\to\infty}\mathbf{1}_{A_{n_i}}f\big(\mathcal{E}_{\tau,\gamma}^g[\xi]\big) \le \lim\limits_{i\to\infty}Y_\tau^{\mathbf{1}_{A_{n_i}}f(\xi),g_\gamma} = Y_\tau^{f(\xi),g_\gamma} = \mathcal{E}_{\tau,\gamma}^g\big[f(\xi)\big]$, $P-$a.s. □

A Appendix: A Monotonic Limit Theorem of jump diffusion processes over $\mathbb{D}^p$

In this appendix, we will extend the monotonic limit theorem of [79] to jump diffusion processes over $\mathbb{D}^p$, which is crucial for the decomposition of $g-$supermartingale (Theorem 4.1).

Fix $p\in(1,2]$. We consider a series of jump diffusion processes $\{Y^n\}_{n\in\mathbb{N}}$ in form of

$$Y_t^n = Y_0^n - \int_0^t b_s^n ds - K_t^n + \int_0^t Z_s^n dB_s + \int_{(0,t]}\int_{\mathcal{X}} U_s^n(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\,t\in[0,T], \tag{A.1}$$

where
(i) $\big\{(b^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)\times\mathbb{Z}^{2,p}\times\mathbb{U}^p$, i.e. there exists a $C_\sharp>0$ such that

$$\bigg(E\int_0^T|b^n_t|^p dt\bigg)^{\frac{1}{p}}+\|Z^n\|_{\mathbb{Z}^{2,p}}+\|U^n\|_{\mathbb{U}^p}\le C_\sharp,\qquad\forall\, n\in\mathbb{N};\tag{A.2}$$

(ii) For any $n\in\mathbb{N}$, K^n is an $\mathbf{F}-$adapted, continuous increasing process with $K^n_0=0$ and $K^n_T\in L^p(\mathcal{F}_T)$;
(iii) Y^n is an increasing sequence that is bounded above by some $X\in\mathbb{D}^p$, i.e. $P\{Y^n_t\le Y^{n+1}_t\le X_t,\ \forall\, t\in[0,T]\}=1$ for any $n\in\mathbb{N}$.

The Burkholder-Davis-Gundy Inequality, (1.7), Hölder's inequality as well as (1.5) imply that

$$E\big[(Y^1_*)^p\big]\le 5^{p-1}c_0E\bigg[|Y^1_0|^p+T^{p-1}\int_0^T|b^1_s|^p ds+(K^n_T)^p+\Big(\int_0^t|Z^n_s|^2ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|U^n_s(x)|^p\nu(dx)ds\bigg]<\infty,\tag{A.3}$$

which shows that $Y^1\in\mathbb{D}^p$. It follows from (iii) that $\{Y^n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{D}^p$ $\Big(\sup\limits_{n\in\mathbb{N}}\|Y^n\|_{\mathbb{D}^p}\le\|Y^1\|_{\mathbb{D}^p}+\|X\|_{\mathbb{D}^p}<\infty\Big)$.

Define a $[-\infty,\infty]-$valued, $\mathbf{F}-$optional process $Y_t:=\varlimsup\limits_{n\to\infty}Y^n_t$, $t\in[0,T]$. The monotone convergence theorem and (iii) imply that

$$P\big\{Y_t=\lim_{n\to\infty}\uparrow Y^n_t\le X_t,\ \forall\, t\in[0,T]\big\}=1.\tag{A.4}$$

So one can regard Y as a real-valued, $\mathbf{F}-$optional process.

Our generalized monotonic limit theorem of jump diffusion processes over $\mathbb{D}^p$ is stated as follows:

Theorem A.1. *Given $p\in(1,2]$, let assumptions $(i)-(iii)$ hold. Then Y belongs to $\mathbb{D}^p$ and has the following decomposition: There exists $(b,Z,U,K)\in\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)\times\mathbb{Z}^{2,p}\times\mathbb{U}^p\times\mathbb{K}^p$ such that $P-$a.s.*

$$Y_t=Y_0-\int_0^t b_s ds-K_t+\int_0^t Z_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\qquad\forall\, t\in[0,T],\tag{A.5}$$

and that for any $\varpi\in(2/p,2)$

$$\lim_{n\to\infty}E\bigg[\Big(\int_0^T|Z^n_s-Z_s|^\varpi ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|U^n_s(x)-U_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds\bigg]=0.\tag{A.6}$$

Moreover, if Y has only inaccessible jumps, then K is a continuous process and

$$\lim_{n\to\infty}E\Big[\big(\int_0^T|Z^n_s-Z_s|^2ds\big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|U^n_s(x)-U_s(x)|^p\nu(dx)ds\Big]=0.\tag{A.7}$$

Before proving this theorem, let us first cite two auxiliary results from [9] and [79] respectively.

Lemma A.1. (*Lemma A.3 of [9]*) *Let K be a real-valued, $\mathbf{F}-$optional process with $P-$a.s. right upper semi-continuous paths $\big($i.e., it holds $P-$a.s. that $K_t(\omega)\ge\varlimsup\limits_{s\searrow t}K_s(\omega)$, for any $t\in[0,T)\big)$. If $K_\tau\le K_\gamma$, $P-$a.s. holds for any $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$, then K is an increasing process.*

Lemma A.2. (*Lemma 2.2 of [79]*) *Let $\{\mathcal{Y}^n\}_{n\in\mathbb{N}}$ be a sequence of real-valued, $\mathbf{F}-$adapted càdlàg processes, let J be a real-valued, $\mathbf{F}-$adapted càdlàg process, and let K be an $\mathbf{F}-$adapted increasing process with $K_0=0$ and with $K_T<\infty$, $P-$a.s. If it holds $P-$a.s. that $\lim\limits_{n\to\infty}\uparrow\mathcal{Y}^n_t=J_t-K_t$ for any $t\in[0,T]$, then K is also an càdlàg process.*

The demonstration of Theorem A.1 also relies on the following extensions of Lemma A.1 and Lemma 2.3 of [79].

Lemma A.3. *Let $p\in(1,2]$, $K\in\mathbb{K}^p$ and $\delta>0$. There exists a finite number of $\mathbf{F}-$predictable stopping times $0=\tau_0<\tau_1\le\tau_2\le\cdots\le\tau_N\le\tau_{N+1}=T$ such that $\tau_i<\tau_{i+1}$ on $\{\tau_i<T\}$ for $i=1\cdots N$ and that*

$$\sum_{i=0}^N E\bigg[\sum_{s\in(\tau_i,\tau_{i+1})}(\Delta K_s)^p\bigg]<\delta.\tag{A.8}$$

Proof: Set $\alpha := (\delta/3)^{\frac{1}{p-1}} \big(E[K_T^p]\big)^{-\frac{1}{p(p-1)}}$. As process $K_t^\alpha := K_t - \sum\limits_{s\in(0,t]} \Delta K_s \mathbf{1}_{\{\Delta K_s > \alpha\}}$, $t\in[0,T]$ retains only those jumps of K whose sizes are smaller than α, one can deduce from Hölder's inequality that

$$E\bigg[\sum_{s\in(0,T]} (\Delta K_s^\alpha)^p\bigg] \le \alpha^{p-1} E\bigg[\sum_{s\in(0,T]} \Delta K_s^\alpha\bigg] \le \alpha^{p-1} E K_T \le \alpha^{p-1}\big(E[K_T^p]\big)^{\frac{1}{p}} = \delta/3. \tag{A.9}$$

Set $\gamma_0 := 0$. For any $j\in\mathbb{N}$, we inductively define $\gamma_j := \inf\big\{t\in(\gamma_{j-1},T] : \Delta K_t > \alpha\big\}\wedge T$, and (1.7) implies that $\sum_{s\in(\gamma_j,T)} (\Delta K_s)^p \le \big(\sum_{s\in(\gamma_j,T)} \Delta K_s\big)^p \le K_T^p$, $P-$a.s. Since $\lim\limits_{j\to\infty}\downarrow \sum_{s\in(\gamma_j,T)}(\Delta K_s)^p = 0$, $P-$a.s. the dominated convergence theorem implies that $\lim\limits_{j\to\infty}\downarrow E\big[\sum_{s\in(\gamma_j,T)}(\Delta K_s)^p\big] = 0$. So one can find an $\ell\in\mathbb{N}$ such that

$$E\bigg[\sum_{s\in(\gamma_\ell,T)} (\Delta K_s)^p\bigg] < \delta/3. \tag{A.10}$$

In light of Proposition I.2.24 of [47] (see also "Complements to Chapter IV" of [34]), the jumps of $\mathbf{F}-$predictable càdlàg process K are exhausted by a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ of $\mathbf{F}-$predictable stopping times, i.e. $\mathfrak{K} := \{(t,\omega)\in[0,T]\times\Omega : \Delta K_t(\omega) > 0\}$ is a union of graphs $[\![\tau_i]\!]$ and these graphs are disjoint on period $(0,T)$.

As $E[K_T^p] < \infty$, there exists a $\lambda = \lambda(\delta) > 0$ such that $E[\mathbf{1}_A K_T^p] < \delta/3$ holds for any $A\in\mathcal{F}_T$ with $P(A) < \lambda$. Let $j = 1,\cdots,\ell$. For any $n\in\mathbb{N}$, define $A_n^j := \Big\{\gamma_j \in \overset{n}{\underset{i=1}{\cup}} [\![\tau_i]\!]\Big\} = \big\{\omega\in\Omega : \gamma_j(\omega) = \tau_i(\omega) \text{ for some } i\in\{1,\cdots,n\}\big\}\in\mathcal{F}_T$. Since $[\![\gamma_j]\!]\subset\mathfrak{K} = \underset{i\in\mathbb{N}}{\cup}[\![\tau_i]\!]$, we see that $\lim\limits_{i\to\infty}\uparrow A_n^j = \Omega$, there exists $n_j\in\mathbb{N}$ such that $P(A_{n_j}^j) > 1-\lambda/\ell$.

