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Doubly reflected BSDEs with integrable parameters and related Dynkin games[☆]

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Abstract

We study a doubly reflected backward stochastic differential equation (BSDE) with integrable parameters and the related Dynkin game. When the lower obstacle L and the upper obstacle U of the equation are completely separated, we construct a unique solution of the doubly reflected BSDE by pasting local solutions, and show that the Y-component of the unique solution represents the value process of the corresponding Dynkin game under g-evaluation, a nonlinear expectation induced by BSDEs with the same generator g as the doubly reflected BSDE concerned. In particular, the first time τ^* when process Y meets L and the first time γ^* when process Y meets U form a saddle point of the Dynkin game.

Keywords: BSDEs; Reflected BSDEs; Doubly reflected BSDEs; g-evaluation/expectation; Penalization; Optimal stopping problems; Pasting local solutions; Dynkin games; Saddle points

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1. Introduction

In this paper, we study a doubly reflected backward stochastic differential equation with generator g, integrable terminal data ξ and two integrable obstacles L, U

$$\begin{cases} Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds + K_T - K_t - J_T + J_t - \int_t^T Z_s dB_s, \quad t \in [0, T], \\ L_t \le Y_t \le U_t, \quad t \in [0, T], \\ \int_0^T (Y_t - L_t) dK_t = \int_0^T (U_t - Y_t) dJ_t = 0 \quad \text{(flat-off conditions)}. \end{cases} \tag{1.1}$$

A solution of such an equation consists of four adapted processes: a continuous process Y, a locally square-integrable process Z and two continuous increasing processes K and J. Klimsiak [38] studied the same problem but assumed an extended *Mokobodzki's condition*: there exists a semi-martingale between L and U, which is practically difficult to verify. Instead, we only require the two obstacles L, U to be completely separable, i.e. $L_t < U_t$, $\forall\, t \in [0, T]$.

Backward stochastic differential equations (BSDEs) were introduced in linear case by Bismut [9] as the adjoint equations for the stochastic Pontryagin maximum principle in control theory. Later, Pardoux and Peng [42] extended them to a fully nonlinear version

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \tag{1.2}$$

and showed that the BSDE admits a unique solution (Y, Z) when generator g is Lipschitz continuous in (y, z) and terminal datum ξ is square-integrable. Since then, the theory of BSDEs has rapidly grown and been applied in many areas such as mathematical finance, theoretical economics, stochastic control, stochastic differential games, partial differential equations (see e.g. the references in [21] or in [15]).

As a variation of BSDEs, a BSDE with one reflecting obstacle (say lower obstacle L)

$$\begin{cases} L_t \le Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \quad t \in [0, T], \\ \int_0^T (Y_t - L_t) dK_t = 0 \quad \text{(flat-off condition)} \end{cases} \tag{1.3}$$

was first studied by El Karoui et al. [20]. If g is Lipschitz continuous in (y, z) and if both terminal datum ξ and lower obstacle L are square-integrable, these authors showed that the reflected BSDE has a unique solution (Y, Z, K) and that the Y-component of the unique solution is the Snell envelope of the reward process L in the related optimal stopping problem under g-evaluation (for a more general statement, see e.g. Appendix A of [13], Section 7 of [7]). As a nonlinear expectation induced by BSDEs with the same generator g as the reflected BSDE, the g-evaluation possesses many (martingale) properties of the classic linear expectation and thus become a very useful tool in nonlinear analysis. In particular, the g-evaluation is closely related to risk measures in mathematical finance.

Based on [20], Cvitanić and Karatzas [14] extended the research of BSDEs to those with two reflecting obstacles. They showed that a doubly reflected BSDE with Lipschitz generator, square-integrable terminal datum and square-integrable obstacles admits a unique solution under Mokobodzki's condition (there exists a quasimartingale between two obstacles) or certain regularity condition on one of the obstacles (see assumption (H) of [28] for a simplified form). Cvitanić and Karatzas also found that the Y-component of the unique solution is exactly the value process of the related Dynkin game, a zero-sum stochastic differential game of optimal stopping, under g-evaluation (for a more general statement, see e.g. [17]). From a perspective of mathematical finance, this discovery is significant for the evaluation of American game options or *Israeli* options, see e.g. Hamadène [24]. Later, Hamadène et al. [29,27,24] added controls into a doubly reflected BSDE and the drift coefficient of the associated state process to analyze a *mixed* zero-sum controller and stopper game as well as the corresponding *saddle* point problem. For the literature and the recent advances of Dynkin games, see e.g. [36,48,32,4]. As to the history and latest development of controller and stopper games, see e.g. [35,37,5,6,3,2,18,41,8].

Among other development in doubly reflected BSDEs, Lepeltier and San Martín [39] obtained the existence result when g is only continuous and has linear growth in variables (y, z); Xu [49] got the wellposedness result when the Lipschitz continuity of g in y-variable is relaxed to a monotonicity condition; and Bahlali et al. [1], Essaky et al. [23,22] analyzed the existence of a maximal solution when g has quadratic growth in z-variable.

All the above articles on doubly reflected BSDEs, except [24], assumed either (extended) Mokobodzki's condition or the aforementioned regularity condition. According to [24]'s observation that the existence of local solutions of a doubly reflected BSDE relies on neither of these two conditions, Hamadène and Hassani [25] pasted local solutions to form a unique solution of a doubly reflected BSDE with two distinct obstacles. Since then, the complete separation of obstacles has been postulated by most of the subsequent papers including [12,19,26,31] as well as the present one.

During the evolution of the BSDE theory, some efforts were made to weaken the square integrability on terminal data so as to match up with the fact that linear BSDEs are well-posed for integrable terminal data: El Karoui et al. [21] demonstrated that for any p-integrable terminal datum with $p \in (1, \infty)$, a BSDE with Lipschitz generator admits a unique p-integrable solution. This wellposedness result was later upgraded by Briand et al. [10,11] who reduced the Lipschitz condition of generator g on y-variable to a monotonicity condition on y. After Hamadène and Popier [30] extended [11]'s results for reflected BSDEs, Hamadène et al. [19] make a further generalization for doubly reflected BSDEs with two completely separate obstacles.

We dedicate this paper to the solvability of the doubly reflected BSDE (1.1) with integrable parameters and will discuss the related Dynkin game. Besides the monotonicity condition on y-variable and the Lipschitz condition on z-variable, if the generator g additionally has a growth condition on z-variable of order $\alpha \in (0, 1)$ (see (H7) of [11] or (H5) in the current paper), then the BSDE with integrable terminal datum admits a unique solution (Y, Z) such that both Y and Z are p-integrable processes for any $p \in (0, 1)$ and that Y is of class (D). So the corresponding g-evaluation is well-defined for each integrable random variable. Under the same hypotheses on generator g as Section 6 of [11], we will demonstrate a similar wellposedness result for doubly reflected BSDEs with integrable terminal data and two distinct integrable obstacles. Though we follow the approach of [25,19] on pasting local solutions, the estimations used for L^p-solutions, $p > 1$ are no longer valid in the $p = 1$ or class (D) case. We managed to derive some novel estimation and approximation scheme.

To construct a unique solution of a reflected BSDE with integrable terminal datum ξ and integrable lower obstacle L, we use the *penalization* method introduced in [20] together with a *localization* technique. This is because the approximating solutions are only p-integrable $(\forall\, p \in (0, 1))$: Given $n \in \mathbb{N}$, we compensate the generator g by n times the distance that y-variable is below L_t, i.e. $g_n(t, y, z) := g(t, y, z) + n(y - L_t)^-$. The BSDE with generator g_n

and terminal datum ξ

$$Y_t^n = \xi + \int_t^T g\left(s, Y_s^n, Z_s^n\right) ds + n \int_t^T \left(Y_s^n - L_s\right)^- ds - \int_t^T Z_s^n dB_s, \quad t \in [0, T], \tag{1.4}$$

has a unique p-integrable ($\forall\, p \in (0, 1)$) solution (Y^n, Z^n) such that Y^n is of class (D). The monotonicity of $\{g_n\}_{n\in\mathbb{N}}$ implies that of $\{Y^n\}_{n\in\mathbb{N}}$, thanks to a general comparison result (Proposition 3.2). Then we can find a stopping time τ_ℓ such that $|Y^n|$ is uniformly bounded by ℓ over the stochastic interval $[\![0, \tau_\ell]\!]$. By a local estimation (Lemma A.2), the local $\mathbb{L}^2$-norms of Z^n's are uniformly bounded by a multiple of ℓ^2. So up to a subsequence, Z^n weakly converges to some $\mathcal{Z}^\ell$. Consequently, we can deduce that $K_t^n := n \int_0^t \left(Y_s^n - L_s\right)^- ds$ converges to $\mathcal{K}_t^\ell := Y_0 - Y_t - \int_0^t g(s, Y_s, \mathcal{Z}_s^\ell) ds + \int_0^t \mathcal{Z}_s^\ell dB_s$ uniformly over $[\![0, \tau_\ell]\!]$. Letting $n \to \infty$ in (1.4) shows that $(Y, \mathcal{Z}^\ell, \mathcal{K}^\ell)$ is a local solution of (1.3) over $[\![0, \tau_\ell]\!]$. Pasting up $(Y, \mathcal{Z}^\ell, \mathcal{K}^\ell)$'s over stochastic intervals $]\!]\tau_{\ell-1}, \tau_\ell]\!]$'s we obtain a global p-integrable ($\forall\, p \in (0, 1)$) solution (Y, Z, K) of (1.3). The uniqueness of such a solution follows from a comparison result (Proposition 5.3) of reflected BSDEs, which is a corollary of Proposition 3.2.

Applying Proposition 3.2 again shows that with respect to the corresponding g-evaluation, the Y-component of the unique solution of (1.3) is a supermartingale and even a martingale up to the first time when process Y meets the lower obstacle L. Consequently, Y is the Snell envelope of the reward process L in the related optimal stopping problem in which the player is trying to select a best exit time from the game so as to maximize her expected reward under g-expectation.

Based on the wellposedness result for reflected BSDEs with integrable parameters, we next take [25]'s approach of pasting local solutions to construct a global solution of (1.1): Let (Y^n, Z^n, K^n) be the unique p-integrable ($\forall\, p \in (0, 1)$) solution of a reflected BSDE with the penalized generator g_n and the upper obstacle U. We first show that the increasing limit Y of Y^n's, together with some processes (Z^ℓ, K^ℓ), solves (1.3) over some stochastic intervals $[\![\nu_\ell, \nu'_\ell]\!]$ for any $\ell \in \mathbb{N}$. A reverse conclusion can be obtained for the limit $\widetilde{Y}$ of a decreasing scheme that involves reflected BSDEs with generator $\widetilde{g}_n(t, y, z) := g(t, y, z) - n(y - U_t)^+$ and the lower obstacle L: For some processes $(\widetilde{Z}^\ell, \widetilde{J}^\ell)$, $(\widetilde{Y}, \widetilde{Z}^\ell, \widetilde{J}^\ell)$ solves a reflected BSDE with upper obstacle U over some stochastic interval $[\![\nu'_\ell, \nu_{\ell+1}]\!]$ for any $\ell \in \mathbb{N}$. Then pasting $(Y, Z^\ell, K^\ell, 0)$ and $(\widetilde{Y}, \widetilde{Z}^\ell, 0, \widetilde{J}^\ell)$ alternatively over $[\![\nu_\ell, \nu'_\ell]\!]$ and $[\![\nu'_\ell, \nu_{\ell+1}]\!]$ yields a global p-integrable ($\forall\, p \in (0, 1)$) solution of the doubly reflected BSDE (1.1).

Leveraging Proposition 3.2 once again shows that with respect to the corresponding g-evaluation, the Y-component of the solution of (1.1) just constructed is a submartingale up to the first time τ^* when Y meets the lower obstacle L, and is a supermartingale up to time γ^* when Y meets the upper obstacle U. Consequently, Y is the value process of the related Dynkin game under g-evaluation in which L (resp. U) is the amount process a player will receive from her opponent when she stops the game earlier (resp. not earlier) than her opponent. The uniqueness result of (1.1) then easily follows. Moreover, the pair (τ^*, γ^*) forms a saddle point of such a Dynkin game.

Since dealing mostly with p-integrable ($\forall\, p \in (0, 1)$) solutions, we cannot apply Doob's martingale inequality and many well-known estimates in BSDE theory without using localization first, which increases the technical difficulty. Also, to overcome technical subtleties we encounter when proving the p-integrability ($\forall\, p \in (0, 1)$) of the limit Y in the penalization scheme, we appropriately exploit Tanaka–Itô's formula, Hypothesis (H5) and other tricks, see in particular the proof of (6.14).

The rest of the paper is organized as follows: After listing necessary notations, we give the definition of doubly reflected BSDEs and make some assumptions on their generators g in Section 1. We first present in Section 2 the main result of our paper, a wellposedness result of doubly reflected BSDEs with integrable parameters as well as the g-martingale characterization of the Y-component of the unique solution, the latter of which implies that Y is a value process of the related Dynkin games under g-evaluation. Section 3 recalls a wellposedness result of BSDEs with integrable terminal data and gives a general comparison result for BSDEs over stochastic intervals, which plays an important role in our analysis. The unique solutions of BSDEs with generator g and integrable terminal data induce a widely-defined nonlinear expectation, called "g-evaluation/expectation", whose properties will be discussed in Section 4. In Section 5, to construct a unique solution for a reflected BSDE with integrable parameters as a preparation for our main result, we use the *penalization* method which involves two auxiliary monotonicity results. And we show that the Y-component of the unique solution of the reflected BSDE is exactly the Snell envelope in the related optimal stopping problem under g-evaluation. Section 6 contains proofs of our results while the demonstration of some technical claims are deferred to the Appendix.

1.1. Notation and definitions

Throughout this paper, we fix a time horizon $T \in (0, \infty)$, and let B be a d-dimensional standard Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The augmented filtration generated by B

$$\mathbf{F} = \{\mathcal{F}_t := \sigma\left(\sigma\left(B_s; s \in [0, t]\right) \cup \mathscr{N}\right)\}_{t\in[0,T]}$$

satisfies the *usual hypothesis*, where $\mathscr{N}$ collects all $\mathbb{P}$-null sets in $\mathcal{F}$.

