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$\mathbb{L}^p$ solutions of backward stochastic differential equations with jumps☆

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Abstract

Given $p \in (1, 2)$, we study $\mathbb{L}^p$ solutions of a multi-dimensional backward stochastic differential equation with jumps (BSDEJ) whose generator may not be Lipschitz continuous in (y, z)-variables. We show that such a BSDEJ with p-integrable terminal data admits a unique $\mathbb{L}^p$ solution by approximating the monotonic generator by a sequence of Lipschitz generators via convolution with mollifiers and using a stability result.

Keywords: Backward stochastic differential equations with jumps; $\mathbb{L}^p$ solutions; Monotonic generators; Convolution with mollifiers

1. Introduction

Let $p \in (1, 2)$ and $T \in (0, \infty)$. In this paper, we study $\mathbb{L}^p$ solutions of a multi-dimensional backward stochastic differential equation with jumps (BSDEJ)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s \, dB_s - \int_{(t,T]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T] \tag{1.1}$$

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over a probability space $(\Omega, \mathcal{F}, P)$ on which B is a Brownian motion and $\mathfrak{p}$ is an $\mathcal{X}$-valued Poisson point process independent of B. Practically speaking, if the Brownian motion stands for the noise from the financial market, then the Poisson random measure can be interpreted as the randomness of insurance claims. In the BSDEJ (1.1) with generator f and terminal data ξ, a solution consists of an adapted càdlàg process Y, a locally square-integrable predictable process Z and a locally p-integrable predictable random field U.

The backward stochastic equation (BSDE) was introduced by Bismut [7] as the adjoint equation for the Pontryagin maximum principle in stochastic control theory. Later, Pardoux and Peng [45] commenced a systematical research of BSDEs. Since then, the BSDE theory has grown rapidly and has been applied to various areas such as mathematical finance, theoretical economics, stochastic control and optimization, partial differential equations, differential geometry and etc., (see the references in [25,20]).

Li and Tang [51] introduced into the BSDE a jump term that is driven by a Poisson random measure independent of the Brownian motion. These authors obtained the existence of a unique solution to a BSDEJ with a Lipschitz generator and square-integrable terminal data. Then Barles, Buckdahn and Pardoux [13,5] showed that the wellposedness of BSDEJs gives rise to a viscosity solution of a semilinear parabolic partial integro-differential equation (PIDE) and thus provides a probabilistic interpretation of such a PIDE. Later, Pardoux [44] relaxed the Lipschitz condition of the generator on variable y by assuming a monotonicity condition on variable y instead. Situ [50] and Mao and Yin [57] even degenerated the monotonicity condition of the generator to a weaker version so as to remove the Lipschitz condition on variable z.

During the development of the BSDE theory, some efforts were made in relaxing the square integrability on the terminal data so as to be compatible with the fact that linear BSDEs are well-posed for integrable terminal data or that linear expectations have $\mathbb{L}^1$ domains: El Karoui et al. [25] showed that for any p-integrable terminal data, the BSDE with a Lipschitz generator admits a unique $\mathbb{L}^p$-solution. Then Briand and Carmona [9] reduced the Lipschitz condition of the generator on variable y by a strong monotonicity condition as well as a polynomial growth condition on variable y. Later, Briand et al. [10] found that the polynomial growth condition is not necessary if one uses the monotonicity condition similar to that of [44].

In the present paper, assuming that the generator f satisfies monotonicity conditions (H6) and (H3) on (y, z); that f has a general growth condition (H2) on y, a linear growth condition (H4) on z; and that f is Lipschitz continuous in u, we show in Theorem 2.1 that for any p-integrable terminal data ξ, the BSDEJ (1.1) admits a unique $\mathbb{L}^p$-solution (Y, Z, U) (see the notations in Section 1.1). Consequently, we obtain a general martingale representation theorem for p-integrable martingales in the jump case (Corollary 2.1).

To demonstrate Theorem 2.1, we start with an inequality (3.2) about the difference of two local p-integrable solutions to BSDEJs with different parameters under a general monotonicity condition (3.1). The basic inequality (3.2) gives rise to an *a priori* estimate (3.3) of the $\mathbb{L}^p$-norm of a solution (Y, Z, U) of a BSDEJ with parameter (ξ, f) in terms of the L^p norms of $|\xi| + \int_0^T |f(t, 0, 0, 0)|dt$. The inequality (3.2) also leads to a stability result of $\mathbb{L}^p$-solutions of BSDEJs (Proposition 3.2), which claims that a sequence of solutions to BSDEJs is a Cauchy sequence under the L^p-norm if their terminal data is a Cauchy sequence under the L^p-norm and if the solutions satisfy an asymptotic monotonicity condition (3.4). Then the uniqueness of the $\mathbb{L}^p$-solution to BSDEJ (1.1) immediately follows.

For the existence of an $\mathbb{L}^p$-solution to BSDEJ (1.1), we first deal with the case when the monotonic generator f has linear growth (H2$'$) in y and when the random variable

$|\xi| + \int_0^T |f(t,0,0,0)|dt$ is bounded. In Proposition 3.3, we exploit *convolution with mollifiers* to approach the monotonic generator f by a sequence of Lipschitz generators, and utilize the stability result (Proposition 3.2) to show that the $\mathbb{L}^2$-solutions of the BSDEJs with the approximating Lipschitz generators and the bounded terminal data are actually a Cauchy sequence in $\mathbb{S}^p$ whose limit solves the BSDEJ (1.1). Then by truncating the generator f and the terminal data ξ respectively, we employ the stability result again to obtain the general existence result in Theorem 2.1.

When the generator f is Lipschitz in (y,z,u), one can use the classic *fixed-point* argument to demonstrate the existence of a unique $\mathbb{L}^p$-solution of BSDEJ (1.1) with p-integrable terminal data ξ, see Remark 4.1. Our ArXiv version [53] contains a detailed proof of this result as well as the related generator representation.

Main contributions.

Given $U \in \mathbb{U}^2_{loc}$, unlike the case of Brownian stochastic integrals, the Burkholder–Davis–Gundy inequality is not applicable for the $p/2$th power of the Poisson stochastic integral $\int_{(0,t]}\int_{\mathcal{X}} Y_s U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t \in [0,T]$ (see e.g. Theorem VII.92 of [22]): i.e. $E\Big[\sup_{t\in[0,T]}\Big(\int_{(0,t]}\int_{\mathcal{X}} Y_s U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\Big)^{\frac{p}{2}}\Big]$ cannot be dominated by $E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}} |Y_s|^2 |U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{4}}\Big]$. So to derive an *a priori* $\mathbb{L}^p$ estimate for BSDEJs, we could not follow the classical argument in the proof of [10, Proposition 3.2], neither could we employ the space $\mathbb{U}^{2,p} := \Big\{U : E\Big[\Big(\int_0^T\int_{\mathcal{X}} |U_t(x)|^2\nu(dx)dt\Big)^{\frac{p}{2}}\Big] < \infty\Big\}$ or the space $\widetilde{\mathbb{U}}^{2,p} := \Big\{U : E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}} |U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big] < \infty\Big\}$ (Actually one may not be able to compare $E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}} |U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big]$ with $E\Big[\Big(\int_0^T\int_{\mathcal{X}} |U_t(x)|^2\nu(dx)dt\Big)^{\frac{p}{2}}\Big]$).

To address these technical difficulties, we first generalize the Poisson stochastic integral for a random field $U \in \mathbb{U}^p$ by constructing in Lemma 1.1 a càdlàg uniformly integrable martingale $M^U_t := \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t \in [0,T]$, whose quadratic variation $[M^U, M^U]$ is still $\int_{(0,t]}\int_{\mathcal{X}} |U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)$, $t \in [0,T]$. Our inequality (5.1) shows that

$$E\Big[[M^U, M^U]^{\frac{p}{2}}\Big] \le E\int_{(0,T]}\int_{\mathcal{X}} |U_t(x)|^p N_{\mathfrak{p}}(dt,dx) = E\int_0^T\int_{\mathcal{X}} |U_t(x)|^p\nu(dx)dt. \quad (1.2)$$

In deriving the key $\mathbb{L}^p$-type inequality (3.2) about the difference $Y = Y^1 - Y^2$ of two local p-integrable solutions to BSDEJs with different parameters, our delicate analysis showed that the variational jump part $\sum_s\Big(|Y_s|^p - |Y_{s-}|^p - p\big\langle|Y_{s-}|^{p-1}, \Delta Y_s\big\rangle\Big)$ in the dynamics of $|Y|^p$ will eventually boil down to the term $E\int_0^T\int_{\mathcal{X}} |U^1_t(x) - U^2_t(x)|^p\nu(dx)dt$, which justifies our choice of $\mathbb{U}^p$ over $\mathbb{U}^{2,p}$ or $\widetilde{\mathbb{U}}^{2,p}$ as the space for jump diffusion. The estimation course of the variational jump is full of analytical subtleties, but we manage to overcome them by leveraging Taylor's expansion, (1.2) and some new techniques (see (5.11)–(5.21) for details).

It is also worth mentioning that although our "convolution with mollifiers" approach seems similar to that of [50], some special treatments are necessary along the way to overcome various technical hurdles raising in the $\mathbb{L}^p$-jump case; and some auxiliary results, like Lemmas A.2 and A.6, are interesting in their own right.

The financial significance of the present paper lies in the fact that it allows us to study many mathematical finance problems for a large class of p-integrable financial positions (which may not be square-integrable) under nonlinear evaluation criteria or risk measurement in a market with jumps. In particular, the paper provides a solid technical ground for our accompanying articles [55,56,54]:

Given a real-valued p-integrable ξ, the wellposedness result (Theorem 2.1 or Remark 4.1) shows that the BSDEJ with a generator g and the terminal data ξ admits a unique solution, whose Y-component Y^ξ can be regarded as the so-called "(conditional) g-expectation" of ξ: $\mathcal{E}_g[\xi|\mathcal{F}_t] := Y^\xi_t$, $t \in [0,T]$. In [55], we show that the g-expectations, as nonlinear expectations with $\mathbb{L}^p$ domains under jump filtration, inherit many basic properties from the classic linear expectations and are closely related to axiom-based coherent and convex risk measures (see [2,26,48]) in mathematical finance.

In [56], we study a general class of jump-filtration consistent nonlinear expectations $\mathcal{E}$ with $\mathbb{L}^p$-domains, which includes many coherent or convex time-consistent risk measures $\rho = \{\rho_t\}_{t\in[0,T]}$. Under certain domination condition, we demonstrate that the nonlinear expectation $\mathcal{E}$ can be represented by some g-expectation. Consequently, one can utilize the BSDEJ theory to systematically analyze the risk measure ρ with $\mathbb{L}^p$-domains and employ numerical schemes of BSDEJs to run simulation for financial problems involving ρ in a financial market with jumps.

Moreover, we analyze in [54] a BSDEJ with a p-integrable reflecting barrier $\mathfrak{L}$ whose generator g is Lipschitz continuous in (y,z,u). We show that such a reflected BSDEJ with p-integrable parameters admits a unique $\mathbb{L}^p$ solution, and thus solves the corresponding optimal stopping problem under the g-expectation or some dominated risk measure with $\mathbb{L}^p$-domain.

Relevant literature.

Besides the aforementioned works, we would like to outline some recent research on BSDEJs:
(1) Kruse and Popier [38] lately studied a similar $\mathbb{L}^p$-solution problem of BSDE under a right-continuous filtration which may be larger than the jump filtration:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s\, dB_s - \int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) - \int_t^T dM_s, \quad t \in [0,T], \tag{1.3}$$

where M is a local martingale orthogonal to the jump filtration. However, their wellposedness result requires a relatively stronger monotone condition and Lipschitz continuity of f in z (see (H1) and (H3) therein).

Klimsiak studied $\mathbb{L}^p$ solutions of reflected BSDEs under a general right-continuous filtration in [36], and analyzed $\mathbb{L}^p$ solutions to BSDEs with monotone generators and two irregular reflecting barriers in [35].

(2) The researches on BSDEs over general filtered probability spaces have recently attracted more and more attention. A series of works [12,23,25,11,14,39,15] are dedicated to the theory of BSDEs (1.3) but driven by a càdlàg martingale under a right-continuous filtration that is also quasi-left continuous. Lately, [8,43] removed the quasi-left continuity assumption from the filtration so that the quadratic variation of the driving martingale does not need to be absolutely continuous. On the other hand, based on a general martingale representation result due to Davis and Varaiya [21], Cohen and Elliott [16,17] discussed the case where the driving martingales are not *a priori* chosen but imposed by the filtration; see Hassani and Ouknine [29] for a similar approach on a BSDE in the form of a generic map from a space of semimartingales to the spaces of martingales and those of finite-variation processes. Also, Mania and Tevzadze [40] and

Jeanblanc et al. [31] studied BSDEs for semimartingales and their applications to mean–variance hedging.

As to BSDEs driven by other discontinuous random sources, Xia [52] and Bandini [4] studied BSDEs driven by a random measure; Confortola et al. [18,19] considered BSDEs driven by a marked point process; [42,3,47,28] analyzed BSDEs driven by Lévy processes; [1,49,33] discussed BSDEs driven by a process with a finite number of marked jumps.

(3) There are also plenty of researches on quadratic BSDEJs and BSDEJs in other interesting directions, for example [6,41,24,32,34,37,27] among others. See [43] or our ArXiv version [53] for a synopsis of these topics.

The rest of the paper is organized as follows: In Section 1, we list necessary notations, and we generalize the Poisson stochastic integral for $U \in \mathbb{U}^p$ so as to define BSDEJs in $\mathbb{L}^p$ sense. After making some assumptions on generator f (including the monotonicity conditions in (y, z)), we present in Section 2, the main result of our paper, the existence and uniqueness of an $\mathbb{L}^p$-solution to a BSDEJ with p-terminal data, which gives rise to a general martingale representation theorem for p-integrable martingales in the jump case. In Section 3, we give an inequality about the difference of two local p-integrable solutions to BSDEJs as well as two consequences of it: an *a priori* estimate and a stability result of $\mathbb{L}^p$-solutions of BSDEJs, both are important to prove Theorem 2.1. Section 3 also includes a basic existence result of $\mathbb{L}^p$-solutions to BSDEJs with bounded parameters, which is also crucial for Theorem 2.1. Section 4 further discusses the wellposedness of BSDEJs with Lipschitz generators in $\mathbb{L}^p$ sense. The proofs of our results are deferred to Section 5, and the Appendix contains some necessary technical lemmata.

1.1. Notation and preliminaries

Throughout this paper, we fix a time horizon $T \in (0, \infty)$ and consider a complete probability space $(\Omega, \mathcal{F}, P)$ on which a d-dimensional Brownian motion B is defined.

For a generic càdlàg process X, we denote its corresponding jump process by $\Delta X_t := X_t - X_{t-}$, $t \in [0, T]$ with $X_{0-} := X_0$. Given a measurable space $(\mathcal{X}, \mathcal{F}_{\mathcal{X}})$, let $\mathfrak{p}$ be an $\mathcal{X}$-valued Poisson point process on $(\Omega, \mathcal{F}, P)$ that is independent of B. For any scenario $\omega \in \Omega$, let $D_{\mathfrak{p}(\omega)}$ collect all jump times of the path $\mathfrak{p}(\omega)$, which is a countable subset of $(0, T]$ (see e.g.Section 1.9 of [30]). We assume that for some finite measure ν on $\big(\mathcal{X}, \mathcal{F}_{\mathcal{X}}\big)$, the counting measure $N_{\mathfrak{p}}(dt, dx)$ of $\mathfrak{p}$ on $[0, T] \times \mathcal{X}$ has compensator $E\big[N_{\mathfrak{p}}(dt, dx)\big] = \nu(dx)dt$. The corresponding compensated Poisson random measure $\widetilde{N}_{\mathfrak{p}}$ is $\widetilde{N}_{\mathfrak{p}}(dt, dx) := N_{\mathfrak{p}}(dt, dx) - \nu(dx)dt$.

For any $t \in [0, T]$, we define sigma-fields

$$\mathcal{F}_t^B := \sigma\big\{B_s; s \le t\big\}, \quad \mathcal{F}_t^N := \sigma\big\{N_{\mathfrak{p}}\big((0, s], A\big); s \le t, A \in \mathcal{F}_{\mathcal{X}}\big\}, \quad \mathcal{F}_t := \sigma\big(\mathcal{F}_t^B \cup \mathcal{F}_t^N\big)$$

and augment them by all P-null sets in $\mathcal{F}$. Clearly, the jump filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is complete and right-continuous (i.e. satisfies the *usual hypotheses*, see e.g., [46]). Let $\mathscr{P}$ (resp. $\widehat{\mathscr{P}}$) denote the $\mathbf{F}$-progressively measurable (resp. $\mathbf{F}$-predictable) sigma-field on $[0, T] \times \Omega$, and let $\mathcal{T}$ collect all $\mathbf{F}$-stopping times.

For a generic Euclidean space $\mathbb{E}$ with norm $\|\cdot\|$, we define:

$$\mathscr{D}(x) := \mathbf{1}_{\{x \ne 0\}} \frac{1}{\|x\|} x \quad \text{and} \quad \pi_r(x) := \frac{r}{r \vee \|x\|} x, \qquad \forall\, x \in \mathbb{E}, \quad \forall\, r \in (0, T].$$

See Lemmas A.4 and A.6 for the properties of these two functions.

Given $l \in \mathbb{N}$, the following spaces of functions will be used in the sequel:
(1) For any $p \in [1,\infty)$, let $L^p_+[0,T]$ be the space of all measurable functions $\psi : [0,T] \to [0,\infty)$ with $\int_0^T \big(\psi(t)\big)^p dt < \infty$.
(2) For $p \in (1,2]$, let $L^p_\nu := L^p(\mathcal{X},\mathcal{F}_\mathcal{X},\nu;\mathbb{R}^l)$ be the space of all $\mathbb{R}^l$-valued, $\mathcal{F}_\mathcal{X}$-measurable functions u with $\|u\|_{L^p_\nu} := \big(\int_\mathcal{X} |u(x)|^p \nu(dx)\big)^{\frac{1}{p}} < \infty$. For any $u_1, u_2 \in L^p_\nu$, we say $u_1 = u_2$ if $u_1(x) = u_2(x)$ for ν-a.s. $x \in \mathcal{X}$.
(3) For any sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$, let

- $L^0_+(\mathcal{G})$ be the space of all real-valued non-negative $\mathcal{G}$-measurable random variables;
- $L^p_+(\mathcal{G}) := \Big\{\xi \in L^0_+(\mathcal{G}) : \|\xi\|_{L^p_+(\mathcal{G})} := \big\{E\big[\xi^p\big]\big\}^{\frac{1}{p}} < \infty\Big\}$ for all $p \in [1,2)$;
- $L^\infty_+(\mathcal{G}) := \Big\{\xi \in L^0_+(\mathcal{G}) : \|\xi\|_{L^\infty_+(\mathcal{G})} := \operatorname{esssup}_{\omega\in\Omega} \xi(\omega) < \infty\Big\}$;
- $L^0(\mathcal{G})$ be the space of all $\mathbb{R}^l$-valued, $\mathcal{G}$-measurable random variables;
- $L^p(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^p(\mathcal{G})} := \big\{E\big[|\xi|^p\big]\big\}^{\frac{1}{p}} < \infty\Big\}$ for all $p \in [1,2)$;
- $L^\infty(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^\infty(\mathcal{G})} := \operatorname{esssup}_{\omega\in\Omega} |\xi(\omega)| < \infty\Big\}$.

(4) Let $\mathbb{D}^0$ be the space of all $\mathbb{R}^l$-valued, $\mathbf{F}$-adapted càdlàg processes, and let $\mathbb{D}^\infty$ be the space of all $\mathbb{R}^l$-valued, $\mathbf{F}$-adapted càdlàg processes X with $\|X\|_{\mathbb{D}^\infty} := \operatorname{esssup}_{(t,\omega)\in[0,T]\times\Omega} |X_t(\omega)| = \operatorname{esssup}_{\omega\in\Omega} X_*(\omega) < \infty$, where $X_*(\omega) := \sup_{t\in[0,T]} \big|X_t(\omega)\big|$.
(5) Set $\mathbb{Z}^2_{\rm loc} := L^2_{\rm loc}\big([0,T]\times\Omega, \widehat{\mathscr{P}}, dt\times dP; \mathbb{R}^{l\times d}\big)$, the space of all $\mathbb{R}^{l\times d}$-valued, $\mathbf{F}$-predictable processes Z with $\int_0^T |Z_t|^2 dt < \infty$, P-a.s.
(6) For any $p \in [1,2]$, we let

- $\mathbb{D}^p := \Big\{X \in \mathbb{D}^0 : \|X\|_{\mathbb{D}^p} := \big\{E[X^p_*]\big\}^{\frac{1}{p}} < \infty\Big\}$.
- $\mathbb{Z}^{2,p} := \Big\{Z \in \mathbb{Z}^2_{\rm loc} : \|Z\|_{\mathbb{Z}^{2,p}} := \Big\{E\Big[\big(\int_0^T |Z_t|^2 dt\big)^{\frac{p}{2}}\Big]\Big\}^{\frac{1}{p}} < \infty\Big\}$. We will simply denote $\mathbb{Z}^{2,2}$ by $\mathbb{Z}^2$. For any $Z \in \mathbb{Z}^{2,p}$, the Burkholder–Davis–Gundy inequality implies that

$$E\bigg[\sup_{t\in[0,T]} \Big|\int_0^t Z_s dB_s\Big|^p\bigg] \le c_{p,l} E\bigg[\Big(\int_0^T \big|Z_s\big|^2 ds\Big)^{\frac{p}{2}}\bigg] < \infty \tag{1.4}$$

for some constant $c_{p,l} > 0$ depending on p and l. So $\big\{\int_0^t Z_s dB_s\big\}_{t\in[0,T]}$ is a uniformly integrable martingale.

- $\mathbb{U}^p_{\rm loc} := L^p_{\rm loc}\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_\mathcal{X}, dt\times dP\times\nu(dx); \mathbb{R}^l\big)$ be the space of all $\widehat{\mathscr{P}}\otimes\mathcal{F}_\mathcal{X}$-measurable random fields $U : [0,T]\times\Omega\times\mathcal{X} \to \mathbb{R}^l$ such that $\int_0^T \int_\mathcal{X} |U_t(x)|^p \nu(dx)dt = \int_0^T \|U_t\|^p_{L^p_\nu} dt < \infty$, P-a.s. For any $U \in \mathbb{U}^p_{\rm loc}$, it is clear that $U(t,\omega) \in L^p_\nu$ for $dt\times dP$-a.s. $(t,\omega) \in [0,T]\times\Omega$.
- $\mathbb{U}^p := \Big\{U \in \mathbb{U}^p_{\rm loc} : \|U\|_{\mathbb{U}^p} := \big\{E\int_0^T \int_\mathcal{X} |U_t(x)|^p \nu(dx)dt\big\}^{\frac{1}{p}} < \infty\Big\} = L^p\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_\mathcal{X}, dt\times dP\times\nu(dx); \mathbb{R}^l\big)$.
- Let us simply denote $\mathbb{D}^p \times \mathbb{Z}^{2,p} \times \mathbb{U}^p$ by $\mathbb{S}^p$.

In this paper, we use the convention $\inf\emptyset := \infty$ and let $c_{p,l}$ denote a generic constant depending only on p and l (in particular, c_l stands for a generic constant depending only on l), whose form may vary from line to line.

1.2. Generalization of Poisson stochastic integrals

The stochastic integral with respect to the compensated Poisson random measure $\widetilde{N}_{\mathfrak{p}}(dt, dx)$ (or simply "Poisson stochastic integral") is usually defined for locally square integrable random fields $U \in \mathbb{U}^2_{\mathrm{loc}}$. In this subsection, we will generalize such kind of stochastic integral for random fields in $\cup_{p\in[1,2)} \mathbb{U}^p_{\mathrm{loc}}$ in spirit of [22, VIII.75].

Let $\mathbb{M}^1$ be the space of all càdlàg local martingales $M = \{M_t\}_{t\in[0,T]}$ with $\|M\|_{\mathbb{M}^1} := E\big\{\big[M, M\big]_T^{\frac{1}{2}}\big\} < \infty$. According to [22, VII.81–VII.92], $\|\cdot\|_{\mathbb{M}^1}$ is a norm on $\mathbb{M}^1$ that is equivalent to $\|\cdot\|_{\mathbb{D}^1}$, thus $\big(\mathbb{M}^1, \|\cdot\|_{\mathbb{M}^1}\big)$ is a Banach space.

Let $p \in [1, 2)$ and $U \in \mathbb{U}^p$. For any $n \in \mathbb{N}$, since $E\int_0^T \int_{\mathcal{X}} \mathbf{1}_{\{|U_s(x)|\le n\}}|U_s(x)|^2 \nu(dx)ds \le n^{2-p} E\int_0^T \int_{\mathcal{X}} |U_s(x)|^p \nu(dx)ds < \infty$, $M^{U,n}_t := \int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{|U_s(x)|\le n\}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx)$, $t \in [0, T]$ defines a square integrable martingale.

Lemma 1.1. *Let $p \in [1, 2)$. For any $U \in \mathbb{U}^p$, $\{M^{U,n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(\mathbb{M}^1, \|\cdot\|_{\mathbb{M}^1}\big)$, whose limit M^U is a càdlàg uniformly integrable martingale with quadratic variation $[M^U, M^U]_t = \int_{(0,t]}\int_{\mathcal{X}} |U_s(x)|^2 N_{\mathfrak{p}}(ds, dx)$, $t \in [0, T]$. The jump process of M^U satisfies that for P-a.s. $\omega \in \Omega$,*

$$\Delta M^U_t(\omega) = \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}} U\big(t, \omega, \mathfrak{p}_t(\omega)\big), \quad \forall\, t \in (0, T]. \tag{1.5}$$

Moreover, $U \to M^U$ is a linear mapping on $\mathbb{U}^p$.

We shall assign M^U as the Poisson stochastic integral

$$\int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds, dx), \quad t \in [0, T] \tag{1.6}$$

of $U \in \mathbb{U}^p$. Analogous to the classic extension of Poisson stochastic integrals from $\mathbb{U}^2$ to $\mathbb{U}^2_{\mathrm{loc}}$, one can define the stochastic integral (1.6) (or simply M^U) for any $U \in \mathbb{U}^p_{\mathrm{loc}}$, which is a càdlàg local martingale with quadratic variation $\int_0^t \int_{\mathcal{X}} |U_s(x)|^2 N_{\mathfrak{p}}(ds, dx)$, $t \in [0, T]$ and whose jump process satisfies (1.5) also. This generalized Poisson stochastic integral is still linear in $U \in \mathbb{U}^p_{\mathrm{loc}}$.

1.3. BSDEs with jumps

From now on, let us fix $p \in (1, 2)$. A mapping $f : [0, T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^p_\nu \to \mathbb{R}^l$ is called a *p-generator* if it is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^l) \otimes \mathscr{B}\big(\mathbb{R}^{l\times d}\big) \otimes \mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable. For any $\tau \in \mathcal{T}$,

$$f_\tau(t, \omega, y, z, u) := \mathbf{1}_{\{t<\tau(\omega)\}} f(t, \omega, y, z, u),$$
$$\forall\, (t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^p_\nu$$

is also $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^l) \otimes \mathscr{B}(\mathbb{R}^{l\times d}) \otimes \mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable.