Set $\tau_0 := 0$, $N := \max\limits_{j=1,\cdots,\ell} n_j$ and $\mathcal{A} := \overset{\ell}{\underset{j=1}{\cap}} A_{n_j}^j$. Also, we reset $\gamma_{\ell+1} := T$ and $\tau_{N+1} := T$. Given $\omega\in\mathcal{A} := \overset{\ell}{\underset{j=1}{\cap}} A_{n_j}^j$, as $\gamma_j(\omega)\in\big\{\tau_1(\omega),\cdots,\tau_{n_j}(\omega)\big\}\subset\big\{\tau_1(\omega),\cdots,\tau_N(\omega)\big\}$ for any $j = 1,\cdots,\ell$, we have that

$$\overset{N}{\underset{i=0}{\cup}}\big(\tau_i(\omega),\tau_{i+1}(\omega)\big) = (0,T)\backslash\big\{\tau_1(\omega),\cdots,\tau_N(\omega)\big\}\subset(0,T)\backslash\big\{\gamma_1(\omega),\cdots,\gamma_\ell(\omega)\big\} = \overset{\ell}{\underset{j=0}{\cup}}\big(\gamma_j(\omega),\gamma_{j+1}(\omega)\big).$$

Since $P(\mathcal{A}^c) = P\Big(\overset{\ell}{\underset{j=1}{\cup}}\big(A_{n_j}^j\big)^c\Big) \le \sum\limits_{j=1}^{\ell} P\Big(\big(A_{n_j}^j\big)^c\Big) < \lambda$ and since $\Delta K_s = \Delta K_s^\alpha$ for any $s\in\overset{\ell-1}{\underset{j=0}{\cup}}(\gamma_j,\gamma_{j+1})$, we can deduce from (1.7), (A.9) and (A.10) that

$$\begin{aligned} E\bigg[\sum_{i=0}^{N}\sum_{s\in(\tau_i,\tau_{i+1})}(\Delta K_s)^p\bigg] &\le E\bigg[\mathbf{1}_{\mathcal{A}^c}\bigg(\sum_{i=0}^{N}\sum_{s\in(\tau_i,\tau_{i+1})}\Delta K_s\bigg)^p\bigg] + E\bigg[\mathbf{1}_{\mathcal{A}}\sum_{j=0}^{\ell}\sum_{s\in(\gamma_j,\gamma_{j+1})}(\Delta K_s)^p\bigg] \\ &\le E[\mathbf{1}_{\mathcal{A}^c}K_T^p] + E\bigg[\sum_{j=0}^{\ell-1}\sum_{s\in(\gamma_j,\gamma_{j+1})}(\Delta K_s^\alpha)^p + \sum_{s\in(\gamma_\ell,T)}(\Delta K_s)^p\bigg] < \frac{2}{3}\delta + E\bigg[\sum_{s\in(0,T]}(\Delta K_s^\alpha)^p\bigg] < \delta. \end{aligned}$$

□

Lemma A.4. *Let $p\in(1,2]$, $K\in\mathbb{K}^p$ and $\varepsilon,\delta>0$. There exists a finite sequence of $\mathbf{F}-$predictable stopping times $0 = \tau_0 < \gamma_0 \le \tau_1 < \gamma_1 \le \cdots \le \tau_N < \gamma_N \le \tau_{N+1} = T$ such that*

$$\sum_{i=0}^{N} E\big[(\tau_{i+1}-\gamma_i) + (\tau_{i+1}-\gamma_i)^{\frac{p}{2}}\big] < \varepsilon \quad \textit{and} \quad \sum_{i=0}^{N} E\bigg[\sum_{s\in(\tau_i,\gamma_i]}(\Delta K_s)^p\bigg] < \delta. \tag{A.11}$$

Proof: According to Lemma A.3, (A.8) holds for a finite number of $\mathbf{F}-$predictable stopping times $0 = \tau_0 < \tau_1 \le \tau_2 \le \cdots \le \tau_N \le \tau_{N+1} = T$ such that

$$\tau_i < \tau_{i+1} \text{ on } \{\tau_i < T\} \text{ for } i = 1\cdots N. \tag{A.12}$$

Let $i = 0,\cdots,N$. The PFA Theorem or *foretelling* Theorem (see e.g. Theorem IV.77 of [33]) shows that the $\mathbf{F}-$predictable stopping time τ_{i+1} can be approximated by an increasing sequence $\{\zeta_n^i\}_{n\in\mathbb{N}}$ of $\mathbf{F}-$predictable stopping times: i.e. $\lim\limits_{n\to\infty}\uparrow\zeta_n^i = \tau_{i+1}$, $P-$a.s., and for any $n\in\mathbb{N}$,

$$\zeta_n^i < \tau_{i+1}, \ P-\text{a.s. on } \{\tau_{i+1} > 0\}. \tag{A.13}$$

So one can find an $n(i)\in\mathbb{N}$ such that $E\Big[\big(\tau_{i+1}-\zeta^i_{n(i)}\big)+\big(\tau_{i+1}-\zeta^i_{n(i)}\big)^{\frac{p}{2}}\Big]<\dfrac{\varepsilon}{N+1}$. Consequently, the $\mathbf{F}-$predictable stopping times $\gamma_i:=\tau_i\vee\zeta^i_{n(i)}\le\tau_{i+1}$, $i=0\cdots N$ satisfy the first inequality of (A.11).

For $P-$a.s. $\omega\in\Omega$, if $\tau_i(\omega)<T$ and for $i=0,\cdots,N$, (A.12) shows that $\tau_{i+1}(\omega)>\tau_i(\omega)\ge 0$. Then we see from (A.13) that $\zeta^i_{n(i)}(\omega)<\tau_{i+1}(\omega)$ and thus $\gamma_i(\omega)<\tau_{i+1}(\omega)$. It follows from (A.8) that

$$\sum_{i=0}^N E\bigg[\sum_{s\in(\tau_i,\gamma_i]}(\Delta K_s)^p\bigg]\le\sum_{i=0}^N E\bigg[\sum_{s\in(\tau_i,\tau_{i+1})}(\Delta K_s)^p\bigg]<\delta. \qquad\Box$$

Proof of Theorem A.1: For $n\in\mathbb{N}$, we set $\xi_n:=\int_0^T Z^n_t dB_t$. The Burkholder-Davis-Gundy inequality and condition (i) imply that $E\big[|\xi_n|^p\big]\le c_pE\Big[\big(\int_0^T|Z^n_t|^2dt\big)^{\frac{p}{2}}\Big]=c_p\|Z^n\|^p_{\mathbb{Z}^{2,p}}\le c_pC^p_\sharp$, which shows that $\{\xi_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(\mathcal{F}_T)$. As $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$, $L^p(\mathcal{F}_T)$ and $\mathbb{U}^p$ are reflexive spaces, we know from e.g. Theorem 5.2.1 of [97] that $\big\{(b^n,\xi_n,U^n)\big\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence $\big($we still denote it by $\big\{(b^n,\xi_n,U^n)\big\}_{n\in\mathbb{N}}\big)$ with limit $(b,\xi,U)\in\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)\times L^p(\mathcal{F}_T)\times\mathbb{U}^p$. By Corollary (2.1), there exists $\big(Z,\mathfrak{U}\big)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P-$a.s., $E[\xi|\mathcal{F}_t]=E[\xi]+\int_0^t Z_sdB_s+\int_{(0,t]}\int_{\mathcal{X}}\mathfrak{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $\forall\,t\in[0,T]$.

1) *Let $\Phi\in\mathbb{Z}^{2,q}$. We first show that* $\displaystyle\lim_{n\to\infty}E\int_0^T\Phi_t(Z^n_t-Z_t)dt=0.$ (A.14)

Define a martingale $M^\Phi_t:=\int_0^t\Phi_sdB_s$, $t\in[0,T]$. The Burkholder-Davis-Gundy inequality shows that

$$E\big[(M^\Phi_*)^q\big]\le c_qE\bigg[\Big(\int_0^T|\Phi_s|^2ds\Big)^{\frac{q}{2}}\bigg]<\infty \tag{A.15}$$

for some $c_q>0$, thus $M^\Phi_T\in L^q(\mathcal{F}_T)$.

Fix $n\in\mathbb{N}$ and define $\Gamma^n_t:=\int_0^t(Z^n_s-Z_s)dB_s-\int_{(0,t]}\int_{\mathcal{X}}\mathfrak{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. The Burkholder-Davis-Gundy inequality and (1.5) imply that

$$E\big[(\Gamma^n_*)^p\big]\le c_pE\bigg[\Big(\int_0^T|Z^n_s-Z_s|^2ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}\big|\mathfrak{U}_s(x)\big|^p\nu(dx)ds\bigg]<\infty. \tag{A.16}$$

Also, integrating by parts yields that $P-$a.s.

$$M^\Phi_t\Gamma^n_t=\int_0^t\big(\Gamma^n_s\Phi_s+M^\Phi_s(Z^n_s-Z_s)\big)dB_s-\int_{(0,t]}\int_{\mathcal{X}}M^\Phi_s\mathfrak{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)+\int_0^t\Phi_s(Z^n_s-Z_s)ds,\quad t\in[0,T]. \tag{A.17}$$

For any $i\in\mathbb{N}$, we set $\zeta^n_i:=\inf\big\{t\in[0,T]:\int_0^t|\Phi_s|^2ds+\int_0^t|Z^n_s-Z_s|^2ds+\int_0^t\int_{\mathcal{X}}\big|\mathfrak{U}_s(x)\big|^p\nu(dx)ds>i\big\}\wedge T\in\mathcal{T}$ and $\Upsilon^{n,i}_t:=\int_0^{\zeta^n_i\wedge t}\big(\Gamma^n_s\Phi_s+M^\Phi_s(Z^n_s-Z_s)\big)dB_s-\int_{(0,\zeta^n_i\wedge t]}\int_{\mathcal{X}}M^\Phi_s\mathfrak{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. Applying the Burkholder-Davis-Gundy inequality, we can deduce from (1.7), (1.5), (A.16), (A.15) and Hölder's inequality that

$$\begin{aligned}E\Big[\sup_{t\in[0,T]}\big|\Upsilon^{n,i}_t\big|^p\Big]&\le c_pE\bigg[\Big(\int_0^{\zeta^n_i}\big|\Gamma^n_s\Phi_s\big|^2ds\Big)^{\frac{p}{2}}+\Big(\int_0^{\zeta^n_i}\big|M^\Phi_s(Z^n_s-Z_s)\big|^2ds\Big)^{\frac{p}{2}}+\int_0^{\zeta^n_i}\int_{\mathcal{X}}|M^\Phi_s\mathfrak{U}_s(x)|^p\nu(dx)ds\bigg]\\&\le c_pE\bigg[(\Gamma^n_*)^p\Big(\int_0^{\zeta^n_i}\big|\Phi_s\big|^2ds\Big)^{\frac{p}{2}}+\big(M^\Phi_*\big)^p\Big(\int_0^{\zeta^n_i}\big|Z^n_s-Z_s\big|^2ds\Big)^{\frac{p}{2}}+\big(M^\Phi_*\big)^p\int_0^{\zeta^n_i}\int_{\mathcal{X}}|\mathfrak{U}_s(x)|^p\nu(dx)ds\bigg]\\&\le c_p\Big\{i^{\frac{p}{2}}E\big[(\Gamma^n_*)^p\big]+\big(i+i^{\frac{p}{2}}\big)\big(E\big[(M^\Phi_*)^q\big]\big)^{\frac{p}{q}}\Big\}<\infty.\end{aligned} \tag{A.18}$$