Let $\mathcal{T}$ be the set of all $\mathbf{F}$-stopping times τ taking values in $[0, T]$. For any $\nu, \tau \in \mathcal{T}$ with $\nu \le \tau$, we set $\mathcal{T}_{\nu,\tau} := \{\gamma \in \mathcal{T} : \nu \le \gamma \le \tau\}$. An increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$ is called "stationary" if for $\mathbb{P}$-a.s. $\omega \in \Omega$, $T = \tau_n(\omega)$ for some $n = n(\omega) \in \mathbb{N}$. As usual, we say that a $\mathscr{B}\big([0, T]\big) \otimes \mathcal{F}$-measurable process X is of *class (D)*, with respect to $(\mathcal{T}, \mathbb{P})$, if $\{X_\tau\}_{\tau\in\mathcal{T}}$ is $\mathbb{P}$-uniformly integrable. Moreover, we let $\mathscr{P}$ denote the $\mathbf{F}$-progressively measurable σ-field on $[0, T] \times \Omega$ and will use the convention $\inf \emptyset := \infty$.

Let $p \in (0, \infty)$. It holds for any finite subset $\{a_1, \dots, a_n\}$ of $(0, \infty)$ that

$$\left(1 \wedge n^{p-1}\right) \sum_{i=1}^{n} a_i^p \le \left(\sum_{i=1}^{n} a_i\right)^p \le \left(1 \vee n^{p-1}\right) \sum_{i=1}^{n} a_i^p. \tag{1.5}$$

And for any $p' \in (p, \infty)$, one has

$$x^p \le 1 + x^{p'}, \quad \forall\, x \in (0, \infty). \tag{1.6}$$

The following spaces will be frequently used in the sequel.

(1) For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, let $L^0(\mathcal{G})$ be the space of all real-valued, $\mathcal{G}$-measurable random variables ξ and set $L^p(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^p(\mathcal{G})} := \big\{\mathbb{E}[|\xi|^p]\big\}^{1\wedge\frac{1}{p}} < \infty\Big\}$.

(2) We need the following subspaces of $\mathbb{S}^0$, which denotes all real-valued, $\mathbf{F}$-adapted continuous processes:
- $\mathbb{S}^p := \Big\{X \in \mathbb{S}^0 : \|X\|_{\mathbb{S}^p} := \big\{\mathbb{E}\left[(X_*)^p\right]\big\}^{1\wedge\frac{1}{p}} < \infty\Big\}$, where $X_* := \sup_{t\in[0,T]} |X_t|$;
- $\mathbb{S}^p_+ := \Big\{X \in \mathbb{S}^0 : X^+ = X \vee 0 \in \mathbb{S}^p\Big\}$ and $\mathbb{S}^p_- := \Big\{X \in \mathbb{S}^0 : X^- = (-X) \vee 0 \in \mathbb{S}^p\Big\}$;

- $\mathbb{V}^0 := \{X \in \mathbb{S}^0 : X \text{ is of finite variation}\}$;
- $\mathbb{K}^0 := \{X \in \mathbb{S}^0 : X \text{ is an increasing process with } X_0 = 0\} \subset \mathbb{V}^0$;
- $\mathbb{K}^p := \{X \in \mathbb{K}^0 : X_T \in \mathbb{L}^p(\mathcal{F}_T)\}$.

(3) Let $\widetilde{\mathbb{H}}^{2,0}$ (resp. $\mathbb{H}^{2,0}$) denote the space of all $\mathbb{R}^d$-valued, **F**-progressively measurable (resp. **F**-predictable) processes X with $\int_0^T |X_t|^2 dt < \infty$, $\mathbb{P}$-a.s. and set $\mathbb{H}^{2,p} := \Big\{X \in \mathbb{H}^{2,0} : \|X\|_{\mathbb{H}^{2,p}} := \Big\{\mathbb{E}\Big[\big(\int_0^T |X_t|^2 dt\big)^{p/2}\Big]\Big\}^{1\wedge(1/p)} < \infty\Big\}$.

In the above notations, if $p \ge 1$, $\|\cdot\|_{\Xi^p}$ is a norm on $\Xi^p = L^p(\mathcal{G}), \mathbb{S}^p, \mathbb{H}^{2,p}$. And if $p \in (0,1)$, $(X, X') \to \|X - X'\|_{\Xi^p}$ defines a distance on Ξ^p, under which Ξ^p is a complete metric space.

Let us recall the notions of backward stochastic differential equations (BSDEs), reflected BSDEs and doubly reflected BSDEs: A (basic) parameter pair (ξ, g) consists of a real-valued, $\mathcal{F}_T$-measurable random variable ξ and a function $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ that is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable.

Definition 1.1. Given a parameter pair (ξ, g), let $L, U \in \mathbb{S}^0$ such that $\mathbb{P}\{L_t \le U_t, \forall\, t \in [0,T]\} = 1$ and $L_T \le \xi \le U_T$, $\mathbb{P}$-a.s. We say that (1) $(Y, Z) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0}$ is a solution of a BSDE with terminal data ξ and generator g (BSDE (ξ, g) for short) if (1.2) holds $\mathbb{P}$-a.s. (2) A triplet $(Y, Z, K) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0} \times \mathbb{K}^0$ is a solution of a reflected BSDE with terminal data ξ, generator g and (lower) obstacle L (RBSDE (ξ, g, L) for short) if (1.3) holds $\mathbb{P}$-a.s. (3) A quadruplet $(Y, Z, K, J) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0} \times \mathbb{K}^0 \times \mathbb{K}^0$ is a solution of a doubly reflected BSDE with terminal data ξ, generator g, lower obstacle L and upper obstacle U (DRBSDE (ξ, g, L, U) for short) if (1.1) holds $\mathbb{P}$-a.s.

Remark 1.1. Given a parameter pair (ξ, g),

$$g_-(t,\omega,y,z) := -g(t,\omega,-y,-z), \quad \forall\, (t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \tag{1.7}$$

clearly defines a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function. For any $L \in \mathbb{S}^0$ with $L_T \le \xi$, $\mathbb{P}$-a.s., $(Y, Z, K) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0} \times \mathbb{K}^0$ solves RBSDE(ξ, g, L) if and only if $(\widetilde{Y}, \widetilde{Z}, \widetilde{J}) = (-Y, -Z, K) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0} \times \mathbb{K}^0$ is a solution of the following reflected BSDE with terminal data $\widetilde{\xi} = -\xi$, generator g_- and upper obstacle $U = -L$:

$$\begin{cases} U_t \ge \widetilde{Y}_t = \widetilde{\xi} + \displaystyle\int_t^T g_-(s, \widetilde{Y}_s, \widetilde{Z}_s)ds - \widetilde{J}_T + \widetilde{J}_t - \int_t^T \widetilde{Z}_s dB_s, \quad t \in [0,T], \\ \displaystyle\int_0^T (U_t - \widetilde{Y}_t)d\widetilde{J}_t = 0 \quad \text{(flat-off condition)}. \end{cases} \tag{1.8}$$

Let $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function. To study doubly reflected BSDEs with generator g and integrable parameters (ξ, L, U), we will make the following assumptions on function g:

Standing assumptions on g.

Let $\kappa > 0$, $\lambda \in \mathbb{R}$, $\alpha \in (0,1)$ and let $\{h_t\}_{t\in[0,T]}$ be a non-negative integrable process (i.e. $h \in \mathbb{L}^1([0,T] \times \Omega, \mathscr{B}([0,T]) \otimes \mathcal{F}, dt \otimes \mathbb{P})$). It holds $dt \otimes d\mathbb{P}$-a.s. that

(H1) $|g(t,\omega,y,z) - g(t,\omega,y,z')| \le \kappa|z - z'|$, $\forall\, y \in \mathbb{R}$, $\forall\, z, z' \in \mathbb{R}^d$;

(H2) $\mathrm{sgn}(y - y') \cdot \big(g(t,\omega,y,z) - g(t,\omega,y',z)\big) \le \lambda|y - y'|$, $\forall\, y, y' \in \mathbb{R}$, $\forall\, z \in \mathbb{R}^d$;

(H3) $y \to g(t,\omega,y,z)$ is continuous, $\forall\, z \in \mathbb{R}^d$;

(H4) $|g(t,\omega,y,0)| \le h_t(\omega) + \kappa|y|$, $\forall\, y \in \mathbb{R}$;

(H5) $|g(t,\omega,y,z) - g(t,\omega,y,0)| \le \kappa(h_t(\omega) + |y| + |z|)^\alpha$, $\forall\, (y,z) \in \mathbb{R} \times \mathbb{R}^d$.

From now on, for any $p \in [0, \infty)$ we let C_p be a generic constant depending on p, κ, λ^+, T and $\mathbb{E}\int_0^T h_t dt$ (in particular, C_0 will denote a generic constant depending on κ, λ^+, T and $\mathbb{E}\int_0^T h_t dt$), whose form may vary from line to line. For convenience, we will call a function $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ a "generator" if it is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable and satisfies (H1)–(H5).

Remark 1.2. If a function $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous in y (i.e. for some $\widetilde{\kappa} > 0$, it holds $dt \otimes d\mathbb{P}$-a.s. that $|g(t, \omega, y, z) - g(t, \omega, y', z)| \le \widetilde{\kappa}|y - y'|$, $\forall\, y, y' \in \mathbb{R}$, $\forall\, z \in \mathbb{R}^d$), then (H2) automatically holds and (H4) will be replaced by $|g(t, \omega, 0, 0)| \le h_t(\omega)$, $dt \otimes d\mathbb{P}$-a.s.

Remark 1.3. Let g be a generator.

(1) The function g_- defined in (1.7) is also a generator.
(2) Given $\tau \in \mathcal{T}$, since $\{\mathbf{1}_{\{t \le \tau\}}\}_{t \in [0,T]}$ is an $\mathbf{F}$-adapted càglàd process (and thus $\mathbf{F}$-predictable), the measurability of g implies that

$$g_\tau(t, \omega, y, z) := \mathbf{1}_{\{t \le \tau(\omega)\}} g(t, \omega, y, z), \qquad \forall\, (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \quad (1.9)$$

defines a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function. And one can deduce that g_τ also satisfies (H1)–(H5) (actually, it satisfies (H2) with $\widetilde{\lambda} = \lambda \vee 0$).
(3) If g' is another generator, so is $ag + bg'$ for any $a, b > 0$.
(4) Given $L \in \mathbb{S}^1_+$, $g_L(t, \omega, y) := (y - L_t(\omega))^-$, $(t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}$ is clearly a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable function that is Lipschitz continuous in y and satisfies $\mathbb{E}\int_0^T g_L(t, 0)dt = \mathbb{E}\int_0^T L_t^+ dt \le T\|L^+\|_{\mathbb{S}^1} < \infty$. By Remark 1.2, g_L satisfies (H2)–(H4). Then part (3) shows that for any $n \in \mathbb{N}$

$$g_n(t, \omega, y, z) := g(t, \omega, y, z) + n\,(y - L_t(\omega))^-, \quad \forall\, (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \quad (1.10)$$

defines a generator.

2. Main result: doubly reflected BSDEs with integrable parameters and related Dynkin games

The contribution of this paper is the following wellposedness result of a doubly reflected BSDE with integrable parameters in which the Y-component of the unique solution represents the value of the related Dynkin game under a so-called "g-evaluation" (see Section 4), a nonlinear expectation induced by BSDEs with the same generator g as the doubly reflected BSDE. Like [25], we assume the complete separation of the lower and upper obstacles in the doubly reflected BSDE instead of the traditional Mokobodzki condition which is quite difficult to check in practice.

Theorem 2.1. *Let g be a generator. For any $\xi \in L^1(\mathcal{F}_T)$, $L \in \mathbb{S}^1_+$ and $U \in \mathbb{S}^1_-$ such that $\mathbb{P}\{L_T \le \xi \le U_T\} = \mathbb{P}\{L_t < U_t,\ \forall\, t \in [0, T]\} = 1$, DRBSDE (ξ, g, L, U) admits a unique solution $(Y, Z, K, J) \in \big(\cap_{p \in (0,1)} \mathbb{S}^p\big) \times \mathbb{H}^{2,0} \times \mathbb{K}^0 \times \mathbb{K}^0$ such that Y is of class (D).*

Define $R(\tau, \gamma) := \mathbf{1}_{\{\tau < \gamma\}} L_\tau + \mathbf{1}_{\{\gamma \le \tau\} \cap \{\gamma < T\}} U_\gamma + \mathbf{1}_{\{\tau = \gamma = T\}} \xi$, $\forall\, \tau, \gamma \in \mathcal{T}$. Let $\nu \in \mathcal{T}$, $\tau^*_\nu := \inf\big\{t \in [\nu, T] : Y_t = \mathbf{1}_{\{t < T\}} L_t + \mathbf{1}_{\{t = T\}} \xi\big\} \in \mathcal{T}_{\nu,T}$ and $\gamma^*_\nu := \inf\big\{t \in [\nu, T] : Y_t =$

$\mathbf{1}_{\{t<T\}}U_t + \mathbf{1}_{\{t=T\}}\xi\big\} \in \mathcal{T}_{\nu,T}$. *It holds for any* $\tau, \gamma \in \mathcal{T}_{\nu,T}$ *that*

$$\mathcal{E}^g_{\nu,\tau\wedge\gamma^*_\nu}\big[Y_{\tau\wedge\gamma^*_\nu}\big] \le Y_\nu \le \mathcal{E}^g_{\nu,\tau^*_\nu\wedge\gamma}\big[Y_{\tau^*_\nu\wedge\gamma}\big], \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

Consequently, it holds $\mathbb{P}$*-a.s. that*

$$\begin{aligned}\operatorname*{esssup}_{\tau\in\mathcal{T}_{\nu,T}} \mathcal{E}^g_{\nu,\tau\wedge\gamma^*_\nu}\big[R(\tau,\gamma^*_\nu)\big] = Y_\nu = \mathcal{E}^g_{\nu,\tau^*_\nu\wedge\gamma^*_\nu}\big[R\big(\tau^*_\nu,\gamma^*_\nu\big)\big] \\ = \operatorname*{essinf}_{\gamma\in\mathcal{T}_{\nu,T}} \mathcal{E}^g_{\nu,\tau^*_\nu\wedge\gamma}\big[R(\tau^*_\nu,\gamma)\big].\end{aligned} \tag{2.2}$$

In particular, we have

$$Y_\nu = \operatorname*{esssup}_{\tau\in\mathcal{T}_{\nu,T}} \operatorname*{essinf}_{\gamma\in\mathcal{T}_{\nu,T}} \mathcal{E}^g_{\nu,\tau\wedge\gamma}\big[R(\tau,\gamma)\big] = \operatorname*{essinf}_{\gamma\in\mathcal{T}_{\nu,T}} \operatorname*{esssup}_{\tau\in\mathcal{T}_{\nu,T}} \mathcal{E}^g_{\nu,\tau\wedge\gamma}\big[R(\tau,\gamma)\big], \quad \mathbb{P}\text{-a.s.} \tag{2.3}$$

Remark 2.1. (1) For any $\nu, \zeta \in \mathcal{T}$ with $0 \le \nu \le \zeta \le \tau^*_0$, it is clear that $\tau^*_\nu = \tau^*_0$, $\mathbb{P}$-a.s. Then (2.1) shows that $Y_\nu \le \mathcal{E}^g_{\nu,\tau^*_\nu\wedge\zeta}\big[Y_{\tau^*_\nu\wedge\zeta}\big] = \mathcal{E}^g_{\nu,\gamma}\big[Y_\gamma\big]$, $\mathbb{P}$-a.s., which shows that the Y-component of the unique solution of DRBSDE(ξ, g, L, U) is a g-submartingale up to time τ^*_0 (see (4.3) for definition of g-martingales). Similarly, Y is a g-supermartingale up to time γ^*_0. Consequently, Y is a g-martingale up to time $\tau^*_0 \wedge \gamma^*_0$.
(2) In (2.3), if we regard L (resp. U) as the amount process a player will receive from, or pay to if the amount is negative, her opponent when the time τ she chooses to stop the game is earlier (resp. not earlier) than the stopping time γ selected by her opponent, then the Y-component of the unique solution of DRBSDE(ξ, g, L, U) is exactly the player's value of the Dynkin game under the g-evaluation. If the game starts at $\nu \in \mathcal{T}$, (2.2) shows that the first time τ^*_ν when the value process Y meets L after ν and the first time γ^*_ν when Y meets U after ν form a saddle point of the game.