Definition 1.1. Given $p \in (1, 2)$, let $\xi \in L^0(\mathcal{F}_T)$ and f be a p-generator. A triplet of processes $(Y, Z, U) \in \mathbb{D}^0 \times \mathbb{Z}^2_{\mathrm{loc}} \times \mathbb{U}^p_{\mathrm{loc}}$ is called a solution of a backward stochastic differential equation with jumps that has terminal data ξ and generator f (BSDEJ (ξ, f) for short) if $\int_0^T |f(s, Y_s, Z_s, U_s)|ds < \infty$, P-a.s. and if (1.1) holds P-a.s.

Remark 1.1. Let $p \in (1,2)$.
(1) Let $U \in \mathbb{U}^p_{\rm loc}$. For any $\tau \in \mathcal{T}$, since $\{\mathbf{1}_{\{t\le\tau\}}\}_{t\in[0,T]}$ is an **F**-adapted càglàd process (and thus **F**-predictable), the process $\{\mathbf{1}_{\{t\le\tau\}}U_t\}_{t\in[0,T]}$ also belongs to $\mathbb{U}^p_{\rm loc}$. By Section 1.2, integral $\int_{(0,\tau]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) = \int_{(0,T]}\int_{\mathcal{X}} \mathbf{1}_{\{s\le\tau\}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$ is well defined. More general, the stochastic integral $\int_{(\tau,\gamma]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$ is valid for any $\tau,\gamma\in\mathcal{T}$ with $\tau\le\gamma$, P-a.s.
(2) Given $\xi \in L^0(\mathcal{F}_T)$ and a p-generator f, let (Y,Z,U) be a solution of BSDEJ (ξ,f) as described in Definition 1.1. For P-a.s. $\omega\in\Omega$, we see from (1.1) and (1.5) that

$$\Delta Y_t(\omega) = \Delta M^U_t(\omega) = \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}} U\big(t,\omega,\mathfrak{p}_t(\omega)\big), \quad \forall\, t\in[0,T], \tag{1.7}$$

which implies that

$$\big\{t\in[0,T] : Y_{t-}(\omega)\neq Y_t(\omega)\big\} \subset D_{\mathfrak{p}(\omega)} \text{ is a countable subset of } [0,T]. \tag{1.8}$$

2. Main result

In the rest of this paper, we set $q := \frac{p}{p-1} > 2$ and let β be a $[0,\infty)$-valued, **F**-progressively measurable process with $\int_0^T \beta_t dt \in L^\infty_+(\mathcal{F}_T)$. We make the following assumptions on p-generators f:
(H1) For each $(t,\omega,u)\in[0,T]\times\Omega\times L^p_\nu$, the mapping $(y,z)\to f(t,\omega,y,z,u)$ is continuous.
(H2) For any $\delta>0$, there exists a $[0,\infty)$-valued, **F**-progressively measurable process ϕ^δ with $E\int_0^T \phi^\delta_t dt<\infty$ such that $\sup_{|y|\le\delta}\big|f(t,y,0,0)-f(t,0,0,0)\big|\le\phi^\delta_t$, $dt\times dP$-a.s.
(H3) It holds for $dt\times dP$-a.s.$(t,\omega)\in[0,T]\times\Omega$ that

$$\big\langle y, f(t,\omega,y,0,0)-f(t,\omega,0,0,0)\big\rangle \le \beta(t,\omega)|y|^2, \quad \forall\, y\in\mathbb{R}^l.$$

(H4) For some $c_1(\cdot)\in L^2_+[0,T]$, it holds for $dt\times dP$-a.s. $(t,\omega)\in[0,T]\times\Omega$ that

$$|f(t,\omega,y,z,0)-f(t,\omega,y,0,0)| \le \beta(t,\omega)+c_1(t)|z|, \quad \forall\,(y,z)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}.$$

(H5) For some $c_2(\cdot)\in L^q_+[0,T]$, it holds for $dt\times dP$-a.s. $(t,\omega)\in[0,T]\times\Omega$ that

$$\big|f(t,\omega,y,z,u_1)-f(t,\omega,y,z,u_2)\big| \le c_2(t)\|u_1-u_2\|_{L^p_\nu}, \quad \forall\,(y,z,u_1,u_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times L^p_\nu\times L^p_\nu.$$

(H6) It holds for $dt\times dP$-a.s.$(t,\omega)\in[0,T]\times\Omega$ that

$$\begin{aligned}&|y_1-y_2|^{p-1}\big\langle \mathscr{D}(y_1-y_2), f(t,\omega,y_1,z_1,u)-f(t,\omega,y_2,z_2,u)\big\rangle \le \lambda(t)\,\theta\big(|y_1-y_2|^p\big)\\&\quad+\Phi(t,\omega)|y_1-y_2|^p+\Lambda(t,\omega)|y_1-y_2|^{p-1}|z_1-z_2|,\\&\quad\forall\,(y_1,z_1),(y_2,z_2)\in\mathbb{R}^l\times\mathbb{R}^{l\times d},\ \forall\, u\in L^p_\nu,\end{aligned}$$

where $\lambda(\cdot)\in L^1_+[0,T]$; $\theta:[0,\infty)\to[0,\infty)$ is an increasing concave function satisfying $\int_{0+}^1 \frac{1}{\theta(t)}dt=\infty$; and Φ, Λ are two $[0,\infty)$-valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T$-measurable process such that $\int_0^T(\Phi_t\vee\Lambda_t^2)dt\in L^\infty_+(\mathcal{F}_T)$ and $E\int_0^T \Lambda_t^{2+\mathfrak{e}}dt<\infty$ for some $\mathfrak{e}\in(0,1)$.

Remark 2.1. Given $p\in(1,2)$, let f be a p-generator satisfying (H2), (H4), (H5) and that $\int_0^T|f(t,0,0,0)|dt<\infty$, P-a.s. Then it holds for any $(Y,Z,U)\in\mathbb{D}^1\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$ that $\int_0^T|f(t,Y_t,Z_t,U_t)|dt<\infty$, P-a.s.

For simplicity, set $\overline{C} := \left(\int_0^T \left(c_1(t)\right)^2 dt\right) \vee \left(\int_0^T \left(c_2(t)\right)^q dt\right)$, $C_\beta := \left\| \int_0^T \beta_t dt \right\|_{L_+^\infty(\mathcal{F}_T)}$, $C_\Phi := \left\| \int_0^T \Phi_t dt \right\|_{L_+^\infty(\mathcal{F}_T)}$ and $C_\Lambda := \left\| \int_0^T \Lambda_t^2 dt \right\|_{L_+^\infty(\mathcal{F}_T)}$.

Our main goal is the following existence and uniqueness result of BSDEJs for case "$p \in (1,2)$".

Theorem 2.1. *Given $p \in (1,2)$, let $\xi \in L^p(\mathcal{F}_T)$ and let f be a p-generator satisfying* (H1)–(H6) *such that $\int_0^T |f(t,0,0,0)|dt \in L_+^p(\mathcal{F}_T)$ and that the parameter $c_2(\cdot) \in L_+^{q'}[0,T]$ for some $q' \in (q,\infty)$. Then the BSDEJ (ξ, f) admits a unique solution $(Y,Z,U) \in \mathbb{S}^p$.*

This wellposedness gives rise to a general martingale representation theorem in the jump case as follows:

Corollary 2.1. *Let $p \in (1,2)$. For any $\xi \in L^p(\mathcal{F}_T)$, there exists a unique pair $(Z,U) \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$ such that P-a.s.*

$$E[\xi|\mathcal{F}_t] = E[\xi] + \int_0^t Z_s dB_s + \int_{(0,t]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t \in [0,T]. \tag{2.1}$$

3. A priori estimate and stability result

To prove Theorem 2.1, we started with an inequality about the difference of two local p-integrable solutions to BSDEJs with different parameters under a general monotonicity condition.

Lemma 3.1. *Let $p \in (1,2)$. For $i = 1,2$, let $\xi_i \in L^0(\mathcal{F}_T)$, let f_i be a p-generator, and let $(Y^i, Z^i, U^i) \in \mathbb{D}^0 \times \mathbb{Z}^2_{\rm loc} \times \mathbb{U}^p_{\rm loc}$ be a solution of BSDEJ (ξ_i, f_i) such that $Y^1 - Y^2 \in \mathbb{D}^p$. Assume that $ds \times dP$-a.s.*

$$\begin{aligned} &|Y_s^1 - Y_s^2|^{p-1} \big\langle \mathscr{D}(Y_s^1 - Y_s^2), f_1(s, Y_s^1, Z_s^1, U_s^1) - f_2(s, Y_s^2, Z_s^2, U_s^2) \big\rangle \\ &\quad \le |Y_s^1 - Y_s^2|^{p-1} \Big[g_s + \Phi_s |Y_s^1 - Y_s^2| + \Lambda_s |Z_s^1 - Z_s^2| + \Gamma_s \big\| U_s^1 - U_s^2 \big\|_{L_\nu^p} \Big] + \Upsilon_s, \end{aligned} \tag{3.1}$$

where g, Φ, Λ, Υ, Γ are five $[0,\infty)$-valued, $\mathscr{B}[0,T] \otimes \mathcal{F}_T$-measurable processes satisfying $\int_0^T (\Phi_t \vee \Lambda_t^2 \vee \Gamma_t^q) dt \in L_+^\infty(\mathcal{F}_T)$ and $E\big[(\int_0^T g_s ds)^p + \int_0^T \Upsilon_s ds\big] < \infty$. Then for some constant $\mathfrak{C}$ depending on T, $\nu(\mathcal{X})$, p, C_Φ, C_Λ and $C_\Gamma := \left\| \int_0^T \Gamma_t^q dt \right\|_{L_+^\infty(\mathcal{F}_T)}$,

$$\begin{aligned} &E\bigg[\sup_{s \in [t,T]} |Y_s^1 - Y_s^2|^p + \Big(\int_t^T \big|Z_s^1 - Z_s^2\big|^2 ds \Big)^{\frac{p}{2}} + \int_t^T \int_{\mathcal{X}} |U_s^1(x) - U_s^2(x)|^p \nu(dx) ds \bigg] \\ &\quad \le \mathfrak{C}\, E\bigg[|\xi_1 - \xi_2|^p + \Big(\int_t^T g_s ds \Big)^p + \int_t^T \Upsilon_s ds \bigg], \quad \forall\, t \in [0,T]. \end{aligned} \tag{3.2}$$

This basic inequality gives rise to an *a priori* estimate and a stability result of $\mathbb{L}^p$-solutions of BSDEJs, both of which will play important roles in the demonstration of Theorem 2.1.

Proposition 3.1. *Given $p \in (1,2)$, let $\xi \in L^p(\mathcal{F}_T)$ and f be a p-generator satisfying* (H3)–(H5) *and $\int_0^T |f(t,0,0,0)|dt \in L_+^p(\mathcal{F}_T)$. If $(Y,Z,U) \in \mathbb{D}^p \times \mathbb{Z}^2_{\rm loc} \times \mathbb{U}^p_{\rm loc}$ solves BSDEJ*

(ξ, f), *then*

$$\|Y\|^p_{\mathbb{D}^p} + \|Z\|^p_{\mathbb{Z}^{2,p}} + \|U\|^p_{\mathbb{U}^p} \le \mathcal{C} E\bigg[1 + |\xi|^p + \Big(\int_0^T |f(t,0,0,0)|dt\Big)^p\bigg] < \infty \tag{3.3}$$

for some constant $\mathcal{C}$ *depending on* T, $\nu(\mathcal{X})$, p, $\overline{C}$ *and* C_β.

Proposition 3.2. *Given* $p \in (1,2)$, *let* $\{\xi_n\}_{n\in\mathbb{N}}$ *be a Cauchy sequence in* $L^p(\mathcal{F}_T)$. *For each* $n \in \mathbb{N}$, *let* f_n *be a p-generator and let* $(Y^n, Z^n, U^n) \in \mathbb{S}^p$ *be a solution of BSDEJ* (ξ_n, f_n). *Assume that for any* $m, n \in \mathbb{N}$ *with* $m > n$, $(Y^{m,n}, Z^{m,n}, U^{m,n}) := (Y^m - Y^n, Z^m - Z^n, U^m - U^n)$ *satisfies that* $ds \times dP$*-a.s.*

$$\begin{aligned} &|Y^{m,n}_s|^{p-1}\big\langle \mathscr{D}(Y^{m,n}_s), f_m\big(s, Y^m_s, Z^m_s, U^m_s\big) - f_n\big(s, Y^n_s, Z^n_s, U^n_s\big)\big\rangle \\ &\quad \le \lambda(s)\,\theta\big(|Y^{m,n}_s|^p + \eta_n\big) + \Phi_s|Y^{m,n}_s|^p \\ &\qquad + |Y^{m,n}_s|^{p-1}\Big[\Lambda_s|Z^{m,n}_s| + c(s)\big\|U^{m,n}_s\big\|_{L^p_\nu}\Big] + \Upsilon^{m,n}_s, \end{aligned} \tag{3.4}$$

where
(i) $\lambda(\cdot) \in L^1_+[0,T]$ *and* $\theta : [0,\infty) \to [0,\infty)$ *is an increasing concave function satisfying* $\int_{0+}^1 \frac{1}{\theta(t)}dt = \infty$;
(ii) $c(\cdot) \in L^q_+[0,T]$ *and* Φ, Λ *are two* $[0,\infty)$*-valued,* $\mathscr{B}[0,T]\otimes\mathcal{F}_T$*-measurable processes with* $\int_0^T (\Phi_t \vee \Lambda_t^2)dt \in L^\infty_+(\mathcal{F}_T)$;
(iii) $\eta_n \in L^1_+(\mathcal{F}_T)$ *and* $\Upsilon^{m,n}$ *is a* $[0,\infty)$*-valued,* $\mathscr{B}[0,T]\otimes\mathcal{F}_T$*-measurable process such that*

$$\lim_{n\to\infty} E[\eta_n] = \lim_{n\to\infty} \sup_{m>n} E\int_0^T \Upsilon^{m,n}_t dt = 0. \tag{3.5}$$

If $\int_0^T \lambda(t)dt > 0$, *we further assume that*

$$\sup_{n\in\mathbb{N}}\Big(\|Y^n\|_{\mathbb{D}^p} + \|Z^n\|_{\mathbb{Z}^{2,p}} + \|U^n\|_{\mathbb{U}^p}\Big) < \infty. \tag{3.6}$$

Then $\big\{(Y^n, Z^n, U^n)\big\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathbb{S}^p$.

The following result shows that a BSDEJ with bounded terminal data has a solution, which will also play a key role in the proof of Theorem 2.1.

Proposition 3.3. *Given* $p \in (1,2)$, *let* $\xi \in L^\infty(\mathcal{F}_T)$ *and* f *be a p-generator satisfying* (H1), (H3)–(H6) *and that*

(**H2′**) *For some* $\kappa_0 \in (0,\infty)$, *it holds for* $dt \times dP$*-a.s.* $(t,\omega) \in [0,T]\times\Omega$ *that*

$$\big|f(t,\omega,y,0,0) - f(t,\omega,0,0,0)\big| \le \kappa_0\big(1+|y|\big), \quad \forall\, y \in \mathbb{R}^l.$$

If $\int_0^T |f(t,0,0,0)|dt \in L^\infty_+(\mathcal{F}_T)$, *then the BSDEJ* (ξ, f) *has a solution* $(Y,Z,U) \in \mathbb{D}^\infty \times \mathbb{Z}^{2,p} \times \mathbb{U}^p$.

4. Wellposedness with Lipschitz generators

When the p-generator is Lipschitz continuous in (y,z,u), the condition (H1) is not necessary to derive a unique solution for the corresponding BSDE with jump. One can demonstrate this using a fixed-point argument, Theorem 2.1 as well as similar techniques to those developed in the proof of Lemma 3.1:

Remark 4.1. Given $p \in (1,2)$, let $\xi \in L^p(\mathcal{F}_T)$ and let f be a p-generator with $\int_0^T |f(t,0,0,0)|dt \in L^p_+(\mathcal{F}_T)$. If there exists two $[0,\infty)$-valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T$-measurable processes $\widetilde{\beta}$, Λ with $\int_0^T (\widetilde{\beta}^q_t \vee \Lambda^2_t)dt \in L^\infty_+(\mathcal{F}_T)$ such that for $dt\times dP$-a.s. $(t,\omega)\in[0,T]\times\Omega$

$$\begin{aligned} &\big|f(t,\omega,y_1,z_1,u_1)-f(t,\omega,y_2,z_2,u_2)\big| \le \widetilde{\beta}(t,\omega)\big(|y_1-y_2|+\|u_1-u_2\|_{L^p_\nu}\big)\\ &\quad +\Lambda(t,\omega)|z_1-z_2|, \qquad \forall\,(y_i,z_i,u_i)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}\times L^p_\nu,\ i=1,2. \end{aligned} \tag{4.1}$$

Then BSDEJ (ξ,f) admits a unique solution $(Y,Z,U)\in\mathbb{S}^p$.

As a consequence of Theorem 2.1 and Remark 4.1, we have the following result on BSDEJs whose generator f is null after some stopping time τ.

Corollary 4.1. *Given $p\in(1,2)$, let f be a p-generator with $\int_0^T |f(t,0,0,0)|dt \in L^p_+(\mathcal{F}_T)$ such that either* (H1)–(H6) *or* (4.1) *holds. For any $\tau\in\mathcal{T}$ and $\xi\in L^p(\mathcal{F}_\tau)$, the unique solution $\big(Y,Z,U\big)$ of the BSDEJ (ξ,f_τ) in $\mathbb{S}^p$ satisfies that $P\big\{Y_t=Y_{\tau\wedge t},\ t\in[0,T]\big\}=1$ and that $\big(Z_t,U_t\big)=\mathbf{1}_{\{t\le\tau\}}\big(Z_t,U_t\big)$, $dt\times dP$-a.s.*

See our ArXiv version [53] for detailed proofs of Remark 4.1 and Corollary 4.1.

5. Proofs

Proof of Lemma 1.1. (1) Let $U\in\mathbb{U}^p$. Given $\omega\in\Omega$, we denote the countable set $D_{\mathfrak{p}(\omega)}$ by $\{t_i(\omega)\}_{i\in\mathbb{N}}$. For any $j\in\mathbb{N}$, Lemma A.1 shows that

$$\begin{aligned} &\Big(\sum_{i=1}^{j}\big|U\big(t_i(\omega),\omega,\mathfrak{p}_{t_i(\omega)}(\omega)\big)\big|^2\Big)^{\frac{p}{2}} \le \sum_{i=1}^{j}\big|U\big(t_i(\omega),\omega,\mathfrak{p}_{t_i(\omega)}(\omega)\big)\big|^p\\ &\quad\le \sum_{t\in D_{\mathfrak{p}(\omega)}}\big|U\big(t,\omega,\mathfrak{p}_t(\omega)\big)\big|^p = \Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^p N_{\mathfrak{p}}(dt,dx)\Big)(\omega). \end{aligned}$$

Letting $j\to\infty$ on the left-hand-side yields that

$$\begin{aligned} \Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}(\omega) &= \Big(\sum_{t\in D_{\mathfrak{p}(\omega)}}\big|U\big(t,\omega,\mathfrak{p}_t(\omega)\big)\big|^2\Big)^{\frac{p}{2}}\\ &\le \Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^p N_{\mathfrak{p}}(dt,dx)\Big)(\omega). \end{aligned} \tag{5.1}$$

It follows that

$$\begin{aligned} &E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{1}{2}}\Big]\\ &\quad\le 1+E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big]\\ &\quad\le 1+E\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt<\infty, \end{aligned} \tag{5.2}$$

which implies that $\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)<\infty$, P-a.s.

For any $k, n \in \mathbb{N}$ with $k > n$, since $\big[M^{U,k} - M^{U,n}, M^{U,k} - M^{U,n}\big]_T = \int_0^T \int_{\mathcal{X}} \mathbf{1}_{\{n<|U_s(x)|\le k\}} |U_s(x)|^2 N_{\mathfrak{p}}(ds, dx)$, one has

$$\sup_{k\ge n} E\Big\{\big[M^{U,k} - M^{U,n}, M^{U,k} - M^{U,n}\big]_T^{\frac{1}{2}}\Big\}$$
$$\le E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}} \mathbf{1}_{\{|U_t(x)|>n\}} |U_t(x)|^2 N_{\mathfrak{p}}(dt, dx)\Big)^{\frac{1}{2}}\bigg].$$

As $n \to \infty$, (5.2) and the monotone convergence theorem show that $\{M^{U,n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(\mathbb{M}^1, \|\cdot\|_{\mathbb{M}^1}\big)$. Let M^U be its limit.

(2) By Kunita–Watanabe inequality,

$$\begin{aligned}
&\big|[M^{U,n}, M^{U,n}]_t - [M^U, M^U]_t\big| \\
&= \big|[M^{U,n} - M^U, M^{U,n} - M^U]_t - 2[M^{U,n} - M^U, M^{U,n}]_t\big| \\
&\le [M^{U,n} - M^U, M^{U,n} - M^U]_t \\
&\quad + 2\Big([M^{U,n} - M^U, M^{U,n} - M^U]_t\Big)^{\frac{1}{2}}\Big([M^{U,n}, M^{U,n}]_t\Big)^{\frac{1}{2}} \\
&= [M^{U,n} - M^U, M^{U,n} - M^U]_t \\
&\quad + 2\Big([M^{U,n} - M^U, M^{U,n} - M^U]_t\Big)^{\frac{1}{2}}\Big(\int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{|U_s(x)|\le n\}} |U_s(x)|^2 N_{\mathfrak{p}}(ds, dx)\Big)^{\frac{1}{2}}, \\
&\quad \forall\, t \in [0, T].
\end{aligned}$$

Then Lemma A.1 and Hölder's inequality imply that

$$\begin{aligned}
&E\bigg[\sup_{t\in[0,T]} \big|[M^{U,n}, M^{U,n}]_t - [M^U, M^U]_t\big|^{\frac{1}{2}}\bigg] \\
&\le E\Big\{\big[M^{U,n} - M^U, M^{U,n} - M^U\big]_T^{\frac{1}{2}}\Big\} \\
&\quad + \sqrt{2}E\bigg[\Big([M^{U,n} - M^U, M^{U,n} - M^U]_T\Big)^{\frac{1}{4}}\Big(\int_{(0,T]}\int_{\mathcal{X}} |U_t(x)|^2 N_{\mathfrak{p}}(dt, dx)\Big)^{\frac{1}{4}}\bigg] \\
&\le \big\|M^{U,n} - M^U\big\|_{\mathbb{M}^1} \\
&\quad + \sqrt{2}\|M^{U,n} - M^U\|_{\mathbb{M}^1}^{\frac{1}{2}}\bigg(E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}} |U_t(x)|^2 N_{\mathfrak{p}}(dt, dx)\Big)^{\frac{1}{2}}\bigg]\bigg)^{\frac{1}{2}}.
\end{aligned}$$

Letting $n \to \infty$ yields that $\lim_{n\to\infty} E\Big[\sup_{t\in[0,T]} \big|[M^{U,n}, M^{U,n}]_t - [M^U, M^U]_t\big|^{\frac{1}{2}}\Big] = 0$. So there exists a subsequence of $\{M^{U,n}\}_{n\in\mathbb{N}}$ (we still denote it by $\{M^{U,n}\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty} \sup_{t\in[0,T]} \big|[M^{U,n}, M^{U,n}]_t - [M^U, M^U]_t\big| = 0$, P-a.s., which together with the monotone convergence theorem yields that for P-a.s. $\omega \in \Omega$

$$[M^U, M^U]_t(\omega) = \lim_{n\to\infty} [M^{U,n}, M^{U,n}]_t(\omega)$$
$$= \lim_{n\to\infty} \uparrow \Big(\int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{|U_s(x)|\le n\}} |U_s(x)|^2 N_{\mathfrak{p}}(ds, dx)\Big)(\omega)$$

$$= \lim_{n\to\infty} \uparrow \sum_{s\in D_{\mathfrak{p}(\omega)}\cap(0,t]} \mathbf{1}_{\{|U(s,\omega,\mathfrak{p}_s(\omega))|\le n\}} \big|U\big(s,\omega,\mathfrak{p}_s(\omega)\big)\big|^2$$
$$= \sum_{s\in D_{\mathfrak{p}(\omega)}\cap(0,t]} \big|U\big(s,\omega,\mathfrak{p}_s(\omega)\big)\big|^2$$
$$= \Big(\int_{(0,t]}\int_{\mathcal{X}} |U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\Big)(\omega), \quad \forall\, t\in[0,T].$$

Then the Burkholder–Davis–Gundy inequality and (5.2) show that

$$E\Big[\sup_{t\in[0,T]} |M^U_t|^p\Big] \le c_{p,l}\Big[\big[M^U,M^U\big]_T^{\frac{p}{2}}\Big]$$
$$= c_{p,l} E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}} |U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big] < \infty,$$

which implies that M^U is a uniformly integrable martingale.

(3) As $\|\cdot\|_{\mathbb{M}^1}$ is equivalent to $\|\cdot\|_{\mathbb{D}^1}$ on $\mathbb{M}^1$, we see that $\lim_{n\to\infty} E\Big[\sup_{t\in[0,T]}\big|M^{U,n}_t - M^U_t\big|\Big] = 0$. So there exists a subsequence of $\{M^{U,n}\}_{n\in\mathbb{N}}$ (we still denote it by $\{M^{U,n}\}_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty}\sup_{t\in[0,T]}\big|M^{U,n}_t - M^U_t\big| = 0$ except on a P-null set $\mathcal{N}$. We also assume that for any $\omega\in\mathcal{N}^c$, the paths $M^U(\omega)$ and $M^{U,n}(\omega)$, $n\in\mathbb{N}$ are càdlàg.

Let $\omega\in\mathcal{N}^c$, $t\in(0,T]$ and $\varepsilon>0$. One can find $N=N(\omega)\in\mathbb{N}$ such that $\sup_{t\in[0,T]}\big|M^{U,n}_t - M^U_t\big| < \varepsilon/2$ for any $n\ge N$. Also, there exists $\delta=\delta(t,\omega)\in(0,t)$ such that $\big|M^U_s(\omega) - M^U_{t-}(\omega)\big| < \varepsilon/2$ for any $s\in(t-\delta,t)$. Then for any $n\ge N$, we have $\big|M^{U,n}_s(\omega) - M^U_{t-}(\omega)\big| \le \big|M^{U,n}_s(\omega) - M^U_s(\omega)\big| + \big|M^U_s(\omega) - M^U_{t-}(\omega)\big| < \varepsilon$, $\forall\, s\in(t-\delta,t)$. Letting $s\nearrow t$ yields that $\big|M^{U,n}_{t-}(\omega) - M^U_{t-}(\omega)\big| \le \varepsilon$, which shows that $\lim_{n\to\infty} M^{U,n}_{t-}(\omega) = M^U_{t-}(\omega)$. It follows that

$$\Delta M^U_t(\omega) = M^U_t(\omega) - M^U_{t-}(\omega) = \lim_{n\to\infty}\big(M^{U,n}_t(\omega) - M^{U,n}_{t-}(\omega)\big) = \lim_{n\to\infty}\Delta M^{U,n}_t(\omega)$$
$$= \lim_{n\to\infty} \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\mathbf{1}_{\{|U(t,\omega,\mathfrak{p}_t(\omega))|\le n\}} U\big(t,\omega,\mathfrak{p}_t(\omega)\big) = \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}} U\big(t,\omega,\mathfrak{p}_t(\omega)\big).$$

(4) Let $U^1, U^2\in\mathbb{U}^p$ and $n\in\mathbb{N}$. For $i=1,2$, define

$$X^{i,n}_t := \int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{|U^1_s(x)+U^2_s(x)|\le n\}} U^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \text{ and}$$
$$\widetilde{X}^{i,n}_t := \int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{|U^1_s(x)+U^2_s(x)|\le n,\, |U^i_s(x)|\le n\}} U^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T].$$

We can deduce that

$$\big\|M^{U^1+U^2,n} - M^{U^1,n} - M^{U^2,n}\big\|_{\mathbb{M}^1} = \big\|X^{1,n} + X^{2,n} - M^{U^1,n} - M^{U^2,n}\big\|_{\mathbb{M}^1}$$
$$\le \sum_{i=1,2}\Big(\big\|X^{i,n} - \widetilde{X}^{i,n}\big\|_{\mathbb{M}^1} + \big\|\widetilde{X}^{i,n} - M^{U^i,n}\big\|_{\mathbb{M}^1}\Big)$$
$$= \sum_{i=1,2} E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}} \mathbf{1}_{\{|U^1_t(x)+U^2_t(x)|\le n,\, |U^i_t(x)|>n\}} |U^i_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{1}{2}}\Big]$$

$$+\sum_{i=1,2}E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|U^1_t(x)+U^2_t(x)|>n,|U^i_t(x)|\le n\}}|U^i_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac12}\bigg]$$

$$\le\sum_{i=1,2}E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|U^i_t(x)|>n\}}|U^i_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac12}$$

$$+\Big(\int_{(0,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|U^1_t(x)+U^2_t(x)|>n\}}|U^i_t(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac12}\bigg].$$

As $n\to\infty$, (5.2) and the monotone convergence theorem show that $\lim_{n\to\infty}\big\|M^{U^1+U^2,n}-M^{U^1,n}-M^{U^2,n}\big\|_{\mathbb{M}^1}=0$, which implies that $M^{U^1+U^2}=M^{U^1}+M^{U^2}$.