So $\Upsilon^{n,i}$ is a uniformly integrable martingale. Taking $t=\zeta^n_i$ in (A.17) and then taking expectation yield that

$$E\big[M^\Phi_{\zeta^n_i}\Gamma^n_{\zeta^n_i}\big]=E\int_0^{\zeta^n_i}\Phi_s(Z^n_s-Z_s)ds. \tag{A.19}$$

As $(\Phi,Z^n-Z,U)\in\mathbb{Z}^{2,q}\times\mathbb{Z}^{2,p}\times\mathbb{U}^p$, it holds for all $\omega\in\Omega$ except on a $P-$null set $\mathcal{N}_n$ that $\zeta^n_{\mathfrak{i}}(\omega)=T$ for some $\mathfrak{i}=\mathfrak{i}(n,\omega)\in\mathbb{N}$. For any $\omega\in\mathcal{N}^c_n$, one has

$$\lim_{i\to\infty}\Gamma^n(\zeta^n_i(\omega),\omega)=\Gamma^n(T,\omega)=\xi_n(\omega)-\xi(\omega)+E[\xi]=\xi_n(\omega)-\xi(\omega), \tag{A.20}$$

although the path $\Gamma^n_\cdot(\omega)$ may not be left-continuous. Since Hölder's inequality, (A.15) and (A.16) show that $E[M^\Phi_*\Gamma^n_*]\le\big\{E\big[(M^\Phi_*)^q\big]\big\}^{\frac1q}\big\{E\big[(\Gamma^n_*)^p\big]\big\}^{\frac1p}<\infty$ and that $E\int_0^T|\Phi_t||Z^n_t-Z_t|dt\le E\big[(\int_0^T|\Phi_t|^2dt)^{\frac12}(\int_0^T|Z^n_t-Z_t|^2dt)^{\frac12}\big]\le\Big\{E\big[(\int_0^T|\Phi_t|^2dt)^{\frac q2}\big]\Big\}^{\frac1q}\times\Big\{E\big[(\int_0^T|Z^n_t-Z_t|^2dt)^{\frac p2}\big]\Big\}^{\frac1p}<\infty$, letting $i\to\infty$ in (A.19), we can deduce from the dominated convergence theorem and (A.20) that $E\big[M^\Phi_T(\xi_n-\xi)\big]=E\big[M^\Phi_T\Gamma^n_T\big]=E\int_0^T\Phi_t(Z^n_t-Z_t)dt$. As $M^\Phi_T\in L^q(\mathcal{F}_T)$, letting $n\to\infty$, we obtain (A.14) from the weak convergence of ξ_n's to ξ in $L^p(\mathcal{F}_T)$.

2) *Define a real-valued, $\mathbf{F}-$optional process*

$$K_t:=Y_0-Y_t-\int_0^t b_sds+\int_0^t Z_sdB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T]. \tag{A.21}$$

In this step, we show that K_τ is a weak limit of K^n_τ's in $L^p(\mathcal{F}_\tau)$ for any $\tau\in\mathcal{T}$.

The Burkholder-Davis-Gundy inequality, (1.7), Hölder's inequality and (1.5) imply that

$$E\big[K^p_T\big]\le c_pE\bigg[(Y^1_*)^p+X^p_*+T^{p-1}\int_0^T|b_t|^pdt+\Big(\int_0^T|Z_t|^2dt\Big)^{\frac p2}+\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt\bigg]<\infty. \tag{A.22}$$

Let $\tau\in\mathcal{T}$ and $\eta\in L^q(\mathcal{F}_\tau)\subset L^2(\mathcal{F}_\tau)$. We know from the regular martingale representation theorem that there exists $(Z^\eta,U^\eta)\in\mathbb{Z}^{2,2}\times\mathbb{U}^2$ such that $P-$a.s., $\mathcal{M}^\eta_t:=E[\eta|\mathcal{F}_t]=E[\eta]+\int_0^{\tau\wedge t}Z^\eta_sdB_s+\int_{(0,\tau\wedge t]}\int_{\mathcal{X}}U^\eta_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ \forall\,t\in[0,T]$. Similar to (A.16), the Burkholder-Davis-Gundy inequality, (1.7) and (1.5) imply that

$$E\big[(\mathcal{M}^\eta_*)^p\big]\le c_p\bigg\{\big(E[\eta]\big)^p+E\bigg[\Big(\int_0^\tau|Z^\eta_s|^2ds\Big)^{\frac p2}+\int_0^\tau\int_{\mathcal{X}}\big|U^\eta_s(x)\big|^p\nu(dx)ds\bigg]\bigg\}<\infty. \tag{A.23}$$

Given $\omega\in\Omega$, we denote the countable set $D_{\mathfrak{p}(\omega)}$ by $\{t_j(\omega)\}_{i\in\mathbb{N}}$. For any $\mathfrak{J}\in\mathbb{N}$, (1.7) shows that

$$\sum_{j=1}^{\mathfrak{J}}\mathbf{1}_{\{t\le\tau(\omega)\}}\big|U^\eta\big(t_j(\omega),\omega,\mathfrak{p}_{t_j(\omega)}(\omega)\big)\big|^q\le\Big(\sum_{j=1}^{\mathfrak{J}}\mathbf{1}_{\{t\le\tau(\omega)\}}\big|U^\eta\big(t_j(\omega),\omega,\mathfrak{p}_{t_j(\omega)}(\omega)\big)\big|^2\Big)^{\frac q2}\le\Big(\sum_{t\in D_{\mathfrak{p}(\omega)}}\mathbf{1}_{\{t\le\tau(\omega)\}}\big|U^\eta\big(t,\omega,\mathfrak{p}_t(\omega)\big)\big|^2\Big)^{\frac q2}$$
$$=\Big(\int_{(0,\tau]}\int_{\mathcal{X}}|U^\eta_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac q2}(\omega).$$

Letting $\mathfrak{J}\to\infty$ on the left-hand-side yields that

$$\Big(\int_{(0,\tau]}\int_{\mathcal{X}}|U^\eta_t(x)|^qN_{\mathfrak{p}}(dt,dx)\Big)(\omega)=\sum_{t\in D_{\mathfrak{p}(\omega)}}\mathbf{1}_{\{t\le\tau(\omega)\}}\big|U^\eta\big(t,\omega,\mathfrak{p}_t(\omega)\big)\big|^q\le\Big(\int_{(0,\tau]}\int_{\mathcal{X}}|U^\eta_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac q2}(\omega).$$

Then (1.7), the Burkholder-Davis-Gundy inequality and the Doob's martingale inequality imply that

$$E\bigg[\Big(\int_0^\tau|Z^\eta_s|^2ds\Big)^{\frac q2}+\int_{(0,\tau]}\int_{\mathcal{X}}|U^\eta_t(x)|^qN_{\mathfrak{p}}(dt,dx)\bigg]\le E\bigg[\Big(\int_0^\tau|Z^\eta_s|^2ds+\int_{(0,\tau]}\int_{\mathcal{X}}|U^\eta_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac q2}\bigg]$$
$$=E\Big\{\big[\mathcal{M}^\eta,\mathcal{M}^\eta\big]^{\frac q2}_T\Big\}\le\widetilde{c}_qE\big[(\mathcal{M}^\eta_*)^q\big]\le\widetilde{c}_qp^qE\big[|\mathcal{M}^\eta_T|^q\big]=\widetilde{c}_qp^qE\big[|\eta|^q\big]<\infty \tag{A.24}$$

for some $\widetilde{c}_q>0$, thus $\big\{\big(\mathbf{1}_{\{t\le\tau\}}Z^\eta_t,\mathbf{1}_{\{t\le\tau\}}U^\eta_t\big)\big\}_{t\in[0,T]}\in\mathbb{Z}^{2,q}\times\mathbb{U}^q$.