3. BSDEs with integrable parameters

The derivation of Theorem 2.1 is based on the wellposedness result of BSDEs with integrable terminal data, i.e. Theorems 6.2 and 6.3 of [11] cited below as Proposition 3.1. Then in Section 5, we will exploit the *penalization* method to construct a unique solution of the corresponding reflected BSDEs with integrable parameters, with which we can adopt [25]'s approach of pasting local solutions to obtain Theorem 2.1.

Proposition 3.1. *Let g be a generator. For any $\xi \in L^1(\mathcal{F}_T)$, BSDE (ξ, g) admits a unique solution $(Y, Z) \in \cap_{p\in(0,1)}(\mathbb{S}^p \times \mathbb{H}^{2,p})$ such that Y is of class (D).*

This wellposedness result leads to a general martingale representation theorem:

Corollary 3.1. *For any $\xi \in L^1(\mathcal{F}_T)$, there exists a unique $Z \in \cap_{p\in(0,1)} \mathbb{H}^{2,p}$ such that $\mathbb{P}$-a.s.*

$$\mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s dB_s, \quad t \in [0, T]. \tag{3.1}$$

Proposition 3.1 also gives rise to "g-evaluation/expectation" (see next section), a nonlinear expectation under which the value of optimal stopping problem (resp. Dynkin game) solves the corresponding reflected BSDE (resp. double reflected BSDE) with generator g, see (5.2) (resp. (2.3)).

To derive a corresponding comparison result of Proposition 3.1 (which is crucial for the penalty method in solving reflected BSDEs with integrable parameters), we need the following mere generalization of Lemma 2.2 of [11] (cf. Corollary 1 of [30]):

Lemma 3.1. *Given $V \in \mathbb{V}^0$, if $(Y, Z) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0}$ satisfies that $\mathbb{P}$-a.s.*

$$Y_t = Y_0 + V_t - V_0 + \int_0^t Z_s dB_s, \quad t \in [0, T],$$

then it holds for any $p \in (1, \infty)$ that $\mathbb{P}$-a.s.

$$|Y_t|^p = |Y_0|^p + p\int_0^t sgn(Y_s)|Y_s|^{p-1} dV_s + p\int_0^t sgn(Y_s)|Y_s|^{p-1} Z_s dB_s + \frac{p(p-1)}{2}\int_0^t \mathbf{1}_{\{Y_s \neq 0\}}|Y_s|^{p-2}|Z_s|^2 ds, \quad t \in [0, T].$$

With help of Lemma 3.1, we can deduce a general comparison result for BSDEs over stochastic intervals, which is critical in proving Theorem 5.1 and our main result, Theorem 2.1:

Proposition 3.2. *Given $\nu, \tau \in \mathcal{T}$ with $\nu \le \tau$, for $i = 1, 2$ let $g^i : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be an $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})-$ measurable function and let $\left(Y^i, Z^i, V^i\right) \in \mathbb{S}^0 \times \mathbb{H}^{2,0} \times \mathbb{V}^0$ such that $\{Y^i_\gamma\}_{\gamma \in \mathcal{T}_{\nu,\tau}}$ is uniformly integrable, that $\mathbb{E}\Big[\big(\int_\nu^\tau |Z^i_t|^2 dt\big)^{p/2}\Big] < \infty$ for some $p \in (\alpha, 1)$, and that $\mathbb{P}$-a.s.*

$$Y^i_t = Y^i_\tau + \int_t^\tau g^i(s, Y^i_s, Z^i_s) ds + V^i_\tau - V^i_t - \int_t^\tau Z^i_s dB_s, \quad \forall\, t \in [\nu, \tau]. \tag{3.2}$$

Assume that $Y^1_\tau \le Y^2_\tau$, $\mathbb{P}$-a.s. and that $\mathbb{P}$-a.s.

$$\int_t^s \mathbf{1}_{\{Y^1_r > Y^2_r\}}(dV^1_r - dV^2_r) \le 0, \quad \forall\, t, s \in [\nu, \tau] \text{ with } t < s. \tag{3.3}$$

For either $i = 1$ or $i = 2$, if g^i satisfies (H1), (H2), (H5) *and if $g^1(t, Y^{3-i}_t, Z^{3-i}_t) \le g^2(t, Y^{3-i}_t, Z^{3-i}_t)$, $dt \otimes d\mathbb{P}$-a.s. on the stochastic interval $[\![\nu, \tau]\!] := \big\{(t, \omega) \in [0, T] \times \Omega : \nu(\omega) \le t \le \tau(\omega)\big\}$, then it holds $\mathbb{P}$-a.s. that $Y^1_t \le Y^2_t$ for any $t \in [\nu, \tau]$.*

Applying Proposition 3.2 over period $[0, T]$ with $V^1 = V^2 \equiv 0$, we obtain the following comparison result for BSDEs whose Y-solutions are of class (D) and whose Z-solutions are of $\mathbb{H}^{2,p}$ for some $p \in (\alpha, 1)$.

Proposition 3.3. *For $i = 1, 2$, given parameter pair $\left(\xi_i, g^i\right)$ with $\xi_1 \le \xi_2$, $\mathbb{P}$-a.s., let $\left(Y^i, Z^i\right)$ be a solution of BSDE $\left(\xi_i, g^i\right)$ such that Y^i is of class (D) and $Z^i \in \cup_{p\in(\alpha,1)} \mathbb{H}^{2,p}$. For either $i = 1$ or $i = 2$, if g^i satisfies* (H1),(H2),(H5) *and if $g^1(t, Y^{3-i}_t, Z^{3-i}_t) \le g^2(t, Y^{3-i}_t, Z^{3-i}_t)$, $dt \otimes d\mathbb{P}$-a.s., then it holds $\mathbb{P}$-a.s. that $Y^1_t \le Y^2_t$ for any $t \in [0, T]$.*

4. g-evaluations and g-expectations

Let g be a generator. For any $\tau \in \mathcal{T}$, since the function g_τ defined in (1.9) is a generator, Proposition 3.1 shows that for any $\xi \in L^1(\mathcal{F}_T)$, the BSDE$(\xi, g_\tau)$ admits a unique solution

$$\left(Y^{\tau,\xi}, Z^{\tau,\xi}\right) \in \cap_{p\in(0,1)}(\mathbb{S}^p \times \mathbb{H}^{2,p}) \tag{4.1}$$

such that $Y^{\tau,\xi}$ is of class (D). Then we can introduce the notion of "g-evaluation/expectation", which slightly generalizes the one initiated in [43,45]:

Definition 4.1. A family of operators $\mathcal{E}^g_{\nu,\tau} : L^0(\mathcal{F}_\tau) \to L^0(\mathcal{F}_\nu)$, $\nu \in \mathcal{T}, \tau \in \mathcal{T}_{\nu,T}$ is called a "g-evaluation" if for any $\nu, \tau \in \mathcal{T}$ with $\nu \le \tau$ and any $\xi \in L^0(\mathcal{F}_\tau)$,

$$\mathcal{E}^g_{\nu,\tau}[\xi] := \begin{cases} Y^{\tau,\xi}_\nu \in L^1(\mathcal{F}_\nu) & \text{if } \xi \in L^1(\mathcal{F}_\tau); \\ -\infty, & \text{if } \mathbb{E}[\xi^-] = \infty; \\ \infty, & \text{if } \mathbb{E}[\xi^-] < \infty \text{ and } \mathbb{E}[\xi^+] = \infty. \end{cases}$$

In particular, for any $\nu \in \mathcal{T}$ and $\xi \in L^0(\mathcal{F}_T)$ we refer to $\mathcal{E}^g[\xi|\mathcal{F}_\nu] := \mathcal{E}^g_{\nu,T}[\xi]$ as "g-expectation" of ξ conditional on the σ-field $\mathcal{F}_\nu$.

Remark 4.1. If g is independent of (y, z), i.e., if $\{g_t\}_{t\in[0,T]}$ is an $\mathbf{F}$-progressively measurable process with $\mathbb{E}\int_0^T |g_t|dt < \infty$, then for any $\nu \in \mathcal{T}, \tau \in \mathcal{T}_{\nu,T}$

$$\mathcal{E}^g_{\nu,\tau}[\xi] = \mathbb{E}\left[\xi + \int_\nu^\tau g_t dt \,\middle|\, \mathcal{F}_\nu\right], \mathbb{P}\text{-a.s.}, \qquad \forall\, \xi \in L^0(\mathcal{F}_\tau). \tag{4.2}$$

When $g \equiv 0$, the g-expectation degenerates into the classic linear expectation, i.e. for any $\nu \in \mathcal{T}$ and $\xi \in L^0(\mathcal{F}_T)$, $\mathcal{E}^g[\xi|\mathcal{F}_\nu] = \mathbb{E}[\xi|\mathcal{F}_\nu]$, $\mathbb{P}$-a.s.

In light of Proposition 3.3 and the uniqueness result in Proposition 3.1, one can deduce that g-evaluation with domain $L^1(\mathcal{F}_T)$ inherits the following basic properties from the classic linear expectation: Let $\nu, \tau \in \mathcal{T}$ with $\nu \le \tau$

(1) "Monotonicity": For any $\xi, \eta \in L^0(\mathcal{F}_\tau)$ with $\xi \le \eta$, $\mathbb{P}$-a.s. we have $\mathcal{E}^g_{\nu,\tau}[\xi] \le \mathcal{E}^g_{\nu,\tau}[\eta]$, $\mathbb{P}$-a.s.;

(2) "Time-consistency": For any $\gamma \in \mathcal{T}_{\nu,\tau}$ and $\xi \in L^1(\mathcal{F}_\tau)$, $\mathcal{E}^g_{\nu,\gamma}\big[\mathcal{E}^g_{\gamma,\tau}[\xi]\big] = \mathcal{E}^g_{\nu,\tau}[\xi]$, $\mathbb{P}$-a.s.;

(3) "Constant-Preserving": If it holds $dt \otimes d\mathbb{P}$-a.s. that $g(t, y, 0) = 0$, $\forall\, y \in \mathbb{R}$, then $\mathcal{E}^g_{\nu,\tau}[\xi] = \xi$, $\mathbb{P}$-a.s. for any $\xi \in L^1(\mathcal{F}_\nu)$;

(4) "Zero–one Law": For any $\xi \in L^1(\mathcal{F}_\tau)$ and $A \in \mathcal{F}_\nu$, we have $\mathbf{1}_A\mathcal{E}^g_{\nu,\tau}[\mathbf{1}_A\xi] = \mathbf{1}_A\mathcal{E}^g_{\nu,\tau}[\xi]$, $\mathbb{P}$-a.s.; In addition, if $g(t, 0, 0) = 0$, $dt \otimes d\mathbb{P}$-a.s., then $\mathcal{E}^g_{\nu,\tau}[\mathbf{1}_A\xi] = \mathbf{1}_A\mathcal{E}^g_{\nu,\tau}[\xi]$, $\mathbb{P}$-a.s.;

(5) "Translation Invariant": If g is independent of y, then $\mathcal{E}^g_{\nu,\tau}[\xi + \eta] = \mathcal{E}^g_{\nu,\tau}[\xi] + \eta$, $\mathbb{P}$-a.s. for any $\xi \in L^0(\mathcal{F}_\tau)$ and $\eta \in L^1(\mathcal{F}_\nu)$.

We can define the corresponding g-martingales as usual: A $\mathscr{B}\big([0, T]\big)\otimes\mathcal{F}$-measurable process X of class (D) is called an g-submartingale (resp. g-supermartingale or g-martingale) if for any $0 \le t \le s \le 0$

$$\mathcal{E}^g_{t,s}[X_s] \ge (\text{resp. } \le \text{ or } =)\, X_t, \qquad \mathbb{P}\text{-a.s.} \tag{4.3}$$

The g-martingales possess many classic martingale properties such as *Upcrossing inequality*, *Optional sampling theorem*, *Doob–Meyer decomposition* and etc., which relate the g-evaluation closely to risk measures in mathematical finance (see [46,47] for the case of Lipschitz g-evaluation with domain $L^2(\mathcal{F}_T)$ and see [40,34] for the case of quadratic g-evaluation with domain $L^\infty(\mathcal{F}_T)$). Due to the page limitation, we will elaborate neither on the martingale properties of our g-evaluation with domain $L^1(\mathcal{F}_T)$ nor on the connection of this g-evaluation to risk measures in the present paper.

5. Reflected BSDEs with integrable parameters and related optimal stopping problems

With Propositions 3.1 and 3.3, we can employ the *penalization* method to obtain, as an intermediate step towards our goal (Theorem 2.1), the following wellposedness result of a reflected BSDE with integrable parameters, in which the Y-component of the unique solution stands for the value of the related optimal stopping problem under g-evaluation.