Next, let $U\in\mathbb{U}^p$, $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$. One has

$$\big\|M^{\alpha U,n}-\alpha M^{U,n}\big\|_{\mathbb{M}^1}$$

$$=E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}}\mathbf{1}_{\{(1\wedge|\alpha|^{-1})n<|U_s(x)|\le(1\vee|\alpha|^{-1})n\}}|\alpha U_s(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac12}\bigg]$$

$$\le|\alpha|E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|U_s(x)|>(1\wedge|\alpha|^{-1})n\}}|U_s(x)|^2N_{\mathfrak{p}}(dt,dx)\Big)^{\frac12}\bigg].$$

Letting $n\to\infty$, using (5.2) and the monotone convergence theorem again yield that $\lim_{n\to\infty}\big\|M^{\alpha U,n}-\alpha M^{U,n}\big\|_{\mathbb{M}^1}=0$, which implies that $M^{\alpha U}=\alpha M^U$. Therefore $U\to M^U$ is a linear mapping on $\mathbb{U}^p$. □

Proof of Remark 2.1. Let $(Y,Z,U)\in\mathbb{D}^1\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$. Fix $n,k\in\mathbb{N}$. Define

$$\tau_n:=\inf\bigg\{t\in[0,T]:\int_0^t|f(s,0,0,0)|ds+\int_0^t\big|Z_s\big|^2ds$$

$$+\int_0^t\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds>n\bigg\}\wedge T\in\mathcal{T}$$

and $A_k:=\{Y_*\le k\}\in\mathcal{F}_T$.

Since $|Y_t|\le k$, $\forall\,t\in[0,T]$ on A_k, (H2), (H4), (H5) and Hölder's inequality imply that

$$E\bigg[\mathbf{1}_{A_k}\int_0^{\tau_n}\big|f(t,Y_t,Z_t,U_t)\big|dt\bigg]$$

$$\le E\int_0^{\tau_n}\big(|f(t,0,0,0)|+\phi^k_t+\beta_t+c_1(t)|Z_t|+c_2(t)\|U_t\|_{L^p_\nu}\big)dt$$

$$\le n+C_\beta+E\int_0^T\phi^k_tdt+\Big(E\int_0^{\tau_n}\big(c_1(t)\big)^2dt\Big)^{\frac12}\Big(E\int_0^{\tau_n}|Z_t|^2dt\Big)^{\frac12}$$

$$+\Big(E\int_0^{\tau_n}\big(c_2(t)\big)^qdt\Big)^{\frac1q}\Big(E\int_0^{\tau_n}\|U_t\|^p_{L^p_\nu}dt\Big)^{\frac1p}$$

$$\le n+C_\beta+E\int_0^T\phi^k_tdt+\sqrt{n}\Big(\int_0^T\big(c_1(t)\big)^2dt\Big)^{\frac12}+n^{\frac1p}\Big(\int_0^T\big(c_2(t)\big)^qdt\Big)^{\frac1q}<\infty,$$

which shows that $\mathbf{1}_{A_k}\int_0^{\tau_n}\big|f(t,Y_t,Z_t,U_t)\big|dt<\infty$, P-a.s. As $Y_*<\infty$, P-a.s., letting $k\to\infty$, we see that $\int_0^{\tau_n}\big|f(t,Y_t,Z_t,U_t)\big|dt<\infty$ except on a P-null set $\mathcal{N}_n$. Since $\int_0^T|f(t,0,0,0)|dt<\infty$, P-a.s. and since $(Z,U)\in\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, there exists a P-null set $\mathcal{N}_0$ such that for any $\omega\in\mathcal{N}_0^c$,

$\tau_{\mathfrak{n}}(\omega) = T$ for some $\mathfrak{n} = \mathfrak{n}(\omega) \in \mathbb{N}$. Now, for any $\omega \in \cap_{n \in \mathbb{N} \cup \{0\}} \mathcal{N}_n^c$, one can deduce that $\int_0^T \big| f(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega)) \big| dt = \int_0^{\tau_{\mathfrak{n}}(\omega)} \big| f(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega)) \big| dt < \infty$. □

Proof of Lemma 3.1. Set $\wp := \big(2^{p-4} p(p-1)\big)^{\frac{1}{p}}$ and define processes

$$a_t := \Phi_t + \frac{\Lambda_t^2}{p-1} + \frac{p-1}{p} \wp^{-q} \Gamma_s^q + \frac{1}{p} \wp^p \nu(\mathcal{X}) \quad \text{and} \quad A_t := p \int_0^t a_s ds, \quad t \in [0, T].$$

Then $C_A := \|A_T\|_{L_+^\infty(\mathcal{F}_T)} \le p C_\Phi + q C_\Lambda + (p-1)\wp^{-q} C_\Gamma + \wp^p \nu(\mathcal{X}) T$. In this proof, we let $\mathfrak{C}$ denote a generic constant depending on T, $\nu(\mathcal{X})$, p, C_Φ, C_Λ and C_Γ, whose form may vary from line to line.

(1) *Denote* $(Y, Z, U) := (Y^1 - Y^2, Z^1 - Z^2, U^1 - U^2)$. *We first apply Itô's formula to derive the dynamics of the approximate pth power of process Y*: $\varphi_\varepsilon(Y_t) := \big(|Y_t|^2 + \varepsilon\big)^{\frac{1}{2}}$.

Let us fix $t_0 \in [0, T]$, $n \in \mathbb{N}$ and define

$$\tau_n := \inf\left\{ t \in [0, T] : \int_0^t |Z_s|^2 ds + \int_0^t \int_{\mathcal{X}} |U_s(x)|^p \nu(dx) ds > n \right\} \wedge T \in \mathcal{T}. \tag{5.3}$$

For any $\varepsilon \in (0, 1]$, the function $\varphi_\varepsilon(x) := \big(|x|^2 + \varepsilon\big)^{\frac{1}{2}}$, $x \in \mathbb{R}^l$ has the following derivatives of its pth power:

$$D_i \varphi_\varepsilon^p(x) = p \varphi_\varepsilon^{p-2}(x) x_i \quad \text{and} \quad D_{ij}^2 \varphi_\varepsilon^p(x) = p \varphi_\varepsilon^{p-2}(x) \delta_{ij} + p(p-2) \varphi_\varepsilon^{p-4}(x) x_i x_j, \quad \forall i, j \in \{1, \dots, l\}. \tag{5.4}$$

We also set $\mathfrak{S}_t^\varepsilon = \mathfrak{S}_t^{n,\varepsilon} := \sup_{s \in [\tau_n \wedge t, \tau_n]} \varphi_\varepsilon(Y_s)$, $t \in [t_0, T]$. By Lemma A.1,

$$E\Big[\big(\mathfrak{S}_{t_0}^\varepsilon\big)^p\Big] \le E\Big[\sup_{s \in [0,T]} \varphi_\varepsilon^p(Y_s)\Big] \le E\Big[\sup_{t \in [0,T]} |Y_t|^p\Big] + \varepsilon^{\frac{p}{2}} = \|Y\|_{\mathbb{D}^p}^p + \varepsilon^{\frac{p}{2}} < \infty. \tag{5.5}$$

Now, let us fix $(t, \varepsilon) \in [t_0, T] \times (0, 1]$. Applying Itô's formula (see e.g. [22, Theorem VIII.27] or [46, Theorem II.32]) to process $e^{A_s} \varphi_\varepsilon^p(Y_s)$ over the interval $[\tau_n \wedge t, \tau_n]$ and using (1.8) yield that

$$\begin{aligned}
& e^{A_{\tau_n \wedge t}} \varphi_\varepsilon^p(Y_{\tau_n \wedge t}) + \frac{1}{2} \int_{\tau_n \wedge t}^{\tau_n} e^{A_s} \operatorname{trace}\big(Z_s Z_s^T D^2 \varphi_\varepsilon^p(Y_s)\big) ds \\
& \quad + \sum_{s \in (\tau_n \wedge t, \tau_n]} e^{A_s} \Big(\varphi_\varepsilon^p(Y_s) - \varphi_\varepsilon^p(Y_{s-}) - \big\langle D\varphi_\varepsilon^p(Y_{s-}), \Delta Y_s \big\rangle \Big) \\
& = e^{A_{\tau_n}} \varphi_\varepsilon^p(Y_{\tau_n}) + p \int_{\tau_n \wedge t}^{\tau_n} e^{A_s} \big[\varphi_\varepsilon^{p-2}(Y_s) \big\langle Y_s, f_1(s, Y_s^1, Z_s^1, U_s^1) \\
& \quad - f_2(s, Y_s^2, Z_s^2, U_s^2) \big\rangle - a_s \varphi_\varepsilon^p(Y_s) \big] ds \\
& \quad - p(M_T - M_t + \mathcal{M}_T - \mathcal{M}_t), \qquad P\text{-a.s.},
\end{aligned} \tag{5.6}$$

where $M_s := M_s^\varepsilon = \int_0^{\tau_n \wedge s} \mathbf{1}_{\{r > t_0\}} e^{A_r} \varphi_\varepsilon^{p-2}(Y_{r-}) \langle Y_{r-}, Z_r dB_r \rangle$ and $\mathcal{M}_s := \mathcal{M}_s^\varepsilon = \int_{(0, \tau_n \wedge s]} \int_{\mathcal{X}} \mathbf{1}_{\{r > t_0\}} e^{A_r} \varphi_\varepsilon^{p-2}(Y_{r-}) \langle Y_{r-}, U_r(x) \rangle \widetilde{N}_{\mathfrak{p}}(dr, dx)$, $\forall s \in [0, T]$. Since an analogy to (5.1) shows that for any $t \in [0, T]$

$$E\bigg[\bigg(\int_{(\tau_n\wedge t,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\bigg)^{\frac{p}{2}}\bigg] \le E\int_{(\tau_n\wedge t,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^p N_{\mathfrak{p}}(ds,dx)$$
$$= E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p \nu(dx)ds \le n, \qquad (5.7)$$

we can deduce from the Burkholder–Davis–Gundy inequality, Young's inequality, (5.5) and (5.3) that

$$E\bigg[\sup_{s\in[0,T]}|M_s| + \sup_{s\in[0,T]}|\mathcal{M}_s|\bigg]$$
$$\le c_l e^{C_A} E\bigg[(\mathfrak{S}^{\varepsilon}_{t_0})^{p-1}\Big(\int_0^{\tau_n}|Z_s|^2 ds\Big)^{\frac12} + (\mathfrak{S}^{\varepsilon}_{t_0})^{p-1}\Big(\int_{(0,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\Big)^{\frac12}\bigg]$$
$$\le c_{p,l} e^{C_A} E\bigg[(\mathfrak{S}^{\varepsilon}_{t_0})^{p} + \Big(\int_0^{\tau_n}|Z_s|^2 ds\Big)^{\frac p2} + \Big(\int_{(0,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\Big)^{\frac p2}\bigg]$$
$$\le c_{p,l} e^{C_A}\big(\varepsilon^{\frac p2} + \|Y\|^p_{\mathbb{D}^p} + n^{\frac p2} + n\big) < \infty. \qquad (5.8)$$

So both M and $\mathcal{M}$ are uniformly integrable martingales.

(2) *Next, we use Taylor's expansion and some new analytic techniques to estimate the jump series* $\sum_{s\in(\tau_n\wedge t,\tau_n]} e^{A_s}\Big(\varphi^p_\varepsilon(Y_s) - \varphi^p_\varepsilon(Y_{s-}) - \big\langle D\varphi^p_\varepsilon(Y_{s-}), \Delta Y_s\big\rangle\Big)$ *and thus Eq.* (5.6).

Given $s \in [0,T]$, (5.4) implies that

$$\mathrm{trace}\big(Z_s Z_s^T D^2\varphi^p_\varepsilon(Y_s)\big) = p\,\varphi^{p-2}_\varepsilon(Y_s)|Z_s|^2 + p(p-2)\varphi^{p-4}_\varepsilon(Y_s)\cdot\sum_{j=1}^d\Big(\sum_{i=1}^l Y^i_s Z^{ij}_s\Big)^2$$
$$\ge p\,\varphi^{p-2}_\varepsilon(Y_s)|Z_s|^2 + p(p-2)\varphi^{p-4}_\varepsilon(Y_s)|Y_s|^2|Z_s|^2$$
$$\ge p(p-1)\varphi^{p-2}_\varepsilon(Y_s)|Z_s|^2. \qquad (5.9)$$

Setting $Y^\alpha_s := Y_{s-} + \alpha\Delta Y_s$, $\alpha \in [0,1]$, we can deduce from Taylor's Expansion Theorem and (5.4) that

$$\varphi^p_\varepsilon\big(Y_s\big) - \varphi^p_\varepsilon\big(Y_{s-}\big) - \big\langle D\varphi^p_\varepsilon\big(Y_{s-}\big), \Delta Y_s\big\rangle = \int_0^1 (1-\alpha)\big\langle \Delta Y_s, D^2\varphi^p_\varepsilon\big(Y^\alpha_s\big)\Delta Y_s\big\rangle d\alpha$$
$$= p\int_0^1(1-\alpha)\Big[\varphi^{p-2}_\varepsilon\big(Y^\alpha_s\big)|\Delta Y_s|^2 + (p-2)\varphi^{p-4}_\varepsilon\big(Y^\alpha_s\big)\big\langle\Delta Y_s, Y^\alpha_s\big\rangle^2\Big]d\alpha$$
$$\ge p(p-1)|\Delta Y_s|^2\int_0^1(1-\alpha)\varphi^{p-2}_\varepsilon(Y^\alpha_s)d\alpha. \qquad (5.10)$$

When $|Y_{s-}| \le |\Delta Y_s|$, one has $\varphi^{p-2}_\varepsilon(Y^\alpha_s) \ge \big((|Y_{s-}|+\alpha|\Delta Y_s|)^2+\varepsilon\big)^{\frac p2 -1} \ge \big(4|\Delta Y_s|^2+\varepsilon\big)^{\frac p2-1} \ge 2^{p-2}\big(|\Delta Y_s|^2+\varepsilon\big)^{\frac p2-1}$, $\forall\,\alpha\in[0,1]$. So it follows from (5.10) and (1.7) that for P-a.s. $\omega\in\Omega$

$$\sum_{s\in(\tau_n(\omega)\wedge t,\tau_n(\omega)]} e^{A_s(\omega)}\Big(\varphi^p_\varepsilon\big(Y_s(\omega)\big) - \varphi^p_\varepsilon\big(Y_{s-}(\omega)\big) - \big\langle D\varphi^p_\varepsilon\big(Y_{s-}(\omega)\big), \Delta Y_s(\omega)\big\rangle\Big)$$
$$\ge 2^{p-3}p(p-1)\sum_{s\in(\tau_n(\omega)\wedge t,\tau_n(\omega)]}\mathbf{1}_{\{|Y_{s-}(\omega)|\le|\Delta Y_s(\omega)|\}}e^{A_s(\omega)}$$
$$\times\big|\Delta Y_s(\omega)\big|^2\big(\big|\Delta Y_s(\omega)\big|^2+\varepsilon\big)^{\frac p2-1}$$

$$
\begin{aligned}
&= 2^{p-3}p(p-1) \sum_{s \in D_{\mathfrak{p}(\omega)} \cap (\tau_n(\omega)\wedge t, \tau_n(\omega)]} \mathbf{1}_{\{|Y_{s-}(\omega)| \le |U(s,\omega,\mathfrak{p}_s(\omega))|\}} e^{A_s(\omega)} \\
&\quad \times \big|U(s,\omega,\mathfrak{p}_s(\omega))\big|^2 \big(\big|U(s,\omega,\mathfrak{p}_s(\omega))\big|^2 + \varepsilon\big)^{\frac{p}{2}-1} \\
&= 2^{p-3}p(p-1)\bigg(\int_{(\tau_n\wedge t, \tau_n]} \int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}| \le |U_s(x)|\}} e^{A_s} \big|U_s(x)\big|^2 \\
&\quad \times \big(|U_s(x)|^2 + \varepsilon\big)^{\frac{p}{2}-1} N_{\mathfrak{p}}(ds,dx)\bigg)(\omega).
\end{aligned} \tag{5.11}
$$

Multiplying $\left(\frac{|Y_s|}{\varphi_\varepsilon(Y_s)}\right)^{2-p} \le 1$ to (3.1) and applying Young's inequality yield that P-a.s.

$$
\begin{aligned}
&\varphi_\varepsilon^{p-2}(Y_s)\big\langle Y_s, f_1(s,Y_s^1,Z_s^1,U_s^1) - f_2(s,Y_s^2,Z_s^2,U_s^2)\big\rangle \\
&\quad \le \varphi_\varepsilon^{p-2}(Y_s)|Y_s|\big(g_s + \Phi_s|Y_s|\big) + \Lambda_s \varphi_\varepsilon^{p-2}(Y_s)|Y_s||Z_s| \\
&\quad\quad + \Gamma_s \varphi_\varepsilon^{p-2}(Y_s)|Y_s|\|U_s\|_{L^p_\nu} + \Upsilon_s \\
&\quad \le g_s\varphi_\varepsilon^{p-1}(Y_s) + \Phi_s\varphi_\varepsilon^{p}(Y_s) + \frac{\Lambda_s^2}{p-1}\varphi_\varepsilon^{p-2}(Y_s)|Y_s|^2 + \frac{p-1}{4}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2 \\
&\quad\quad + \Gamma_s\varphi_\varepsilon^{p-1}(Y_s)\|U_s\|_{L^p_\nu} + \Upsilon_s \\
&\quad \le g_s\varphi_\varepsilon^{p-1}(Y_s) + \Big(\Phi_s + \frac{\Lambda_s^2}{p-1} + \frac{p-1}{p}\wp^{-q}\Gamma_s^q\Big)\varphi_\varepsilon^{p}(Y_s) + \frac{p-1}{4}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2 \\
&\quad\quad + \frac{1}{p}\wp^p\|U_s\|^p_{L^p_\nu} + \Upsilon_s \ \text{ for a.e. } s \in [0,T].
\end{aligned}
$$

Since

$$
\begin{aligned}
\|U_s\|^p_{L^p_\nu} &= \int_{\mathcal{X}} |U_s(x)|^p \nu(dx) \le \int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}| > |U_s(x)|\}} \varphi_\varepsilon^p(Y_{s-})\nu(dx) \\
&\quad + \int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}| \le |U_s(x)|\}} |U_s(x)|^p \nu(dx) \\
&\le \varphi_\varepsilon^p(Y_{s-})\nu(\mathcal{X}) + \int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}| \le |U_s(x)|\}} |U_s(x)|^p \nu(dx), \qquad \forall\, s \in [0,T],
\end{aligned} \tag{5.12}
$$

it then follows from (1.8) that P-a.s.

$$
\begin{aligned}
&\varphi_\varepsilon^{p-2}(Y_s)\big\langle Y_s, f_1(s,Y_s^1,Z_s^1,U_s^1) - f_2(s,Y_s^2,Z_s^2,U_s^2)\big\rangle \\
&\quad \le g_s\varphi_\varepsilon^{p-1}(Y_s) + a_s\varphi_\varepsilon^{p}(Y_s) + \frac{p-1}{4}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2 \\
&\quad\quad + \frac{1}{p}\wp^p \int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}| \le |U_s(x)|\}} |U_s(x)|^p \nu(dx) + \Upsilon_s \ \text{ for a.e. } s \in [0,T].
\end{aligned} \tag{5.13}
$$

Plugging this inequality together with (5.9), (5.11) into (5.6) leads to that for any $t \in [t_0, T]$

$$
\begin{aligned}
&e^{A_{\tau_n\wedge t}}\varphi_\varepsilon^p(Y_{\tau_n\wedge t}) + \frac{p}{4}(p-1)\int_{\tau_n\wedge t}^{\tau_n} e^{A_s}\,\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2 ds \\
&\quad + 2\wp^p \int_{(\tau_n\wedge t, \tau_n]} \int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}| \le |U_s(x)|\}} e^{A_s} \big|U_s(x)\big|^2 \big(|U_s(x)|^2 + \varepsilon\big)^{\frac{p}{2}-1} N_{\mathfrak{p}}(ds,dx)
\end{aligned}
$$

$$\leq \eta_t^\varepsilon + \wp^p \int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}|U_s(x)|^p \nu(dx)ds$$
$$- p\,(M_T - M_t + \mathcal{M}_T - \mathcal{M}_t), \qquad P\text{-a.s.}, \tag{5.14}$$

where $\eta_t^\varepsilon = \eta_t^{n,\varepsilon} := e^{C_A}\Big(\varphi_\varepsilon^p(Y_{\tau_n}) + p\int_{\tau_n\wedge t}^{\tau_n} g_s \varphi_\varepsilon^{p-1}(Y_s)ds + p\int_{t_0}^T \Upsilon_s ds\Big)$. Young's inequality and (5.5) show that

$$E[\eta_t^\varepsilon] \leq e^{C_A} E\bigg[\big(\mathfrak{S}_{t_0}^\varepsilon\big)^p + p\big(\mathfrak{S}_{t_0}^\varepsilon\big)^{p-1}\int_{\tau_n\wedge t_0}^{\tau_n} g_s ds + p\int_{t_0}^T \Upsilon_s ds\bigg]$$
$$\leq e^{C_A} E\bigg[p\big(\mathfrak{S}_{t_0}^\varepsilon\big)^p + \bigg(\int_{t_0}^T g_s ds\bigg)^p + p\int_{t_0}^T \Upsilon_s ds\bigg] < \infty. \tag{5.15}$$

As M and $\mathcal{M}$ are uniformly integrable martingales, taking expectation in (5.14) gives that

$$\frac{p}{4}(p-1)E\int_{\tau_n\wedge t}^{\tau_n} e^{A_s}\,\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2 ds$$
$$+ 2\wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}\big|U_s(x)\big|^2\big(|U_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds$$
$$\leq E[\eta_t^\varepsilon] + \wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}|U_s(x)|^p \nu(dx)ds. \tag{5.16}$$

(3) *We continue our deduction, in which the analysis of* $\mathbb{L}^p$*-norm of random field* U *is quite technically involved.*

Clearly, $\lim_{\varepsilon\to 0}\uparrow\ |U(s,\omega,x)|^2\big(|U(s,\omega,x)|^2+\varepsilon\big)^{\frac{p}{2}-1} = |U(s,\omega,x)|^p$, $\forall\,(s,\omega,x)\in[0,T]\times\Omega\times\mathcal{X}$, so the monotone convergence theorem implies that

$$\lim_{\varepsilon\to 0}\uparrow\ E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}|U_s(x)|^2\big(|U_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds$$
$$= E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}|U_s(x)|^p \nu(dx)ds.$$

On the other hand, since $\eta_t^\varepsilon \leq \eta_t^1$, $\forall\,\varepsilon\in(0,1]$ and since $E[\eta_t^1] < \infty$ by (5.15), the dominated convergence theorem shows that $\lim_{\varepsilon\to 0} E[\eta_t^\varepsilon] = E\big[\widetilde{\eta}_t\big]$, where $\widetilde{\eta}_t := e^{C_A}\big(|Y_{\tau_n}|^p + p\int_{\tau_n\wedge t}^{\tau_n} g_s|Y_s|^{p-1}ds + p\int_{t_0}^T \Upsilon_s ds\big)$.