Fix $n\in\mathbb{N}$. An analogy to (A.16) shows that the process $\widetilde{\Gamma}^n_t:=\int_0^t(Z^n_s-Z_s)dB_s+\int_{(0,t]}\int_{\mathcal{X}}\big(U^n_s(x)-U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$ is of $\mathbb{D}^p$. Also, integrating by parts yields that $P-$a.s.

$$\mathcal{M}^\eta_t\widetilde{\Gamma}^n_t=\int_0^{\tau\wedge t}Z^\eta_s(Z^n_s-Z_s)ds+\int_0^t\big(\mathbf{1}_{\{s\le\tau\}}\widetilde{\Gamma}^n_sZ^\eta_s+\mathcal{M}^\eta_s(Z^n_s-Z_s)\big)dB_s+\int_{(0,t]}\int_{\mathcal{X}}\big(\mathbf{1}_{\{s\le\tau\}}\widetilde{\Gamma}^n_s\,U^\eta_s(x)+\mathcal{M}^\eta_s(U^n_s(x)-U_s(x))\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)$$
$$+\int_{(0,\tau\wedge t]}\int_{\mathcal{X}}U^\eta_s(x)\big(U^n_s(x)-U_s(x)\big)N_{\mathfrak{p}}(ds,dx),\quad t\in[0,T]. \tag{A.25}$$

For any $i\in\mathbb{N}$, we set

$$\varsigma^n_i:=\inf\bigg\{t\in[0,T]:\int_0^t\big(\mathbf{1}_{\{s\le\tau\}}|Z^\eta_s|^2+|Z^n_s-Z_s|^2\big)ds+\int_0^t\int_{\mathcal{X}}\big(\mathbf{1}_{\{s\le\tau\}}|U^\eta_s(x)|^p+|U^n_s(x)-U_s(x)|^p\big)\nu(dx)ds>i\bigg\}\wedge T\in\mathcal{T},$$

and $\widetilde{\Upsilon}^{n,i}_t := \int_0^{\varsigma^n_i \wedge t}\big(\mathbf{1}_{\{s\le\tau\}}\widetilde{\Gamma}^n_s Z^\eta_s + \mathcal{M}^\eta_s(Z^n_s - Z_s)\big)dB_s + \int_{(0,\varsigma^n_i\wedge t]}\int_{\mathcal{X}}\big(\mathbf{1}_{\{s\le\tau\}}\widetilde{\Gamma}^n_s U^\eta_s(x) + \mathcal{M}^\eta_s(U^n_s(x) - U_s(x))\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. An analogy to (A.18) and (A.24) imply that

$$E\bigg[\sup_{t\in[0,T]}\big|\widetilde{\Upsilon}^{n,i}_t\big|^p\bigg] \le c_p E\bigg[\big(\widetilde{\Gamma}^n_*\big)^p\Big(\int_0^{\varsigma^n_i\wedge\tau}\big|Z^\eta_s\big|^2 ds\Big)^{\frac{p}{2}} + \big(\mathcal{M}^\eta_*\big)^p\Big(\int_0^{\varsigma^n_i}\big|Z^n_s - Z_s\big|^2 ds\Big)^{\frac{p}{2}} + \big(\widetilde{\Gamma}^n_*\big)^p\int_0^{\varsigma^n_i\wedge\tau}\int_{\mathcal{X}}|U^\eta_s(x)|^p\nu(dx)ds$$
$$+\big(\mathcal{M}^\eta_*\big)^p\int_0^{\varsigma^n_i}\int_{\mathcal{X}}|U^n_s(x) - U_s(x)|^p\nu(dx)ds\bigg] \le c_p\big(i + i^{\frac{p}{2}}\big)\Big\{E\big[(\widetilde{\Gamma}^n_*)^p\big] + \big(E\big[(\mathcal{M}^\eta_*)^q\big]\big)^{\frac{p}{q}}\Big\} < \infty.$$

So $\widetilde{\Upsilon}^{n,i}$ is a uniformly integrable martingale. Taking $t=\varsigma^n_i$ in (A.25) and then taking expectation yield that

$$E\Big[\mathcal{M}^\eta_{\varsigma^n_i}\widetilde{\Gamma}^n_{\varsigma^n_i}\Big] = E\int_0^{\varsigma^n_i}\mathbf{1}_{\{s\le\tau\}}Z^\eta_s(Z^n_s - Z_s)ds + E\int_0^{\varsigma^n_i}\int_{\mathcal{X}}\mathbf{1}_{\{s\le\tau\}}U^\eta_s(x)\big(U^n_s(x) - U_s(x)\big)\nu(dx)ds. \tag{A.26}$$

As $\big\{\big(\mathbf{1}_{\{t\le\tau\}}Z^\eta_t, \mathbf{1}_{\{t\le\tau\}}U^\eta_t\big)\big\}_{t\in[0,T]} \in \mathbb{Z}^{2,q}\times\mathbb{U}^q$ and $(Z^n - Z, U^n - U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$, it holds for all $\omega\in\Omega$ except on a $P-$null set $\widetilde{\mathcal{N}}_n$ that $\varsigma^n_{\mathfrak{i}'}(\omega) = T$ for some $\mathfrak{i}' = \mathfrak{i}'(n,\omega)\in\mathbb{N}$. For any $\omega\in\widetilde{\mathcal{N}}^c_n$, we have

$$\lim_{i\to\infty}\mathcal{M}^\eta(\varsigma^n_i(\omega),\omega) = \mathcal{M}^\eta(T,\omega) = \eta(\omega) \quad \text{and} \quad \lim_{i\to\infty}\widetilde{\Gamma}^n(\varsigma^n_i(\omega),\omega) = \widetilde{\Gamma}^n(T,\omega),$$

although the paths $\mathcal{M}^\eta_\cdot(\omega)$ and $\widetilde{\Gamma}^n_\cdot(\omega)$ may not be left-continuous. Since Hölder's inequality and (A.23) show that $E[\mathcal{M}^\eta_*\widetilde{\Gamma}^n_*] \le \big\{E\big[(\mathcal{M}^\eta_*)^q\big]\big\}^{\frac{1}{q}}\big\{E\big[(\widetilde{\Gamma}^n_*)^p\big]\big\}^{\frac{1}{p}} < \infty$, that $E\int_0^\tau|Z^\eta_t||Z^n_t - Z_t|dt \le E\Big[\big(\int_0^\tau|Z^\eta_t|^2dt\big)^{\frac{1}{2}}\big(\int_0^T|Z^n_t - Z_t|^2dt\big)^{\frac{1}{2}}\Big] \le \Big\{E\big[\big(\int_0^\tau|Z^\eta_t|^2dt\big)^{\frac{q}{2}}\big]\Big\}^{\frac{1}{q}}\Big\{E\big[\big(\int_0^T|Z^n_t - Z_t|^2dt\big)^{\frac{p}{2}}\big]\Big\}^{\frac{1}{p}} < \infty$ and that $E\int_0^\tau\int_{\mathcal{X}}|U^\eta_t(x)||U^n_t(x) - U_t(x)|\nu(dx)dt \le \big(E\int_0^\tau\int_{\mathcal{X}}|U^\eta_t(x)|^q \nu(dx)dt\big)^{\frac{1}{q}}\big(E\int_0^\tau\int_{\mathcal{X}}|U^n_t(x) - U_t(x)|^p\nu(dx)dt\big)^{\frac{1}{p}} < \infty$, letting $i\to\infty$ in (A.26), we can deduce from the dominated convergence theorem $E\big[\eta\widetilde{\Gamma}^n_T\big] = E\int_0^T\mathbf{1}_{\{t\le\tau\}}Z^\eta_t(Z^n_t - Z_t)dt + E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{t\le\tau\}}U^\eta_t(x)\big(U^n_t(x) - U_t(x)\big)\nu(dx)dt$. Since $\big\{\big(\mathbf{1}_{\{t\le\tau\}}Z^\eta_t, \mathbf{1}_{\{t\le\tau\}}U^\eta_t\big)\big\}_{t\in[0,T]} \in \mathbb{Z}^{2,q}\times\mathbb{U}^q$, letting $n\to\infty$, we see from (A.14) and the weak convergence of U^n's to U in $\mathbb{U}^p$ that

$$\lim_{n\to\infty}E\big[\eta\widetilde{\Gamma}^n_T\big] = 0. \tag{A.27}$$

For any $\Phi\in\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$, we define $f_\eta(\Phi) := E\big[\eta\int_0^T\Phi_t dt\big]$. As $\eta\in L^q(\mathcal{F}_\tau)$, we see from Hölder's Inequality that $|f_\eta(\Phi)| \le \|\eta\|_{L^q(\mathcal{F}_\tau)}\Big\{E\Big[\big(\int_0^T|\Phi_t|dt\big)^p\Big]\Big\}^{\frac{1}{p}} \le T^{\frac{1}{q}}\|\eta\|_{L^q(\mathcal{F}_\tau)}\|\Phi\|_{\mathbb{L}^p([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R})}$. So f_η is a bounded a linear functional on $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$. In light of Riesz's representation theorem, there exists a $\Psi\in\mathbb{L}^q\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$ such that $f_\eta(\Phi) = E\int_0^T\Psi_t\Phi_t dt$, $\forall\,\Phi\in\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$. It then follows from the weak convergence b^n's to b in $\mathbb{L}^p\big([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R}\big)$ that

$$\lim_{n\to\infty}E\bigg[\eta\int_0^\tau(b^n_t - b_t)dt\bigg] = \lim_{n\to\infty}f_\eta\Big(\big\{\mathbf{1}_{\{t\le\tau\}}(b^n_t - b_t)\big\}_{t\in[0,T]}\Big) = \lim_{n\to\infty}E\int_0^T\mathbf{1}_{\{t\le\tau\}}\Psi_t(b^n_t - b_t)dt = 0. \tag{A.28}$$

Moreover, since $|Y^n_0 - Y_0 - Y^n_\tau + Y_\tau| \le 2(Y^1_* + X_*)$ and since $E[|\eta|(Y^1_* + X_*)] \le \|\eta\|_{L^q(\mathcal{F}_\tau)}\big(\|Y^1\|_{\mathbb{D}^p} + \|X\|_{\mathbb{D}^p}\big) < \infty$ by Hölder's inequality, the dominated convergence theorem and condition (iii) imply that $\lim_{n\to\infty}E\big[\eta(Y^n_0 - Y_0 - Y^n_\tau + Y_\tau)\big] = 0$, which together with (A.27) and (A.28) leads to that

$$\lim_{n\to\infty}E\big[\eta(K^n_\tau - K_\tau)\big] = \lim_{n\to\infty}E\big[\eta(Y^n_0 - Y_0 - Y^n_\tau + Y_\tau)\big] - \lim_{n\to\infty}E\bigg[\eta\int_0^\tau(b^n_t - b_t)dt\bigg] + \lim_{n\to\infty}E\big[\eta\widetilde{\Gamma}^n_T\big] = 0.$$

Hence, K^n_τ's weakly converge to K_τ in $L^p(\mathcal{F}_\tau)$ for any $\tau\in\mathcal{T}$.