Theorem 5.1. *Let g be a generator. For any $\xi \in L^1(\mathcal{F}_T)$ and $L \in \mathbb{S}^1_+$ with $L_T \le \xi$, $\mathbb{P}$-a.s., RBSDE (ξ, g, L) admits a unique solution $(Y, Z, K) \in \cap_{p\in(0,1)}(\mathbb{S}^p \times \mathbb{H}^{2,p} \times \mathbb{K}^p)$ such that Y is of class (D).*

Define $\mathcal{R}_t := \mathbf{1}_{\{t<T\}} L_t + \mathbf{1}_{\{t=T\}}\xi$, $t \in [0, T]$. Let $\nu \in \mathcal{T}$ and $\tau_\sharp(\nu) := \inf\big\{t \in [\nu, T] : Y_t = \mathcal{R}_t\big\} \in \mathcal{T}_{\nu,T}$. It holds for any $\gamma \in \mathcal{T}_{\nu,T}$ that

$$\mathcal{E}^g_{\nu,\gamma}\big[Y_\gamma\big] \le Y_\nu = \mathcal{E}^g_{\nu,\tau_\sharp(\nu)\wedge\gamma}\big[Y_{\tau_\sharp(\nu)\wedge\gamma}\big], \quad \mathbb{P}\text{-a.s.} \tag{5.1}$$

In particular, we have

$$Y_\nu = \operatorname*{esssup}_{\gamma\in\mathcal{T}_{\nu,T}} \mathcal{E}^g_{\nu,\gamma}\big[\mathcal{R}_\gamma\big] = \mathcal{E}^g_{\nu,\tau_\sharp(\nu)}\big[\mathcal{R}_{\tau_\sharp(\nu)}\big], \quad \mathbb{P}\text{-a.s.} \tag{5.2}$$

Remark 5.1. (1) In view of (5.1), the Y-component of the unique solution of RBSDE(ξ, g, L) is a g-supermartingale. For any $\nu, \tau \in \mathcal{T}$ with $0 \le \nu \le \tau \le \tau_\sharp(0)$, it is clear that $\tau_\sharp(\nu) = \tau_\sharp(0)$, $\mathbb{P}$-a.s. Then we have $Y_\nu = \mathcal{E}^g_{\nu,\tau_\sharp(\nu)\wedge\gamma}\big[Y_{\tau_\sharp(\nu)\wedge\gamma}\big] = \mathcal{E}^g_{\nu,\gamma}\big[Y_\gamma\big]$, $\mathbb{P}$-a.s., which shows that Y is a g-martingale up to time $\tau_\sharp(0)$.
(2) In (5.2), if we regard $\mathcal{R}$ as a reward process that include a running reward L and a terminal reward ξ, then the Y-component of the unique solution of RBSDE(ξ, g, L) is exactly the Snell envelope of $\mathcal{R}$ under the g-evaluation. Given a start time $\nu \in \mathcal{T}$, the first time $\tau_\sharp(\nu)$ when Y meets $\mathcal{R}$ after ν is an optimal stopping time for a player to choose if she is aimed to maximize her expected reward under g-expectation.

To derive the existence result in Theorem 5.1, we will use penalization method which can be summarized in the following two monotonicity results:

Proposition 5.1. *Let $L \in \mathbb{S}^1_+$ and let $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function satisfying* (H1), (H4) *and* (H5). *For any $n \in \mathbb{N}$, consider the function g_n defined in (1.10) and let $(Y^n, Z^n, J^n) \in \big(\cap_{p\in(0,1)} \mathbb{S}^p\big) \times \mathbb{H}^{2,0} \times \mathbb{K}^0$ such that Y^n is of class (D) and that $\mathbb{P}$-a.s.*

$$Y^n_t = Y^n_T + \int_t^T g_n(s, Y^n_s, Z^n_s)ds - J^n_T + J^n_t - \int_t^T Z^n_s dB_s, \quad t \in [0, T].$$

If $\{Y^n\}_{n\in\mathbb{N}}$ is an increasing sequence of processes, then its limit $Y_t := \lim_{n\to\infty} \uparrow Y^n_t$, $t \in [0, T]$ is an $\mathbf{F}$-predictable process of class (D) that satisfies $\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^p] < \infty$, $\forall\, p \in (0, 1)$.

Proposition 5.2. *Let $L \in \mathbb{S}^1_+$, let $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function satisfying* (H1)–(H4)*, and let $\nu, \tau \in \mathcal{T}$ with $\nu \le \tau$. For any $n \in \mathbb{N}$, consider the function g_n defined in (1.10) and let $(Y^n, Z^n) \in \mathbb{S}^0 \times \mathbb{H}^{2,0}$ satisfies that $\mathbb{P}$-a.s.*

$$Y^n_t = Y^n_\tau + \int_t^\tau g_n(s, Y^n_s, Z^n_s) ds - \int_t^\tau Z^n_s dB_s, \quad \forall\, t \in [\nu, \tau]. \tag{5.3}$$

If $\left\{ \mathbf{1}_{\{t \ge \nu\}} Y^n_{\tau \wedge t} \right\}_{t \in [0,T]}$, $n \in \mathbb{N}$ is an increasing sequence of processes whose limit $Y_t := \lim_{n \to \infty} \uparrow \mathbf{1}_{\{t \ge \nu\}} Y^n_{\tau \wedge t}$, $t \in [0,T]$ satisfies $\mathbb{P}\{Y_\tau \ge L_\tau\} = \mathbb{P}\Big\{ \sup_{t \in [\nu,\tau]} \big((Y^1_t)^- + Y^+_t \big) < \infty \Big\} = 1$, then process $\{Y_{\nu \vee t}\}_{t \in [0,T]}$ has $\mathbb{P}$-a.s. continuous paths and there exist $(Z, K) \in \mathbb{H}^{2,0} \times \mathbb{K}^0$ such that $\mathbb{P}$-a.s.

$$\begin{cases} L_t \le Y_t = Y_\tau + \displaystyle\int_t^\tau g(s, Y_s, Z_s) ds + K_\tau - K_t - \int_t^\tau Z_s dB_s, & \forall\, t \in [\nu, \tau], \\ \displaystyle\int_\nu^\tau (Y_t - L_t) dK_t = 0. \end{cases} \tag{5.4}$$

On the other hand, the uniqueness result in Theorem 5.1 follows from the following comparison result for reflected BSDEs whose Y-solutions are of class (D) and whose Z-solutions are of $\mathbb{H}^{2,p}$ for some $p \in (\alpha, 1)$.

Proposition 5.3. *For $i = 1, 2$, given parameter pair (ξ_i, g^i) and $L^i \in \mathbb{S}^1_+$ such that $\mathbb{P}\{L^i_T \le \xi_i\} = \mathbb{P}\{\xi_1 \le \xi_2\} = \mathbb{P}\{L^1_t \le L^2_t,\ \forall\, t \in [0,T]\} = 1$, let (Y^i, Z^i, K^i) be a solution of RBSDE (ξ_i, g^i, L^i) such that Y^i is of class (D) and $Z^i \in \cup_{p \in (\alpha,1)} \mathbb{H}^{2,p}$. For either $i = 1$ or $i = 2$, if g^i satisfies* (H1), (H2), (H5) *and if $g^1(t, Y^{3-i}_t, Z^{3-i}_t) \le g^2(t, Y^{3-i}_t, Z^{3-i}_t)$, $dt \otimes d\mathbb{P}$-a.s., then it holds $\mathbb{P}$-a.s. that $Y^1_t \le Y^2_t$ for any $t \in [0,T]$.*

Remark 5.2. By Remark 1.3(1), one can apply Theorem 5.1, Propositions 5.2 and 5.3 to g_- (defined in (1.7)) to obtain a version of them for the reflected BSDE with upper obstacle like (1.8).

6. Proofs

6.1. Proofs of the results in Sections 3 and 4

Proof of Proposition 3.1. As condition (H7) of [11] is automatically satisfied, it suffices to verify condition (H5) therein, i.e., Given $r \ge 0$,

the process $\psi^r_t(\omega) := \sup_{|y| \le r} |g(t,\omega,y,0) - g(t,\omega,0,0)|$, $(t,\omega) \in [0,T] \times \Omega$ is integrable.

By (H3), it holds $dt \otimes d\mathbb{P}$-a.s. that $\psi^r_t(\omega) = \sup_{y \in [-r,r] \cap \mathbb{Q}} |g(t,\omega,y,0) - g(t,\omega,0,0)|$, which implies that ψ^r is $\mathbf{F}$-progressively measurable. Also, (H4) shows that $dt \otimes d\mathbb{P}$-a.s., $\psi^r_t(\omega) \le |g(t,\omega,0,0)| + \sup_{|y| \le r} |g(t,\omega,y,0)| \le 2h_t(\omega) + \kappa r$. It follows that ψ^r_t belongs to $\mathbb{L}^1([0,T] \times \Omega, \mathscr{B}([0,T]) \otimes \mathcal{F}, dt \otimes \mathbb{P})$. □

Proof of Corollary 3.1. Clearly, $\mathfrak{g}(t,\omega,y,z) := 0$, $\forall\, (t,\omega,y,z) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ is a generator. In light of Proposition 3.1, BSDE$(\xi, 0)$ admits a unique solution $(Y, Z) \in \cap_{p \in (0,1)}$

$(\mathbb{S}^p \times \mathbb{H}^{2,p})$ such that Y is of class (D). For any $n \in \mathbb{N}$, we define stopping time $\tau_n := \inf\big\{t \in [0,T] : \int_0^t |Z_s|^2 ds > n\big\} \wedge T \in \mathcal{T}$, and see from $Z \in \cap_{p\in(0,1)} \mathbb{H}^{2,p} \subset \mathbb{H}^{2,0}$ that $\{\tau_n\}_{n\in\mathbb{N}}$ is stationary.

Let $t \in [0,T]$ and $n \in \mathbb{N}$. Since $Y_{\tau_n \wedge t} = Y_{\tau_n} - \int_{\tau_n\wedge t}^{\tau_n} Z_s dB_s$, $\mathbb{P}$-a.s., taking conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ yields that

$$Y_{\tau_n\wedge t} = \mathbb{E}[Y_{\tau_n}|\mathcal{F}_t], \quad \mathbb{P}\text{-a.s.} \tag{6.1}$$

As $\{\tau_n\}_{n\in\mathbb{N}}$ is stationary, letting $n \to \infty$ in (6.1), we can deduce from the continuity of Y and the uniform integrability of $\{Y_\tau\}_{\tau\in\mathcal{T}}$ that $Y_t = \mathbb{E}[Y_T|\mathcal{F}_t] = \mathbb{E}[\xi|\mathcal{F}_t]$, $\mathbb{P}$-a.s. In particular, $Y_0 = \mathbb{E}[\xi]$. Then

$$\mathbb{E}[\xi|\mathcal{F}_t] = Y_t = Y_0 + \int_0^t Z_s dB_s = \mathbb{E}[\xi] + \int_0^t Z_s dB_s, \quad \mathbb{P}\text{-a.s.}$$

This together with the continuity of processes $\big\{\mathbb{E}[\xi|\mathcal{F}_\cdot]\big\}_{t\in[0,T]}$ and $\big\{\int_0^t Z_s dB_s\big\}_{t\in[0,T]}$ leads to (3.1) while the uniqueness of process Z is clear. □

Proof of Proposition 3.2. Without loss of generality, suppose that g^1 satisfies (H1), (H2), (H5) and that

$$g^1(t, Y_t^2, Z_t^2) \le g^2(t, Y_t^2, Z_t^2), \quad dt \otimes d\mathbb{P}\text{-a.s. on } [\![\nu, \tau]\!]. \tag{6.2}$$

Set $(\mathcal{Y}, \mathcal{Z}) := (Y^1 - Y^2, Z^1 - Z^2)$ and $q := p/\alpha \in (1, 1/\alpha)$.

(1) *We first show that* $\mathbb{E}[\sup_{t\in[\nu,\tau]} (\mathcal{Y}_t^+)^q] < \infty$.

Since $\mathbb{E}[|\mathcal{Y}_\nu|] \le \mathbb{E}\big[|Y_\nu^1|\big] + \mathbb{E}\big[|Y_\nu^2|\big] < \infty$ by the uniform integrability of $\{Y_\gamma^i\}_{\gamma\in\mathcal{T}_{\nu,\tau}}$, $i = 1, 2$, Corollary 3.1 implies that there exists a unique $\widetilde{\mathcal{Z}} \in \cap_{p'\in(0,1)} \mathbb{H}^{2,p'}$ such that $\mathbb{P}\big\{\mathbb{E}[\mathcal{Y}_\nu|\mathcal{F}_t] = \mathbb{E}[\mathcal{Y}_\nu] + \int_0^t \widetilde{\mathcal{Z}}_s dB_s, \ \forall\, t \in [0,T]\big\} = 1$. This together with (3.2) shows that $\mathbb{P}$-a.s.

$$\begin{aligned}
\widetilde{\mathcal{Y}}_t &:= \mathbb{E}[\mathcal{Y}_\nu|\mathcal{F}_{\nu\wedge t}] + \mathcal{Y}_{\nu\vee(\tau\wedge t)} - \mathcal{Y}_\nu = \mathbb{E}[\mathcal{Y}_\nu] + \int_0^{\nu\wedge t} \widetilde{\mathcal{Z}}_s dB_s \\
&\quad - \int_\nu^{\nu\vee(\tau\wedge t)} \Delta g_s ds - V^1_{\nu\vee(\tau\wedge t)} + V^1_\nu + V^2_{\nu\vee(\tau\wedge t)} - V^2_\nu + \int_\nu^{\nu\vee(\tau\wedge t)} \mathcal{Z}_s dB_s \\
&= \mathbb{E}[\mathcal{Y}_\nu] - \int_0^t \mathbf{1}_{\{\nu<s\le\tau\}} \Delta g_s ds - \int_0^t \mathbf{1}_{\{\nu<s\le\tau\}} (dV_s^1 - dV_s^2) \\
&\quad + \int_0^t \big(\mathbf{1}_{\{s\le\nu\}} \widetilde{\mathcal{Z}}_s + \mathbf{1}_{\{\nu<s\le\tau\}} \mathcal{Z}_s\big) dB_s, \quad t \in [0,T],
\end{aligned} \tag{6.3}$$

where $\Delta g_s := g^1(s, Y_s^1, Z_s^1) - g^2(s, Y_s^2, Z_s^2)$. So $\widetilde{\mathcal{Y}}$ is an $\mathbf{F}$-adapted continuous process, i.e. $\widetilde{\mathcal{Y}} \in \mathbb{S}^0$. Applying Itô–Tanaka's formula to process $\widetilde{\mathcal{Y}}^+$ yields that $\mathbb{P}$-a.s.

$$\begin{aligned}
\widetilde{\mathcal{Y}}_t^+ &= \big(\mathbb{E}[\mathcal{Y}_\nu]\big)^+ - \int_0^t \mathbf{1}_{\{\widetilde{\mathcal{Y}}_s>0\}} \mathbf{1}_{\{\nu<s\le\tau\}} \Delta g_s ds + \frac{1}{2}\mathfrak{L}_t \\
&\quad - \int_0^t \mathbf{1}_{\{\widetilde{\mathcal{Y}}_s>0\}} \mathbf{1}_{\{\nu<s\le\tau\}} (dV_s^1 - dV_s^2) \\
&\quad + \int_0^t \mathbf{1}_{\{\widetilde{\mathcal{Y}}_s>0\}} \big(\mathbf{1}_{\{s\le\nu\}} \widetilde{\mathcal{Z}}_s + \mathbf{1}_{\{\nu<s\le\tau\}} \mathcal{Z}_s\big) dB_s, \quad t \in [0,T],
\end{aligned} \tag{6.4}$$

where $\mathfrak{L}$ is an $\mathbf{F}$-adapted, continuous increasing process known as the "local time" of $\widetilde{\mathcal{Y}}$ at 0.