Then letting $\varepsilon\to 0$ in (5.16) yields that

$$2\wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}\big|U_s(x)\big|^p \nu(dx)ds$$
$$\leq E\big[\widetilde{\eta}_t\big] + \wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}\big|U_s(x)\big|^p \nu(dx)ds.$$

As $E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}\big|U_s(x)\big|^p \nu(dx)ds \leq e^{C_A}E\int_0^{\tau_n}\int_{\mathcal{X}}\big|U_s(x)\big|^p\nu(dx)ds \leq e^{C_A}n < \infty$, we obtain that

$$\wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}}\big|U_s(x)\big|^p \nu(dx)ds$$
$$\leq \wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}} \mathbf{1}_{\{|Y_{s-}|\leq |U_s(x)|\}} e^{A_s}\big|U_s(x)\big|^p \nu(dx)ds \leq E\big[\widetilde{\eta}_t\big]. \tag{5.17}$$

Now, fix $\varepsilon \in (0,1]$ again. As $\widetilde{\eta}_t \le \eta_t^\varepsilon$, (5.16) and (5.17) show that

$$\frac{p}{4}(p-1)E\int_{\tau_n\wedge t}^{\tau_n}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2ds \le \frac{p}{4}(p-1)E\int_{\tau_n\wedge t}^{\tau_n}e^{A_s}\,\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2ds \le E\big[\widetilde{\eta}_t+\eta_t^\varepsilon\big]\le 2E\big[\eta_t^\varepsilon\big]. \quad (5.18)$$

Also, (5.14) and (5.17) imply that

$$\begin{aligned} E\big[(\mathfrak{S}_t^\varepsilon)^p\big] &\le E\Big[\sup_{s\in[\tau_n\wedge t,\tau_n]}e^{A_s}\varphi_\varepsilon^p(Y_s)\Big]\le E[\eta_t^\varepsilon] \\ &\quad +\wp^p E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}e^{A_s}|U_s(x)|^p\nu(dx)ds \\ &\quad +2p\,E\Big[\sup_{s\in[t,T]}|M_s|+\sup_{s\in[t,T]}|\mathcal{M}_s|\Big] \\ &\le 2E\big[\eta_t^\varepsilon\big]+2p\,E\Big[\sup_{s\in[t,T]}|M_s|+\sup_{s\in[t,T]}|\mathcal{M}_s|\Big]. \end{aligned} \quad (5.19)$$

Similar to (5.8), one can deduce from the Burkholder–Davis–Gundy inequality, Young's inequality, (1.8), (5.7) and (5.18) that

$$\begin{aligned} 2pE\Big[\sup_{s\in[t,T]}|M_s|+\sup_{s\in[t,T]}|\mathcal{M}_s|\Big] &\le c_l\,pe^{C_A}E\bigg[\big(\mathfrak{S}_t^\varepsilon\big)^{\frac{p}{2}}\Big(\int_{\tau_n\wedge t}^{\tau_n}\varphi_\varepsilon^{p-2}(Y_{s-})|Z_s|^2ds\Big)^{\frac12} \\ &\quad +\big(\mathfrak{S}_t^\varepsilon\big)^{p-1}\Big(\int_{(\tau_n\wedge t,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)\Big)^{\frac12}\bigg] \\ &\le \frac12 E\big[(\mathfrak{S}_t^\varepsilon)^p\big]+c_l\,p^2e^{2C_A}E\int_{\tau_n\wedge t}^{\tau_n}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2ds \\ &\quad +c_{p,l}\,e^{pC_A}E\bigg[\Big(\int_{(\tau_n\wedge t,\tau_n]}\int_{\mathcal{X}}|U_s(x)|^2N_{\mathfrak{p}}(ds,dx)\Big)^{\frac{p}{2}}\bigg] \\ &\le \frac12 E\big[(\mathfrak{S}_t^\varepsilon)^p\big]+\mathfrak{C}E[\eta_t^\varepsilon]+\mathfrak{C}E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds. \end{aligned} \quad (5.20)$$

By (5.17) again,

$$\begin{aligned} E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds &\le E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}\mathbf{1}_{\{|Y_{s-}|\le|U_s(x)|\}}|U_s(x)|^p\nu(dx)ds \\ &\quad +E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}\mathbf{1}_{\{|U_s(x)|<|Y_{s-}|\}}|Y_{s-}|^p\nu(dx)ds \\ &\le \wp^{-p}E\big[\widetilde{\eta}_t\big]+\nu(\mathcal{X})E\int_{\tau_n\wedge t}^{\tau_n}|Y_{s-}|^pds. \end{aligned}$$

Since (1.8) and Fubini's Theorem imply that $E\int_{\tau_n\wedge t}^{\tau_n}|Y_{s-}|^pds = E\int_{\tau_n\wedge t}^{\tau_n}|Y_s|^pds \le E\int_{\tau_n\wedge t}^{\tau_n}\big(\mathfrak{S}_s^\varepsilon\big)^pds\le E\int_t^T\big(\mathfrak{S}_s^\varepsilon\big)^pds=\int_t^T E\big[(\mathfrak{S}_s^\varepsilon)^p\big]ds$,

$$E\int_{\tau_n\wedge t}^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds\le\wp^{-p}E\big[\eta_t^\varepsilon\big]+\nu(\mathcal{X})\int_t^T E\big[(\mathfrak{S}_s^\varepsilon)^p\big]ds. \quad (5.21)$$

(4) *The remaining argument is relatively routine (c.f. Proposition 3.2 of* [10]).

As $E\big[(\mathfrak{S}^\varepsilon_t)^p\big] \le E\big[(\mathfrak{S}^\varepsilon_{t_0})^p\big] < \infty$ by (5.5), plugging (5.21) back into (5.20) and (5.19), we can deduce from Lemma A.1 and Young's inequality that

$$\begin{aligned} E\big[(\mathfrak{S}^\varepsilon_t)^p\big] &\le \mathfrak{C} E\big[\eta^\varepsilon_t\big] + \mathfrak{C}\int_t^T E\big[(\mathfrak{S}^\varepsilon_s)^p\big]ds \\ &\le \mathfrak{C} E\bigg[\big(|Y_{\tau_n}|^2+\varepsilon\big)^{\frac{p}{2}} + \big(\mathfrak{S}^\varepsilon_t\big)^{p-1}\int_{\tau_n\wedge t}^{\tau_n} g_s ds + \int_{t_0}^T \Upsilon_s ds\bigg] \\ &\quad + \mathfrak{C}\int_t^T E\big[(\mathfrak{S}^\varepsilon_s)^p\big]ds \\ &\le \frac{1}{2}E\big[(\mathfrak{S}^\varepsilon_t)^p\big] + \mathfrak{C}\mathscr{J}_\varepsilon + \mathfrak{C}\int_t^T E\big[(\mathfrak{S}^\varepsilon_s)^p\big]ds, \end{aligned} \tag{5.22}$$

where $\mathscr{J}_\varepsilon = \mathscr{J}^n_\varepsilon := \varepsilon^{\frac{p}{2}} + E\Big[|Y_{\tau_n}|^p + \big(\int_{t_0}^T g_s ds\big)^p + \int_{t_0}^T \Upsilon_s ds\Big] < \infty$. So an application of Gronwall's inequality shows that

$$E\big[\big(\mathfrak{S}^\varepsilon_t\big)^p\big] \le \mathfrak{C}\mathscr{J}_\varepsilon e^{\mathfrak{C}T} = \mathfrak{C}\mathscr{J}_\varepsilon, \qquad \forall\, t\in[t_0,T].$$

Then we see from (5.22) and (5.21) that

$$E\bigg[\sup_{s\in[\tau_n\wedge t_0,\tau_n]}|Y_s|^p\bigg] \le E\big[(\mathfrak{S}^\varepsilon_{t_0})^p\big] \le \mathfrak{C}\mathscr{J}_\varepsilon,$$

$$E\big[\eta^\varepsilon_{t_0}\big] \le \mathfrak{C}E\big[(\mathfrak{S}^\varepsilon_{t_0})^p\big] + \mathfrak{C}\mathscr{J}_\varepsilon + \mathfrak{C}\int_{t_0}^T E\big[(\mathfrak{S}^\varepsilon_s)^p\big]ds \le \mathfrak{C}\mathscr{J}_\varepsilon \quad \text{and}$$

$$E\int_{\tau_n\wedge t_0}^{\tau_n}\int_{\mathcal{X}} |U_s(x)|^p\nu(dx)ds \le \mathfrak{C}\mathscr{J}_\varepsilon.$$

These inequalities together with Young's inequality and (5.18) imply that

$$\begin{aligned} E\bigg[\Big(\int_{\tau_n\wedge t_0}^{\tau_n}|Z_s|^2ds\Big)^{\frac{p}{2}}\bigg] &\le E\bigg[\big(\mathfrak{S}^\varepsilon_{t_0}\big)^{\frac{p(2-p)}{2}}\Big(\int_{\tau_n\wedge t_0}^{\tau_n}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2ds\Big)^{\frac{p}{2}}\bigg] \\ &\le \frac{2-p}{2}E\big[\big(\mathfrak{S}^\varepsilon_{t_0}\big)^p\big] + \frac{p}{2}E\int_{\tau_n\wedge t_0}^{\tau_n}\varphi_\varepsilon^{p-2}(Y_s)|Z_s|^2ds \le \mathfrak{C}\mathscr{J}_\varepsilon. \end{aligned} \tag{5.23}$$

Letting $\varepsilon\to 0$, we obtain that

$$\begin{aligned} &E\bigg[\sup_{s\in[\tau_n\wedge t_0,\tau_n]}|Y_s|^p + \Big(\int_{\tau_n\wedge t_0}^{\tau_n}|Z_s|^2ds\Big)^{\frac{p}{2}} + \int_{\tau_n\wedge t_0}^{\tau_n}\int_{\mathcal{X}}|U_s(x)|^p\nu(dx)ds\bigg] \\ &\quad\le \mathfrak{C}E\bigg[|Y_{\tau_n}|^p + \Big(\int_{t_0}^T g_s ds\Big)^p + \int_{t_0}^T \Upsilon_s ds\bigg]. \end{aligned} \tag{5.24}$$

As $(Z,U)\in\mathbb{Z}^2_{\text{loc}}\times\mathbb{U}^p_{\text{loc}}$, it holds for all $\omega\in\Omega$ except on a P-null set $\mathcal{N}$ that $\tau_{\mathfrak{n}}(\omega)=T$ for some $\mathfrak{n}=\mathfrak{n}(\omega)\in\mathbb{N}$. Then

$$\lim_{n\to\infty} Y(\tau_n(\omega),\omega) = Y(T,\omega) = \xi_1(\omega)-\xi_2(\omega), \qquad \forall\,\omega\in\mathcal{N}^c.$$

(One can alternatively show this statement as follows: Since the compensator $\nu(dx)dt$ of the counting measure $N_{\mathfrak{p}}(dt,dx)$ is absolutely continuous with respect to dt, P-almost surely process Y does not have a jump at time T. Thus $\lim_{n\to\infty} Y_{\tau_n} = Y_{T-} = Y_T$, P-a.s.) Eventually, letting $n \to \infty$ in (5.24), we can derive (3.2) from the monotone convergence theorem and the dominated convergence theorem. □

Proof of Proposition 3.1. By (H3)–(H5), it holds $ds \times dP$-a.s. that

$$\begin{aligned}
&|Y_s|^{p-1}\big\langle \mathscr{D}(Y_s), f(s,Y_s,Z_s,U_s)\big\rangle \\
&\quad = |Y_s|^{p-1}\Big(\big\langle \mathscr{D}(Y_s), f(s,0,0,0)\big\rangle + \big\langle \mathscr{D}(Y_s), f(s,Y_s,0,0)-f(s,0,0,0)\big\rangle \\
&\qquad + \big\langle \mathscr{D}(Y_s), f(s,Y_s,Z_s,U_s)-f(s,Y_s,0,0)\big\rangle\Big) \\
&\quad \le |Y_s|^{p-1}\big(|f(s,0,0,0)| + \beta_s|Y_s| + |f(s,Y_s,Z_s,U_s)-f(s,Y_s,0,0)|\big) \\
&\quad \le |Y_s|^{p-1}\big(|f(s,0,0,0)| + \beta_s|Y_s| + \beta_s + c_1(s)|Z_s| + c_2(s)\|U_s\|_{L^p_\nu}\big).
\end{aligned}$$

Clearly, $(0,0,0)$ is the solution to the BSDEJ $(0,0)$, applying Lemma 3.1 with $(\xi_1, f_1, Y^1, Z^1, U^1) = (\xi, f, Y, Z, U)$, $(\xi_2, f_2, Y^2, Z^2, U^2) = (0,0,0,0,0)$ and $(g_s, \Phi_s, \Lambda_s, \Gamma_s, \Upsilon_s) = \big(\beta_s + |f(s,0,0,0)|, \beta_s, c_1(s), c_2(s), 0\big)$, $s \in [0,T]$ yields the inequality (3.3). □

Proof of Proposition 3.2. Given $m, n \in \mathbb{N}$ with $m > n$, we set

$$\Xi^{m,n}_t := \sup_{s\in[t,T]} |Y^{m,n}_s|^p + \Big(\int_t^T |Z^{m,n}_s|^2 ds\Big)^{\frac{p}{2}} + \int_t^T \int_{\mathcal{X}} |U^{m,n}_s(x)|^p \nu(dx)ds, \quad t \in [0,T].$$

Applying Lemma 3.1 with $(\xi_1, f_1, Y^1, Z^1, U^1) = (\xi_m, f_m, Y^m, Z^m, U^m)$, $(\xi_2, f_2, Y^2, Z^2, U^2) = (\xi_n, f_n, Y^n, Z^n, U^n)$ and $(g_s, \Gamma_s, \Upsilon_s) = (0, c(s), \lambda(s)\,\theta(|Y^{m,n}_s|^p + \eta_n) + \Upsilon^{m,n}_s)$, $s \in [0,T]$, we can deduce from Fubini Theorem, the concavity of θ and Jensen's inequality that for some constant $\mathfrak{C}$ depending on T, $\nu(\mathcal{X})$, p, C_Φ, C_Λ and $\int_0^T \big(c(t)\big)^q dt$

$$\begin{aligned}
E\big[\Xi^{m,n}_t\big] &\le \mathfrak{C}\Big(E\big[|\xi_m-\xi_n|^p\big] + \int_t^T \lambda(s) E\big[\theta(\Xi^{m,n}_s + \eta_n)\big]ds + E\int_t^T \Upsilon^{m,n}_s ds\Big) \\
&\le \mathfrak{C}\Big(E\big[|\xi_m-\xi_n|^p\big] + \int_t^T \lambda(s)\,\theta\big(E\big[\Xi^{m,n}_s\big] + E[\eta_n]\big)ds + E\int_0^T \Upsilon^{m,n}_s ds\Big), \\
&\quad t \in [0,T].
\end{aligned}$$

Hence, it holds for any $n \in \mathbb{N}$ and $t \in [0,T]$ that

$$\begin{aligned}
\sup_{m>n} E\big[\Xi^{m,n}_t\big] \le \mathfrak{C}\Big(&\sup_{m>n} E\big[|\xi_m-\xi_n|^p\big] + \int_t^T \lambda(s)\,\theta\Big(\sup_{m>n} E\big[\Xi^{m,n}_s\big] + E[\eta_n]\Big)ds \\
&+ \sup_{m>n} E\int_0^T \Upsilon^{m,n}_s ds\Big).
\end{aligned} \tag{5.25}$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathcal{F}_T)$, one has

$$\lim_{n\to\infty} \sup_{m>n} E\big[|\xi_m-\xi_n|^p\big] = 0. \tag{5.26}$$

If $\int_0^T \lambda(t)dt = 0$, then $\int_0^T \lambda(s)\,\theta\Big(\sup_{m>n} E\big[\Xi^{m,n}_s\big] + E[\eta_n]\Big)ds = 0$. Taking $t = 0$ and letting $n \to \infty$ in (5.25), we see from (5.26) and (3.5) that

$$\lim_{n\to\infty} \sup_{m>n} E\big[\Xi^{m,n}_0\big] = 0. \tag{5.27}$$

On the other hand, suppose that $\int_0^T \lambda(t)dt > 0$. Lemma A.1 implies that

$$\sup_{m>n} E\big[\Xi_s^{m,n}\big] \le \sup_{m>n} E\big[\Xi_0^{m,n}\big] \le \sup_{m>n}\Big(\|Y^{m,n}\|_{\mathbb{D}^p} + \|Z^{m,n}\|_{\mathbb{Z}^{2,p}} + \|U^{m,n}\|_{\mathbb{U}^p}\Big)^p$$
$$\le \Big\{2\sup_{i\in\mathbb{N}}\Big(\|Y^i\|_{\mathbb{D}^p} + \|Z^i\|_{\mathbb{Z}^{2,p}} + \|U^i\|_{\mathbb{U}^p}\Big)\Big\}^p < \infty, \quad \forall (s,n) \in [0,T]\times\mathbb{N}. \tag{5.28}$$

Since $\lambda \in L^1_+[0,T]$ and since $\sup_{n\in\mathbb{N}} E[\eta_n] < \infty$ by (3.5), Fatou's Lemma, the monotonicity and the continuity of θ (real-valued concave functions are continuous) imply that for any $t \in [0,T]$

$$\overline{\lim}_{n\to\infty}\int_t^T \lambda(s)\,\theta\Big(\sup_{m>n} E\big[\Xi_s^{m,n}\big] + E[\eta_n]\Big)ds$$
$$\le \int_t^T \lambda(s)\overline{\lim}_{n\to\infty}\theta\Big(\sup_{m>n} E\big[\Xi_s^{m,n}\big] + E[\eta_n]\Big)ds$$
$$\le \int_t^T \lambda(s)\,\theta\Big(\overline{\lim}_{n\to\infty}\sup_{m>n} E\big[\Xi_s^{m,n}\big]\Big)ds.$$

Letting $n \to \infty$ in (5.25), we can deduce from (5.26) and (3.5) that

$$\overline{\lim}_{n\to\infty}\sup_{m>n} E\big[\Xi_t^{m,n}\big] \le \mathfrak{C}\int_t^T \lambda(s)\,\theta\Big(\overline{\lim}_{n\to\infty}\sup_{m>n} E\big[\Xi_s^{m,n}\big]\Big)ds, \quad t \in [0,T].$$

As $\theta : [0,\infty) \to [0,\infty)$ is an increasing concave function, it is easy to see that either $\theta \equiv 0$ or $\theta(t) > 0$ for any $t > 0$. Moreover, one can deduce from (5.28) that the function $\chi(t) := \overline{\lim}_{n\to\infty}\sup_{m>n} E\big[\Xi_t^{m,n}\big]$, $t \in [0,T]$ is bounded. Then Bihari's inequality (see Lemma A.3) and (5.28) imply that $\lim_{n\to\infty}\sup_{m>n} E\big[\Xi_t^{m,n}\big] = 0$, $\forall t \in [0,T]$. Therefore, (5.27) always holds, which shows that $\big\{(Y^n, Z^n, U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p$. □

Proof of Proposition 3.3. Let us make the following settings first:

• Set $C_f := \big\|\int_0^T |f(t,0,0,0)|dt\big\|_{L^\infty_+(\mathcal{F}_T)}$, $C_{p,\mathcal{X}} := \big(\nu(\mathcal{X})\big)^{\frac{2-p}{2p}}$ and

$$R := 2 + \exp\Big\{T + C_f + 4C_\beta + 2\int_0^T \big(c_1(t)\big)^2 dt + C^2_{p,\mathcal{X}}\int_0^T \big(c_2(t)\big)^2 dt\Big\}$$
$$\times\sqrt{\|\xi\|^2_{L^\infty(\mathcal{F}_T)} + 5T + C_f/2 + 7C_\beta/2}. \tag{5.29}$$

Let $\psi : \mathbb{R}^l \to [0,1]$ be a smooth function such that $\psi(x) = 1$ (resp. $\psi(x) = 0$) if $|x| \le R-1$ (resp. $|x| \ge R$).

• Let $\rho : \mathbb{R}^{l+l\times d} \to \mathbb{R}^+$ be a smooth function that vanishes outside the unit open ball $\mathcal{B}_1(0)$ of $\mathbb{R}^{l+l\times d}$ and satisfies $\int_{\mathbb{R}^{l+l\times d}} \rho(x)dx = 1$. For any $r \in (0,\infty)$, we set $\rho_r(x) := r^{l(1+d)}\rho(rx)$, $\forall x \in \mathbb{R}^{l+l\times d}$.

• We say that $\big\{O_i\big\}_{i=1}^m$ is a partition of the unit closed ball $\overline{\mathcal{B}}_1(0)$ of $\mathbb{R}^{l+l\times d}$ if O_i, $i = 1,\dots,m$ are simple-connected, open subsets of $\mathcal{B}_1(0)$ that are pairwisely disjoint, and if $\cup_{i=1}^m \overline{O}_i = \overline{\mathcal{B}}_1(0)$. Let $\big\{O_i^k\big\}_{i=1}^{2^k}$, $k \in \mathbb{N}$ be partitions of $\overline{\mathcal{B}}_1(0)$ such that $\overline{O}_i^k = \overline{O}_{2i-1}^{k+1} \cup \overline{O}_{2i}^{k+1}$ holds for any $k \in \mathbb{N}$ and $i = 1,\dots,2^k$. For each O_i^k, we pick up a $(y_i^k, z_i^k) \in O_i^k$ with $y_i^k \in \mathbb{R}^l$, and let $\|O_i^k\|$ denote the volume of O_i^k.

(1) *To apply the existing wellposedness result on $\mathbb{L}^p$-solutions of BSDEJs with Lipschitz generator, we first approximate the monotonic generator f by a sequence of Lipschitz generators $\{f_n\}_{n\in\mathbb{N}}$ via convolution with mollifiers $\{\rho_n\}_{n\in\mathbb{N}}$.*

Fix $n \in \mathbb{N}$ with $n > \kappa_0$. For any $u \in L^2_\nu$, since Hölder's inequality shows that u also belongs to L^p_ν with $\|u\|_{L^p_\nu} \le \big(\nu(\mathcal{X})\big)^{\frac{2-p}{2p}} \|u\|_{L^2_\nu} = C_{p,\mathcal{X}}\|u\|_{L^2_\nu}$, we define

$$\zeta_n(u) := \Big(\frac{n}{n \vee \|u\|_{L^p_\nu}}\Big) u \in L^p_\nu.$$

Applying Lemma A.5 with $(\mathbb{E}, \|\cdot\|, r, x, y) = \big(L^p_\nu, \|\cdot\|_{L^p_\nu}, n, u_1, u_2\big)$ yields that

$$\|\zeta_n(u_1) - \zeta_n(u_2)\|_{L^p_\nu} \le 2\|u_1 - u_2\|_{L^p_\nu} \le 2C_{p,\mathcal{X}}\|u_1 - u_2\|_{L^2_\nu}, \quad \forall\, u_1, u_2 \in L^2_\nu, \tag{5.30}$$

which shows that the mapping $\zeta_n : L^2_\nu \to L^p_\nu$ is $\mathscr{B}(L^2_\nu)/\mathscr{B}(L^p_\nu)$-measurable. (Note: As the space L^p_ν may not have an inner product, one may not apply Lemma A.4.)

Since $\beta^n_t := \frac{n}{n\vee\beta_t\vee|f(t,0,0,0)|} \in (0, 1], t \in [0, T]$ is an $\mathbf{F}$-progressively measurable process, we can deduce from (A.3), (5.30) and the $\mathscr{P} \otimes \mathscr{B}\big(\mathbb{R}^l\big) \otimes \mathscr{B}\big(\mathbb{R}^{l\times d}\big) \otimes \mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurability of f that the mapping

$$f^0_n(t, \omega, y, z, u) := \beta^n(t, \omega)\psi(y) f\big(t, \omega, y, \pi_n(z), \zeta_n(u)\big),$$
$$\forall\, (t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^2_\nu$$

is $\mathscr{P} \otimes \mathscr{B}\big(\mathbb{R}^l\big) \otimes \mathscr{B}\big(\mathbb{R}^{l\times d}\big) \otimes \mathscr{B}\big(L^2_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable. Given $(t, \omega, y, z, u) \in [0, T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^2_\nu$, we further define

$$f_n(t, \omega, y, z, u) := \big(f^0_n(t, \omega, \cdot, \cdot, u) * \rho_n\big)(y, z).$$

By (H1), the continuity of mapping $f(t, \omega, \cdot, \cdot, u)$ implies the continuity of mapping $f^0_n(t, \omega, \cdot, \cdot, u)$. Hence, $f_n(t, \omega, y, z, u)$ is indeed a Riemann integral:

$$\begin{aligned} f_n(t, \omega, y, z, u) &= \int_{|(\tilde{y},\tilde{z})|\le 1} f^0_n\Big(t, \omega, y - \frac{1}{n}\tilde{y}, z - \frac{1}{n}\tilde{z}, u\Big)\rho(\tilde{y}, \tilde{z})d\tilde{y}d\tilde{z} \\ &= \lim_{k\to\infty} \sum_{i=1}^{2^k} f^0_n\Big(t, \omega, y - \frac{1}{n}y^k_i, z - \frac{1}{n}z^k_i, u\Big)\rho(y^k_i, z^k_i)\|O^k_i\|, \end{aligned} \tag{5.31}$$

from which one can deduce that f_n is also $\mathscr{P}\otimes\mathscr{B}\big(\mathbb{R}^l\big)\otimes\mathscr{B}\big(\mathbb{R}^{l\times d}\big)\otimes\mathscr{B}\big(L^2_\nu\big)/\mathscr{B}(\mathbb{R}^l)$-measurable.

Now, set $c_n(t) := n\big(3 + R + c_1(t) + c_2(t)\big)$, $t \in [0, T]$, which is clearly of $L^2_+[0, T]$. As $\beta^n_t\big(\beta_t \vee |f(t, 0, 0, 0)|\big) \le n$, (H2$'$), (H4) and (H5) show that $dt \times dP$-a.s.

$$\begin{aligned} |f^0_n(t, y, z, u)| &\le \beta^n_t\psi(y)\big(|f(t, 0, 0, 0)| + \kappa_0(1 + |y|) + \beta_t + c_1(t)\big|\pi_n(z)\big| \\ &\qquad + c_2(t)\big\|\zeta_n(u)\big\|_{L^p_\nu}\big) \\ &\le c_n(t), \quad \forall\, (y, z, u) \in \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^2_\nu. \end{aligned} \tag{5.32}$$

This implies that for $dt \times dP$-a.s. $(t, \omega) \in [0, T] \times \Omega$ and any $u \in L^2_\nu$, $f_n(t, \omega, \cdot, \cdot, u)$ is a smooth function on $\mathbb{R}^l \times \mathbb{R}^{l\times d}$ via convolution.

Let $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ and set $\mathfrak{y}_\alpha := \alpha y_1 + (1 - \alpha)y_2$, $\mathfrak{z}_\alpha := \alpha z_1 + (1 - \alpha)z_2$, $\forall\, \alpha \in (0, 1)$. Since

$$\rho_n(y_1 - \widetilde{y}, z_1 - \widetilde{z}) - \rho_n(y_2 - \widetilde{y}, z_2 - \widetilde{z}) = \rho_n(\mathfrak{y}_1 - \widetilde{y}, \mathfrak{z}_1 - \widetilde{z}) - \rho_n(\mathfrak{y}_0 - \widetilde{y}, \mathfrak{z}_0 - \widetilde{z})$$

$$= \int_0^1 d\rho_n(\mathfrak{y}_\alpha - \widetilde{y}, \mathfrak{z}_\alpha - \widetilde{z})$$
$$= \int_0^1 \Big\langle \big(y_1 - y_2, z_1 - z_2\big), \nabla \rho_n\big(\mathfrak{y}_\alpha - \tilde{y}, \mathfrak{z}_\alpha - \tilde{z}\big)\Big\rangle d\alpha, \quad \forall (\widetilde{y}, \widetilde{z}) \in \mathbb{R}^l \times \mathbb{R}^{l\times d},$$

(5.32) also yields that $dt \times dP$-a.s.

$$\begin{aligned} &|f_n(t, y_1, z_1, u) - f_n(t, y_2, z_2, u)| \\ &= \Big| \int_{\mathbb{R}^{l+l\times d}} \big(\rho_n(y_1 - \widetilde{y}, z_1 - \widetilde{z}) - \rho_n(y_2 - \widetilde{y}, z_2 - \widetilde{z})\big) f_n^0(t, \tilde{y}, \tilde{z}, u) d\tilde{y} d\tilde{z} \Big| \\ &= \Big| \int_{\mathbb{R}^{l+l\times d}} \Big(\int_0^1 \Big\langle \big(y_1 - y_2, z_1 - z_2\big), \nabla \rho_n\big(\mathfrak{y}_\alpha - \tilde{y}, \mathfrak{z}_\alpha - \tilde{z}\big)\Big\rangle d\alpha \Big) f_n^0(t, \tilde{y}, \tilde{z}, u) d\tilde{y} d\tilde{z} \Big| \\ &\le c_n(t) \int_0^1 \int_{\mathbb{R}^{l+l\times d}} \big|(y_1 - y_2, z_1 - z_2)\big| \cdot \big|\nabla \rho_n\big(\mathfrak{y}_\alpha - \tilde{y}, \mathfrak{z}_\alpha - \tilde{z}\big)\big| d\tilde{y} d\tilde{z} d\alpha \\ &\le \kappa_\rho^n c_n(t)\big(|y_1 - y_2| + |z_1 - z_2|\big), \ \ \forall (y_1, z_1), (y_2, z_2) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}, \ \ \forall u \in L^2_\nu, \end{aligned} \tag{5.33}$$

where $\kappa_\rho^n := \int_{\mathbb{R}^{l+l\times d}} |\nabla \rho_n(x)| dx < \infty$ is a constant determined by ρ and n.