3) By the right-continuity of Y^n's, it holds for $P-$a.s. $\omega\in\Omega$ that

$$\begin{aligned}\varliminf_{s\searrow t}Y_s(\omega) &= \lim_{n\to\infty}\uparrow\inf_{s\in(t,(t+2^{-n})\wedge T]}Y_s(\omega) = \lim_{n\to\infty}\uparrow\inf_{s\in(t,(t+2^{-n})\wedge T]}\lim_{m\to\infty}\uparrow Y^m_s(\omega) \ge \lim_{m\to\infty}\uparrow\lim_{n\to\infty}\uparrow\inf_{s\in(t,(t+2^{-n})\wedge T]}Y^m_s(\omega)\\ &= \lim_{m\to\infty}\uparrow\varliminf_{s\searrow t}Y^m_s(\omega) = \lim_{m\to\infty}\uparrow Y^m_t(\omega) = Y_t(\omega), \quad \forall\, t\in[0,T].\end{aligned}$$

So process Y has $P-$a.s. right lower semi-continuous paths, which together with the right-continuity of process $\big\{\int_0^t Z_s dB_s + \int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\big\}_{t\in[0,T]}$ shows that process K has $P-$a.s. right upper semi-continuous paths.

Let $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$. For any $n\in\mathbb{N}$, since K^n is an increasing process, it holds $P-$a.s. that

$$K^n_\tau\le K^n_\gamma\,. \tag{A.29}$$

We claim that $K_\tau\le K_\gamma$, $P-$a.s.: *Assume not*, i.e. the $P-$measure of set $A:=\big\{K_\tau>K_\gamma\big\}\in\mathcal{F}_T$ strictly larger than 0, it would follow that $E\big[\mathbf{1}_A K_\tau\big]>E\big[\mathbf{1}_A K_\gamma\big]$. However, one can deduce from part (2) and (A.29) that $E\big[\mathbf{1}_A K_\tau\big]=\lim\limits_{n\to\infty}E\big[\mathbf{1}_A K^n_\tau\big]\le\lim\limits_{n\to\infty}E\big[\mathbf{1}_A K^n_\gamma\big]=E\big[\mathbf{1}_A K_\gamma\big]$. An contradiction appears. Thus, $K_\tau\le K_\gamma$, $P-$a.s. Then Lemma A.1 shows that K is an increasing process. Applying Lemma A.2 with $\mathcal{Y}^n=Y^n$ and $J_t=Y_0-\int_0^t b_s ds+\int_0^t Z_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$, we see from (A.22), (A.4) and (1.7) that both Y and K are càdlàg processes and thus that $E[Y^p_*]\le 2^{p-1}E[(Y^1_*)^p+X^p_*]<\infty$.

The monotonicity of K implies that $\widehat{Y}_t:=Y_t-Y_0+\int_0^t b_s ds$, $t\in[0,T]$ is a càdlàg supermartingale. By (1.7) and Hölder's inequality, $E\Big[\widehat{Y}^p_*\Big]\le 3^{p-1}E\big[2Y^p_*+T^{p-1}\int_0^T|b_t|^p dt\big]<\infty$. In virtue of Theorem VII.12 of [34] (or Theorem III.3.8 of [82]), there exist a uniformly integrable càdlàg martingale $\widehat{M}$ and an $\mathbf{F}-$predictable càdlàg increasing process $\widehat{K}$ with $\widehat{K}(0)=0$ such that $P-$a.s.

$$\widehat{Y}_t=\widehat{M}_t-\widehat{K}_t,\quad t\in[0,T]. \tag{A.30}$$

By the supermartingality of $\widehat{Y}$, $\widehat{\mathscr{Y}_t}:=\widehat{Y}_t-E\Big[\widehat{Y}_T\big|\mathcal{F}_t\Big]$, $t\in[0,T]$ is a nonnegative càdlàg supermartingale whose corresponding Doob-Meyer decomposition is $\widehat{\mathscr{Y}}=\widehat{\mathscr{M}}-\widehat{K}$ with $\widehat{\mathscr{M}_t}:=\widehat{M}_t-E\Big[\widehat{Y}_T\big|\mathcal{F}_t\Big]$. Since (1.7) and Doob's martingale inequality show that $E\Big[\widehat{\mathscr{Y}^p_*}\Big]\le 2^{p-1}E\bigg[\widehat{Y}^p_*+\sup\limits_{t\in[0,T]}\big|E[\widehat{Y}_T|\mathcal{F}_t]\big|^p\bigg]\le 2^{p-1}E\Big[\widehat{Y}^p_*+q^p\big|\widehat{Y}_T\big|^p\Big]<\infty$, we can deduce from the estimate (VII.15.1) of [34] that

$$E\Big[\widehat{K}^p_T\Big]\le p^p E\Big[\widehat{\mathscr{Y}^p_*}\Big]<\infty, \tag{A.31}$$

so $\widehat{K}\in\mathbb{K}^p$. It follows from (A.30) and (1.7) that $E\Big[\widehat{M}^p_*\Big]\le 2^{p-1}E\Big[\widehat{Y}^p_*+\widehat{K}^p_T\Big]<\infty$. An application of Corollary 2.1 again yields that for some $\big(\widehat{Z},\widehat{U}\big)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$, it holds $P-$a.s. that

$$\widehat{M}_t=E\Big[\widehat{M}_T\big|\mathcal{F}_t\Big]=\int_0^t\widehat{Z}_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}\widehat{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

Putting it back into (A.30), we obtain that $P-$a.s.

$$-K_t+\int_0^t Z_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)=\widehat{Y}_t=-\widehat{K}_t+\int_0^t\widehat{Z}_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}\widehat{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T]. \tag{A.32}$$

Comparing the continuous martingale parts of both sides gives that

$$\widehat{Z}_t=Z_t,\quad dt\times dP-\text{a.s.} \tag{A.33}$$

We will eventually see that $\widehat{K}=K$.

4) Fix $\varpi\in(2/p,2)$ and $\lambda>0$. As $\widehat{K}\in\mathbb{K}^p$, Lemma A.4 and (1.7) imply that there exists a finite sequence of $\mathbf{F}-$predictable stopping times $0=\tau_0<\gamma_0\le\tau_1<\gamma_1\le\cdots\le\tau_N<\gamma_N\le\tau_{N+1}=T$ such that

$$E\left[\sum_{i=0}^N(\tau_{i+1}-\gamma_i)+\bigg(\sum_{i=0}^N(\tau_{i+1}-\gamma_i)\bigg)^{\frac{p}{2}}\right]<\lambda\quad\text{and}\quad\sum_{i=0}^N E\left[\sum_{s\in(\tau_i,\gamma_i]}\big(\Delta\widehat{K}_s\big)^p\right]<\lambda^{1+p(1-\frac{\varpi}{2})}, \tag{A.34}$$

where N depends on λ.

Fix $n\in\mathbb{N}$ and set $(\mathcal{Y}^n,\mathcal{Z}^n,\mathcal{U}^n):=\big(Y^n-Y,Z^n-Z,U^n-\widehat{U}\big)$. Subtracting (A.32) from (A.1) and using (A.33) yields that $P-$a.s., $\mathcal{Y}^n_t=\mathcal{Y}^n_0-\int_0^t(b^n_s-b_s)ds-\big(K^n_t-\widehat{K}_t\big)+\int_0^t\mathcal{Z}^n_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}\mathcal{U}^n_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $\forall\,t\in[0,T]$. An analogy to (6.52) and the continuity of K^n show that for $P-$a.s. $\omega\in\Omega$

$$\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\Delta\widehat{K}_t(\omega)=0\quad\text{and}\quad\Delta\mathcal{Y}^n_t(\omega)=\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\mathcal{U}^n\big(t,\omega,\mathfrak{p}_t(\omega)\big)+\mathbf{1}_{\{t\notin D_{\mathfrak{p}(\omega)}\}}\Delta\widehat{K}_t(\omega),\quad\forall\,t\in[0,T]. \tag{A.35}$$

Since the càdlàg increasing process $\widehat{K}$ and the Poisson stochastic integral $M^{\mathcal{U}^n}$ jump countably many times along their $P-$a.s. paths, an analogy to (6.53) shows that

$$\big\{t \in [0,T] : \mathcal{Y}^n_{t-}(\omega) \neq \mathcal{Y}^n_t(\omega)\big\} \text{ is a countable subset of } [0,T] \text{ for } P\text{-a.s. } \omega \in \Omega. \tag{A.36}$$

Next, fix $i \in \{0, \cdots, N\}$, $(t,\varepsilon) \in [0,T] \times (0,1]$ and let $\varphi_\varepsilon(\cdot)$ be the function defined in (6.54). We see from (1.7) that $\mathfrak{Y}^{n,i,\varepsilon}_t := \sup\limits_{s\in[\gamma_i\wedge t,\gamma_i]} \varphi_\varepsilon(\mathcal{Y}^n_s)$, $t \in [0,T]$ satisfies

$$E\Big[\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^p\Big] \le E\Big[\sup_{s\in[0,T]} \varphi^p_\varepsilon(\mathcal{Y}^n_s)\Big] \le E\Big[\sup_{s\in[0,T]} |\mathcal{Y}^n_s|^p\Big] + \varepsilon^{\frac{p}{2}} = \|\mathcal{Y}^n\|^p_{\mathbb{D}^p} + \varepsilon^{\frac{p}{2}} < \infty. \tag{A.37}$$