Let $n \in \mathbb{N}$. We define a stopping time $\tau_n := \inf\{t \in [\nu, \tau] : \int_\nu^t |\mathcal{Z}_s|^2 ds > n\} \wedge \tau \in \mathcal{T}_{\nu,\tau}$, and integrate by parts the process $\{e^{\lambda^+(\tau_n \wedge t)} \widetilde{\mathcal{Y}}^+_{\tau_n \wedge t}\}_{t\in[0,T]}$ to obtain that $\mathbb{P}$-a.s.

$$\begin{aligned} e^{\lambda^+(\tau_n \wedge t)} \widetilde{\mathcal{Y}}^+_{\tau_n \wedge t} &= e^{\lambda^+ \tau_n} \mathcal{Y}^+_{\tau_n} + \int_{\tau_n \wedge t}^{\tau_n} \mathbf{1}_{\{\mathcal{Y}_s > 0\}} \mathbf{1}_{\{s>\nu\}} e^{\lambda^+ s} \Delta g_s ds \\ &\quad + \int_{\tau_n \wedge t}^{\tau_n} \mathbf{1}_{\{\mathcal{Y}_s > 0\}} \mathbf{1}_{\{s>\nu\}} e^{\lambda^+ s} (dV_s^1 - dV_s^2) \\ &\quad - \frac{1}{2} \int_{\tau_n \wedge t}^{\tau_n} e^{\lambda^+ s} d\mathfrak{L}_s - \lambda^+ \int_{\tau_n \wedge t}^{\tau_n} e^{\lambda^+ s} \widetilde{\mathcal{Y}}^+_s ds \\ &\quad - \int_{\tau_n \wedge t}^{\tau_n} \mathbf{1}_{\{\widetilde{\mathcal{Y}}_s > 0\}} e^{\lambda^+ s} \left(\mathbf{1}_{\{s\le\nu\}} \widetilde{\mathcal{Z}}_s + \mathbf{1}_{\{s>\nu\}} \mathcal{Z}_s\right) dB_s, \quad t \in [0, T]. \end{aligned} \tag{6.5}$$

Here we used the fact that $\widetilde{\mathcal{Y}}_{\nu \vee (\tau \wedge t)} = \mathbb{E}[\mathcal{Y}_\nu | \mathcal{F}_\nu] + \mathcal{Y}_{\nu\vee(\tau\wedge t)} - \mathcal{Y}_\nu = \mathcal{Y}_{\nu\vee(\tau\wedge t)}$, $\forall\, t \in [0, T]$, i.e.

$$\widetilde{\mathcal{Y}}_t = \mathcal{Y}_t, \qquad \forall\, t \in [\nu, \tau]. \tag{6.6}$$

Since g^1 satisfies (H2) and (H5), it holds $ds \otimes d\mathbb{P}$-a.s. on $[\![\nu, \tau]\!]$ that

$$\begin{aligned} &\mathbf{1}_{\{\mathcal{Y}_s(\omega)>0\}} \Big(g^1\left(s, \omega, Y_s^1(\omega), Z_s^1(\omega)\right) - g^1\left(s, \omega, Y_s^2(\omega), Z_s^1(\omega)\right)\Big) \\ &\quad \le \mathbf{1}_{\{\mathcal{Y}_s(\omega)>0\}} \lambda \mathcal{Y}_s^+(\omega) \le \lambda^+ \mathcal{Y}_s^+(\omega), \end{aligned} \tag{6.7}$$

and that

$$\begin{aligned} &\left| g^1\left(s, \omega, Y_s^2(\omega), Z_s^1(\omega)\right) - g^1\left(s, \omega, Y_s^2(\omega), Z_s^2(\omega)\right)\right| \\ &\quad \le \kappa \left(h_s(\omega) + |Y_s^2(\omega)| + |Z_s^1(\omega)|\right)^\alpha + \kappa\left(h_s(\omega) + |Y_s^2(\omega)| + |Z_s^2(\omega)|\right)^\alpha. \end{aligned}$$

Plugging them back into (6.5) and taking $t = \nu \vee t$ there, we see from (3.3) and (6.2) that $\mathbb{P}$-a.s.

$$\begin{aligned} &e^{\lambda^+(\nu\vee(\tau_n\wedge t))} \mathcal{Y}^+_{\nu\vee(\tau_n\wedge t)} \le e^{\lambda^+\tau_n} \mathcal{Y}^+_{\tau_n} + 2\kappa e^{\lambda^+ T} \eta - \int_{\nu\vee(\tau_n\wedge t)}^{\tau_n} \mathbf{1}_{\{\mathcal{Y}_s>0\}} e^{\lambda^+ s} \mathcal{Z}_s dB_s, \\ &t \in [0, T], \end{aligned} \tag{6.8}$$

where $\eta := \int_\nu^\tau \left(h_t + |Y_t^2| + |Z_t^1| + |Z_t^2|\right)^\alpha dt$.

Let $t \in [0, T]$. Taking conditional expectation $\mathbb{E}\big[\cdot|\mathcal{F}_{\nu\vee(\tau_n\wedge t)}\big]$ in (6.8) yields that $e^{\lambda^+(\nu\vee(\tau_n\wedge t))}$ $\mathcal{Y}^+_{\nu\vee(\tau_n\wedge t)} \le \mathbb{E}\left[e^{\lambda^+\tau_n}\mathcal{Y}^+_{\tau_n} + 2\kappa e^{\lambda^+ T}\eta \big| \mathcal{F}_{\nu\vee(\tau_n\wedge t)}\right]$, $\mathbb{P}$-a.s., and it follows that

$$\mathbf{1}_{\{\nu\le t\le\tau_n\}} e^{\lambda^+ t} \mathcal{Y}_t^+ \le \mathbf{1}_{\{\nu\le t\le\tau_n\}} \mathbb{E}\left[e^{\lambda^+\tau_n}\mathcal{Y}^+_{\tau_n} + 2\kappa e^{\lambda^+ T}\eta \big| \mathcal{F}_t\right], \quad \mathbb{P}\text{-a.s.} \tag{6.9}$$

By (1.5) and Hölder's inequality,

$$\begin{aligned} \eta &\le \int_\nu^\tau \left(h_t + |Y_t^2|\right)^\alpha dt + \sum_{i=1}^2 \int_\nu^\tau |Z_t^i|^\alpha dt \le T^{1-\alpha}\left(\int_\nu^\tau \left(h_t + |Y_t^2|\right) dt\right)^\alpha \\ &\quad + T^{1-\alpha/2} \sum_{i=1}^2 \left(\int_\nu^\tau |Z_t^i|^2 dt\right)^{\alpha/2}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Fubini's Theorem and the uniform integrability of $\{Y^2_\gamma\}_{\gamma\in\mathcal{T}_{\nu,\tau}}$ imply that

$$\mathbb{E}\int_\nu^\tau |Y^2_t|dt = \mathbb{E}\int_\nu^\tau |Y^2_{\nu\vee(\tau\wedge t)}|dt \le \mathbb{E}\int_0^T \left|Y^2_{\nu\vee(\tau\wedge t)}\right|dt = \int_0^T \mathbb{E}\left[\left|Y^2_{\nu\vee(\tau\wedge t)}\right|\right]dt$$
$$\le T \sup_{\gamma\in\mathcal{T}_{\nu,\tau}} \mathbb{E}\left[|Y^2_\gamma|\right] < \infty.$$

As $q = p/\alpha$, applying (1.5) and Hölder's inequality again yields that

$$\mathbb{E}[\eta^q] \le 3^{q-1}T^{(1-\alpha)q}\left\{\mathbb{E}\int_\nu^\tau \left(h_t + |Y^2_t|\right)dt\right\}^p$$
$$+ 3^{q-1}T^{(1-\alpha/2)q}\sum_{i=1}^2 \mathbb{E}\left[\left(\int_\nu^\tau |Z^i_t|^2 dt\right)^{p/2}\right] < \infty. \tag{6.10}$$

We see from $\mathbb{E}\Big[\big(\int_\nu^\tau |Z^i_t|^2dt\big)^{p/2}\Big] < \infty$, $i = 1, 2$ that for $\mathbb{P}$-a.s. $\omega \in \Omega$, $\tau(\omega) = \tau_{N_\omega}(\omega)$ for some $N_\omega \in \mathbb{N}$. For any $t \in [0, T]$, since the uniform integrability of $\{Y^i_\gamma\}_{\gamma\in\mathcal{T}_{\nu,\tau}}$, $i = 1, 2$ implies that of $\{e^{\lambda^+\gamma}\,\mathcal{Y}^+_\gamma\}_{\gamma\in\mathcal{T}_{\nu,\tau}}$, letting $n \to \infty$ in (6.9) yields that $\mathbb{P}$-a.s.

$$\mathbf{1}_{\{\nu\le t\le\tau\}}\mathcal{Y}^+_t \le \mathbf{1}_{\{\nu\le t\le\tau\}}e^{\lambda^+t}\mathcal{Y}^+_t \le \mathbf{1}_{\{\nu\le t\le\tau\}}2\kappa e^{\lambda^+T}\mathbb{E}\left[e^{\lambda^+\tau}(Y^1_\tau - Y^2_\tau)^+ + \eta|\mathcal{F}_t\right]$$
$$= \mathbf{1}_{\{\nu\le t\le\tau\}}2\kappa e^{\lambda^+T}\mathbb{E}[\eta|\mathcal{F}_t].$$

Using the continuity of $\mathcal{Y}^+$ and that of process $\{\mathbb{E}[\eta|\mathcal{F}_t]\}_{t\in[0,T]}$, one gets $\mathbb{P}\{\mathcal{Y}^+_t \le 2\kappa e^{\lambda^+T}\,\mathbb{E}[\eta|\mathcal{F}_t],\ \forall t \in [\nu, \tau]\} = 1$. Then Doob's martingale inequality and (6.10) lead to that

$$\mathbb{E}\left[\sup_{t\in[\nu,\tau]}(\mathcal{Y}^+_t)^q\right] \le (2\kappa)^q e^{q\lambda^+T}\mathbb{E}\left[\sup_{t\in[0,T]}(\mathbb{E}[\eta|\mathcal{F}_t])^q\right]$$
$$\le \left(\frac{q}{q-1}\right)^q (2\kappa)^q e^{q\lambda^+T}\mathbb{E}[\eta^q] < \infty. \tag{6.11}$$

(2) *Next, we show that* $\mathbb{E}[\sup_{t\in[\nu,\tau]}(\mathcal{Y}^+_t)^q] = 0$ *indeed; then the conclusion easily follows.*
According to (6.4), applying Lemma 3.1 yields that $\mathbb{P}$-a.s.

$$(\widetilde{\mathfrak{Y}}^+_t)^q = \left(\left(\mathbb{E}[\mathcal{Y}_\nu]\right)^+\right)^q - q\int_0^t \mathbf{1}_{\{\widetilde{\mathfrak{Y}}_s>0\}}\mathbf{1}_{\{\nu<s\le\tau\}}\left(\widetilde{\mathfrak{Y}}^+_s\right)^{q-1}\Delta g_s ds + \frac{q}{2}\int_0^t \left(\widetilde{\mathfrak{Y}}^+_s\right)^{q-1}d\mathfrak{L}_s$$
$$- q\int_0^t \mathbf{1}_{\{\widetilde{\mathfrak{Y}}_s>0\}}\mathbf{1}_{\{\nu<s\le\tau\}}\left(\widetilde{\mathfrak{Y}}^+_s\right)^{q-1}(dV^1_s - dV^2_s) + q\int_0^t \mathbf{1}_{\{\widetilde{\mathfrak{Y}}_s>0\}}\left(\widetilde{\mathfrak{Y}}^+_s\right)^{q-1}$$
$$\times\left(\mathbf{1}_{\{s\le\nu\}}\widetilde{\mathcal{Z}}_s + \mathbf{1}_{\{\nu<s\le\tau\}}\mathcal{Z}_s\right)dB_s + \frac{q(q-1)}{2}\int_0^t \mathbf{1}_{\{\widetilde{\mathfrak{Y}}_s>0\}}\left(\widetilde{\mathfrak{Y}}^+_s\right)^{q-2}$$
$$\times\left(\mathbf{1}_{\{s\le\nu\}}\big|\widetilde{\mathcal{Z}}_s\big|^2 + \mathbf{1}_{\{\nu<s\le\tau\}}|\mathcal{Z}_s|^2\right)ds, \quad t \in [0, T].$$

Set $a := \lambda^+ + \frac{\kappa^2}{1\wedge(q-1)}$ and let $n \in \mathbb{N}$. We define a stopping time $\gamma_n := \inf\Big\{t \in [\nu, \tau] : \sup_{s\in[\nu,t]}\mathcal{Y}^+_s + \int_\nu^t |\mathcal{Z}_s|^2ds > n\Big\} \wedge \tau \in \mathcal{T}_{\nu,\tau}$, and integrate by parts the process $\Big\{e^{aq(\gamma_n\wedge t)}$

$\big(\widetilde{\mathcal{Y}}^+_{\gamma_n\wedge t}\big)^q\Big\}_{t\in[0,T]}$ to obtain that $\mathbb{P}$-a.s.