On the other hand, (5.31), (H5) and (5.30) imply that $dt \times dP$-a.s.

$$\begin{aligned} &|f_n(t, y, z, u_1) - f_n(t, y, z, u_2)| \\ &= \Big| \int_{|(\tilde{y},\tilde{z})|\le 1} \beta_t^n \psi\Big(y - \frac{1}{n}\tilde{y}\Big)\Big(f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \zeta_n(u_1)\Big) \\ &\quad - f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \zeta_n(u_2)\Big)\Big)\rho(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} \Big| \\ &\le \int_{|(\tilde{y},\tilde{z})|\le 1} \Big| f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \zeta_n(u_1)\Big) \\ &\quad - f\Big(t, y - \frac{1}{n}\tilde{y}, \pi_n\Big(z - \frac{1}{n}\tilde{z}\Big), \zeta_n(u_2)\Big)\Big| \rho(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} \\ &\le c_2(t)\big\|\zeta_n(u_1) - \zeta_n(u_2)\big\|_{L^p_\nu} \le 2c_2(t) C_{p,\mathcal{X}} \|u_1 - u_2\|_{L^2_\nu}, \\ &\quad \forall (y, z, u_1, u_2) \in \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^2_\nu \times L^2_\nu, \end{aligned}$$

which together with (5.33) shows that f_n is Lipschitz continuous in $(y, z, u) \in \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^2_\nu$ with $L^2_+[0, T]$-coefficients.

Moreover, (5.31), (H2$'$) and (H4) imply that $dt \times dP$-a.s.

$$\begin{aligned} |f_n(t, 0, 0, 0)| &\le \int_{|(\tilde{y},\tilde{z})|\le 1} \beta_t^n \Big| f\Big(t, -\frac{1}{n}\tilde{y}, \pi_n\Big(-\frac{1}{n}\tilde{z}\Big), 0\Big)\Big| \rho(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} \\ &\le \int_{|(\tilde{y},\tilde{z})|\le 1} \beta_t^n \Big(|f(t, 0, 0, 0)| + \kappa_0 + \frac{\kappa_0}{n}|\tilde{y}| + \beta_t + c_1(t)\Big|\pi_n\Big(-\frac{1}{n}\tilde{z}\Big)\Big|\Big) \rho(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} \\ &\le \int_{|(\tilde{y},\tilde{z})|\le 1} \Big(n + \kappa_0 + 1 + n + \frac{1}{n}c_1(t)\Big)\rho(\tilde{y}, \tilde{z}) d\tilde{y} d\tilde{z} = 2n + \kappa_0 + 1 + \frac{1}{n}c_1(t), \end{aligned}$$

so $E\Big[\big(\int_0^T |f_n(t, 0, 0, 0)| dt\big)^2\Big] \le \big((2n + \kappa_0 + 1)T + \frac{1}{n}\int_0^T c_1(t) dt\big)^2 < \infty$. Then we know from the classical wellposedness result of BSDEJs in $\mathbb{L}^2$-case (see e.g. Lemma 2.2 of [58]) that the BSDEJ (ξ, f_n) has a unique solution $(Y^n, Z^n, U^n) \in \mathbb{D}^2 \times \mathbb{Z}^2 \times \mathbb{U}^2$.

(2) *In this part, we will use regular argument to show that the $\mathbb{L}^2$-norms of $\{(Y^n, Z^n, U^n)\}_{n\in\mathbb{N}}$ are bounded.*

Next, we define $a_t := 1+|f(t,0,0,0)|+4\beta_t+2\big(c_1(t)\big)^2+C^2_{p,\mathcal{X}}\big(c_2(t)\big)^2$ and $A_t := 2\int_0^t a_s ds$, $t\in[0,T]$. Clearly, $A_T \in L^\infty_+(\mathcal{F}_T)$ with $C_A := \|A_T\|_{L^\infty_+(\mathcal{F}_T)} \le 2T + 2C_f + 8C_\beta + 4\int_0^T \big(c_1(t)\big)^2 dt + 2C^2_{p,\mathcal{X}}\int_0^T \big(c_2(t)\big)^2 dt < \infty$.

Fix $n\in\mathbb{N}$ with $n>\kappa_0$ and fix $t\in[0,T]$. Applying Itô's formula to process $e^{A_s}|Y^n_s|^2$ over interval $[t,T]$ and using (1.8) yield that

$$
\begin{aligned}
&e^{A_t}|Y^n_t|^2 + \int_t^T e^{A_s}|Z^n_s|^2 ds + \int_{(t,T]}\int_{\mathcal{X}} e^{A_s}|U^n_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\\
&= e^{A_T}|\xi|^2 + 2\int_t^T e^{A_s}\big[\big\langle Y^n_s, f_n(s,Y^n_s,Z^n_s,U^n_s)\big\rangle - a_s|Y^n_s|^2\big]ds\\
&\quad -2\big(M_T - M_t + \mathcal{M}_T - \mathcal{M}_t\big),\ P\text{-a.s.,}
\end{aligned}
\tag{5.34}
$$

where $M_s := \int_0^s e^{A_r}\langle Y^n_{r-}, Z^n_r dB_r\rangle$ and $\mathcal{M}_s := \int_{(0,s]}\int_{\mathcal{X}} e^{A_r}\langle Y^n_{r-}, U^n_r(x)\rangle \widetilde{N}_{\mathfrak{p}}(dr,dx)$, $\forall s\in[0,T]$.

Since (H2$'$) and (H3) show that $dt\times dP$-a.s.

$$
\begin{aligned}
&\Big\langle Y^n_s, f\Big(s, Y^n_s - \frac{1}{n}y, 0, 0\Big) - f(s,0,0,0)\Big\rangle\\
&\le \beta_s\Big|Y^n_s - \frac{1}{n}y\Big|^2 + \frac{1}{n}|y|\Big|f\Big(s, Y^n_s - \frac{1}{n}y, 0, 0\Big) - f(s,0,0,0)\Big|\\
&\le \beta_s\Big|Y^n_s - \frac{1}{n}y\Big|^2 + |y|\Big(1+\Big|Y^n_s - \frac{1}{n}y\Big|\Big), \quad \forall (y,z)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}
\end{aligned}
$$

and since $\big\|\zeta_n(U^n_s)\big\|_{L^p_\nu} \le \|U^n_s\|_{L^p_\nu} \le C_{p,\mathcal{X}}\|U^n_s\|_{L^2_\nu}$, we can deduce from (5.31), (H4) and (H5) that P-a.s.

$$
\begin{aligned}
&\langle Y^n_s, f_n(s,Y^n_s,Z^n_s,U^n_s)\rangle\\
&= \int_{|(y,z)|\le 1} \beta^n_s \psi\Big(Y^n_s - \frac{1}{n}y\Big)\Big\langle Y^n_s, f\Big(s, Y^n_s - \frac{1}{n}y, \pi_n\Big(Z^n_s - \frac{1}{n}z\Big), \zeta_n\big(U^n_s\big)\Big)\Big\rangle \rho(y,z)dydz\\
&\le \int_{|(y,z)|\le 1}\Big\{|Y^n_s|\Big[|f(s,0,0,0)| + \beta_s + c_1(s)\Big|\pi_n\Big(Z^n_s - \frac{1}{n}z\Big)\Big|\\
&\quad + c_2(s)\big\|\zeta_n(U^n_s)\big\|_{L^p_\nu}\Big] + \beta_s\Big|Y^n_s - \frac{1}{n}y\Big|^2 + |y|\Big(1+\Big|Y^n_s - \frac{1}{n}y\Big|\Big)\Big\}\rho(y,z)dydz\\
&\le 2 + \beta_s + |Y^n_s|\Big(1 + |f(s,0,0,0)| + 3\beta_s + c_1(s)\big(1+|Z^n_s|\big) + c_2(s)C_{p,\mathcal{X}}\|U^n_s\|_{L^2_\nu}\Big)\\
&\quad + \beta_s|Y^n_s|^2\\
&\le \frac{5}{2} + \frac{1}{4}|f(s,0,0,0)| + \frac{7}{4}\beta_s + a_s|Y^n_s|^2 + \frac{1}{4}|Z^n_s|^2 + \frac{1}{4}\|U^n_s\|^2_{L^2_\nu} \quad \text{for a.e. } s\in[0,T],
\end{aligned}
\tag{5.35}
$$

where we used the inequality $\alpha \le \frac{1}{4} + \alpha^2$, $\forall \alpha\in[0,\infty)$.

Moreover, Burkholder–Davis–Gundy inequality and Hölder's inequality imply that

$$
E\Big[\sup_{s\in[0,T]}|M_s| + \sup_{s\in[0,T]}|\mathcal{M}_s|\Big] \le c_l e^{C_A} E\left[Y^n_*\Big(\int_0^T |Z^n_s|^2 ds\Big)^{\frac{1}{2}}\right.
$$

$$+ Y^n_* \bigg(\int_0^T \int_{\mathcal{X}} \big|U^n_s(x)\big|^2 N_{\mathfrak{p}}(ds, dx)\bigg)^{\frac{1}{2}}\Bigg]$$
$$\le c_l e^{C_A} \|Y^n\|_{\mathbb{D}^2}\big(\|Z^n\|_{\mathbb{Z}^2} + \|U^n\|_{\mathbb{U}^2}\big) < \infty,$$

which shows that both M and $\mathcal{M}$ are uniformly integrable martingales. Since

$$E\bigg[\int_{(t,T]}\int_{\mathcal{X}} e^{A_s}|U^n_s(x)|^2 N_{\mathfrak{p}}(ds, dx)\bigg|\mathcal{F}_t\bigg] = E\bigg[\int_t^T \int_{\mathcal{X}} e^{A_s}|U^n_s(x)|^2 \nu(dx)\, ds\bigg|\mathcal{F}_t\bigg]$$
$$= E\bigg[\int_t^T e^{A_s}\|U^n_s\|^2_{L^2_\nu} ds\bigg|\mathcal{F}_t\bigg], \quad P\text{-a.s.},$$

taking conditional expectation $E[\cdot|\mathcal{F}_t]$ in (5.34), one can deduce from (5.35) that P-a.s.

$$|Y^n_t|^2 + \frac{1}{2}E\bigg[\int_t^T \Big(|Z^n_s|^2 + \|U^n_s\|^2_{L^2_\nu}\Big) ds\bigg|\mathcal{F}_t\bigg]$$
$$\le e^{A_t}|Y^n_t|^2 + \frac{1}{2}E\bigg[\int_t^T e^{A_s}\Big(|Z^n_s|^2 + \|U^n_s\|^2_{L^2_\nu}\Big) ds\bigg|\mathcal{F}_t\bigg]$$
$$\le e^{C_A}\Big(\|\xi\|^2_{L^\infty(\mathcal{F}_T)} + 5T + C_f/2 + 7C_\beta/2\Big) \le (R-2)^2.$$

This together with the right-continuity of Y^n implies that

$$\|Y^n\|_{\mathbb{D}^\infty} \le R-2 \quad \text{and} \quad \big\|Z^n\big\|^2_{\mathbb{Z}^2} + \|U^n\|^2_{\mathbb{U}^2} \le 2(R-2)^2, \qquad \forall n \in \mathbb{N}. \tag{5.36}$$

(3) *Next, we carefully verify conditions (3.4) and (3.5) for (Y^n, Z^n, U^n)'s, so the sequence has a limit (Y, Z, U) according to Proposition 3.2.*

For any $(t, \omega) \in [0, T] \times \Omega$ except on a $dt \times dP$-null set $\mathfrak{N}$, we may assume that (H2$'$), (H4)–(H6) hold, that $|Y^n_t(\omega)| \le R-2$, $\forall n \in \mathbb{N}$, and that $U^n_t(\omega) \in L^2_\nu \subset L^p_\nu$, $\forall n \in \mathbb{N}$.

Fix $(t, \omega) \in \mathfrak{N}^c$. By (H5) and (H6), it holds for any $(y_1, z_1, u_1), (y_2, z_2, u_2) \in \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^p_\nu$ that

$$|y_1 - y_2|^{p-1}\big\langle \mathscr{D}(y_1-y_2), f(t,\omega,y_1,z_1,u_1) - f(t,\omega,y_2,z_2,u_2)\big\rangle$$
$$\le |y_1-y_2|^{p-1}\Big(\big\langle \mathscr{D}(y_1-y_2), f(t,\omega,y_1,z_1,u_1) - f(t,\omega,y_2,z_2,u_1)\big\rangle$$
$$+ \big|f(t,\omega,y_2,z_2,u_1) - f(t,\omega,y_2,z_2,u_2)\big|\Big)$$
$$\le \lambda(t)\,\theta\big(|y_1-y_2|^p\big) + \Phi(t,\omega)|y_1-y_2|^p$$
$$+ |y_1-y_2|^{p-1}\big(\Lambda(t,\omega)|z_1-z_2| + c_2(t)\|u_1-u_2\|_{L^p_\nu}\big). \tag{5.37}$$

Let us also fix $m, n \in \mathbb{N}$ with $m > n$. Since (Y^m, Z^m, U^m) is the unique solution of BSDEJ (ξ, f_m) and since $\psi(x) \equiv 1$ for all $|x| \le R-1$, (5.31) and (5.36) show that $(Y^{m,n}, Z^{m,n}, U^{m,n}) := (Y^m - Y^n, Z^m - Z^n, U^m - U^n)$ satisfies

$$|Y^{m,n}_t(\omega)|^{p-1}\big\langle \mathscr{D}(Y^{m,n}_t(\omega)), f_m(t,\omega,Y^m_t(\omega),Z^m_t(\omega),U^m_t(\omega))$$
$$- f_n(t,\omega,Y^n_t(\omega),Z^n_t(\omega),U^n_t(\omega))\big\rangle$$
$$= \int_{|(\tilde{y},\tilde{z})|\le 1} |Y^{m,n}_t(\omega)|^{p-1}\big\langle \mathscr{D}(Y^{m,n}_t(\omega)), \beta^m_t(\omega)h^m_{t,\omega}(\tilde{y},\tilde{z})$$
$$- \beta^n_t(\omega)h^n_{t,\omega}(\tilde{y},\tilde{z})\big\rangle \rho(\tilde{y},\tilde{z})d\tilde{y}d\tilde{z}, \tag{5.38}$$

where $h^n_{t,\omega}(\tilde y,\tilde z) := f\big(t,\omega,Y^n_t(\omega)-\frac1n\tilde y,\pi_n(Z^n_t(\omega)-\frac1n\tilde z),\zeta_n(U^n_t(\omega))\big)$. Next, we fix $(\tilde y,\tilde z)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}$ with $|(\tilde y,\tilde z)|<1$ and set $(\widetilde y_{m,n},\widetilde z_{m,n}) := \Big(\big(\frac1m-\frac1n\big)\tilde y,\big(\frac1m-\frac1n\big)\tilde z\Big)$. Consider the following decomposition:

$$\begin{aligned}
&|Y^{m,n}_t(\omega)|^{p-1}\big\langle\mathscr{D}(Y^{m,n}_t(\omega)),\beta^m_t(\omega)h^m_{t,\omega}(\tilde y,\tilde z)-\beta^n_t(\omega)h^n_{t,\omega}(\tilde y,\tilde z)\big\rangle\\
&\quad=\beta^m_t(\omega)\big|Y^{m,n}_t(\omega)-\widetilde y_{m,n}\big|^{p-1}\big\langle\mathscr{D}\left(Y^{m,n}_t(\omega)-\widetilde y_{m,n}\right),h^m_{t,\omega}(\tilde y,\tilde z)-h^n_{t,\omega}(\tilde y,\tilde z)\big\rangle\\
&\qquad+\beta^m_t(\omega)\big\langle|Y^{m,n}_t(\omega)|^{p-1}\mathscr{D}\left(Y^{m,n}_t(\omega)\right)\\
&\qquad-\big|Y^{m,n}_t(\omega)-\widetilde y_{m,n}\big|^{p-1}\mathscr{D}\left(Y^{m,n}_t(\omega)-\widetilde y_{m,n}\right),h^m_{t,\omega}(\tilde y,\tilde z)-h^n_{t,\omega}(\tilde y,\tilde z)\big\rangle\\
&\qquad+|Y^{m,n}_t(\omega)|^{p-1}\big\langle\mathscr{D}\left(Y^{m,n}_t(\omega)\right),(\beta^m_t(\omega)-\beta^n_t(\omega))h^n_{t,\omega}(\tilde y,\tilde z)\big\rangle\\
&\quad:=I^1_{t,\omega}(\tilde y,\tilde z)+I^2_{t,\omega}(\tilde y,\tilde z)+I^3_{t,\omega}(\tilde y,\tilde z).
\end{aligned}$$

(3a) We see from (5.37) that

$$\begin{aligned}
I^1_{t,\omega}(\tilde y,\tilde z)&\le\lambda(t)\theta(|Y^{m,n}_t(\omega)-\widetilde y_{m,n}|^p)+\Phi_t(\omega)|Y^{m,n}_t(\omega)-\widetilde y_{m,n}|^p\\
&\quad+|Y^{m,n}_t(\omega)-\widetilde y_{m,n}|^{p-1}\\
&\quad\times\left(\Lambda_t(\omega)\Big|\pi_m\Big(Z^m_t(\omega)-\frac1m\tilde z\Big)-\pi_n(Z^n_t(\omega)-\tfrac1n\tilde z)\Big|\right.\\
&\quad\left.+c_2(t)\big\|\zeta_m\big(U^m_t(\omega)\big)-\zeta_n\big(U^n_t(\omega)\big)\big\|_{L^p_\nu}\right).
\end{aligned}\tag{5.39}$$

Applying Lemma A.2 with $(b,c)=\big(|Y^{m,n}_t(\omega)-\widetilde y_{m,n}|,|Y^{m,n}_t(\omega)|\big)$ and $p=p-1$ (then $p=p$) yields that

$$|Y^{m,n}_t(\omega)-\widetilde y_{m,n}|^{p-1}\le\big|Y^{m,n}_t(\omega)\big|^{p-1}+|\widetilde y_{m,n}|^{p-1}\le\big|Y^{m,n}_t(\omega)\big|^{p-1}+n^{1-p}\tag{5.40}$$

$$\begin{aligned}
\text{and}\quad&|Y^{m,n}_t(\omega)-\widetilde y_{m,n}|^p\le\big|Y^{m,n}_t(\omega)\big|^p+p\Big(|Y^{m,n}_t(\omega)|+|\widetilde y_{m,n}|\Big)^{p-1}\big|\widetilde y_{m,n}\big|\\
&\le\big|Y^{m,n}_t(\omega)\big|^p+\eta_n
\end{aligned}\tag{5.41}$$

with $\eta_n:=\frac pn(2R-3)^{p-1}$. Also, (A.3) implies that

$$\begin{aligned}
&\Big|\pi_m\Big(Z^m_t(\omega)-\frac1m\tilde z\Big)-\pi_n(Z^n_t(\omega)-\tfrac1n\tilde z)\Big|\\
&\quad\le\Big|\pi_m\Big(Z^m_t(\omega)-\frac1m\tilde z\Big)-\pi_m(Z^n_t(\omega)-\tfrac1n\tilde z)\Big|+\Big|\pi_m(Z^n_t(\omega)-\tfrac1n\tilde z)\\
&\qquad-\pi_n(Z^n_t(\omega)-\tfrac1n\tilde z)\Big|\\
&\quad\le|Z^{m,n}_t(\omega)-\widetilde z_{m,n}|+\mathbf{1}_{\{|Z^n_t(\omega)-\frac1n\tilde z|\,>\,n\}}|Z^n_t(\omega)-\tfrac1n\tilde z|\\
&\quad\le|Z^{m,n}_t(\omega)|+\frac2n+\mathbf{1}_{\{|Z^n_t(\omega)|>n-1\}}|Z^n_t(\omega)|.
\end{aligned}\tag{5.42}$$

For any $u\in L^2_\nu$, since $\frac{k}{k\vee\|u\|_{L^p_\nu}}=\frac{1}{1\vee(\|u\|_{L^p_\nu}/k)}\nearrow1$ as $k\to\infty$, one can deduce that

$$\big\|\zeta_m(u)-\zeta_n(u)\big\|_{L^p_\nu}=\mathbf{1}_{\{\|u\|_{L^p_\nu}>n\}}\Big(\frac{m}{m\vee\|u\|_{L^p_\nu}}-\frac{n}{n\vee\|u\|_{L^p_\nu}}\Big)\|u\|_{L^p_\nu}\le\mathbf{1}_{\{\|u\|_{L^p_\nu}>n\}}\|u\|_{L^p_\nu},$$

which together with the first inequality of (5.30) implies that

$$\begin{aligned}
&\big\|\zeta_m\big(U_t^m(\omega)\big)-\zeta_n\big(U_t^n(\omega)\big)\big\|_{L^p_\nu}\\
&\quad\le\big\|\zeta_m\big(U_t^m(\omega)\big)-\zeta_m\big(U_t^n(\omega)\big)\big\|_{L^p_\nu}+\big\|\zeta_m\big(U_t^n(\omega)\big)-\zeta_n\big(U_t^n(\omega)\big)\big\|_{L^p_\nu}\\
&\quad\le 2\big\|U_t^{m,n}(\omega)\big\|_{L^p_\nu}+\mathbf{1}_{\{\|U_t^n(\omega)\|_{L^p_\nu}>n\}}\big\|U_t^n(\omega)\big\|_{L^p_\nu}\\
&\quad\le 2\big\|U_t^{m,n}(\omega)\big\|_{L^p_\nu}+n^{\frac{p-2}{p}}\big\|U_t^n(\omega)\big\|_{L^p_\nu}^{\frac{2}{p}}\\
&\quad\le 2\big\|U_t^{m,n}(\omega)\big\|_{L^p_\nu}+n^{\frac{p-2}{p}}C_{p,\mathcal{X}}^{\frac{2}{p}}\big\|U_t^n(\omega)\big\|_{L^2_\nu}^{\frac{2}{p}}. \qquad (5.43)
\end{aligned}$$

Since $\big\|U_t^{m,n}(\omega)\big\|_{L^p_\nu}\le C_{p,\mathcal{X}}\big\|U_t^{m,n}(\omega)\big\|_{L^2_\nu}$, plugging this inequality and (5.40)–(5.42) into (5.39), we can deduce from the monotonicity of function θ that

$$\begin{aligned}
I^1_{t,\omega}(\tilde y,\tilde z)&\le\lambda(t)\theta\big(|Y_t^{m,n}(\omega)|^p+\eta_n\big)+\Phi_t(\omega)\big(|Y_t^{m,n}(\omega)|^p+\eta_n\big)\\
&\quad+\big(|Y_t^{m,n}(\omega)|^{p-1}+n^{1-p}\big)\times\Big[\Psi_t^n(\omega)+\Lambda_t(\omega)|Z_t^{m,n}(\omega)|+2c_2(t)\|U_t^{m,n}(\omega)\|_{L^p_\nu}\Big]\\
&\le\lambda(t)\theta\big(|Y_t^{m,n}(\omega)|^p+\eta_n\big)+\Phi_t(\omega)|Y_t^{m,n}(\omega)|^p\\
&\quad+\eta_n\Phi_t(\omega)+\big[1+(2R-4)^{p-1}\big]\Psi_t^n(\omega)+|Y_t^{m,n}(\omega)|^{p-1}\Big[\Lambda_t(\omega)|Z_t^{m,n}(\omega)|\\
&\quad+2c_2(t)\|U_t^{m,n}(\omega)\|_{L^p_\nu}\Big]\\
&\quad+\frac{1}{2}n^{1-p}\Big(\Lambda_t^2(\omega)+|Z_t^{m,n}(\omega)|^2+4C^2_{p,\mathcal{X}}(c_2(t))^2+\big\|U_t^{m,n}(\omega)\big\|^2_{L^2_\nu}\Big), \qquad (5.44)
\end{aligned}$$

where $\Psi_t^n(\omega):=\Lambda_t(\omega)\big(\frac{2}{n}+\mathbf{1}_{\{|Z_t^n(\omega)|>n-1\}}|Z_t^n(\omega)|\big)+n^{\frac{p-2}{p}}c_2(t)C_{p,\mathcal{X}}^{\frac{2}{p}}\big\|U_t^n(\omega)\big\|_{L^2_\nu}^{\frac{2}{p}}\le\frac{2}{n}\Lambda_t(\omega)+(n-1)^{\frac{-\mathfrak{e}}{2+\mathfrak{e}}}\Lambda_t(\omega)|Z_t^n(\omega)|^{\frac{2+2\mathfrak{e}}{2+\mathfrak{e}}}+n^{\frac{p-2}{p}}c_2(t)C_{p,\mathcal{X}}^{\frac{2}{p}}\big\|U_t^n(\omega)\big\|_{L^2_\nu}^{\frac{2}{p}}$.