Applying Itô's formula to $\varphi^p_\varepsilon(\mathcal{Y}^n_t)$ on the interval $\big[(\tau_i \vee t)\wedge\gamma_i, \gamma_i\big]$ and using (A.36) yield that

$$\begin{aligned}
&\varphi^p_\varepsilon(\mathcal{Y}^n_{(\tau_i\vee t)\wedge\gamma_i}) + \frac{1}{2}\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} D^2_x\varphi^p_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2 ds + \sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} \Big(\varphi^p_\varepsilon(\mathcal{Y}^n_s) - \varphi^p_\varepsilon(\mathcal{Y}^n_{s-}) - D_x\varphi^p_\varepsilon(\mathcal{Y}^n_{s-})\Delta\mathcal{Y}^n_s\Big)\\
&= \varphi^p_\varepsilon(\mathcal{Y}^n_{\gamma_i}) + p\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)\mathcal{Y}^n_s\,(b^n_s - b_s)ds + p\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)\mathcal{Y}^n_s dK^n_s\\
&\quad - p\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})\mathcal{Y}^n_{s-} d\widehat{K}_s - p\big(M^n_{\gamma_i} - M^n_{(\tau_i\vee t)\wedge\gamma_i} + \mathcal{M}^n_{\gamma_i} - \mathcal{M}^n_{(\tau_i\vee t)\wedge\gamma_i}\big), \quad P\text{-a.s.},
\end{aligned} \tag{A.38}$$

where $M^n_t := M^{n,\varepsilon}_t = \int_0^t \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})\mathcal{Y}^n_{s-}\mathcal{Z}^n_s dB_s$ and $\mathcal{M}^n_t := \mathcal{M}^{n,\varepsilon}_t = \int_{(0,t]}\int_{\mathcal{X}} \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})\mathcal{Y}^n_{s-}\mathcal{U}^n_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t \in [0,T]$. We can deduce from the Burkholder-Davis-Gundy inequality, Young's inequality, (A.37) and (1.5) that

$$\begin{aligned}
&E\Big[\sup_{t\in[0,T]}|M^n_t| + \sup_{t\in[0,T]}|\mathcal{M}^n_t|\Big] \le c_0 E\Big[\Big(\sup_{s\in[0,T]}\varphi^{p-1}_\varepsilon(\mathcal{Y}^n_s)\Big)\Big(\int_0^T|\mathcal{Z}^n_s|^2ds\Big)^{\frac{1}{2}} + \Big(\sup_{s\in[0,T]}\varphi^{p-1}_\varepsilon(\mathcal{Y}^n_s)\Big)\Big(\int_{(0,T]}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^2N_{\mathfrak{p}}(ds,dx)\Big)^{\frac{1}{2}}\Big]\\
&\le c_p E\Big[\sup_{s\in[0,T]}\varphi^p_\varepsilon(\mathcal{Y}^n_s) + \Big(\int_0^T|\mathcal{Z}^n_s|^2ds\Big)^{\frac{p}{2}} + \int_0^T\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\Big] < \infty.
\end{aligned} \tag{A.39}$$

So both M^n and $\mathcal{M}^n$ are uniformly integrable martingales.

Analogous to (6.57), we can deduce from Taylor's Expansion Theorem and (A.35) that $P-$a.s.

$$\begin{aligned}
&\sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} \Big(\varphi^p_\varepsilon(\mathcal{Y}^n_s) - \varphi^p_\varepsilon(\mathcal{Y}^n_{s-}) - D\varphi^p_\varepsilon(\mathcal{Y}^n_{s-})\Delta\mathcal{Y}^n_s\Big)\\
&\qquad \ge 2^{p-3}p(p-1)\int_{((\tau_i\vee t)\wedge\gamma_i,\gamma_i]}\int_{\mathcal{X}} \mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}N_{\mathfrak{p}}(ds,dx).
\end{aligned} \tag{A.40}$$

Since it holds $P-$a.s. that

$$-\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})\mathcal{Y}^n_{s-} = \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})|\mathcal{Y}^n_{s-}| \le |\mathcal{Y}^n_{s-}|^{p-1} = |\mathcal{Y}^n_s - \Delta\mathcal{Y}^n_s|^{p-1} \le \big(|\mathcal{Y}^n_s| + |\Delta\mathcal{Y}^n_s|\big)^{p-1} \le |\mathcal{Y}^n_s|^{p-1} + |\Delta\mathcal{Y}^n_s|^{p-1}, \quad s \in [0,T],$$

(A.35) implies that $P-$a.s.

$$\begin{aligned}
-\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} \varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})\mathcal{Y}^n_{s-}d\widehat{K}_s &\le \int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} |\mathcal{Y}^n_{s-}|^{p-1}d\widehat{K}_s = \int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} |\mathcal{Y}^n_s|^{p-1}d\widehat{K}^c_s + \sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} |\mathcal{Y}^n_{s-}|^{p-1}\Delta\widehat{K}_s\\
&\le \int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} |\mathcal{Y}^n_s|^{p-1}d\widehat{K}^c_s + \sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} |\mathcal{Y}^n_s|^{p-1}\Delta\widehat{K}_s + \sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} |\Delta\mathcal{Y}^n_s|^{p-1}\Delta\widehat{K}_s\\
&= \int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i} |\mathcal{Y}^n_s|^{p-1}d\widehat{K}^c_s + \sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} |\mathcal{Y}^n_s|^{p-1}\Delta\widehat{K}_s + \sum_{s\in((\tau_i\vee t)\wedge\gamma_i,\gamma_i]} \big(\Delta\widehat{K}_s\big)^p,
\end{aligned} \tag{A.41}$$

where $\widehat{K}^c$ denotes the continuous part of $\widehat{K}$. As the condition (iii) shows that

$$P\{\mathcal{Y}^n_s \le 0, \ \forall\, s \in [0,T]\} = 1, \tag{A.42}$$

plugging (A.40) and (A.41) into (A.38), we see from (6.54) that

$$\varphi^p_\varepsilon(\mathcal{Y}^n_{(\tau_i\vee t)\wedge\gamma_i})+\frac{p}{2}(p-1)\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i}\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+2^{p-3}p(p-1)\int_{(\tau_i\vee t)\wedge\gamma_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}N_{\mathfrak{p}}(ds,dx)$$
$$\le\xi^n_{i,\varepsilon}+p\eta^n_i-p\big(M^n_{\gamma_i}-M^n_{(\tau_i\vee t)\wedge\gamma_i}+\mathcal{M}^n_{\gamma_i}-\mathcal{M}^n_{(\tau_i\vee t)\wedge\gamma_i}\big),\quad P\text{-a.s.,}\tag{A.43}$$

where $\xi^n_{i,\varepsilon}:=\varphi^p_\varepsilon(\mathcal{Y}^n_{\gamma_i})+p\int_{\tau_i}^{\gamma_i}\varphi^{p-1}_\varepsilon(\mathcal{Y}^n_s)|b^n_s-b_s|ds$ and $\eta^n_i:=\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^{p-1}d\widehat{K}^c_s+\sum_{s\in(\tau_i,\gamma_i]}|\mathcal{Y}^n_s|^{p-1}\Delta\widehat{K}_s+\sum_{s\in(\tau_i,\gamma_i]}\big(\Delta\widehat{K}_s\big)^p$.
5) Since M^n and $\mathcal{M}^n$ are uniformly integrable martingales, taking $t=0$ and taking expectation in (A.43) yield that

$$E\int_{\tau_i}^{\gamma_i}\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+2^{p-2}E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds\le\frac{2E\big[\xi^n_{i,\varepsilon}+p\eta^n_i\big]}{p(p-1)}.\tag{A.44}$$

Clearly, $\lim\limits_{\varepsilon\to0}\uparrow|\mathcal{U}^n(s,\omega,x)|^2\big(|\mathcal{U}^n(s,\omega,x)|^2+\varepsilon\big)^{\frac{p}{2}-1}=|\mathcal{U}^n(s,\omega,x)|^p,\ \forall\,(s,\omega,x)\in[0,T]\times\Omega\times\mathcal{X}$, so the monotone convergence theorem implies that

$$\lim_{\varepsilon\to0}\uparrow E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}|\mathcal{U}^n_s(x)|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds=E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds.\tag{A.45}$$

On the other hand, since $\xi^n_{i,\varepsilon}\le\xi^n_{i,1},\ \forall\,\varepsilon\in(0,1]$ and since Young's inequality implies that $E[\xi^n_{i,1}]\le\big(1+(p-1)T\big)E\Big[\big(\mathfrak{Y}^{n,i,1}_{\tau_i}\big)^p\Big]+E\int_0^T|b^n_s-b_s|^pds<\infty$ by (A.37), the dominated convergence theorem shows that

$$\lim_{\varepsilon\to0}E\big[\xi^n_{i,\varepsilon}\big]=E\Big[\widetilde{\xi}^n_i\Big],\tag{A.46}$$

where $\widetilde{\xi}^n_i:=|\mathcal{Y}^n_{\gamma_i}|^p+p\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^{p-1}|b^n_s-b_s|ds$.

Letting $\varepsilon\to0$ in (A.44) yields that $2^{p-2}E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^p\nu(dx)ds\le\frac{2E\big[\widetilde{\xi}^n_i+p\eta^n_i\big]}{p(p-1)}$. And it follows from (A.36) that

$$E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\le E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds+E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{U}^n_s(x)|<|\mathcal{Y}^n_{s-}|\}}|\mathcal{Y}^n_{s-}|^p\nu(dx)ds$$
$$\le\frac{2^{3-p}}{p(p-1)}E\big[\widetilde{\xi}^n_i+p\eta^n_i\big]+\nu(\mathcal{X})E\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_{s-}|^pds=\frac{2^{3-p}}{p(p-1)}E\big[\widetilde{\xi}^n_i+p\eta^n_i\big]+\nu(\mathcal{X})E\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds.\tag{A.47}$$

Now, fix $\varepsilon\in(0,1]$ again. We can deduce from (A.43) that

$$E\Big[\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^p\Big]=E\Big[\sup_{t\in[0,T]}\varphi^p_\varepsilon\big(\mathcal{Y}^n_{(\tau_i\vee t)\wedge\gamma_i}\big)\Big]\le E\big[\xi^n_{i,\varepsilon}+p\eta^n_i\big]+2p\,E\Big[\sup_{s\in[\tau_i,\gamma_i]}|M^n_s|+\sup_{s\in[\tau_i,\gamma_i]}|\mathcal{M}^n_s|\Big].\tag{A.48}$$

Similar to (A.39), the Burkholder-Davis-Gundy inequality, Young's inequality, (A.36), (1.5), (A.44) and (A.47) imply that

$$2pE\Big[\sup_{s\in[\tau_i,\gamma_i]}|M^n_s|+\sup_{s\in[\tau_i,\gamma_i]}|\mathcal{M}^n_s|\Big]\le c_0pE\left[\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^{\frac{p}{2}}\Big(\int_{\tau_i}^{\gamma_i}\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_{s-})|\mathcal{Z}^n_s|^2ds\Big)^{\frac12}+\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^{p-1}\Big(\int_{(\tau_i,\gamma_i]}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^2N_{\mathfrak{p}}(ds,dx)\Big)^{\frac12}\right]$$
$$\le\frac12E\big[(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i})^p\big]+c_0p^2E\int_{\tau_i}^{\gamma_i}\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+c_pE\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds$$
$$\le\frac12E\big[(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i})^p\big]+c_pE\big[\xi^n_{i,\varepsilon}+\widetilde{\xi}^n_i+\eta^n_i\big]+c_p\nu(\mathcal{X})E\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds.\tag{A.49}$$