$$\begin{aligned}
&e^{aq(\gamma_n\wedge t)}\left(\widetilde{\mathcal{Y}}^+_{\gamma_n\wedge t}\right)^q+\frac{q(q-1)}{2}\int_{\gamma_n\wedge t}^{\gamma_n}\mathbf{1}_{\{\widetilde{\mathcal{Y}}_s>0\}}e^{aqs}\left(\widetilde{\mathcal{Y}}^+_s\right)^{q-2}\\
&\quad\times\left(\mathbf{1}_{\{s\le\nu\}}\big|\widetilde{\mathcal{Z}}_s\big|^2+\mathbf{1}_{\{s>\nu\}}|\mathcal{Z}_s|^2\right)ds\\
&\quad=e^{aq\gamma_n}\left(\mathcal{Y}^+_{\gamma_n}\right)^q+q\int_{\gamma_n\wedge t}^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}\cap\{s>\nu\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}\Delta g_s ds-aq\int_{\gamma_n\wedge t}^{\gamma_n}e^{aqs}\left(\widetilde{\mathcal{Y}}^+_s\right)^q ds\\
&\quad-\frac{q}{2}\int_{\gamma_n\wedge t}^{\gamma_n}e^{aqs}\left(\widetilde{\mathcal{Y}}^+_s\right)^{q-1}d\mathfrak{L}_s\\
&\quad+q\int_{\gamma_n\wedge t}^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}\cap\{s>\nu\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}(dV^1_s-dV^2_s)\\
&\quad-q\int_{\gamma_n\wedge t}^{\gamma_n}\mathbf{1}_{\{\widetilde{\mathcal{Y}}_s>0\}}e^{aqs}\left(\widetilde{\mathcal{Y}}^+_s\right)^{q-1}\left(\mathbf{1}_{\{s\le\nu\}}\widetilde{\mathcal{Z}}_s+\mathbf{1}_{\{s>\nu\}}\mathcal{Z}_s\right)dB_s,\quad t\in[0,T].
\end{aligned}$$

Then (6.6), (6.2), (6.7) and (3.3) imply that $\mathbb{P}$-a.s.

$$\begin{aligned}
&e^{aqt}\left(\mathcal{Y}^+_t\right)^q+\frac{q(q-1)}{2}\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-2}|\mathcal{Z}_s|^2ds\\
&\quad=e^{aq\gamma_n}\left(\mathcal{Y}^+_{\gamma_n}\right)^q+q\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}\Delta g_s ds-aq\int_t^{\gamma_n}e^{aqs}\left(\mathcal{Y}^+_s\right)^q ds\\
&\quad-\frac{q}{2}\int_t^{\gamma_n}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}d\mathfrak{L}_s\\
&\quad+q\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}(dV^1_s-dV^2_s)-q\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}\mathcal{Z}_s dB_s\\
&\quad\le e^{aq\gamma_n}\left(\mathcal{Y}^+_{\gamma_n}\right)^q+q\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}\left(g^1(s,Y^2_s,Z^1_s)-g^1(s,Y^2_s,Z^2_s)\right)ds\\
&\quad-\frac{q\kappa^2}{1\wedge(q-1)}\int_t^{\gamma_n}e^{aqs}\left(\mathcal{Y}^+_s\right)^q ds\\
&\quad-q\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}\mathcal{Z}_s dB_s,\qquad\forall\, t\in[\nu,\gamma_n].
\end{aligned}$$

Since $\mathbf{1}_{\{\mathcal{Y}_t>0\}}\kappa\left(\mathcal{Y}^+_t\right)^{q-1}|\mathcal{Z}_t|\le\frac{q-1}{4}\mathbf{1}_{\{\mathcal{Y}_t>0\}}\left(\mathcal{Y}^+_t\right)^{q-2}|\mathcal{Z}_t|^2+\frac{\kappa^2}{q-1}\left(\mathcal{Y}^+_t\right)^q$, $\forall\, t\in[\nu,\tau]$, we can deduce from (H1) that $\mathbb{P}$-a.s.

$$\begin{aligned}
&e^{aqt}\left(\mathcal{Y}^+_t\right)^q+\frac{q(q-1)}{4}\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-2}|\mathcal{Z}_s|^2ds\\
&\quad\le e^{aq\gamma_n}\left(\mathcal{Y}^+_{\gamma_n}\right)^q-q\int_t^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-1}\mathcal{Z}_s dB_s,\quad\forall\, t\in[\nu,\gamma_n].
\end{aligned}$$

Taking expectation for $t=\nu$ shows that

$$\frac{q(q-1)}{4}\mathbb{E}\int_\nu^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\left(\mathcal{Y}^+_s\right)^{q-2}|\mathcal{Z}_s|^2ds\le\mathbb{E}\Big[e^{aq\gamma_n}\left(\mathcal{Y}^+_{\gamma_n}\right)^q\Big].\tag{6.12}$$

On the other hand, the Burkholder–Davis–Gundy inequality implies that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[\nu,\gamma_n]}\big(e^{at}\mathcal{Y}_t^+\big)^q\Big]\\
&\le \mathbb{E}\Big[e^{aq\gamma_n}\big(\mathcal{Y}_{\gamma_n}^+\big)^q\Big]+q\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_t^T \mathbf{1}_{\{\nu\le s\le\gamma_n\}}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\big(\mathcal{Y}_s^+\big)^{q-1}\mathcal{Z}_s dB_s\Big|\Big]\\
&\le \mathbb{E}\Big[e^{aq\gamma_n}\big(\mathcal{Y}_{\gamma_n}^+\big)^q\Big]+C_q\mathbb{E}\Big[\Big(\sup_{t\in[\nu,\gamma_n]}\big(e^{at}\mathcal{Y}_t^+\big)^{q/2}\Big)\\
&\quad\cdot\Big(\int_\nu^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\big(\mathcal{Y}_s^+\big)^{q-2}|\mathcal{Z}_s|^2ds\Big)^{1/2}\Big]\\
&\le \mathbb{E}\Big[e^{aq\gamma_n}\big(\mathcal{Y}_{\gamma_n}^+\big)^q\Big]+\frac{1}{2}\mathbb{E}\Big[\sup_{t\in[\nu,\gamma_n]}\big(e^{at}\mathcal{Y}_t^+\big)^q\Big]\\
&\quad+C_q\mathbb{E}\int_\nu^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\big(\mathcal{Y}_s^+\big)^{q-2}|\mathcal{Z}_s|^2ds.
\end{aligned}$$

As $\mathbb{E}[\sup_{t\in[\nu,\gamma_n]}(e^{at}\mathcal{Y}_t^+)^q]\le e^{aqT}\mathbb{E}[\sup_{t\in[\nu,\tau]}(\mathcal{Y}_t^+)^q]<\infty$ by (6.11), it follows from (6.12) that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[\nu,\gamma_n]}\big(\mathcal{Y}_t^+\big)^q\Big]&\le\mathbb{E}\Big[\sup_{t\in[\nu,\gamma_n]}\big(e^{at}\mathcal{Y}_t^+\big)^q\Big]\le 2\mathbb{E}\Big[e^{aq\gamma_n}\big(\mathcal{Y}_{\gamma_n}^+\big)^q\Big]\\
&\quad+C_q\mathbb{E}\int_\nu^{\gamma_n}\mathbf{1}_{\{\mathcal{Y}_s>0\}}e^{aqs}\big(\mathcal{Y}_s^+\big)^{q-2}|\mathcal{Z}_s|^2ds\le C_q\mathbb{E}\Big[e^{aq\gamma_n}\big(\mathcal{Y}_{\gamma_n}^+\big)^q\Big].
\end{aligned}$$

Because of (6.11) and $\mathbb{E}\Big[\big(\int_\nu^\tau|Z_t^i|^2dt\big)^{p/2}\Big]<\infty$, $i=1,2$, it holds for $\mathbb{P}$-a.s. $\omega\in\Omega$ that $\tau(\omega)=\gamma_{N'_\omega}(\omega)$ for some $N'_\omega\in\mathbb{N}$. Letting $n\to\infty$ in the above inequality, we can deduce from the monotone convergence theorem, (6.11) and dominated convergence theorem that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[\nu,\tau]}\big(\mathcal{Y}_t^+\big)^q\Big]&=\lim_{n\to\infty}\uparrow\mathbb{E}\Big[\sup_{t\in[\nu,\gamma_n]}\big(\mathcal{Y}_t^+\big)^q\Big]\le C_q\lim_{n\to\infty}\mathbb{E}\Big[e^{aq\gamma_n}\big(\mathcal{Y}_{\gamma_n}^+\big)^q\Big]\\
&=C_q\mathbb{E}\Big[e^{aq\tau}\big((Y_\tau^1-Y_\tau^2)^+\big)^q\Big]=0.\quad\square
\end{aligned}$$

Proof of Remark 4.1. Let $\nu\in\mathcal{T}$, $\tau\in\mathcal{T}_{\nu,T}$. It suffices to show (4.2) for $\xi\in L^1(\mathcal{F}_\tau)$. Given $n\in\mathbb{N}$, we still define the stopping time γ_n as in (A.3). As $Y_{\nu\wedge\gamma_n}^{\tau,\xi}=Y_{\gamma_n}^{\tau,\xi}+\int_{\nu\wedge\gamma_n}^{\gamma_n}\mathbf{1}_{\{s\le\tau\}}g_sds-\int_{\nu\wedge\gamma_n}^{\gamma_n}Z_s^{\tau,\xi}dB_s$, $\mathbb{P}$-a.s., similar to (A.4), taking conditional expectation $\mathbb{E}\big[\cdot|\mathcal{F}_{\nu\wedge\gamma_n}\big]$ yields that $Y_{\nu\wedge\gamma_n}^{\tau,\xi}=\mathbf{1}_{\{\nu\le\gamma_n\}}\mathbb{E}\big[Y_{\gamma_n}^{\tau,\xi}+\int_{\nu\wedge\gamma_n}^{\tau\wedge\gamma_n}g_sds\big|\mathcal{F}_\nu\big]+\mathbf{1}_{\{\nu>\gamma_n\}}\Big(Y_{\gamma_n}^{\tau,\xi}+\int_{\nu\wedge\gamma_n}^{\tau\wedge\gamma_n}g_sds\Big)$, $\mathbb{P}$-a.s. Since $\{\gamma_n\}_{n\in\mathbb{N}}$ is stationary, letting $n\to\infty$, we can deduce from the uniform integrability of $\big\{Y_\gamma^{\tau,\xi}\big\}_{\gamma\in\mathcal{T}}$ that

$$\begin{aligned}
\mathcal{E}_{\nu,\tau}^g[\xi]=Y_\nu^{\tau,\xi}&=\mathbf{1}_{\{\nu\le T\}}\mathbb{E}\Big[Y_T^{\tau,\xi}+\int_\nu^\tau g_sds\Big|\mathcal{F}_\nu\Big]+\mathbf{1}_{\{\nu>T\}}\Big(Y_T^{\tau,\xi}+\int_\nu^\tau g_sds\Big)\\
&=\mathbb{E}\Big[\xi+\int_\nu^\tau g_sds\Big|\mathcal{F}_\nu\Big],\quad\mathbb{P}\text{-a.s.}\quad\square
\end{aligned}$$

6.2. Proofs of the results in Section 5

Proof of Proposition 5.1. (1) *We first show that* $\mathbb{E}\big[(Y_*)^p\big] < \infty$, $\forall\, p \in (0,1)$.

As the limit of $\mathbf{F}$-adapted continuous processes Y^n's (thus $\mathbf{F}$-predictable), Y is also an $\mathbf{F}$-predictable process. For any $(t,\omega) \in [0,T]\times\Omega$, $Y_t(\omega) = \lim_{n\to\infty}\uparrow Y^n_t(\omega)$ implies $Y^+_t(\omega) = \lim_{n\to\infty}\uparrow Y^{n,+}_t(\omega)$. Then one can deduce that

$$\begin{aligned} Y^+_*(\omega) &= \sup_{t\in[0,T]} Y^+_t(\omega) = \sup_{t\in[0,T]}\sup_{n\in\mathbb{N}} Y^{n,+}_t(\omega) = \sup_{n\in\mathbb{N}}\sup_{t\in[0,T]} Y^{n,+}_t(\omega) = \sup_{n\in\mathbb{N}} Y^{n,+}_*(\omega) \\ &= \lim_{n\to\infty}\uparrow Y^{n,+}_*(\omega), \qquad \forall\,\omega\in\Omega. \end{aligned} \tag{6.13}$$

For any $n \in \mathbb{N}$, the continuity of process Y^n shows that $\mathbb{P}$-a.s., $Y^{n,+}_* = \sup_{t\in[0,T]} Y^{n,+}_t = \sup_{t\in[0,T]\cap\mathbb{Q}} Y^{n,+}_t \in \mathcal{F}_T$, which implies that $Y^{n,+}_*$ is $\mathcal{F}_T$-measurable. Then we see from (6.13) that Y^+_* is also $\mathcal{F}_T$-measurable.

Let $p \in (\alpha,1)$ and set $\eta := \xi^+ + \int_0^T h_s ds + L^+_* \in L^1(\mathcal{F}_T)$. Given $n\in\mathbb{N}$, we claim that $\mathbb{P}$-a.s.

$$(Y^n_t)^+ \le C_\alpha \mathbb{E}\Big[1+\eta+\big(Y^{n,+}_*\big)^\alpha\Big|\mathcal{F}_t\Big], \qquad t\in[0,T], \tag{6.14}$$

(which will be shown in the last part of this proof). Since $M^n_t := \mathbb{E}\Big[1+\eta+\Big(Y^{n,+}_*\Big)^\alpha\Big|\mathcal{F}_t\Big]$, $t\in[0,T]$ is a uniformly integrable martingale, applying Lemma 6.1 of [11], we can deduce from (6.14), (1.5), (1.6), Hölder's inequality and Young's inequality that

$$\begin{aligned} \mathbb{E}\big[(Y^{n,+}_*)^p\big] &\le C^p_\alpha\,\mathbb{E}\Big[\sup_{t\in[0,T]}(M^n_t)^p\Big] \le \frac{C^p_\alpha}{1-p}\big(\mathbb{E}[M^n_T]\big)^p \\ &\le \frac{C^p_\alpha}{1-p}\Big\{1+(\mathbb{E}[\eta])^p+\big(\mathbb{E}\big[(Y^{n,+}_*)^\alpha\big]\big)^p\Big\} \\ &\le \frac{C^p_\alpha}{1-p}\Big\{2+\mathbb{E}[\eta]+\big(\mathbb{E}\big[(Y^{n,+}_*)^p\big]\big)^\alpha\Big\} \\ &\le C_{\alpha,p}\big\{1+\mathbb{E}[\eta]\big\}+\frac{1}{2}\mathbb{E}\big[(Y^{n,+}_*)^p\big]. \end{aligned}$$

As $\mathbb{E}\Big[(Y^{n,+}_*)^p\Big] \le \mathbb{E}\big[(Y^n_*)^p\big] < \infty$, we see that $\mathbb{E}\Big[(Y^{n,+}_*)^p\Big] \le C_{\alpha,p}\big\{1+\mathbb{E}[\eta]\big\}$. When $n\to\infty$, (6.13) and the monotone convergence theorem yield that $\mathbb{E}\big[(Y^+_*)^p\big] \le C_{\alpha,p}\big\{1+\mathbb{E}[\eta]\big\}$. Since

$$|Y_t| = Y^-_t + Y^+_t \le (Y^1_t)^- + Y^+_t \le |Y^1_t| + Y^+_t, \qquad \forall\, t\in[0,T], \tag{6.15}$$

(1.5) implies that $\mathbb{E}\big[(Y_*)^p\big] \le \mathbb{E}\big[(Y^1_*)^p\big] + \mathbb{E}\big[(Y^+_*)^p\big] < \infty$.