(3b) As $\big\|\zeta_n(U_t^n(\omega))\big\|_{L^p_\nu}\le\big\|U_t^n(\omega)\big\|_{L^p_\nu}\le C_{p,\mathcal{X}}\|U_t^n(\omega)\|_{L^2_\nu}$, (H2$'$), (H4) and (H5) show that

$$\begin{aligned}
|h^n_{t,\omega}(\tilde y,\tilde z)|&\le|f(t,\omega,0,0,0)|+\kappa_0(1+|Y_t^n(\omega)-\tfrac{1}{n}\tilde y|)\\
&\quad+\beta_t(\omega)+c_1(t)\big|\pi_n(Z_t^n(\omega)-\tfrac{1}{n}\tilde z)\big|+c_2(t)\big\|\zeta_n(U_t^n(\omega))\big\|_{L^p_\nu}\\
&\le|f(t,\omega,0,0,0)|+\kappa_0R+\beta_t(\omega)+c_1(t)\big(1+|Z_t^n(\omega)|\big)\\
&\quad+c_2(t)C_{p,\mathcal{X}}\|U_t^n(\omega)\|_{L^2_\nu}, \qquad (5.45)
\end{aligned}$$

which together with Lemma A.6 yields that

$$\begin{aligned}
I^2_{t,\omega}(\tilde y,\tilde z)&\le\Big||Y_t^{m,n}(\omega)|^{p-1}\mathscr{D}\big(Y_t^{m,n}(\omega)\big)\\
&\quad-\big|Y_t^{m,n}(\omega)-\widetilde y_{m,n}\big|^{p-1}\mathscr{D}\big(Y_t^{m,n}(\omega)-\widetilde y_{m,n}\big)\Big|\Big(\big|h^m_{t,\omega}(\tilde y,\tilde z)\big|+\big|h^n_{t,\omega}(\tilde y,\tilde z)\big|\Big)\\
&\le(1+2^{p-1})n^{1-p}\Big[2|f(t,\omega,0,0,0)|+2\kappa_0R+2\beta_t(\omega)\\
&\quad+c_1(t)\big(2+|Z_t^m(\omega)|+|Z_t^n(\omega)|\big)\\
&\quad+c_2(t)C_{p,\mathcal{X}}\big(\|U_t^m(\omega)\|_{L^2_\nu}+\|U_t^n(\omega)\|_{L^2_\nu}\big)\Big]:=\widetilde I_t^2(\omega). \qquad (5.46)
\end{aligned}$$

Since $0 < \beta^n_t(\omega) \le \beta^m_t(\omega) \le 1$, $\forall\, t \in [0,T]$, (5.45) also implies that

$$I^3_{t,\omega}(\tilde y, \tilde z) \le (2R-4)^{p-1}(1-\beta^n_t(\omega))\Big[|f(t,\omega,0,0,0)| + \kappa_0 R + \beta_t(\omega) + c_1(t)\big(1+|Z^n_t(\omega)|\big) + c_2(t)C_{p,\mathcal{X}}\|U^n_t(\omega)\|_{L^2_\nu}\Big] := \widetilde{I}^3_t(\omega). \tag{5.47}$$

Putting (5.44), (5.46) and (5.47) back into (5.38) shows that (3.4) is satisfied with $c(\cdot) = 2c_2(\cdot)$ and

$$\Upsilon^{m,n}_t = \eta_n \Phi_t + \big[1+(2R-4)^{p-1}\big]\Psi^n_t + \frac{1}{2}n^{1-p}\Big(\Lambda^2_t(\omega) + |Z^{m,n}_t(\omega)|^2 + 4C^2_{p,\mathcal{X}}(c_2(t))^2 + \big\|U^{m,n}_t(\omega)\big\|^2_{L^2_\nu}\Big) + \widetilde{I}^2_t + \widetilde{I}^3_t, \quad t \in [0,T].$$

Hölder's inequality, Young's inequality and (5.36) give rise to the following four estimates:

$$\text{(a)} \quad \sup_{m>n} E\int_0^T \Upsilon^{m,n}_t dt \le \frac{p}{n}(2R-3)^{p-1}C_\Phi + \big[1+(2R-4)^{p-1}\big]E\int_0^T \Psi^n_t dt + \frac{1}{2}n^{1-p}\Big(C_\Lambda + C^2_{p,\mathcal{X}}\int_0^T (c_2(t))^2 dt + 8(R-2)^2\Big) + E\int_0^T \big(\widetilde{I}^2_t + \widetilde{I}^3_t\big)dt. \tag{5.48}$$

$$\text{(b)} \quad E\int_0^T \Psi^n_t dt \le \frac{2}{n}C^{(1)}_\Lambda + (n-1)^{\frac{-\mathfrak{e}}{2+\mathfrak{e}}}C^{(2)}_\Lambda \|Z^n\|^{\frac{2+2\mathfrak{e}}{2+\mathfrak{e}}}_{\mathbb{Z}^2} + n^{\frac{p-2}{p}}C^{\frac{2}{p}}_{p,\mathcal{X}}\Big(\int_0^T \big(c_2(t)\big)^q dt\Big)^{\frac{1}{q}}\|U^n\|^{\frac{2}{p}}_{\mathbb{U}^2}$$
$$\le \frac{2}{n}C^{(1)}_\Lambda + 2(n-1)^{\frac{-\mathfrak{e}}{2+\mathfrak{e}}}C^{(2)}_\Lambda (R-2)^{\frac{2+2\mathfrak{e}}{2+\mathfrak{e}}} + 2^{\frac{1}{p}}n^{\frac{p-2}{p}}C^{\frac{2}{p}}_{p,\mathcal{X}}\overline{C}^{\frac{1}{q}}(R-2)^{\frac{2}{p}},$$

where $C^{(1)}_\Lambda := E\int_0^T \Lambda_t dt$ and $C^{(2)}_\Lambda := \big(E\int_0^T \Lambda^{2+\mathfrak{e}}_t dt\big)^{\frac{1}{2+\mathfrak{e}}}$.

$$\text{(c)} \quad E\int_0^T \widetilde{I}^2_t dt \le (1+2^{p-1})n^{1-p}\Big\{2C_f + 2\kappa_0 RT + 2C_\beta + \int_0^T \Big(2c_1(t) + \frac{1}{2}(c_1(t))^2 + \frac{1}{2}C^2_{p,\mathcal{X}}(c_2(t))^2\Big)dt + \sum_{i=m,n}\Big(\|Z^i\|^2_{\mathbb{Z}^2} + \|U^i\|^2_{\mathbb{U}^2}\Big)\Big\}$$
$$\le (1+2^{p-1})n^{1-p}\Big\{2C_f + 2\kappa_0 RT + 2C_\beta + \int_0^T \Big(2c_1(t) + \frac{1}{2}(c_1(t))^2 + \frac{1}{2}C^2_{p,\mathcal{X}}(c_2(t))^2\Big)dt + 4(R-2)^2\Big\}.$$

$$\text{(d)} \quad E\int_0^T \widetilde{I}^3_t dt \le (2R-4)^{p-1}\Big\{E\int_0^T (1-\beta^n_t)\big(|f(t,\omega,0,0,0)| + \kappa_0 R + \beta_t + c_1(t)\big)dt + \|Z^n\|_{\mathbb{Z}^2}\Big(E\int_0^T \big(c_1(t)\big)^2(1-\beta^n_t)^2 dt\Big)^{\frac{1}{2}} + C_{p,\mathcal{X}}\|U^n\|_{\mathbb{U}^2}\Big(E\int_0^T \big(c_2(t)\big)^2(1-\beta^n_t)^2 dt\Big)^{\frac{1}{2}}\Big\}$$
$$\le (2R-4)^{p-1}\Big\{E\int_0^T (1-\beta^n_t)\big(|f(t,\omega,0,0,0)| + \kappa_0 R + \beta_t + c_1(t)\big)dt$$

$$+\sqrt{2}(R-2)\Big(E\int_0^T \big(c_1(t)\big)^2(1-\beta_t^n)^2dt\Big)^{\frac{1}{2}}$$
$$+\sqrt{2}C_{p,\mathcal{X}}(R-2)\Big(E\int_0^T \big(c_2(t)\big)^2(1-\beta_t^n)^2dt\Big)^{\frac{1}{2}}\bigg\} := J_n.$$

Because $\beta_t^n = \frac{1}{1\vee(\beta_t/n)\vee(|f(t,0,0,0)|/n)} \nearrow 1$ as $n \to \infty$, $\forall t \in [0,T]$, the dominated convergence theorem shows that $\lim_{n\to\infty} J_n = 0$. Thus, letting $n \to \infty$ in (5.48) yields that $\lim_{n\to\infty}\sup_{m>n} E\int_0^T \Upsilon_t^{m,n}dt = 0$. Moreover, since $\|\cdot\|_{\mathbb{D}^p} \le \|\cdot\|_{\mathbb{D}^\infty}$, $\|\cdot\|_{\mathbb{Z}^{2,p}} \le \|\cdot\|_{\mathbb{Z}^2}$ and $\|\cdot\|_{\mathbb{U}^p} \le \big(\nu(\mathcal{X})T\big)^{\frac{2-p}{2p}}\|\cdot\|_{\mathbb{U}^2}$ by Hölder's inequality, we see from (5.36) that (3.6) also holds. Then Proposition 3.2 shows that $\big\{(Y^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p$. Let (Y,Z,U) be its limit.

(4) *In this part, we will extract an almost-surely convergent and summable subsequence* $\big\{(Y^{m_i},Z^{m_i},U^{m_i})\big\}_{i\in\mathbb{N}}$ *from* $\{(Y^n,Z^n,U^n)\}_{n\in\mathbb{N}}$.

Since

$$\lim_{n\to\infty} E\bigg[\sup_{t\in[0,T]}\big|Y_t^n-Y_t\big|^p+\Big(\int_0^T\big|Z_t^n-Z_t\big|^2dt\Big)^{\frac{p}{2}}$$
$$+\int_0^T\int_{\mathcal{X}}|U_t^n(x)-U_t(x)|^p\nu(dx)dt\bigg]=0, \tag{5.49}$$

we can extract a subsequence $\{m_i\}_{i\in\mathbb{N}}$ from $\mathbb{N}$ such that

$$\text{(i)}\quad \lim_{i\to\infty}\sup_{t\in[0,T]}\big|Y_t^{m_i}-Y_t\big| = \lim_{i\to\infty}\int_0^T\big|Z_t^{m_i}-Z_t\big|^2dt=0,\qquad P\text{-a.s.}, \tag{5.50}$$

$$\text{(ii)}\quad \lim_{i\to\infty}\big\|U_t^{m_i}-U_t\big\|_{L^p_\nu}=0,\qquad dt\times dP\text{-a.s.}, \tag{5.51}$$

$$\text{(iii)}\quad \big\|Y^{m_{i+1}}-Y^{m_i}\big\|_{\mathbb{D}^p}\vee\big\|Z^{m_{i+1}}-Z^{m_i}\big\|_{\mathbb{Z}^{2,p}}\vee\big\|U^{m_{i+1}}-U^{m_i}\big\|_{\mathbb{U}^p}\le 2^{-i},\qquad \forall i\in\mathbb{N}. \tag{5.52}$$

By (5.36), it holds P-a.s. that $\sup_{t\in[0,T]}|Y_t| \le \sup_{t\in[0,T]}\big|Y_t-Y_t^{m_i}\big|+\sup_{t\in[0,T]}\big|Y_t^{m_i}\big| \le \sup_{t\in[0,T]}\big|Y_t^{m_i}-Y_t\big|+R-2$, $\forall i\in\mathbb{N}$. Letting $i\to\infty$, we see from (5.50) that

$$\sup_{t\in[0,T]}|Y_t|\le R-2,\qquad P\text{-a.s.},\qquad \text{thus}\qquad \|Y\|_{\mathbb{D}^\infty}\le R-2. \tag{5.53}$$

For any $i\in\mathbb{N}$, we define two $[0,\infty)$-valued, **F**-predictable processes

$$\mathcal{Z}_t^i := |Z_t|+\sum_{j=1}^i\big|Z_t^{m_j}-Z_t^{m_{j-1}}\big| \quad\text{and}\quad \mathcal{U}_t^i := \|U_t\|_{L^p_\nu}+\sum_{j=1}^i\big\|U_t^{m_j}-U_t^{m_{j-1}}\big\|_{L^p_\nu},$$
$$t\in[0,T]$$

with $Z^{m_0}:=Z$ and $U^{m_0}:=U$. Minkowski's inequality and (5.52) imply that

$$\bigg\{E\bigg[\Big(\int_0^T(\mathcal{Z}_t^i)^2dt\Big)^{\frac{p}{2}}\bigg]\bigg\}^{\frac{1}{p}} \le \big\|Z\big\|_{\mathbb{Z}^{2,p}}+\sum_{j=1}^i\big\|Z^{m_j}-Z^{m_{j-1}}\big\|_{\mathbb{Z}^{2,p}}$$
$$\le 1+\big\|Z\big\|_{\mathbb{Z}^{2,p}}+\big\|Z^{m_1}-Z\big\|_{\mathbb{Z}^{2,p}},\qquad \forall i\in\mathbb{N}. \tag{5.54}$$

As $\{\mathcal{Z}^i\}_{i\in\mathbb{N}}$ is an increasing sequence,

$$\mathcal{Z}_t := \lim_{i\to\infty} \uparrow \mathcal{Z}_t^i = |Z_t| + \sum_{j=1}^{\infty} \big|Z_t^{m_j} - Z_t^{m_{j-1}}\big|, \quad t \in [0,T] \tag{5.55}$$

defines an $[0,\infty]$-valued, **F**-predictable process. The monotone convergence theorem shows that

$$\int_0^T \big(\mathcal{Z}_t(\omega)\big)^2 dt = \lim_{i\to\infty} \uparrow \int_0^T \big(\mathcal{Z}_t^i(\omega)\big)^2 dt \quad \text{and thus} \quad \bigg(\int_0^T \big(\mathcal{Z}_t(\omega)\big)^2 dt\bigg)^{\frac{p}{2}} = \lim_{i\to\infty} \uparrow \bigg(\int_0^T \big(\mathcal{Z}_t^i(\omega)\big)^2 dt\bigg)^{\frac{p}{2}}, \quad \forall\,\omega\in\Omega.$$

Applying the monotone convergence theorem once again, we can deduce from (5.54) and Lemma A.1 that

$$E\bigg[\Big(\int_0^T \mathcal{Z}_t^2 dt\Big)^{\frac{p}{2}}\bigg] = \lim_{i\to\infty} \uparrow E\bigg[\Big(\int_0^T (\mathcal{Z}_t^i)^2 dt\Big)^{\frac{p}{2}}\bigg] \le 3^{p-1}\Big(1 + \|Z\|^p_{\mathbb{Z}^{2,p}} + \|Z^{m_1} - Z\|^p_{\mathbb{Z}^{2,p}}\Big) < \infty. \tag{5.56}$$

Minkowski's inequality and (5.52) also imply that

$$\bigg\{E\int_0^T (\mathcal{U}_t^i)^p dt\bigg\}^{\frac{1}{p}} \le \|U\|_{\mathbb{U}^p} + \sum_{j=1}^{i} \|U^{m_j} - U^{m_{j-1}}\|_{\mathbb{U}^p} \le 1 + \|U\|_{\mathbb{U}^p} + \|U^{m_1} - U\|_{\mathbb{U}^p}, \quad \forall\, i\in\mathbb{N}. \tag{5.57}$$

As $\{\mathcal{U}^i\}_{i\in\mathbb{N}}$ is an increasing sequence,

$$\mathcal{U}_t := \lim_{i\to\infty} \uparrow \mathcal{U}_t^i = \|U_t\|_{L^p_\nu} + \sum_{j=1}^{\infty} \big\|U_t^{m_j} - U_t^{m_{j-1}}\big\|_{L^p_\nu}, \quad t\in[0,T] \tag{5.58}$$

defines an $[0,\infty]$-valued, **F**-predictable process. Applying the monotone convergence theorem again, we can deduce from (5.57) and Lemma A.1 that

$$E\int_0^T \mathcal{U}_t^p dt = \lim_{i\to\infty} \uparrow E\int_0^T (\mathcal{U}_t^i)^p dt \le 3^{p-1}\Big(1 + \|U\|^p_{\mathbb{U}^p} + \|U^{m_1} - U\|^p_{\mathbb{U}^p}\Big) < \infty. \tag{5.59}$$

(5) *Finally, we will send $i\to\infty$ in BSDEJ (ξ, f_{m_i}) to demonstrate that the processes (Y,Z,U) solve BSDEJ (ξ, f).*

Fix $k\in\mathbb{N}$. We define an **F**-stopping time

$$\tau_k := \inf\bigg\{t\in[0,T] : \int_0^t \mathcal{Z}_s^2 ds > k\bigg\} \wedge T. \tag{5.60}$$

Since $\int_0^{\tau_k} |Z_t^{m_i} - Z_t|^2 dt \le \int_0^{\tau_k} (\mathcal{Z}_t^i)^2 dt \le \int_0^{\tau_k} \mathcal{Z}_t^2 dt \le k$, $\forall\,\omega\in\Omega$, the dominated convergence theorem and (5.50) show that

$$\lim_{i\to\infty} E\int_0^{\tau_k} \big|Z_t^{m_i} - Z_t\big|^2 dt = 0. \tag{5.61}$$

Hence, there exists a subsequence $\{m_i^k\}_{i\in\mathbb{N}}$ of $\{m_i\}_{i\in\mathbb{N}}$ such that for $dt\times dP$-a.s. $(t,\omega)\in[0,T]\times\Omega$

$$\lim_{i\to\infty}\mathbf{1}_{\{t\le\tau_k\}}\big|Z_t^{m_i^k}-Z_t\big|=0. \tag{5.62}$$

We shall show that

$$\lim_{i\to\infty}E\int_0^{\tau_k}\Big|f_{m_i^k}\Big(t,Y_t^{m_i^k},Z_t^{m_i^k},U_t^{m_i^k}\Big)-f\big(t,Y_t,Z_t,U_t\big)\Big|dt=0. \tag{5.63}$$

Since $\psi(x)\equiv 1$ for all $|x|\le R-1$, (5.36) implies that for any $i\in\mathbb{N}$

$$\begin{aligned}&E\int_0^{\tau_k}\Big|f_{m_i^k}\Big(t,Y_t^{m_i^k},Z_t^{m_i^k},U_t^{m_i^k}\Big)-f\big(t,Y_t,Z_t,U_t\big)\Big|dt\\&=E\int_0^{\tau_k}\int_{|(\tilde y,\tilde z)|<1}\bigg|\beta_t^{m_i^k}f\bigg(t,Y_t^{m_i^k}-\frac{1}{m_i^k}\tilde y,\pi_{m_i^k}\Big(Z_t^{m_i^k}-\frac{1}{m_i^k}\tilde z\ Bigr),\zeta_{m_i^k}\Big(U_t^{m_i^k}\Big)\bigg)\\&\quad-f\big(t,Y_t,Z_t,U_t\big)\bigg|\rho(\tilde y,\tilde z)d\tilde y d\tilde z dt.\end{aligned} \tag{5.64}$$

For any $(t,\omega)\in[0,T]\times\Omega$ except on a $dt\otimes dP$-null set $\mathfrak{N}_k\supset\mathfrak{N}$, we may assume further that (5.51), (5.62) hold, that $\lim_{i\to\infty}\big|Y_t^{m_i^k}(\omega)-Y_t(\omega)\big|=0$ (by (5.50)), that $|Y_t(\omega)|\le R-2$ (by (5.53)), and that $U_t(\omega)\in L_\nu^p$.

Let $(t,\omega)\in\mathfrak{N}_k^c\cap[\![0,\tau_k]\!]$ and let $(\tilde y,\tilde z)\in\mathbb{R}^l\times\mathbb{R}^{l\times d}$ with $|(\tilde y,\tilde z)|<1$. Since

$$\lim_{i\to\infty}\big|Z_t^{m_i^k}(\omega)-Z_t(\omega)\big|=0 \tag{5.65}$$

by (5.62), Lemma A.4 and the first inequality of (5.30) imply that

(e1) $\Big|Y_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde y-Y_t(\omega)\Big|\le\frac{1}{m_i^k}+\Big|Y_t^{m_i^k}(\omega)-Y_t(\omega)\Big|\to 0$, as $i\to\infty$;

(e2) $\Big|\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde z\Big)-Z_t(\omega)\Big|\le\Big|\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde z\Big)-\pi_{m_i^k}\big(Z_t(\omega)\big)\Big|+\big|\pi_{m_i^k}\big(Z_t(\omega)\big)-Z_t(\omega)\big|\le\Big|Z_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde z-Z_t(\omega)\Big|+\big|\pi_{m_i^k}\big(Z_t(\omega)\big)-Z_t(\omega)\big|\le\frac{1}{m_i^k}+\Big|Z_t^{m_i^k}(\omega)-Z_t(\omega)\Big|+\big|\pi_{m_i^k}\big(Z_t(\omega)\big)-Z_t(\omega)\big|\to 0$, as $i\to\infty$;

(e3) $\Big\|\zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)-U_t(\omega)\Big\|_{L_\nu^p}\le\Big\|\zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)-\zeta_{m_i^k}(U_t(\omega))\Big\|_{L_\nu^p}+\big\|\zeta_{m_i^k}(U_t(\omega))-U_t(\omega)\big\|_{L_\nu^p}\le 2\Big\|U_t^{m_i^k}(\omega)-U_t(\omega)\Big\|_{L_\nu^p}+\Big(1-\frac{m_i^k}{m_i^k\vee\|U_t(\omega)\|_{L_\nu^p}}\Big)\big\|U_t(\omega)\big\|_{L_\nu^p}\to 0$, as $i\to\infty$.

Since the mapping $f\big(t,\omega,\cdot,\cdot,U_t(\omega)\big)$ is continuous by (H1) and since $\lim_{i\to\infty}\uparrow\beta_t^{m_i^k}(\omega)=1$, we can deduce from (e1) and (e2) that

$$\begin{aligned}&\lim_{i\to\infty}\beta_t^{m_i^k}(\omega)f\Big(t,\omega,Y_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde y,\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde z\Big),U_t(\omega)\Big)\\&=f\big(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega)\big).\end{aligned} \tag{5.66}$$

Moreover, (H5) shows that

$$\bigg|f\Big(t,\omega,Y_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde y,\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)-\frac{1}{m_i^k}\tilde z\Big),\zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)$$

$$- f\Big(t, Y_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{y}, \pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{z}\Big), U_t(\omega)\Big)\Big|$$
$$\le c_2(t)\Big\|\zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big) - U_t(\omega)\Big\|_{L^p_\nu},$$

which together with (5.66) and (e3) implies that

$$\lim_{i\to\infty}\Big|\beta_t^{m_i^k}(\omega) f\Big(t, \omega, Y_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{y}, \pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{z}\Big), \zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)$$
$$- f\big(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big| = 0. \tag{5.67}$$

Given $i \in \mathbb{N}$, there exists an $\widehat{j} = \widehat{j}(k, i) \in \mathbb{N}$ such that $m_i^k = m_{\widehat{j}}$. Since

$$\Big|Z_t^{m_i^k}(\omega)\Big| \le \mathcal{Z}_t^{\widehat{j}}(\omega) \le \mathcal{Z}_t(\omega) \quad \text{and}$$
$$\Big\|\zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big\|_{L^p_\nu} \le \Big\|U_t^{m_i^k}(\omega)\Big\|_{L^p_\nu} \le \mathcal{U}_t^{\widehat{j}}(\omega) \le \mathcal{U}_t(\omega), \tag{5.68}$$

one can deduce from (H2$'$), (H4) and (H5) that

$$\Big|\beta_t^{m_i^k}(\omega) f\Big(t, \omega, Y_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{y}, \pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{z}\Big), \zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)$$
$$- f\big(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big|$$
$$\le \Big|f\Big(t, \omega, Y_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{y}, \pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{z}\Big), \zeta_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)\Big|$$
$$+ \Big|f\big(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big|$$
$$\le 2|f(t, \omega, 0, 0, 0)| + \kappa_0\Big(2 + \Big|Y_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{y}\Big| + |Y_t(\omega)|\Big)$$
$$+ 2\beta_t(\omega) + c_1(t)\Big(\Big|Z_t^{m_i^k}(\omega) - \frac{1}{m_i^k}\tilde{z}\Big| + |Z_t(\omega)|\Big)$$
$$+ c_2(t)\Big(\Big\|\zeta_{m_i^k}\big(U_t^{m_i^k}(\omega)\big)\Big\|_{L^p_\nu} + \|U_t(\omega)\|_{L^p_\nu}\Big)$$
$$\le 2|f(t, \omega, 0, 0, 0)| + (2R - 1)\kappa_0 + 2\beta_t(\omega)$$
$$+ c_1(t)\big(1 + \mathcal{Z}_t(\omega) + |Z_t(\omega)|\big) + c_2(t)\big(\mathcal{U}_t(\omega) + \|U_t(\omega)\|_{L^p_\nu}\big) := H_t(\omega).$$

Applying Hölder's inequality, we see from (5.56) and (5.59) that

$$E\int_0^{\tau_k}\int_{|(\tilde{y},\tilde{z})|<1} H_t\,\rho(\tilde{y}, \tilde{z})d\tilde{y}d\tilde{z}dt = E\int_0^{\tau_k} H_t dt \le E\int_0^T H_t dt$$
$$\le C + E\bigg[\Big(\int_0^T (c_1(t))^2 dt\Big)^{\frac{1}{2}}\bigg\{\Big(\int_0^T \mathcal{Z}_t^2 dt\Big)^{\frac{1}{2}} + \Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{1}{2}}\bigg\}\bigg]$$
$$+ \Big(\int_0^T (c_2(t))^q dt\Big)^{\frac{1}{q}}\bigg\{\Big(E\int_0^T \mathcal{U}_t^p dt\Big)^{\frac{1}{p}} + \|U\|_{\mathbb{U}^p}\bigg\}$$

$$\leq C + \overline{C}^{\frac{1}{2}}\bigg\{\bigg(E\bigg[\Big(\int_0^T \mathcal{Z}_t^2 dt\Big)^{\frac{p}{2}}\bigg]\bigg)^{\frac{1}{p}} + \|Z\|_{\mathbb{Z}^{2,p}}\bigg\}$$

$$+ \overline{C}^{\frac{1}{q}}\bigg\{\Big(E\int_0^T \mathcal{U}_t^p dt\Big)^{\frac{1}{p}} + \|U\|_{\mathbb{U}^p}\bigg\} < \infty \tag{5.69}$$

with $C := 2C_f + (2R-1)\kappa_0 T + 2C_\beta + \int_0^T c_1(s)ds < \infty$. Hence, the dominated convergence theorem and (5.67) show that

$$\lim_{i\to\infty} E\int_0^{\tau_k}\int_{|(\tilde{y},\tilde{z})|<1}\bigg|\beta_t^{m_i^k} f\Big(t, Y_t^{m_i^k} - \frac{1}{m_i^k}\tilde{y}, \pi_{m_i^k}\Big(Z_t^{m_i^k} - \frac{1}{m_i^k}\tilde{z}\Big), \zeta_{m_i^k}\Big(U_t^{m_i^k}\Big)\Big)$$

$$- f\big(t, Y_t, Z_t, U_t\big)\bigg|\rho(\tilde{y},\tilde{z})d\tilde{y}d\tilde{z}dt = 0,$$

which together with (5.64) leads to (5.63).