As $E\big[(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i})^p\big]<\infty$ by (A.37), plugging (A.49) back into (A.48) yields that $E\Big[\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^p\Big]\le c_pE\big[\xi^n_{i,\varepsilon}+\widetilde{\xi}^n_i+\eta^n_i\big]+c_p\nu(\mathcal{X})E\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds$. Then Young's inequality, (A.44) and (A.47) imply that

$$E\Big[\Big(\int_{\tau_i}^{\gamma_i}|\mathcal{Z}^n_s|^2ds\Big)^{\frac{p}{2}}+\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\Big]\le E\left[\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^{\frac{p(2-p)}{2}}\Big(\int_{\tau_i}^{\gamma_i}\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds\Big)^{\frac{p}{2}}+\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\right]$$
$$\le\frac{2-p}{2}E\Big[\big(\mathfrak{Y}^{n,i,\varepsilon}_{\tau_i}\big)^p\Big]+\frac{p}{2}E\int_{\tau_i}^{\gamma_i}\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+E\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\tag{A.50}$$
$$\le c_pE\big[\xi^n_{i,\varepsilon}+\widetilde{\xi}^n_i+\eta^n_i\big]+c_p\nu(\mathcal{X})E\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds.$$

Summing up $i\in\{0,\cdots,N\}$ and letting $\varepsilon\to 0$, we see from (A.46), Hölder's inequality and (A.34) that

$$\sum_{i=0}^N E\bigg[\Big(\int_{\tau_i}^{\gamma_i}|\mathcal{Z}^n_s|^2ds\Big)^{\frac{p}{2}}+\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\bigg]\le c_p\sum_{i=0}^N E\big[\widetilde{\xi}^n_i+\eta^n_i\big]+c_p\nu(\mathcal{X})\sum_{i=0}^N E\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds$$
$$\le c_p\sum_{i=0}^N E\big[\vartheta^n_i\big]+c_p\sum_{i=0}^N\bigg\{E\bigg[\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds\bigg]\bigg\}^{\frac{1}{q}}\bigg\{E\bigg[\int_{\tau_i}^{\gamma_i}|b^n_s-b_s|^pds\bigg]\bigg\}^{\frac{1}{p}}+c_p\lambda^{1+p(1-\frac{\varpi}{2})}, \tag{A.51}$$

where $\vartheta^n_i:=|\mathcal{Y}^n_{\gamma_i}|^p+\nu(\mathcal{X})\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds+\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^{p-1}d\widehat{K}^c_s+\sum_{s\in(\tau_i,\gamma_i]}|\mathcal{Y}^n_s|^{p-1}\Delta\widehat{K}_s$.

The Hölder's inequality and (A.2) imply that

$$E\bigg[\Big(\int_0^T\mathbf{1}_{\{s\in\bigcup_{i=0}^N(\gamma_i,\tau_{i+1}]\}}\mathbf{1}_{\{|\mathcal{Z}^n_s|\ge\lambda\}}|\mathcal{Z}^n_s|^\varpi ds\Big)^{\frac{p}{2}}\bigg]+E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{s\in\bigcup_{i=0}^N(\gamma_i,\tau_{i+1}]\}}\mathbf{1}_{\{|\mathcal{U}^n_s(x)|\ge\lambda\}}|\mathcal{U}^n_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds$$
$$\le E\bigg[\bigg(\int_0^T\mathbf{1}_{\{s\in\bigcup_{i=0}^N(\gamma_i,\tau_{i+1}]\}}ds\bigg)^{\frac{p(2-\varpi)}{4}}\Big(\int_0^T|\mathcal{Z}^n_s|^2ds\Big)^{\frac{p\varpi}{4}}\bigg]+\bigg\{E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{s\in\bigcup_{i=0}^N(\gamma_i,\tau_{i+1}]\}}\nu(dx)ds\bigg\}^{1-\frac{\varpi}{2}}\big\|\mathcal{U}^n\big\|^{\frac{p\varpi}{2}}_{\mathbb{U}^p}$$
$$\le\bigg\{E\bigg[\bigg(\sum_{i=0}^N(\tau_{i+1}-\gamma_i)\bigg)^{\frac{p}{2}}\bigg]\bigg\}^{1-\frac{\varpi}{2}}\big\|\mathcal{Z}^n\big\|^{\frac{p\varpi}{2}}_{\mathbb{Z}^{2,p}}+\bigg\{\nu(\mathcal{X})E\bigg[\sum_{i=0}^N(\tau_{i+1}-\gamma_i)\bigg]\bigg\}^{1-\frac{\varpi}{2}}\big\|\mathcal{U}^n\big\|^{\frac{p\varpi}{2}}_{\mathbb{U}^p}$$
$$\le\lambda^{1-\frac{\varpi}{2}}\big(1\vee 2^{\frac{p\varpi}{2}-1}\big)\Big(C_\sharp^{\frac{p\varpi}{2}}+\|Z\|^{\frac{p\varpi}{2}}_{\mathbb{Z}^{2,p}}\Big)+\big(\nu(\mathcal{X})\lambda\big)^{1-\frac{\varpi}{2}}\big(1\vee 2^{\frac{p\varpi}{2}-1}\big)\Big(C_\sharp^{\frac{p\varpi}{2}}+\big\|\widehat{U}\big\|^{\frac{p\varpi}{2}}_{\mathbb{U}^p}\Big):=\varrho(\lambda).$$

Then we can deduce from (A.34), (1.7), (A.2) and (A.51) that

$$E\bigg[\Big(\int_0^T|\mathcal{Z}^n_s|^\varpi ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds\bigg]$$
$$\le E\bigg[\Big(\int_0^T\mathbf{1}_{\{s\in\bigcup_{i=0}^N(\gamma_i,\tau_{i+1}]\}}\mathbf{1}_{\{|\mathcal{Z}^n_s|\ge\lambda\}}|\mathcal{Z}^n_s|^\varpi ds\Big)^{\frac{p}{2}}\bigg]+E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{s\in\bigcup_{i=0}^N(\gamma_i,\tau_{i+1}]\}}\mathbf{1}_{\{|\mathcal{U}^n_s(x)|\ge\lambda\}}|\mathcal{U}^n_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds$$
$$+\sum_{i=0}^N E\bigg[\Big(\int_{\tau_i}^{\gamma_i}\mathbf{1}_{\{|\mathcal{Z}^n_s|\ge\lambda\}}|\mathcal{Z}^n_s|^\varpi ds\Big)^{\frac{p}{2}}+\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{U}^n_s(x)|\ge\lambda\}}|\mathcal{U}^n_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds\bigg]+(\lambda^\varpi T)^{\frac{p}{2}}+\lambda^{\frac{p\varpi}{2}}\nu(\mathcal{X})T$$
$$\le\varrho(\lambda)+\lambda^{p(\frac{\varpi}{2}-1)}\sum_{i=0}^N E\bigg[\Big(\int_{\tau_i}^{\gamma_i}|\mathcal{Z}^n_s|^2ds\Big)^{\frac{p}{2}}+\int_{\tau_i}^{\gamma_i}\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds\bigg]+\lambda^{\frac{p\varpi}{2}}\big(T^{\frac{p}{2}}+\nu(\mathcal{X})T\big)$$
$$\le\varrho(\lambda)+\lambda^{\frac{p\varpi}{2}}\big(T^{\frac{p}{2}}+\nu(\mathcal{X})T\big)+\lambda^{p(\frac{\varpi}{2}-1)}c_p\bigg(\sum_{i=0}^N E\big[\vartheta^n_i\big]+\sum_{i=0}^N\bigg\{E\bigg[\int_{\tau_i}^{\gamma_i}|\mathcal{Y}^n_s|^pds\bigg]\bigg\}^{\frac{1}{q}}\big(C_\sharp+\|b\|_{\mathbb{L}^p([0,T]\times\Omega,\mathscr{P},dt\times dP;\mathbb{R})}\big)\bigg)+c_p\lambda.$$

Since Young's inequality, the monotonicity of sequence $\{Y^n\}_{n\in\mathbb{N}}$ and (1.7) show that

$$\vartheta^n_i\le(1+\nu(X)T)(\mathcal{Y}^n_*)^p+(\mathcal{Y}^n_*)^{p-1}\big(\widehat{K}_{\gamma_i}-\widehat{K}_{\tau_i}\big)\le\Big(1+\frac{1}{q}+\nu(X)T\Big)\sup_{t\in[0,T]}(X_t-Y^1_t)^p+\frac{1}{p}\widehat{K}^p_T$$
$$\le\Big(1+\frac{1}{q}+\nu(X)T\Big)2^{p-1}\big(X^p_*+(Y^1_*)^p\big)+\frac{1}{p}\widehat{K}^p_T,\quad\forall\, n\in\mathbb{N}, \tag{A.52}$$

letting $n\to\infty$, one can deduce from the dominated convergence theorem, (A.4), (A.3) and (A.31) that

$$\varlimsup_{n\to\infty}E\bigg[\Big(\int_0^T|Z^n_s-Z_s|^\varpi ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|U^n_s(x)-\widehat{U}_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds\bigg]\le\varrho(\lambda)+\lambda^{\frac{p\varpi}{2}}\big(T^{\frac{p}{2}}+\nu(\mathcal{X})T\big)+c_p\lambda.$$

As $\lambda\to 0$, we obtain

$$\lim_{n\to\infty}E\bigg[\Big(\int_0^T|Z^n_s-Z_s|^\varpi ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|U^n_s(x)-\widehat{U}_s(x)|^{\frac{p\varpi}{2}}\nu(dx)ds\bigg]=0, \tag{A.53}$$

which shows that U^n's strongly converge and thus weakly converge to $\widehat{U}$ in $\mathbb{U}^{\frac{p\varpi}{2}}$. On the other hand, the weak convergence of U^n's to U in $\mathbb{U}^p$ implies that U^n's also weakly converge to U in $\mathbb{U}^{\frac{p\varpi}{2}}$. So by the uniqueness of weak limit of $\{U^n\}_{n\in\mathbb{N}}$ in $\mathbb{U}^{\frac{p\varpi}{2}}$, one has $\widehat{U}_t(x)=U_t(x)$, $dt\times dP\times\nu(dx)-$a.s. Consequently, (A.6) follows from (A.53), and (A.32) shows that $K=\widehat{K}\in\mathbb{K}^p$. This together with (A.21) eventually leads to (A.5).