Moreover, for any $\widetilde{p} \in (0,\alpha]$, (1.6) shows that $\mathbb{E}\big[(Y_*)^{\widetilde{p}}\big] \le 1+\mathbb{E}\Big[(Y_*)^{\frac{\alpha+1}{2}}\Big] < \infty$. Hence, $\mathbb{E}\big[(Y_*)^p\big] < \infty$, $\forall\, p\in(0,1)$.

(2) *Next, let us show that Y is of class (D).* Since $\mathbb{E}\big[(Y^+_*)^\alpha\big] < \infty$, letting $n\to\infty$ in (6.14), one can deduce from (6.13) and the monotone convergence theorem that for any $t\in[0,T]$, $Y^+_t \le C_\alpha\mathbb{E}\big[1+\eta+(Y^+_*)^\alpha\big|\mathcal{F}_t\big]$, $\mathbb{P}$-a.s. Using the continuity of process Y^+ and process $\big\{\mathbb{E}\big[1+\eta+(Y^+_*)^\alpha\big|\mathcal{F}_t\big]\big\}_{t\in[0,T]}$, we see from (6.15) that $\mathbb{P}$-a.s.

$$|Y_t| \le |Y^1_t| + Y^+_t \le |Y^1_t| + C_\alpha\mathbb{E}\big[1+\eta+(Y^+_*)^\alpha\big|\mathcal{F}_t\big], \qquad t\in[0,T].$$

This implies that Y is of class (D) as Y^1 is of class (D).

(3) *It remains to demonstrate claim (6.14).*

For any $t \in [0,T]$, the continuity of process L shows that $\mathbb{P}$-a.s., $\Gamma_t := \sup_{s\in[0,t]} L_s^+ = \sup_{s\in[0,t]\cap\mathbb{Q}} L_s^+ \in \mathcal{F}_t$, which implies that Γ is an $\mathbf{F}$-adapted, continuous increasing process with $\mathbb{E}[\Gamma_T] < \infty$.

Let $n \in \mathbb{N}$. Since $\int_0^T \mathbf{1}_{\{Y_t^n > \Gamma_t\}} \big(Y_t^n - L_t\big)^- dt = 0$, applying Itô–Tanaka's formula to process $(Y^n - \Gamma)^+$ yields that

$$(Y_t^n - \Gamma_t)^+ = (Y_T^n - \Gamma_T)^+ + \int_t^T \mathbf{1}_{\{Y_s^n > \Gamma_s\}} \big(g(s, Y_s^n, Z_s^n) ds - dJ_s^n - Z_s^n dB_s\big) + \int_t^T \mathbf{1}_{\{Y_s^n > \Gamma_s\}} d\Gamma_s - \frac{1}{2}(\mathfrak{L}_T^n - \mathfrak{L}_t^n), \quad t \in [0,T],$$

where $\mathfrak{L}^n$ is the "local time" of $Y^n - \Gamma$ at 0.

Set $a := 2(\kappa + \kappa^2)$. Given $j \in \mathbb{N}$, we define a stopping time $\gamma_j = \gamma_j^n := \inf\{t \in [0,T] : \int_0^t |Z_s^n|^2 ds > j\} \wedge T \in \mathcal{T}$, and integrate by parts the process $\big\{e^{a(\gamma_j\wedge t)}(Y_{\gamma_j\wedge t}^n - \Gamma_{\gamma_j\wedge t})^+\big\}_{t\in[0,T]}$ to obtain that $\mathbb{P}$-a.s.

$$\begin{aligned} & e^{a(\gamma_j\wedge t)} \Big(Y_{\gamma_j\wedge t}^n - \Gamma_{\gamma_j\wedge t}\Big)^+ + a \int_{\gamma_j\wedge t}^{\gamma_j} e^{as} (Y_s^n - \Gamma_s)^+ ds \\ &\quad = e^{a\gamma_j} (Y_{\gamma_j}^n - \Gamma_{\gamma_j})^+ + \int_{\gamma_j\wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{as} \big(g(s, Y_s^n, Z_s^n) ds - dJ_s^n - Z_s^n dB_s\big) \\ &\qquad + \int_{\gamma_j\wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{as} d\Gamma_s - \frac{1}{2} \int_{\gamma_j\wedge t}^{\gamma_j} e^{as} d\mathfrak{L}_s^n, \quad t \in [0,T]. \end{aligned} \tag{6.16}$$

Since (H4), (H5), (1.5) and (1.6) imply that

$$\begin{aligned} |g(t, Y_t^n, Z_t^n)| &\le |g(t, Y_t^n, 0)| + |g(t, Y_t^n, Z_t^n) - g(t, Y_t^n, 0)| \le h_t + \kappa |Y_t^n| \\ &\quad + \kappa (h_t + |Y_t^n| + |Z_t^n|)^\alpha \\ &\le h_t + \kappa |Y_t^n| + \kappa (h_t + |Y_t^n|)^\alpha + \kappa |Z_t^n|^\alpha \le h_t + \kappa |Y_t^n| \\ &\quad + \kappa (1 + h_t + |Y_t^n|) + \kappa |Z_t^n|^\alpha \\ &\le \kappa + (1+\kappa) h_t + 2\kappa |Y_t^n - \Gamma_t| + 2\kappa \Gamma_t + \kappa |Z_t^n|^\alpha, \quad dt \otimes d\mathbb{P}\text{-a.s.}, \end{aligned} \tag{6.17}$$

taking conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ in (6.16), we can deduce from Hölder's inequality that for any $t \in [0,T]$

$$\begin{aligned} & e^{a(\gamma_j\wedge t)} \Big(Y_{\gamma_j\wedge t}^n - \Gamma_{\gamma_j\wedge t}\Big)^+ \\ &\quad \le \kappa T e^{aT} + \mathbb{E}\bigg[e^{a\gamma_j} (Y_{\gamma_j}^n - \Gamma_{\gamma_j})^+ + (1+\kappa) e^{aT} \int_0^T h_s ds + (1 + 2\kappa T) e^{aT} \Gamma_T \\ &\qquad + \kappa T^{1-\alpha/2} e^{(1-\alpha)aT} \bigg(\int_{\gamma_j\wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} |Z_s^n|^2 ds \bigg)^{\alpha/2} \bigg| \mathcal{F}_t \bigg], \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{6.18}$$

where we used the fact that $\mathbf{1}_{\{Y_t^n > \Gamma_t\}} |Y_t^n - \Gamma_t| = (Y_t^n - \Gamma_t)^+$.

Applying Itô's formula to process $\left\{e^{2a(\gamma_j\wedge t)}\left((Y^n_{\gamma_j\wedge t}-\Gamma_{\gamma_j\wedge t})^+\right)^2\right\}_{t\in[0,T]}$ in (6.16) yields that

$$\begin{aligned}&e^{2a(\gamma_j\wedge t)}\left((Y^n_{\gamma_j\wedge t}-\Gamma_{\gamma_j\wedge t})^+\right)^2+2a\int_{\gamma_j\wedge t}^{\gamma_j}e^{2as}\left((Y^n_s-\Gamma_s)^+\right)^2ds\\&\quad+\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}|Z^n_s|^2ds\\&=e^{2a\gamma_j}\left((Y^n_{\gamma_j}-\Gamma_{\gamma_j})^+\right)^2+2\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}(Y^n_s-\Gamma_s)^+\\&\quad\times\left(g(s,Y^n_s,Z^n_s)ds-dJ^n_s-Z^n_sdB_s\right)\\&\quad+2\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}(Y^n_s-\Gamma_s)^+d\Gamma_s-\int_{\gamma_j\wedge t}^{\gamma_j}e^{2as}(Y^n_s-\Gamma_s)^+d\mathfrak{L}^n_s,\\&t\in[0,T].\end{aligned}\tag{6.19}$$

Since (H1) and (H4) imply that $dt\otimes d\mathbb{P}$-a.s.

$$\begin{aligned}|g(t,Y^n_t,Z^n_t)|&\le|g(t,Y^n_t,0)|+|g(t,Y^n_t,Z^n_t)-g(t,Y^n_t,0)|\le h_t+\kappa|Y^n_t|+\kappa|Z^n_t|\\&\le h_t+\kappa|Y^n_t-\Gamma_t|+\kappa\Gamma_t+\kappa|Z^n_t|,\end{aligned}$$

it holds $dt\otimes d\mathbb{P}$-a.s. that

$$\begin{aligned}\mathbf{1}_{\{Y^n_t>\Gamma_t\}}(Y^n_t-\Gamma_t)^+g(t,Y^n_t,Z^n_t)&\le(Y^n_t-\Gamma_t)^+h_t+(\kappa+2\kappa^2)\left((Y^n_t-\Gamma_t)^+\right)^2\\&\quad+\frac14\mathbf{1}_{\{Y^n_t>\Gamma_t\}}\Gamma_T^2+\frac14\mathbf{1}_{\{Y^n_t>\Gamma_t\}}|Z^n_t|^2.\end{aligned}$$

Set $\Psi^n_t:=\sup_{s\in[t,T]}(Y^n_s-\Gamma_s)^+$, $t\in[0,T]$. It then follows from (6.19) that

$$\begin{aligned}&e^{2a(\gamma_j\wedge t)}\left(\left(Y^n_{\gamma_j\wedge t}-\Gamma_{\gamma_j\wedge t}\right)^+\right)^2+\frac12\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}|Z^n_s|^2ds\\&\le e^{2aT}\left((Y^n_{\gamma_j})^+\right)^2+2e^{2aT}\Psi^n_t\int_t^Th_sds+\frac12Te^{2aT}\Gamma_T^2+2e^{2aT}\Psi^n_t\Gamma_T\\&\quad-2\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}(Y^n_s-\Gamma_s)^+Z^n_sdB_s\\&\le e^{2aT}\left((Y^n_{\gamma_j})^+\right)^2+(\Psi^n_t)^2+C_0\left(\int_0^Th_sds\right)^2+C_0\Gamma_T^2\\&\quad+\left|2\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}(Y^n_s-\Gamma_s)^+Z^n_sdB_s\right|,\quad t\in[0,T].\end{aligned}$$

Taking powers of order $\alpha/2$ on both sides, we see from (1.5) that

$$\begin{aligned}&2^{\alpha/2-1}e^{\alpha a(\gamma_j\wedge t)}\left(\left(Y^n_{\gamma_j\wedge t}-\Gamma_{\gamma_j\wedge t}\right)^+\right)^\alpha+\frac12\left(\int_{\gamma_j\wedge t}^{\gamma_j}\mathbf{1}_{\{Y^n_s>\Gamma_s\}}e^{2as}|Z^n_s|^2ds\right)^{\alpha/2}\\&\le e^{\alpha aT}\left((Y^n_{\gamma_j})^+\right)^\alpha+(\Psi^n_t)^\alpha\end{aligned}$$

$$+ C_\alpha \left(\int_0^T h_s ds \right)^\alpha + C_\alpha \Gamma_T^\alpha + \left| 2 \int_t^T \mathbf{1}_{\{s \le \gamma_j\}} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} (Y_s^n - \Gamma_s)^+ Z_s^n dB_s \right|^{\alpha/2},$$
$$t \in [0, T]. \tag{6.20}$$

Let $t \in [0, T]$. For any $A \in \mathcal{F}_t$, since

$$\begin{aligned}
&\mathbf{1}_A \left| \int_t^T \mathbf{1}_{\{s \le \gamma_j\}} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} (Y_s^n - \Gamma_s)^+ Z_s^n dB_s \right|^{\alpha/2} \\
&= \left| \int_t^T \mathbf{1}_A \mathbf{1}_{\{s \le \gamma_j\}} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} (Y_s^n - \Gamma_s)^+ Z_s^n dB_s \right|^{\alpha/2} \\
&= \left| \int_0^T \mathbf{1}_A \mathbf{1}_{\{t \le s \le \gamma_j\}} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} (Y_s^n - \Gamma_s)^+ Z_s^n dB_s \right|^{\alpha/2},
\end{aligned}$$

multiplying $\mathbf{1}_A$ to (6.20) and taking expectation, we can deduce from the Burkholder–Davis–Gundy inequality and (1.6)

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\mathbf{1}_A \left(\int_{\gamma_j \wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} |Z_s^n|^2 ds \right)^{\alpha/2} \right] \\
&\le C_\alpha \mathbb{E}\Bigg[\mathbf{1}_A \left((Y_{\gamma_j}^n)^+ \right)^\alpha + \mathbf{1}_A (\Psi_t^n)^\alpha + \mathbf{1}_A \left(\int_0^T h_s ds \right)^\alpha + \mathbf{1}_A \Gamma_T^\alpha \\
&\quad + \left(\int_0^T \mathbf{1}_A \mathbf{1}_{\{t \le s \le \gamma_j\}} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{4as} \left((Y_s^n - \Gamma_s)^+ \right)^2 |Z_s^n|^2 ds \right)^{\alpha/4} \Bigg] \\
&\le C_\alpha \mathbb{E}\Bigg[\mathbf{1}_A + \mathbf{1}_A (Y_{\gamma_j}^n)^+ + \mathbf{1}_A (\Psi_t^n)^\alpha + \mathbf{1}_A \int_0^T h_s ds + \mathbf{1}_A \Gamma_T + \mathbf{1}_A (\Psi_t^n)^{\alpha/2} \\
&\quad \cdot \left(\int_t^T \mathbf{1}_{\{s \le \gamma_j\}} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} |Z_s^n|^2 ds \right)^{\alpha/4} \Bigg] \\
&\le C_\alpha \mathbb{E}\left[\mathbf{1}_A + \mathbf{1}_A (Y_{\gamma_j}^n)^+ + \mathbf{1}_A (\Psi_t^n)^\alpha + \mathbf{1}_A \int_0^T h_s ds + \mathbf{1}_A \Gamma_T \right] \\
&\quad + \frac{1}{4}\mathbb{E}\left[\mathbf{1}_A \left(\int_{\gamma_j \wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} |Z_s^n|^2 ds \right)^{\alpha/2} \right].
\end{aligned}$$