Since $\int_{(\tau_k\wedge t,\tau_k]} = \int_{(0,\tau_k]} - \int_{(0,\tau_k\wedge t]}$, $\forall\, t \in [0,T]$, the Burkholder–Davis–Gundy inequality, Hölder's inequality and an analogy to (5.1) imply that

$$E\bigg[\sup_{t\in[0,T]}\bigg|\int_{(\tau_k\wedge t,\tau_k]}\int_{\mathcal{X}}\big(U_s^{m_i^k}(x) - U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\bigg|\bigg]$$

$$\leq 2E\bigg[\sup_{t\in[0,T]}\bigg|\int_{(0,\tau_k\wedge t]}\int_{\mathcal{X}}\big(U_s^{m_i^k}(x) - U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\bigg|\bigg]$$

$$\leq c_l\, E\bigg[\Big(\int_{(0,\tau_k]}\int_{\mathcal{X}}\big|(U_s^{m_i^k}(x) - U_s(x))\big|^2 N_{\mathfrak{p}}(ds,dx)\Big)^{\frac{1}{2}}\bigg]$$

$$\leq c_l\bigg\{E\bigg[\Big(\int_{(0,T]}\int_{\mathcal{X}}\big|(U_s^{m_i^k}(x) - U_s(x))\big|^2 N_{\mathfrak{p}}(ds,dx)\Big)^{\frac{p}{2}}\bigg]\bigg\}^{\frac{1}{p}}$$

$$\leq c_l\bigg\{E\int_{(0,T]}\int_{\mathcal{X}}\big|(U_s^{m_i^k}(x) - U_s(x))\big|^p N_{\mathfrak{p}}(ds,dx)\bigg\}^{\frac{1}{p}}$$

$$= c_l\bigg\{E\int_0^T\int_{\mathcal{X}}\big|(U_s^{m_i^k}(x) - U_s(x))\big|^p \nu(dx)ds\bigg\}^{\frac{1}{p}}$$

$$= c_l\|U^{m_i^k} - U\|_{\mathbb{U}^p} \to 0, \text{ as } i \to \infty, \tag{5.70}$$

and that

$$E\bigg[\sup_{t\in[0,T]}\bigg|\int_{\tau_k\wedge t}^{\tau_k}(Z_s^{m_i^k} - Z_s)dB_s\bigg|\bigg] \leq c_l E\bigg[\Big(\int_0^{\tau_k}\big|Z_s^{m_i^k} - Z_s\big|^2 ds\Big)^{\frac{1}{2}}\bigg]$$

$$\leq c_l\,\big\|Z^{m_i^k} - Z\big\|_{\mathbb{Z}^{2,p}} \to 0, \text{ as } i \to \infty. \tag{5.71}$$

In light of (5.50), (5.63), (5.70) and (5.71), there exists a subsequence $\{\widetilde{m}_i^k\}_{i\in\mathbb{N}}$ of $\{m_i^k\}_{i\in\mathbb{N}}$ such that except on a P-null set $\mathcal{N}_1^k$

$$\lim_{i\to\infty}\bigg\{\sup_{t\in[0,T]}\Big|Y_t^{\widetilde{m}_i^k} - Y_t\Big| + \int_0^{\tau_k}\Big|f_{\widetilde{m}_i^k}\Big(t, Y_t^{\widetilde{m}_i^k}, Z_t^{\widetilde{m}_i^k}, U_t^{\widetilde{m}_i^k}\Big) - f\big(t, Y_t, Z_t, U_t\big)\Big|dt$$

$$+ \sup_{t\in[0,T]} \left| \int_{\tau_k\wedge t}^{\tau_k} \Big(Z_s^{\widetilde{m}_i^k} - Z_s \Big) dB_s \right|$$
$$+ \sup_{t\in[0,T]} \left| \int_{(\tau_k\wedge t,\tau_k]} \int_{\mathcal{X}} \Big(U_s^{\widetilde{m}_i^k}(x) - U_s(x) \Big) \widetilde{N}_{\mathfrak{p}}(ds,dx) \right| \Bigg\} = 0.$$

Since $\big(Y^{\widetilde{m}_i^k}, Z^{\widetilde{m}_i^k}, U^{\widetilde{m}_i^k}\big)$ solves BSDEJ $\big(\xi, f_{\widetilde{m}_i^k}\big)$ for any $i \in \mathbb{N}$, it holds except on a P-null set $\mathcal{N}_2^k$ that

$$Y_{\tau_k\wedge t}^{\widetilde{m}_i^k} = \mathbf{1}_{\{\tau_k<T\}} Y_{\tau_k}^{\widetilde{m}_i^k} + \mathbf{1}_{\{\tau_k=T\}}\xi + \int_{\tau_k\wedge t}^{\tau_k} f_{\widetilde{m}_i^k}\Big(s, Y_s^{\widetilde{m}_i^k}, Z_s^{\widetilde{m}_i^k}, U_s^{\widetilde{m}_i^k}\Big) ds - \int_{\tau_k\wedge t}^{\tau_k} Z_s^{\widetilde{m}_i^k} dB_s$$
$$- \int_{(\tau_k\wedge t,\tau_k]} \int_{\mathcal{X}} U_s^{\widetilde{m}_i^k}(x) \widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t \in [0,T], \quad \forall\, i \in \mathbb{N}.$$

Letting $i \to \infty$, we obtain that over $\Omega_k := \big(\mathcal{N}_1^k\big)^c \cap \big(\mathcal{N}_2^k\big)^c$

$$Y_{\tau_k\wedge t} = \mathbf{1}_{\{\tau_k<T\}} Y_{\tau_k} + \mathbf{1}_{\{\tau_k=T\}}\xi + \int_{\tau_k\wedge t}^{\tau_k} f\big(s, Y_s, Z_s, U_s\big) ds - \int_{\tau_k\wedge t}^{\tau_k} Z_s\, dB_s$$
$$- \int_{(\tau_k\wedge t,\tau_k]} \int_{\mathcal{X}} U_s(x) \widetilde{N}_{\mathfrak{p}}(ds,dx), \; t \in [0,T]. \tag{5.72}$$

By (5.56), it holds for all $\omega \in \Omega$ except on P-null set $\mathcal{N}_Z$ that $\int_0^T \mathcal{Z}_t^2(\omega)dt < \infty$, and thus $\tau_{\mathfrak{k}}(\omega) = T$ for some $\mathfrak{k} = \mathfrak{k}(\omega) \in \mathbb{N}$. Then letting $k \to \infty$ in (5.72) shows that (1.1) holds over $\Big(\cap_{k\in\mathbb{N}} \Omega_k\Big) \cap \mathcal{N}_Z^c$, which together with Remark 2.1 shows that (Y, Z, U) is a solution of BSDEJ (ξ, f). $\square$

Proof of Theorem 2.1 (Uniqueness). Suppose that $(Y, Z, U), (Y', Z', U') \in \mathbb{S}^p$ are two solutions of the BSDEJ (ξ, f). For any $n \in \mathbb{N}$, we set

$$(\xi_n, f_n) := (\xi, f) \quad \text{and} \quad (Y^n, Z^n, U^n) := \begin{cases} (Y, Z, U) & \text{if } n \text{ is odd}, \\ (Y', Z', U') & \text{if } n \text{ is even}. \end{cases}$$

By an analogy to (5.37), the inequality (3.4) holds for $\eta_n = 0$, $c(\cdot) = c_2(\cdot)$ and $\Upsilon^{m,n} \equiv 0$. Proposition 3.2 then shows that $\big\{(Y^n, Z^n, U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p$, which implies that $\|Y - Y'\|_{\mathbb{D}^p} = \|Z - Z'\|_{\mathbb{Z}^{2,p}} = \|U - U'\|_{\mathbb{U}^p} = 0$. Hence, one has that $P\{Y_t = Y_t', \ \forall\, t \in [0,T]\} = 1$, that $Z_t(\omega) = Z_t'(\omega)$ for $dt \times dP$-a.s. $(t,\omega) \in [0,T] \times \Omega$, and that $U(t,\omega,x) = U'(t,\omega,x)$ for $dt \times dP \times \nu(dx)$-a.s. $(t,\omega,x) \in [0,T] \times \Omega \times \mathcal{X}$.

(Existence)

(1) Let us first assume that $\xi \in L^\infty(\mathcal{F}_T)$ and $\int_0^T |f(t,0,0,0)|dt \in L_+^\infty(\mathcal{F}_T)$. We set

$$R := 2 + \exp\Big\{T + C_f + 4C_\beta + 2\int_0^T \big(c_1(t)\big)^2 dt + 4\big(\nu(\mathcal{X})\big)^{\frac{2-p}{p}} \int_0^T \big(c_2(t)\big)^2 dt\Big\}$$
$$\times \sqrt{\|\xi\|_{L^\infty(\mathcal{F}_T)}^2 + 5T + C_f/2 + 7C_\beta/2}, \tag{5.73}$$

and let $\psi : \mathbb{R}^l \to [0,1]$ be a smooth function such that $\psi(x) = 1$ (resp. $\psi(x) = 0$) if $|x| \le R-1$ (resp. $|x| \ge R$).

Let $n \in \mathbb{N}$. For any $u \in L_\nu^p$, we define $\pi_n(u) := \big(\frac{n}{n\vee\|u\|_{L_\nu^p}}\big) u \in L_\nu^p$. An application of Lemma A.5 with $(\mathbb{E}, \|\cdot\|) = \big(L_\nu^p, \|\cdot\|_{L_\nu^p}\big)$ shows that $\big\|\pi_n(u_1) - \pi_n(u_2)\big\|_{L_\nu^p} \le 2\|u_1 - u_2\|_{L_\nu^p}$,

$\forall\, u_1, u_2 \in L^p_\nu$, which together with (A.3) and the $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^l) \otimes \mathscr{B}(\mathbb{R}^{l\times d}) \otimes \mathscr{B}(L^p_\nu)/\mathscr{B}(\mathbb{R}^l)$-measurability of f shows that

$$f_n(t,\omega,y,z,u) := \frac{n}{n \vee \phi^R(t,\omega)} \psi(y)\big(f(t,\omega,y,\pi_n(z),\pi_n(u)) - f(t,\omega,0,0,0)\big) + f(t,\omega,0,0,0),$$

$(t,\omega,y,z,u) \in [0,T] \times \Omega \times \mathbb{R}^l \times \mathbb{R}^{l\times d} \times L^p_\nu$ defines a $\mathbb{R}^l$-valued, $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}^l) \otimes \mathscr{B}(\mathbb{R}^{l\times d}) \otimes \mathscr{B}(L^p_\nu)/\mathscr{B}(\mathbb{R}^l)$-measurable mapping satisfying (H1), (H3)–(H5) with the same coefficients as f except for $c_2^n(\cdot) = 2c_2(\cdot)$. By (H2), it holds for $dt \times dP$-a.s. $(t,\omega) \in [0,T] \times \Omega$ that

$$\begin{aligned}|f_n(t,\omega,y,0,0) - f_n(t,\omega,0,0,0)| &= \frac{n}{n \vee \phi^R_t} \psi(y)\big|f(t,\omega,y,0,0) - f(t,\omega,0,0,0)\big| \\ &\le \frac{n}{n \vee \phi^R_t} \psi(y)\phi^R_t \le n, \quad \forall\, y \in \mathbb{R}^l,\end{aligned}$$

so f_n satisfies (H2$'$) with $\kappa_0 = n$.

Also, let (H2) and (H4)–(H6) hold for f except on a $dt \times dP$-null set $\mathfrak{N}$ and let $(t,\omega) \in \mathfrak{N}^c$. Given $(y_1,z_1), (y_2,z_2) \in \mathbb{R}^l \times \mathbb{R}^{l\times d}$ and $u \in L^p_\nu$, if $|y_1| \wedge |y_2| \ge R$, then we automatically have $f_n(t,\omega,y_1,z_1,u) - f_n(t,\omega,y_2,z_2,u) = 0$ and thus $|y_1 - y_2|^{p-1}\big\langle \mathscr{D}(y_1-y_2), f_n(t,\omega,y_1,z_1,u) - f_n(t,\omega,y_2,z_2,u)\big\rangle = 0$; on the other hand, let us assume without loss of generality that $|y_1| < R$, then (H2), (H4)–(H6) and (A.3) imply that

$$\begin{aligned}&|y_1 - y_2|^{p-1}\big\langle \mathscr{D}(y_1-y_2), f_n(t,\omega,y_1,z_1,u) - f_n(t,\omega,y_2,z_2,u)\big\rangle \\ &= \frac{n}{n \vee \phi^R_t(\omega)}\big(\psi(y_1) - \psi(y_2)\big)|y_1-y_2|^{p-1}\big\langle \mathscr{D}(y_1-y_2), f(t,\omega,y_1,\pi_n(z_1),\pi_n(u)) \\ &\quad - f(t,\omega,0,0,0)\big\rangle + \frac{n}{n \vee \phi^R_t(\omega)}\psi(y_2)|y_1-y_2|^{p-1} \\ &\quad \times \big\langle \mathscr{D}(y_1-y_2), f\big(t,\omega,y_1,\pi_n(z_1),\pi_n(u)\big) - f(t,\omega,y_2,\pi_n(z_2),\pi_n(u)\big)\big\rangle \\ &\le \frac{n}{n \vee \phi^R_t(\omega)} C_\psi |y_1-y_2|^p\big(\phi^R_t(\omega) + \beta_t(\omega) + c_1(t)|\pi_n(z_1)| + c_2(t)\|\pi_n(u)\|_{L^p_\nu}\big) \\ &\quad + \frac{n}{n \vee \phi^R_t(\omega)}\psi(y_2)\Big[\lambda(t)\,\theta\big(|y_1-y_2|^p\big) + \Phi_t(\omega)|y_1-y_2|^p \\ &\quad + \Lambda_t(\omega)|y_1-y_2|^{p-1}\big|\pi_n(z_1) - \pi_n(z_2)\big|\Big] \\ &\le \lambda(t)\,\theta\big(|y_1-y_2|^p\big) + \big[\Phi_t(\omega) + C_\psi\beta_t(\omega) + nC_\psi(1 + c_1(t) + c_2(t))\big]|y_1-y_2|^p \\ &\quad + \Lambda_t(\omega)|y_1-y_2|^{p-1}|z_1 - z_2|,\end{aligned}$$

where C_ψ denotes the Lipschitz coefficient of the smooth function ψ. Hence, f_n satisfies (H6) with the same coefficients as f except for $\Phi^n_t = \Phi_t + C_\psi\beta_t + nC_\psi(1 + c_1(t) + c_2(t))$, $t \in [0,T]$. Clearly, $\int_0^T \Phi^n_t dt \in L^\infty_+(\mathcal{F}_T)$.

Since f_n satisfies (H3)–(H5) with the same coefficients as f except for $c_2^n(\cdot) = 2c_2(\cdot)$ and since $\int_0^T \big|f_n(t,0,0,0)\big|dt = \int_0^T \big|f(t,0,0,0)\big|dt \in L^\infty_+(\mathcal{F}_T)$, the constant R in (5.73) is exactly that for f_n in (5.29). According to the proof of Proposition 3.3 (in particular, (5.53)), the BSDEJ (ξ, f_n) has a solution $(Y^n, Z^n, U^n) \in \mathbb{D}^\infty \times \mathbb{Z}^{2,p} \times \mathbb{U}^p$ such that

$$\|Y^n\|_{\mathbb{D}^\infty} \le R - 2. \tag{5.74}$$

We also see from Proposition 3.1 that

$$\|Y^n\|^p_{\mathbb{D}^p}+\|Z^n\|^p_{\mathbb{Z}^{2,p}}+\|U^n\|^p_{\mathbb{U}^p}$$
$$\le \mathcal{C}\bigg(1+\|\xi\|^p_{L^\infty(\mathcal{F}_T)}+\Big\|\int_0^T|f(t,0,0,0)|dt\Big\|^p_{L^\infty_+(\mathcal{F}_T)}\bigg):=\widehat{\mathcal{C}}, \tag{5.75}$$

where $\mathcal{C}$ is a constant depending on T, $\nu(\mathcal{X})$, p, $\overline{C}$ and C_β.

Set $\varpi := p(1-\frac{1}{q'}) > p(1-\frac{1}{q}) = 1$ and let $m, n \in \mathbb{N}$ with $m > n$. Since $\psi(x) \equiv 1$ for all $|x| \le R-1$ and since an analogy to (5.43) shows that $\big\|\pi_m\big(U^m_t\big)-\pi_n\big(U^n_t\big)\big\|_{L^p_\nu} \le 2\big\|U^m_t-U^n_t\big\|_{L^p_\nu}+\mathbf{1}_{\{\|U^n_t\|_{L^p_\nu}>n\}}\big\|U^n_t\big\|_{L^p_\nu} \le 2\big\|U^m_t-U^n_t\big\|_{L^p_\nu}+n^{1-\varpi}\big\|U^n_t\big\|^{\varpi}_{L^p_\nu}$, $t\in[0,T]$, we can deduce from (5.74), (H2) and (H4)–(H6) that $dt\times dP$-a.s.

$$\begin{aligned}
&\big|Y^m_t-Y^n_t\big|^{p-1}\big\langle\mathscr{D}\big(Y^m_t-Y^n_t\big), f_m\big(t,Y^m_t,Z^m_t,U^m_t\big)-f_n\big(t,Y^n_t,Z^n_t,U^n_t\big)\big\rangle\\
&=\frac{m}{m\vee\phi^R_t}\big|Y^m_t-Y^n_t\big|^{p-1}\big\langle\mathscr{D}\big(Y^m_t-Y^n_t\big), f\big(t,Y^m_t,\pi_m(Z^m_t),\pi_m(U^m_t)\big)\\
&\quad-f\big(t,Y^n_t,\pi_n(Z^n_t),\pi_n(U^n_t)\big)\big\rangle+\Big(\frac{m}{m\vee\phi^R_t}-\frac{n}{n\vee\phi^R_t}\Big)\big|Y^m_t-Y^n_t\big|^{p-1}\\
&\quad\times\big\langle\mathscr{D}\big(Y^m_t-Y^n_t\big), f\big(t,Y^n_t,\pi_n(Z^n_t),\pi_n(U^n_t)\big)-f(t,0,0,0)\big\rangle\\
&\le\lambda(t)\,\theta\big(|Y^m_t-Y^n_t|^p\big)+\Phi_t|Y^m_t-Y^n_t|^p\\
&\quad+|Y^m_t-Y^n_t|^{p-1}\Big[\Lambda_t\big|\pi_m(Z^m_t)-\pi_n(Z^n_t)\big|+c_2(t)\big\|\pi_m(U^m_t)-\pi_n(U^n_t)\big\|_{L^p_\nu}\Big]\\
&\quad+\Big(1-\frac{n}{n\vee\phi^R_t}\Big)(2R-4)^{p-1}\Big(\phi^R_t+\beta_t+c_1(t)|\pi_n(Z^n_t)|+c_2(t)\|\pi_n(U^n_t)\|_{L^p_\nu}\Big)\\
&\le\lambda(t)\,\theta\big(|Y^m_t-Y^n_t|^p\big)+\Phi_t|Y^m_t-Y^n_t|^p\\
&\quad+|Y^m_t-Y^n_t|^{p-1}\Big[\Lambda_t\big|Z^m_t-Z^n_t\big|+2c_2(t)\big\|U^m_t-U^n_t\big\|_{L^p_\nu}\Big]+\Upsilon^{m,n}_t,
\end{aligned}$$

where $\Upsilon^{m,n}_t := \big(1-\frac{n}{n\vee\phi^R_t}\big)(2R-4)^{p-1}\big(\phi^R_t+\beta_t+c_1(t)|Z^n_t|+c_2(t)\|U^n_t\|_{L^p_\nu}\big)+(2R-4)^{p-1}c_2(t)n^{1-\varpi}\big\|U^n_t\big\|^{\varpi}_{L^p_\nu}$. Thus, (3.4) holds for $\eta_n=0$, $c(\cdot)=2c_2(\cdot)$ and the above process $\Upsilon^{m,n}$. By Hölder's inequality and (5.75),

$$\begin{aligned}
&(2R-4)^{1-p}\,E\int_0^T\Upsilon^{m,n}_t dt\le E\int_0^T\Big(1-\frac{n}{n\vee\phi^R_t}\Big)(\phi^R_t+\beta_t)dt\\
&\quad+\bigg\{E\bigg[\bigg(\int_0^T\Big(1-\frac{n}{n\vee\phi^R_t}\Big)^2c_1^2(t)dt\bigg)^{\frac{q}{2}}\bigg]\bigg\}^{\frac{1}{q}}\|Z^n\|_{\mathbb{Z}^{2,p}}\\
&\quad+\bigg\{E\int_0^T\Big(1-\frac{n}{n\vee\phi^R_t}\Big)^q c_2^q(t)dt\bigg\}^{\frac{1}{q}}\|U^n\|_{\mathbb{U}^p}\\
&\quad+n^{1-\varpi}\bigg(\int_0^T\big(c_2(t)\big)^{q'}dt\bigg)^{\frac{1}{q'}}\bigg(E\int_0^T\|U^n_t\|^p_{L^p_\nu}dt\bigg)^{1-\frac{1}{q'}}
\end{aligned}$$

$$\leq E\int_0^T\Big(1-\frac{n}{n\vee\phi_t^R}\Big)(\phi_t^R+\beta_t)dt+\widehat{\mathcal{C}}^{\frac{1}{p}}\bigg\{E\bigg[\Big(\int_0^T\Big(1-\frac{n}{n\vee\phi_t^R}\Big)^2c_1^2(t)dt\Big)^{\frac{q}{2}}\bigg]\bigg\}^{\frac{1}{q}}$$
$$+\widehat{\mathcal{C}}^{\frac{1}{p}}\bigg\{E\int_0^T\Big(1-\frac{n}{n\vee\phi_t^R}\Big)^q c_2^q(t)dt\bigg\}^{\frac{1}{q}}+\widehat{\mathcal{C}}^{1-\frac{1}{q'}}n^{1-\varpi}\Big(\int_0^T\big(c_2(t)\big)^{q'}dt\Big)^{\frac{1}{q'}}:=I_n.$$

Because $\frac{n}{n\vee\phi_t^R}=\frac{1}{1\vee(\phi_t^R/n)}\nearrow 1$ as $n\to\infty$, $\forall\, t\in[0,T]$, the dominated convergence theorem shows that $\lim_{n\to\infty}I_n=0$. It follows that $\lim_{n\to\infty}\sup_{m>n}E\int_0^T\Upsilon_t^{m,n}dt=0$. Since $\sup_{n\in\mathbb{N}}\Big(\|Y^n\|^p_{\mathbb{D}^p}+\|Z^n\|^p_{\mathbb{Z}^{2,p}}+\|U^n\|^p_{\mathbb{U}^p}\Big)\leq\widehat{\mathcal{C}}$ by (5.75), we see from Proposition 3.2 that $\big\{(Y^n,Z^n,U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p$. Let (Y,Z,U) be its limit. As in the proof of Proposition 3.3, one can extract a subsequence $\{m_i\}_{i\in\mathbb{N}}$ from $\mathbb{N}$ such that (5.50)–(5.52) hold, then we still have (5.53). Also, similar to (5.55) and (5.58), we can define two $[0,\infty)$-valued, **F**-predictable processes $\mathcal{Z}$ and $\mathcal{U}$ that satisfy (5.56) and (5.59) respectively.

Fix $k\in\mathbb{N}$ and define the **F**-stopping time τ_k as in (5.60). We can still derive (5.61) from the dominated convergence theorem and (5.50). Hence, there exists a subsequence $\{m_i^k\}_{i\in\mathbb{N}}$ of $\{m_i\}_{i\in\mathbb{N}}$ such that (5.62) holds $dt\times dP$-a.s. For any $(t,\omega)\in[0,T]\times\Omega$ except on a $dt\times dP$-null set $\mathfrak{N}_k$, we may assume that (H2), (H4), (H5), (5.50), (5.51), (5.62) hold, that $|Y_t(\omega)|\leq R-2$, $|Y_t^{m_i^k}(\omega)|\leq R-2$, $\forall\, i\in\mathbb{N}$ (by (5.74), (5.53)), and that $U_t(\omega)\in L^p_\nu$, $U_t^{m_i^k}(\omega)\in L^p_\nu$, $\forall\, i\in\mathbb{N}$.

Let $(t,\omega)\in\mathfrak{N}_k^c\cap[\![0,\tau_k]\!]$. Since $\lim_{i\to\infty}\uparrow\frac{m_i^k}{m_i^k\vee\phi_t^R}=1$ and since $\psi\big(Y_t^{m_i^k}(\omega)\big)=1$,

$$\lim_{i\to\infty}f_{m_i^k}\Big(t,\omega,Y_t^{m_i^k}(\omega),Z_t^{m_i^k}(\omega),U_t^{m_i^k}(\omega)\Big)$$
$$=\lim_{i\to\infty}f\Big(t,\omega,Y_t^{m_i^k}(\omega),\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big),\pi_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big).\tag{5.76}$$

Using (H5), (5.51) and an analogy to the inequality (e3) in Part 5 of Proposition 3.3's proof, we obtain

$$\bigg|f\Big(t,\omega,Y_t^{m_i^k}(\omega),\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big),\pi_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)$$
$$-f\Big(t,\omega,Y_t^{m_i^k}(\omega),\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big),U_t(\omega)\Big)\bigg|$$
$$\leq c_2(t)\Big\|\pi_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)-U_t(\omega)\Big\|_{L^p_\nu}$$
$$\leq c_2(t)\Big(2\Big\|U_t^{m_i^k}(\omega)-U_t(\omega)\Big\|_{L^p_\nu}+\big\|\pi_{m_i^k}\big(U_t(\omega)\big)-U_t(\omega)\big\|_{L^p_\nu}\Big)$$
$$\to 0,\ \text{as } i\to\infty,\tag{5.77}$$

Also, similar to the inequality (e3) in Part 5 of Proposition 3.3's proof, one can deduce from (A.3) and (5.65) that $\Big|\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big)-Z_t(\omega)\Big|\leq\Big|Z_t^{m_i^k}(\omega)-Z_t(\omega)\Big|+\Big|\pi_{m_i^k}\big(Z_t(\omega)\big)-Z_t(\omega)\Big|\to 0$ as $i\to\infty$, which together with (5.50) and the continuity of the mapping $f\big(t,\omega,\cdot,\cdot,U_t(\omega)\big)$, shows that

$$\lim_{i\to\infty}f\Big(t,\omega,Y_t^{m_i^k}(\omega),\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big),U_t(\omega)\Big)=f\big(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega)\big).\tag{5.78}$$

Combining (5.76)–(5.78) leads to that

$$\lim_{i\to\infty}\Big|f_{m_i^k}\Big(t,\omega,Y_t^{m_i^k}(\omega),Z_t^{m_i^k}(\omega),U_t^{m_i^k}(\omega)\Big)-f\big(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega)\big)\Big|=0. \quad (5.79)$$

Given $i\in\mathbb{N}$, since $\psi\big(Y_t^{m_i^k}(\omega)\big)=1$, one can deduce from (H2), (H4), (H5) and an analogy to (5.68) that

$$\begin{aligned}
&\Big|f_{m_i^k}\Big(t,\omega,Y_t^{m_i^k}(\omega),Z_t^{m_i^k}(\omega),U_t^{m_i^k}(\omega)\Big)-f\big(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega)\big)\Big|\\
&=\bigg|\frac{m_i^k}{m_i^k\vee\phi_t^R(\omega)}\Big(f\Big(t,\omega,Y_t^{m_i^k}(\omega),\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big),\pi_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)\\
&\qquad -f(t,\omega,0,0,0)\Big)+f(t,\omega,0,0,0)-f\big(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega)\big)\bigg|\\
&\le\Big|f\Big(t,\omega,Y_t^{m_i^k}(\omega),\pi_{m_i^k}\Big(Z_t^{m_i^k}(\omega)\Big),\pi_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big)-f(t,\omega,0,0,0)\Big|\\
&\qquad+\big|f\big(t,\omega,Y_t(\omega),Z_t(\omega),U_t(\omega)\big)-f(t,\omega,0,0,0)\big|\\
&\le 2\phi_t^R(\omega)+2\beta_t(\omega)+c_1(t)\Big(\Big|Z_t^{m_i^k}(\omega)\Big|+|Z_t(\omega)|\Big)\\
&\qquad+c_2(t)\Big(\Big\|\pi_{m_i^k}\Big(U_t^{m_i^k}(\omega)\Big)\Big\|_{L_\nu^p}+\|U_t(\omega)\|_{L_\nu^p}\Big)\\
&\le 2\phi_t^R(\omega)+2\beta_t(\omega)+c_1(t)\big(\mathcal{Z}_t(\omega)+|Z_t(\omega)|\big)+c_2(t)\big(\mathcal{U}_t(\omega)+\|U_t(\omega)\|_{L_\nu^p}\big)\\
&:=H_t(\omega).
\end{aligned}$$

Analogous to (5.69), we can deduce from Hölder's inequality, (5.56) and (5.59) that

$$\begin{aligned}
E\int_0^T H_t dt &\le 2E\int_0^T\phi_t^R dt+2C_\beta+\overline{C}^{\frac12}\bigg\{\bigg(E\bigg[\Big(\int_0^T\mathcal{Z}_t^2dt\Big)^{\frac p2}\bigg]\bigg)^{\frac1p}+\|Z\|_{\mathbb{Z}^{2,p}}\bigg\}\\
&\quad+\overline{C}^{\frac1q}\bigg\{\Big(E\int_0^T\mathcal{U}_t^p dt\Big)^{\frac1p}+\|U\|_{\mathbb{U}^p}\bigg\}<\infty.
\end{aligned}$$

The dominated convergence theorem and (5.79) yield that

$$\lim_{i\to\infty}E\int_0^{\tau_k}\Big|f_{m_i^k}\Big(t,Y_t^{m_i^k},Z_t^{m_i^k},U_t^{m_i^k}\Big)-f(t,Y_t,Z_t,U_t)\Big|dt=0.$$

Then following similar arguments to Part 5 in the proof of Proposition 3.3, one can show that (Y,Z,U) is a solution of BSDEJ (ξ,f).