6) Suppose further that Y has only inaccessible jumps. As K is an $\mathbf{F}-$predictable process, we see from (A.5) that K has no jump or K is a continuous process.

We fix $n\in\mathbb{N}$ and reset $(\mathcal{Y}^n,\mathcal{Z}^n,\mathcal{U}^n):=\big(Y^n-Y,Z^n-Z,U^n-U\big)$. Analogous to (A.35) and (A.36), it holds for $P-$a.s. $\omega\in\Omega$ that

$$\Delta\mathcal{Y}^n_t(\omega)=\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\mathcal{U}^n\big(t,\omega,\mathfrak{p}_t(\omega)\big),\ \ \forall\, t\in[0,T]\ \text{ and }\ \big\{t\in[0,T]:\mathcal{Y}^n_{t-}(\omega)\neq\mathcal{Y}^n_t(\omega)\big\}\text{ is a countable subset of }[0,T]. \quad (A.54)$$

Let $(t,\varepsilon)\in[0,T]\times(0,1]$. We see from (A.37) that the process $\mathfrak{Y}^{n,\varepsilon}_t:=\sup\limits_{s\in[t,T]}\varphi_\varepsilon(\mathcal{Y}^n_s)$, $t\in[0,T]$ satisfies

$$E\Big[\big(\mathfrak{Y}^{n,\varepsilon}_0\big)^p\Big]\le E\Big[\sup_{s\in[0,T]}|\mathcal{Y}^n_s|^p\Big]+\varepsilon^{\frac{p}{2}}=\|\mathcal{Y}^n\|^p_{\mathbb{D}^p}+\varepsilon^{\frac{p}{2}}<\infty. \quad (A.55)$$

Subtracting (A.5) from (A.1) and applying Itô's formula to $\varphi^p_\varepsilon(\mathcal{Y}^n_t)$ on the interval $[t,T]$ yield that $P-$a.s.

$$\begin{aligned}&\varphi^p_\varepsilon(\mathcal{Y}^n_t)+\frac12\int_t^T D^2_x\varphi^p_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+\sum_{s\in(t,T]}\Big(\varphi^p_\varepsilon(\mathcal{Y}^n_s)-\varphi^p_\varepsilon(\mathcal{Y}^n_{s-})-D_x\varphi^p_\varepsilon(\mathcal{Y}^n_{s-})\Delta\mathcal{Y}^n_s\Big)\\ &\quad=\varphi^p_\varepsilon(\mathcal{Y}^n_T)+p\int_t^T\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)\mathcal{Y}^n_s\,(b^n_s-b_s)ds+p\int_t^T\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)\mathcal{Y}^n_s(dK^n_s-dK_s)-p\big(M^n_T-M^n_t+\mathcal{M}^n_T-\mathcal{M}^n_t\big),\end{aligned} \quad (A.56)$$

where M^n and $\mathcal{M}^n$ are uniformly integrable martingales as defined in (A.38). Similar to (6.57), Taylor's Expansion Theorem and the first part of (A.54) imply that $P-$a.s.

$$\sum_{s\in(t,T]}\Big(\varphi^p_\varepsilon(\mathcal{Y}^n_s)-\varphi^p_\varepsilon(\mathcal{Y}^n_{s-})-D\varphi^p_\varepsilon(\mathcal{Y}^n_{s-})\Delta\mathcal{Y}^n_s\Big)\ge 2^{p-3}p(p-1)\int_{(t,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}N_{\mathfrak{p}}(ds,dx).$$

Then we can deduce from (A.56), (A.42) and (6.54) that

$$\begin{aligned}&\varphi^p_\varepsilon(\mathcal{Y}^n_t)+\frac{p}{2}(p-1)\int_t^T\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+2^{p-3}p(p-1)\int_t^T\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}N_{\mathfrak{p}}(ds,dx)\\ &\quad\le\xi^n_\varepsilon-p\big(M^n_T-M^n_t+\mathcal{M}^n_T-\mathcal{M}^n_t\big),\quad P-\text{a.s.},\end{aligned} \quad (A.57)$$

with $\xi^n_\varepsilon:=\varphi^p_\varepsilon(\mathcal{Y}^n_T)+p\int_0^T\varphi^{p-1}_\varepsilon(\mathcal{Y}^n_s)|b^n_s-b_s|ds+p\int_0^T|\mathcal{Y}^n_s|^{p-1}dK_s$. So letting $t=0$ and taking expectation yield that

$$E\int_0^T\varphi^{p-2}_\varepsilon(\mathcal{Y}^n_s)|\mathcal{Z}^n_s|^2ds+2^{p-2}E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}\big|\mathcal{U}^n_s(x)\big|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds\le\frac{2}{p(p-1)}E\big[\xi^n_\varepsilon\big]. \quad (A.58)$$

Similar to (A.45) and (A.46), the monotone convergence theorem shows that

$$\lim_{\varepsilon\to0}\uparrow E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}|\mathcal{U}^n_s(x)|^2\big(|\mathcal{U}^n_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds=E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds,$$

while Young's inequality, (A.55) and the dominated convergence theorem imply that

$$\lim_{\varepsilon\to0}E\big[\xi^n_\varepsilon\big]=E\big[\widetilde{\xi}_n\big], \quad (A.59)$$

where $\widetilde{\xi}_n:=|\mathcal{Y}^n_T|^p+p\int_t^T|\mathcal{Y}^n_s|^{p-1}|b^n_s-b_s|ds+p\int_0^T|\mathcal{Y}^n_s|^{p-1}dK_s$. So letting $\varepsilon\to0$ in (A.58) and using the second part of (A.54) yields that

$$\begin{aligned}E\int_0^T\int_{\mathcal{X}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds&\le E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{Y}^n_{s-}|\le|\mathcal{U}^n_s(x)|\}}|\mathcal{U}^n_s(x)|^p\nu(dx)ds+E\int_0^T\int_{\mathcal{X}}\mathbf{1}_{\{|\mathcal{U}^n_s(x)|<|\mathcal{Y}^n_{s-}|\}}|\mathcal{Y}^n_{s-}|^p\nu(dx)ds\\ &\le\frac{2^{3-p}}{p(p-1)}E\big[\widetilde{\xi}_n\big]+\nu(\mathcal{X})E\int_0^T|\mathcal{Y}^n_s|^pds.\end{aligned} \quad (A.60)$$

Now, fix $\varepsilon\in(0,1]$ again. Using similar arguments to those that lead to (A.48) and (A.49), we can deduce from (A.57), (A.58) and (A.60) that

$$E\Big[\big(\mathfrak{Y}_0^{n,\varepsilon}\big)^p\Big]\le E\big[\xi_\varepsilon^n\big]+2p\,E\Big[\sup_{s\in[0,T]}|M_s^n|+\sup_{s\in[0,T]}|\mathcal{M}_s^n|\Big]\le\frac{1}{2}E\big[(\mathfrak{Y}_0^{n,\varepsilon})^p\big]+c_pE\big[\xi_\varepsilon^n+\widetilde{\xi}_n\big]+c_p\nu(\mathcal{X})E\int_0^T|\mathcal{Y}_s^n|^pds.$$

As $E\big[(\mathfrak{Y}_0^{n,\varepsilon})^p\big]<\infty$ by (A.55), similar to (A.50), Young's inequality, (A.58) and (A.60) imply that

$$E\bigg[\Big(\int_0^T|\mathcal{Z}_s^n|^2ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|\mathcal{U}_s^n(x)|^p\nu(dx)ds\bigg]\le\frac{2-p}{2}E\Big[\big(\mathfrak{Y}_0^{n,\varepsilon}\big)^p\Big]+\frac{p}{2}E\int_0^T\varphi_\varepsilon^{p-2}(\mathcal{Y}_s^n)|\mathcal{Z}_s^n|^2ds+E\int_0^T\int_{\mathcal{X}}|\mathcal{U}_s^n(x)|^p\nu(dx)ds$$
$$\le c_pE\big[\xi_\varepsilon^n+\widetilde{\xi}_n\big]+c_p\nu(\mathcal{X})E\int_0^T|\mathcal{Y}_s^n|^pds.$$

Letting $\varepsilon\to0$, we see from (A.59) and Hölder's inequality that

$$E\bigg[\Big(\int_0^T|\mathcal{Z}_s^n|^2ds\Big)^{\frac{p}{2}}+\int_0^T\int_{\mathcal{X}}|\mathcal{U}_s^n(x)|^p\nu(dx)ds\bigg]\le c_pE[\vartheta_n]+c_p\bigg\{E\bigg[\int_0^T|\mathcal{Y}_s^n|^pds\bigg]\bigg\}^{\frac{1}{q}}\bigg\{E\bigg[\int_0^T|b_s^n-b_s|^pds\bigg]\bigg\}^{\frac{1}{p}},$$

where $\vartheta_n:=|\mathcal{Y}_T^n|^p+\nu(\mathcal{X})\int_0^T|\mathcal{Y}_s^n|^pds+\int_0^T|\mathcal{Y}_s^n|^{p-1}dK_s$. Since an analogy to (A.52) shows that

$$\vartheta_n\le(1+\nu(X)T)(\mathcal{Y}_*^n)^p+(\mathcal{Y}_*^n)^{p-1}K_T\le\Big(1+\frac{1}{q}+\nu(X)T\Big)2^{p-1}\big(X_*^p+(Y_*^1)^p\big)+\frac{1}{p}K_T^p,\quad\forall\,n\in\mathbb{N},$$

letting $n\to\infty$, one can derive (A.7) from the dominated convergence theorem, (A.4) and (A.3). □

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