(2) Next, let us consider the general case that $\xi\in L^p(\mathcal{F}_T)$ and $\int_0^T|f(t,0,0,0)|dt\in L_+^p(\mathcal{F}_T)$. For any $n\in\mathbb{N}$, we set $\xi_n:=\pi_n(\xi)$ and define

$$\begin{aligned}
\widetilde{f}_n(t,\omega,y,z,u)&:=f(t,\omega,y,z,u)-f(t,\omega,0,0,0)+\pi_n\big(f(t,\omega,0,0,0)\big),\\
&\qquad (t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}^l\times\mathbb{R}^{l\times d}\times L_\nu^p.
\end{aligned}$$

Clearly, $\widetilde{f}_n$ is an $\mathbb{R}^l$-valued, $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^l)\otimes\mathscr{B}\big(\mathbb{R}^{l\times d}\big)\otimes\mathscr{B}\big(L_\nu^p\big)/\mathscr{B}(\mathbb{R}^l)$-measurable mapping satisfying (H1)–(H6) with the same coefficients as f. As $\int_0^T|\widetilde{f}_n(t,0,0,0)|dt=\int_0^T\big|\pi_n(f(t,0,0,0))\big|dt\le nT$, Part 1 shows that the BSDEJ $\big(\xi_n,\widetilde{f}_n\big)$ has a solution $\big(Y^n,Z^n,U^n\big)\in\mathbb{D}^\infty\times\mathbb{Z}^{2,p}\times\mathbb{U}^p$ (For easy reference, we still denote this solution by

(Y^n, Z^n, U^n). Note its difference from the triple considered in Part 1). Also, we know from Proposition 3.1 that

$$\|Y^n\|_{\mathbb{D}^p}^p + \|Z^n\|_{\mathbb{Z}^{2,p}}^p + \|U^n\|_{\mathbb{U}^p}^p \le \mathcal{C}E\bigg[1 + |\xi_n|^p + \Big(\int_0^T \big|\pi_n(f(t,0,0,0))\big|dt\Big)^p\bigg]$$
$$\le \mathcal{C}E\bigg[1 + |\xi|^p + \Big(\int_0^T \big|f(t,0,0,0)\big|dt\Big)^p\bigg] := \widetilde{\mathcal{C}}, \tag{5.80}$$

where $\mathcal{C}$ is a constant depending on T, $\nu(\mathcal{X})$, p, $\overline{C}$ and C_β.

Given $m, n \in \mathbb{N}$ with $m > n$, an analogy to (5.37) shows that (3.4) holds for $f_n = \widetilde{f}_n$, $\eta_n = 0$, $c(\cdot) = c_2(\cdot)$ and

$$\Upsilon_t^{m,n} = \big|Y_t^m - Y_t^n\big|^{p-1}\big|\pi_m\big(f(t,0,0,0)\big) - \pi_n\big(f(t,0,0,0)\big)\big|, \quad \forall\, t \in [0,T].$$

By Hölder's inequality and (5.80),

$$E\int_0^T \Upsilon_t^{m,n}dt \le E\bigg[\sup_{t\in[0,T]}\big|Y_t^m - Y_t^n\big|^{p-1}\int_0^T \big|f(t,0,0,0) - \pi_n\big(f(t,0,0,0)\big)\big|dt\bigg]$$
$$\le \big\|Y^m - Y^n\big\|_{\mathbb{D}^p}^{\frac{p}{q}}\bigg\{E\bigg[\Big(\int_0^T \big|f(t,0,0,0) - \pi_n\big(f(t,0,0,0)\big)\big|dt\Big)^p\bigg]\bigg\}^{\frac{1}{p}}$$
$$\le 2^{\frac{p}{q}}\widetilde{\mathcal{C}}^{\frac{1}{q}}\bigg\{E\bigg[\Big(\int_0^T \big|f(t,0,0,0) - \pi_n\big(f(t,0,0,0)\big)\big|dt\Big)^p\bigg]\bigg\}^{\frac{1}{p}} := \widetilde{I}_n.$$

As $E\Big[\big(\int_0^T |f(t,0,0,0)|dt\big)^p\Big] < \infty$, the dominated convergence theorem implies that $\lim_{n\to\infty}\widetilde{I}_n = 0$. It follows that $\lim_{n\to\infty}\sup_{m>n} E\int_0^T \Upsilon_t^{m,n}dt = 0$. Since $\sup_{n\in\mathbb{N}}\Big(\big\|Y^n\big\|_{\mathbb{D}^p}^p + \big\|Z^n\big\|_{\mathbb{Z}^{2,p}}^p + \big\|U^n\big\|_{\mathbb{U}^p}^p\Big) \le \widetilde{\mathcal{C}}$ by (5.80), we see from Proposition 3.2 that $\big\{(Y^n, Z^n, U^n)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{S}^p$. Let (Y, Z, U) be its limit. As in the proof of Proposition 3.3, one can extract a subsequence $\{m_i\}_{i\in\mathbb{N}}$ from $\mathbb{N}$ such that (5.50)–(5.52) hold.

For any $i \in \mathbb{N}$, we define an $\mathcal{F}_T$-measurable random variable $\mathfrak{y}_i := Y_* + \sum_{j=1}^{i}\big(Y^{m_j} - Y^{m_{j-1}}\big)_*$ with $Y^{n_0} := Y$. Minkowski's inequality and (5.52) show that

$$\|\mathfrak{y}_i\|_{L_+^p(\mathcal{F}_T)} \le \big\|Y\big\|_{\mathbb{D}^p} + \sum_{j=1}^{i}\big\|Y^{m_j} - Y^{m_{j-1}}\big\|_{\mathbb{D}^p} \le 1 + \big\|Y\big\|_{\mathbb{D}^p} + \big\|Y^{m_1} - Y\big\|_{\mathbb{D}^p}. \tag{5.81}$$

Since $\big\{\mathfrak{y}_i\big\}_{i\in\mathbb{N}}$ is an increasing sequence, $\mathfrak{y} := \lim_{i\to\infty}\uparrow \mathfrak{y}_i = Y_* + \sum_{j=1}^{\infty}\big(Y^{m_j} - Y^{m_{j-1}}\big)_*$ defines a $[0,\infty]$-valued, $\mathcal{F}_T$-measurable random variable. Then the monotone convergence theorem and (5.81) imply that

$$\big\|\mathfrak{y}\big\|_{L_+^p(\mathcal{F}_T)} = \lim_{i\to\infty}\uparrow \big\|\mathfrak{y}_i\big\|_{L_+^p(\mathcal{F}_T)} \le 1 + \big\|Y\big\|_{\mathbb{D}^p} + \big\|Y^{n_1} - Y\big\|_{\mathbb{D}^p} < \infty. \tag{5.82}$$

Moreover, as in (5.55) and (5.58), we can define two $[0,\infty)$-valued, $\mathbf{F}$-predictable processes $\mathcal{Z}$ and $\mathcal{U}$ that satisfy (5.56) and (5.59) respectively.

Fix $k \in \mathbb{N}$ and define the $\mathbf{F}$-stopping time τ_k as in (5.60). One can again derive (5.61) from the dominated convergence theorem and (5.50). Hence, there exists a subsequence $\big\{m_i^k\big\}_{i\in\mathbb{N}}$ of $\{m_i\}_{i\in\mathbb{N}}$ such that (5.62) holds $dt \times dP$-a.s. For any $(t,\omega) \in [0,T] \times \Omega$except on a $dt \times dP$-null set $\widetilde{\mathfrak{N}}_k$, we may assume that (H2), (H4), (H5), (5.50), (5.51), (5.62) hold and that $U_t(\omega) \in L^p_\nu$, $U^{m^k_i}_t(\omega) \in L^p_\nu$, $\forall\, i \in \mathbb{N}$.

Let us also fix $\ell \in \mathbb{N}$ and define $A_\ell := \{\mathfrak{y} \vee Y_* \le \ell\} \in \mathcal{F}_T$.

Let $(t,\omega) \in \widetilde{\mathfrak{N}}^c_k \cap [\![0, \tau_k]\!]$. The continuity of the mapping $f\big(t,\omega,\cdot,\cdot,U_t(\omega)\big)$, (5.50) and (5.65) yield that

$$\lim_{i\to\infty} \widetilde{f}_{m^k_i}\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U_t(\omega)\Big) = \lim_{i\to\infty} f\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U_t(\omega)\Big)$$
$$= f\big(t,\omega,Y_t(\omega), Z_t(\omega), U_t(\omega)\big). \tag{5.83}$$

By (H5), it holds for any $i \in \mathbb{N}$ that

$$\begin{aligned}
&\Big|\widetilde{f}_{m^k_i}\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U^{m^k_i}_t(\omega)\Big) - \widetilde{f}_{m^k_i}\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U_t(\omega)\Big)\Big| \\
&= \Big|f\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U^{m^k_i}_t(\omega)\Big) - f\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U_t(\omega)\Big)\Big| \\
&\le c_2(t)\Big\|U^{m^k_i}_t(\omega) - U_t(\omega)\Big\|_{L^p_\nu},
\end{aligned}$$

which together with (5.51) and (5.83) shows that

$$\lim_{i\to\infty} \Big|\widetilde{f}_{m^k_i}\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U^{m^k_i}_t(\omega)\Big) - f\big(t,\omega,Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big| = 0. \tag{5.84}$$

Given $i \in \mathbb{N}$, there exists an $j = j(k,i) \in \mathbb{N}$ such that $m^k_i = m_j$. Since $Y^{m^k_i}_*(\omega) \le Y_*(\omega) + \sum_{j=1}^i \big(Y^{m^k_j} - Y^{m^k_{j-1}}\big)_*(\omega) = \mathfrak{y}_j(\omega) \le \mathfrak{y}(\omega)$, one can deduce from (H2), (H4), (H5) and an analogy to (5.68) that

$$\begin{aligned}
&\mathbf{1}_{A_\ell}\Big|\widetilde{f}_{m^k_i}\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U^{m^k_i}_t(\omega)\Big) - f\big(t,\omega,Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big| \\
&\le \big|f(t,\omega,0,0,0) - \pi_n(f(t,\omega,0,0,0))\big| + \mathbf{1}_{A_\ell}\Big|f\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U^{m^k_i}_t(\omega)\Big) \\
&\quad - f\big(t,Y_t(\omega), Z_t(\omega), U_t(\omega)\big)\Big| \\
&\le \big|f(t,\omega,0,0,0)\big| + \mathbf{1}_{A_\ell}\Big\{\Big|f\Big(t,\omega,Y^{m^k_i}_t(\omega), Z^{m^k_i}_t(\omega), U^{m^k_i}_t(\omega)\Big) \\
&\quad - f(t,\omega,0,0,0)\Big| + \big|f\big(t,Y_t(\omega), Z_t(\omega), U_t(\omega)\big) - f(t,\omega,0,0,0)\big|\Big\} \\
&\le \big|f(t,\omega,0,0,0)\big| + 2\phi^\ell_t(\omega) + 2\beta_t(\omega) + c_1(t)\Big(\Big|Z^{m^k_i}_t(\omega)\Big| + |Z_t(\omega)|\Big) \\
&\quad + c_2(t)\Big(\Big\|U^{m^k_i}_t(\omega)\Big\|_{L^p_\nu} + \|U_t(\omega)\|_{L^p_\nu}\Big) \\
&\le \big|f(t,\omega,0,0,0)\big| + 2\phi^\ell_t(\omega) + 2\beta_t(\omega) + c_1(t)\big(\mathcal{Z}_t(\omega) + |Z_t(\omega)|\big) \\
&\quad + c_2(t)\big(\mathcal{U}_t(\omega) + \|U_t(\omega)\|_{L^p_\nu}\big) := H^\ell_t(\omega).
\end{aligned}$$

Similar to (5.69), we can deduce from Hölder's inequality, (5.56) and (5.59) that

$$E\int_0^T H^\ell_t dt \le E\int_0^T \big(|f(t,0,0,0)| + 2\phi^\ell_t\big)dt + 2C_\beta$$

$$+\overline{C}^{\frac{1}{2}}\left\{\left(E\left[\left(\int_0^T \mathcal{Z}_t^2 dt\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} + \|Z\|_{\mathbb{Z}^{2,p}}\right\}$$
$$+\overline{C}^{\frac{1}{q}}\left\{\left(E\int_0^T \mathcal{U}_t^p dt\right)^{\frac{1}{p}} + \|U\|_{\mathbb{U}^p}\right\} < \infty.$$

Then the dominated convergence theorem and (5.84) yield that

$$\lim_{i\to\infty} E\int_0^{\tau_k} \mathbf{1}_{A_\ell}\left|\widetilde{f}_{m_i^k}\left(t, Y_t^{m_i^k}, Z_t^{m_i^k}, U_t^{m_i^k}\right) - f(t, Y_t, Z_t, U_t)\right| dt = 0. \tag{5.85}$$

In light of (5.50), (5.85), (5.70) and (5.71), there exists a subsequence $\{m_i^{k,\ell}\}_{i\in\mathbb{N}}$ of $\{m_i^k\}_{i\in\mathbb{N}}$ such that except on a P-null set $\mathcal{N}_1^{k,\ell}$

$$\begin{aligned}\lim_{i\to\infty}\Bigg\{ &\sup_{t\in[0,T]}\left|Y_t^{m_i^{k,\ell}} - Y_t\right| \\ &+\mathbf{1}_{A_\ell}\int_0^{\tau_k}\left|\widetilde{f}_{m_i^{k,\ell}}\left(t, Y_t^{m_i^{k,\ell}}, Z_t^{m_i^{k,\ell}}, U_t^{m_i^{k,\ell}}\right) - f\left(t, Y_t, Z_t, U_t\right)\right| dt \\ &+ \sup_{t\in[0,T]}\left|\int_{\tau_k\wedge t}^{\tau_k}\left(Z_s^{m_i^{k,\ell}} - Z_s\right) dB_s\right| \\ &+ \sup_{t\in[0,T]}\left|\int_{(\tau_k\wedge t,\tau_k]}\int_{\mathcal{X}}\left(U_s^{m_i^{k,\ell}}(x) - U_s(x)\right)\widetilde{N}_{\mathfrak{p}}(ds,dx)\right|\Bigg\} = 0.\end{aligned}$$

Since $\left(Y^{m_i^{k,\ell}}, Z^{m_i^{k,\ell}}, U^{m_i^{k,\ell}}\right)$ solves BSDEJ $\left(\xi_{m_i^{k,\ell}}, \widetilde{f}_{m_i^{k,\ell}}\right)$ for any $i\in\mathbb{N}$, it holds except on a P-null set $\mathcal{N}_2^{k,\ell}$ that

$$\begin{aligned}Y_{\tau_k\wedge t}^{m_i^{k,\ell}} &= \mathbf{1}_{\{\tau_k<T\}}Y_{\tau_k}^{m_i^{k,\ell}} + \mathbf{1}_{\{\tau_k=T\}}\pi_{m_i^{k,\ell}}(\xi) \\ &\quad+\int_{\tau_k\wedge t}^{\tau_k}\widetilde{f}_{m_i^{k,\ell}}\left(s, Y_s^{m_i^{k,\ell}}, Z_s^{m_i^{k,\ell}}, U_s^{m_i^{k,\ell}}\right) ds - \int_{\tau_k\wedge t}^{\tau_k} Z_s^{m_i^{k,\ell}} dB_s \\ &\quad-\int_{(\tau_k\wedge t,\tau_k]}\int_{\mathcal{X}} U_s^{m_i^{k,\ell}}(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \qquad \forall\, t\in[0,T], \qquad \forall\, i\in\mathbb{N}.\end{aligned} \tag{5.86}$$

Set $\widetilde{A}_\ell^k := \left(\mathcal{N}_1^{k,\ell}\right)^c \cap \left(\mathcal{N}_2^{k,\ell}\right)^c \cap A_\ell$, which includes the set $\left(\cup_{\ell\in\mathbb{N}}\mathcal{N}_1^{k,\ell}\right)^c \cap \left(\cup_{\ell\in\mathbb{N}}\mathcal{N}_2^{k,\ell}\right)^c \cap A_\ell$. For any $\omega\in\widetilde{A}_\ell^k$, letting $i\to\infty$ in (5.86), we obtain (5.72) over $\widetilde{A}_\ell^k$. As ℓ varies over $\mathbb{N}$, (5.72) further holds over $\Omega_k := \left(\cup_{\ell\in\mathbb{N}}\mathcal{N}_1^{k,\ell}\right)^c \cap \left(\cup_{\ell\in\mathbb{N}}\mathcal{N}_2^{k,\ell}\right)^c \cap \left(\cup_{\ell\in\mathbb{N}} A_\ell\right)$. By (5.82) and $Y\in\mathbb{D}^p$, one has $P\left(\Omega_k\right) = P\left(\cup_{\ell\in\mathbb{N}} A_\ell\right) = 1$.

We see from (5.56) that for all $\omega\in\Omega$ except on P-null set $\mathcal{N}_Z$, $\int_0^T \mathcal{Z}_t^2(\omega)dt < \infty$ and thus $\tau_{\mathfrak{k}}(\omega) = T$ for some $\mathfrak{k} = \mathfrak{k}(\omega)\in\mathbb{N}$. Then letting $k\to\infty$ in (5.72) shows that (1.1) holds over $\left(\cap_{k\in\mathbb{N}}\Omega_k\right)\cap\mathcal{N}_Z^c$, which together with Remark 2.1 shows that (Y, Z, U) is a solution of BSDEJ (ξ, f). □

Proof of Corollary 2.1. Clearly, $f(t,\omega,y,z,u) := 0$, $\forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L_\nu^2$ satisfies (H1)–(H6). In light of Theorem 2.1, BSDEJ$(\xi, 0)$ admits a unique solution $(Y, Z, U)\in\mathbb{S}^p$. Since (1.4) and Lemma 1.1 show that $\int_0^t Z_s dB_s + \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$,

$t \in [0,T]$ is a uniformly integrable martingale, it holds for any $t \in [0,T]$ that $Y_t = E\big[\xi - \int_t^T Z_s dB_s - \int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\big|\mathcal{F}_t\big] = E[\xi|\mathcal{F}_t]$, P-a.s. In particular, $Y_0 = E[\xi]$. Then for any $t \in [0,T]$,

$$\begin{aligned} E[\xi|\mathcal{F}_t] = Y_t &= Y_0 + \int_0^t Z_s dB_s + \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \\ &= E[\xi] + \int_0^t Z_s dB_s + \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad P\text{-a.s.}, \end{aligned}$$

which together with the right continuity of processes $E[\xi|\mathcal{F}_t]$, $\int_0^t Z_s dB_s$ and $\int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t \in [0,T]$ leads to (2.1).

Next, let $(Z',U') \in \mathbb{Z}^{2,p} \times \mathbb{U}^p$ be another pair satisfying (2.1), so one has that P-a.s.

$$\int_0^t (Z_s - Z'_s)dB_s + \int_{(0,t]}\int_{\mathcal{X}} \big(U_s(x) - U'_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx) = 0, \quad t \in [0,T].$$

Clearly, the quadratic variation of the above process is $\int_0^t |Z_s - Z'_s|^2 ds + \int_{(0,t]}\int_{\mathcal{X}} \big|U_s(x) - U'_s(x)\big|^2 N_{\mathfrak{p}}(ds,dx) = 0$, $t \in [0,T]$, which implies that $Z_t(\omega) = Z'_t(\omega)$ for $dt \times dP$-a.s. $(t,\omega) \in [0,T] \times \Omega$, and $U(t,\omega,x) = U'(t,\omega,x)$ for $dt \times dP \times \nu(dx)$-a.s. $(t,\omega,x) \in [0,T] \times \Omega \times \mathcal{X}$. □

Appendix

Lemma A.1. *Let $\{a_i\}_{i\in\mathbb{N}} \subset [0,\infty)$. For any $p \in (0,\infty)$ and $n \in \mathbb{N}$ with $n \ge 2$, we have*

$$\big(1 \wedge n^{p-1}\big)\sum_{i=1}^n a_i^p \le \left(\sum_{i=1}^n a_i\right)^p \le \big(1 \vee n^{p-1}\big)\sum_{i=1}^n a_i^p. \tag{A.1}$$

This result is routine, see e.g. our ArXiv version [53] for a proof.

Lemma A.2. *For any $b, c \in [0,\infty)$, we have*

$$\big|b^p - c^p\big| \le \begin{cases} |b-c|^p, & \text{if } p \in (0,1], \\ p(b \vee c)^{p-1}|b-c|, & \text{if } p \in (1,\infty). \end{cases} \tag{A.2}$$

Proof. It is trivial when $b = c$. Since b and c take the symmetric roles in (A.2), we only need to assume $b < c$ without loss of generality.

- When $p \in (0,1]$, applying Lemma A.1 with $a_1 = b$ and $a_2 = c - b$ yields that $c^p = (a_1 + a_2)^p \le a_1^p + a_2^p = b^p + (c-b)^p$, which implies that $\big|b^p - c^p\big| = c^p - b^p \le (c-b)^p = |b-c|^p$;
- When $p \in (1,\infty)$, one can deduce that $c^p - b^p = p\int_b^c t^{p-1}dt \le p\int_b^c c^{p-1}dt = pc^{p-1}(c-b)$, which leads to that $\big|b^p - c^p\big| = c^p - b^p \le p\,c^{p-1}(c-b) = p(b \vee c)^{p-1}|b-c|$. □

Lemma A.3. *(Bihari's Inequality) Let $\theta : [0,\infty) \to [0,\infty)$ and $\zeta, \chi : [0,T] \to [0,\infty)$ be three functions such that*

(i) *either $\theta \equiv 0$ or $\theta(x) > 0$ for any $x > 0$;*

(ii) *θ is increasing and satisfies $\int_{0+}^1 \frac{1}{\theta(x)}dx = \infty$;*

(iii) *ζ is integrable and χ is bounded.*

If $\chi(t) \le \int_t^T \theta\big(\chi(s)\big)\zeta(s)ds$ for any $t \in [0,T]$, then $\chi \equiv 0$.

See e.g. our ArXiv version [53] for a proof of this lemma. For the next three lemmas, we consider a generic vector space $\mathbb{E}$ with norm $\|\cdot\|$.

Lemma A.4. *Let $\mathbb{E}$ be a vector space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For any $x, y \in \mathbb{E}$, we have*

$$\big\|\pi_r(x)-\pi_r(y)\big\| \le \|x-y\|, \qquad \forall\, r\in(0,\infty). \tag{A.3}$$

Consequently,

$$\|x-y\| \ge \big(\|x\|\wedge\|y\|\big)\big\|\mathscr{D}(x)-\mathscr{D}(y)\big\|. \tag{A.4}$$

Proof. Without loss of generality, we assume that $\|x\|\le\|y\|$ in the whole proof.

To see (A.3), let us discuss by three cases:
(1) When $r>\|y\|$: Since $\pi_r(x)=x$ and $\pi_r(y)=y$, one simply has $\big\|\pi_r(x)-\pi_r(y)\big\|=\|x-y\|$;
(2) When $\|x\|\le r\le\|y\|$: Let us set $c:=\big\langle x,\mathscr{D}(y)\big\rangle$ and $\widehat{y}:=c\mathscr{D}(y)$. Since $\big\langle x-\widehat{y},\mathscr{D}(y)\big\rangle=0$, it holds for any $\alpha\in\mathbb{R}$ that

$$\begin{aligned}\big\|x-\alpha\mathscr{D}(y)\big\|^2 &= \big\|x-\widehat{y}-(\alpha-c)\mathscr{D}(y)\big\|^2=\big\|x-\widehat{y}\big\|^2+\big\|(\alpha-c)\mathscr{D}(y)\big\|^2\\ &= \big\|x-\widehat{y}\big\|^2+(\alpha-c)^2.\end{aligned}$$

Hence, it follows that $\big\|\pi_r(x)-\pi_r(y)\big\|^2=\big\|x-r\mathscr{D}(y)\big\|^2=\big\|x-\widehat{y}\big\|^2+(r-c)^2\le\big\|x-\widehat{y}\big\|^2+\big(\|y\|-c\big)^2=\big\|x-y\big\|^2$, where we used the fact that $c\le\big|\big\langle x,\mathscr{D}(y)\big\rangle\big|\le\|x\|\le r\le\|y\|$ by the Schwarz inequality.
(3) When $r<\|x\|$: We know from (2) that

$$\begin{aligned}q\|x-y\| &\ge \big\|\pi_{\|x\|}(x)-\pi_{\|x\|}(y)\big\|=\big\|x-\|x\|\mathscr{D}(y)\big\|=\|x\|\big\|\mathscr{D}(x)-\mathscr{D}(y)\big\|\\ &\ge r\big\|\mathscr{D}(x)-\mathscr{D}(y)\big\|=\big\|\pi_r(x)-\pi_r(y)\big\|.\end{aligned}$$

If $x=\mathbf{0}$, (A.4) holds trivially. Otherwise, since $\|x\|\le\|y\|$, applying (A.3) with $r=\|x\|$ gives rise to (A.4). $\square$

Lemma A.5. *Let $\mathbb{E}$ be a vector space with norm $\|\cdot\|$ only. For any $x, y\in\mathbb{E}$, we have*

$$\big\|\pi_r(x)-\pi_r(y)\big\|\le 2\|x-y\|, \qquad \forall\, r\in(0,\infty).$$

Proof. Let $x, y\in\mathbb{E}$. Since $|a\vee b-a\vee c|\le|b-c|$ holds for any $a,b,c\in\mathbb{R}$, the triangle inequality implies that

$$\begin{aligned}\big\|\pi_r(x)-\pi_r(y)\big\| &= \left\|\frac{r}{r\vee\|x\|}x-\frac{r}{r\vee\|y\|}y\right\|\le\frac{r}{r\vee\|x\|}\|x-y\|\\ &\quad+\left|\frac{r}{r\vee\|x\|}-\frac{r}{r\vee\|y\|}\right|\|y\|\\ &\le \|x-y\|+\frac{r\|y\|}{(r\vee\|x\|)(r\vee\|y\|)}\big|r\vee\|x\|-r\vee\|y\|\big|\\ &\le \|x-y\|+\big|\|x\|-\|y\|\big|\le 2\|x-y\|. \quad\square\end{aligned}$$

Lemma A.6. *Let $\mathbb{E}$ be a vector space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For any $p\in(0,1]$ and $x, y\in\mathbb{E}$, we have $\big\|\|x\|^p\,\mathscr{D}(x)-\|y\|^p\,\mathscr{D}(y)\big\|\le(1+2^p)\|x-y\|^p$.*

Proof. The case "$p = 1$" is trivial since $\big\| \|x\|\mathscr{D}(x) - \|y\|\mathscr{D}(y) \big\| = \|x - y\|$. For $p \in (0, 1)$, we assume without loss of generality that $\|x\| \le \|y\|$ and discuss by three cases:
(1) When $x = 0$: $\big\| \|y\|^p \mathscr{D}(y) \big\| = \|y\|^p$;
(2) When $0 < \|x\| \le \|x - y\|$: $\big\| \|x\|^p \mathscr{D}(x) - \|y\|^p \mathscr{D}(y) \big\| \le \big\| \|x\|^p \mathscr{D}(x) \big\| + \big\| \|y\|^p \mathscr{D}(y) \big\| = \|x\|^p + \|y\|^p \le \|x\|^p + \big(\|x\| + \|x - y\|\big)^p \le (1 + 2^p)\|x - y\|^p$;
(3) When $\|x\| > \|x - y\|$: As $\|x\| \le \|y\|$, (A.4) and Lemma A.2 show that $\big\| \|x\|^p \mathscr{D}(x) - \|y\|^p \mathscr{D}(y) \big\| \le \|x\|^p \big\| \mathscr{D}(x) - \mathscr{D}(y) \big\| + \big| \|x\|^p - \|y\|^p \big| \le \|x\|^{p-1}\|x - y\| + \big| \|x\| - \|y\| \big|^p < 2\|x - y\|^p$. □

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