Since $\mathbb{E}\left[\left(\int_{\gamma_j \wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} |Z_s^n|^2 ds \right)^{\alpha/2} \right] \le e^{\alpha a T} j^{\alpha/2}$ and since

$$\mathbb{E}\left[(\Psi_0^n)^\alpha \right] = \mathbb{E}\left[\sup_{t \in [0,T]} \left((Y_t^n - \Gamma_t)^+ \right)^\alpha \right] \le \mathbb{E}\left[\sup_{t \in [0,T]} \left((Y_t^n)^+ \right)^\alpha \right] \le \|Y^n\|_{\mathbb{S}^\alpha} < \infty,$$

letting A vary over $\mathcal{F}_t$ yields that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\gamma_j \wedge t}^{\gamma_j} \mathbf{1}_{\{Y_s^n > \Gamma_s\}} e^{2as} |Z_s^n|^2 ds \right)^{\alpha/2} \middle| \mathcal{F}_t \right] \\
&\le C_\alpha \mathbb{E}\left[1 + (Y_{\gamma_j}^n)^+ + (\Psi_t^n)^\alpha + \int_0^T h_s ds + \Gamma_T \middle| \mathcal{F}_t \right].
\end{aligned}$$

Then we see from (6.18) that

$$\Big(Y^n_{\gamma_j\wedge t}-\Gamma_{\gamma_j\wedge t}\Big)^+ \le e^{a(\gamma_j\wedge t)}\Big(Y^n_{\gamma_j\wedge t}-\Gamma_{\gamma_j\wedge t}\Big)^+$$
$$\le C_\alpha \mathbb{E}\bigg[1+(Y^n_{\gamma_j})^+ + (\Psi^n_0)^\alpha+\int_0^T h_s ds+\Gamma_T\bigg|\mathcal{F}_t\bigg],\quad \mathbb{P}\text{-a.s.} \quad (6.21)$$

The uniform integrability of $\{Y^n_\gamma\}_{\gamma\in\mathcal{T}}$ implies that of $\big\{(Y^n_\gamma)^+\big\}_{\gamma\in\mathcal{T}}$. As $Z^n\in\cap_{p\in(0,1)}\mathbb{H}^{2,p}\subset\mathbb{H}^{2,0}$, $\{\gamma_j\}_{j\in\mathbb{N}}$ is stationary. So letting $j\to\infty$ in (6.21), one can deduce from the continuity of process Y^n that

$$(Y^n_t)^+\le \Gamma_t+\big(Y^n_t-\Gamma_t\big)^+\le \Gamma_t+C_\alpha\mathbb{E}\bigg[1+\xi^+ +(\Psi^n_0)^\alpha+\int_0^T h_s ds+\Gamma_T\bigg|\mathcal{F}_t\bigg]$$
$$\le C_\alpha\mathbb{E}\big[1+\eta+(\Psi^n_0)^\alpha\big|\mathcal{F}_t\big],\quad \mathbb{P}\text{-a.s.}$$

Then claim (6.14) follows from the continuity of process $Y^{n,+}$ and of process $\big\{\mathbb{E}\big[1+\eta+(\Psi^n_0)^\alpha\big|\mathcal{F}_t\big]\big\}_{t\in[0,T]}$. □

Proof of Proposition 5.2. The proof is relatively lengthy, see our introduction for a sketch. We will defer the demonstration of some technicalities (those equations with starred labels) to the Appendix.

(1) For any $n\in\mathbb{N}$, $K^n_t:=n\int_0^t \mathbf{1}_{\{\nu<s\le\tau\}}(Y^n_s-L_s)^-ds$, $t\in[0,T]$ is clearly a process of $\mathbb{K}^0$ satisfying

$$K^n_t=0,\qquad \forall\, t\in[0,\nu]. \quad (6.22)$$

As $K^n_\tau-K^n_t=n\int_t^\tau \mathbf{1}_{\{\nu<s\le\tau\}}(Y^n_s-L_s)^-ds=n\int_t^\tau(Y^n_s-L_s)^-ds$, $\forall\, t\in[\nu,\tau]$, (5.3) shows that $\mathbb{P}$-a.s.

$$Y^n_t=Y^n_\tau+\int_t^\tau g(s,Y^n_s,Z^n_s)ds+K^n_\tau-K^n_t-\int_t^\tau Z^n_s dB_s,\qquad \forall\, t\in[\nu,\tau]. \quad (6.23)$$

Since $\big\{\mathbf{1}_{\{t\ge\nu\}}\big\}_{t\in[0,T]}$ is an **F**-adapted càdlàg process and since each $\big\{Y^n_{\tau\wedge t}\big\}_{t\in[0,T]}$ is an **F**-adapted continuous process, we see that Y is an **F**-optional process. (It takes some effort to show the continuity of Y between ν and T, see (6.40) for an intermediate result.) By the Debut theorem,

$$\tau_\ell:=\inf\bigg\{t\in[\nu,\tau]:(Y^1_t)^-+Y^+_t+L^+_t+\int_\nu^t h_s ds>\ell\bigg\}\wedge\tau,\qquad \ell\in\mathbb{N} \quad (6.24)$$

are stopping times with $\nu\le\tau_\ell\le\tau$, i.e. $\tau_\ell\in\mathcal{T}_{\nu,\tau}$. As $\mathbb{E}\Big[L^+_*+\int_0^T h_t dt\Big]<\infty$ and $\mathbb{P}\Big\{\sup_{t\in[\nu,\tau]}\big((Y^1_t)^-+Y^+_t\big)<\infty\Big\}=1$, it holds for any $\omega\in\Omega$ except on a $\mathbb{P}$-null set $\mathcal{N}_1$ that

$$\tau(\omega)=\tau_{N_\omega}(\omega) \text{ for some } N_\omega\in\mathbb{N}.$$

Now, let us fix $\ell\in\mathbb{N}$ for this part as well as next two parts. Let $\mathcal{N}_2:=\cup_{n\in\mathbb{N}}\{\omega\in\Omega:$ the path $Y^n_\cdot(\omega)$ is not continuous$\}$ (which is clearly a $\mathbb{P}$-null set) and set $A_\ell:=\{\nu<\tau_\ell\}\cap\mathcal{N}^c_2\in\mathcal{F}_{\nu\wedge\tau_\ell}\subset\mathcal{F}_\nu$. Given $\omega\in A_\ell$, for any $n\in\mathbb{N}$ we can deduce from (6.24) that $|Y^n_t(\omega)|\le\ell$, $\forall\, t\in\big[\nu(\omega),\tau_\ell(\omega)\big)$, and the continuity of each Y^n implies that $|Y^n_t(\omega)|\le\ell$, $\forall\, t\in\big[\nu(\omega),\tau_\ell(\omega)\big]$. Then it follows from the monotonicity of $\{Y^n\}_{n\in\mathbb{N}}$ that

$$\sup_{n\in\mathbb{N}}\big|Y^n_t(\omega)\big|\le\big|Y^1_t(\omega)\big|\vee\big|Y_t(\omega)\big|\le\ell,\qquad \forall\, t\in\big[\nu(\omega),\tau_\ell(\omega)\big],\ \forall\,\omega\in A_\ell. \quad (6.25)$$

Let $n \in \mathbb{N}$. As $\mathbb{E}\big[|\mathbf{1}_{A_\ell} Y^n_\nu|\big] \le \ell$, Corollary 3.1 shows that there exists a unique $\widetilde{Z}^{\ell,n} \in \cap_{p\in(0,1)} \mathbb{H}^{2,p}$ such that $\mathbb{P}\big\{\mathbb{E}\big[\mathbf{1}_{A_\ell} Y^n_\nu|\mathcal{F}_t\big] = \mathbb{E}\big[\mathbf{1}_{A_\ell} Y^n_\nu\big] + \int_0^t \widetilde{Z}^{\ell,n}_s dB_s,\ \ \forall\, t \in [0,T]\big\} = 1$. Similar to (6.3), we can deduce from (6.23) that $\mathbb{P}$-a.s.

$$\begin{aligned} Y^{\ell,n}_t &:= \mathbb{E}\big[\mathbf{1}_{A_\ell} Y^n_\nu|\mathcal{F}_{\nu\wedge t}\big] + Y^n_{\nu\vee(\tau_\ell\wedge t)} - Y^n_\nu \\ &= \mathbb{E}\big[\mathbf{1}_{A_\ell} Y^n_\nu\big] - \int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}} g(s, Y^n_s, Z^n_s)ds - K^n_{\nu\vee(\tau_\ell\wedge t)} + K^n_\nu \\ &\quad + \int_0^t \Big(\mathbf{1}_{\{s\le\nu\}} \widetilde{Z}^{\ell,n}_s + \mathbf{1}_{\{\nu<s\le\tau_\ell\}} Z^n_s\Big) dB_s, \quad t \in [0,T]. \end{aligned} \tag{6.26}$$

Thus $Y^{\ell,n}$ is an $\mathbf{F}$-adapted continuous process (i.e. $Y^{\ell,n} \in \mathbb{S}^0$) that satisfies

$$\begin{aligned} Y^{\ell,n}_t &= \mathbb{E}\big[\mathbf{1}_{A_\ell} Y^n_\nu|\mathcal{F}_\nu\big] + Y^n_t - Y^n_\nu = \mathbf{1}_{A_\ell} Y^n_\nu + \mathbf{1}_{A_\ell}(Y^n_t - Y^n_\nu) = \mathbf{1}_{A_\ell} Y^n_t, \\ &\forall\, t \in [\nu, \tau_\ell], \end{aligned} \tag{6.27}$$

which together with (6.26) shows that $\mathbb{P}$-a.s.

$$\begin{aligned} Y^{\ell,n}_t &- Y^{\ell,n}_{\tau_\ell} - K^n_{\tau_\ell} + K^n_t + \int_t^{\tau_\ell} Z^n_s dB_s = \int_t^{\tau_\ell} g(s, Y^n_s, Z^n_s)ds \\ &= \mathbf{1}_{A_\ell} \int_t^{\tau_\ell} g(s, Y^{\ell,n}_s, Z^n_s)ds \\ &= \int_t^{\tau_\ell} g(s, Y^{\ell,n}_s, Z^n_s)ds, \quad \forall\, t \in [\nu, \tau_\ell]. \end{aligned} \tag{6.28}$$

Since $\mathbb{E}\big[\big|Y^{\ell,n}_\nu\big|\big] \le \ell$ by (6.27), (6.25) and since $K^n_\nu = 0$ by (6.22), applying Lemma A.2 with $(Y,Z,K) = (Y^{\ell,n}, Z^n, K^n)$ and $(\tau, p) = (\tau_\ell, 2)$, we see from (6.27), (6.25) and (6.24) that

$$\begin{aligned} \mathbb{E}\int_\nu^{\tau_\ell} |Z^n_t|^2 dt + \mathbb{E}\Big[(K^n_{\tau_\ell})^2\Big] &\le C_0 \mathbb{E}\Big[\sup_{t\in[\nu,\tau_\ell]} \big|Y^{\ell,n}_t\big|^2\Big] + C_0 \mathbb{E}\Big[\Big(\int_\nu^{\tau_\ell} h_t dt\Big)^2\Big] \\ &\le C_0 \ell^2. \end{aligned} \tag{6.29}$$

It then follows from (H1) that $\mathbb{E}\int_\nu^{\tau_\ell} |g(t, Y^{\ell,n}_t, Z^n_t) - g(t, Y^{\ell,n}_t, 0)|^2 dt \le \kappa^2\, \mathbb{E}\int_\nu^{\tau_\ell} |Z^n_t|^2 dt \le C_0 \ell^2$. In virtue of Theorem 5.2.1 of [50], $\big\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}} Z^n_t\big\}_{t\in[0,T]}$, $n \in \mathbb{N}$ has a weakly convergent subsequence (we still denote it by $\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}} Z^n_t\}_{t\in[0,T]}$, $n \in \mathbb{N}$) with limit $\mathcal{Z}^\ell \in \mathbb{H}^{2,2}$; and $\big\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}}(g(t, Y^{\ell,n}_t, Z^n_t) - g(t, Y^{\ell,n}_t, 0))\big\}_{t\in[0,T]}$, $n \in \mathbb{N}$ has a weakly convergent subsequence (we still denote it by $\big\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}}(g(t, Y^{\ell,n}_t, Z^n_t) - g(t, Y^{\ell,n}_t, 0))\big\}_{t\in[0,T]}$, $n \in \mathbb{N}$) with limit $\widetilde{h}^\ell \in \mathbb{H}^{2,2}$. It is easy to deduce that

$$\mathcal{Z}^\ell_t = \mathbf{1}_{\{\nu<t\le\tau_\ell\}} \mathcal{Z}^\ell_t \quad \text{and} \quad \widetilde{h}^\ell_t = \mathbf{1}_{\{\nu<t\le\tau_\ell\}} \widetilde{h}^\ell_t, \quad dt \otimes d\mathbb{P}\text{-a.s.} \tag{6.30}$$

The $\mathbf{F}$-optional measurability of Y implies that of stopped processes $\big\{Y_{\nu\wedge t}\big\}_{t\in[0,T]}$ and $\big\{Y_{\tau_\ell\wedge t}\big\}_{t\in[0,T]}$ (see e.g. Corollary 3.24 of [33]). As $A_\ell \cap \{t > \nu\} \in \mathcal{F}_t$ for any $t \in [0,T]$, $\big\{\mathbf{1}_{A_\ell\cap\{t\ge\nu\}}\big\}_{t\in[0,T]}$ is an $\mathbf{F}$-adapted càdlàg process. Then

$$Y_{\nu\vee(\tau_\ell\wedge t)} - Y_\nu = \mathbf{1}_{A_\ell\cap\{t\ge\nu\}} \big(Y_{\tau_\ell\wedge t} - Y_\nu\big) = \mathbf{1}_{A_\ell\cap\{t\ge\nu\}} \big(Y_{\tau_\ell\wedge t} - Y_{\nu\wedge t}\big), \quad t \in [0,T] \tag{6.31}$$

is an **F**-optional process and it follows that

$$\widetilde{K}^\ell_t := Y_\nu - Y_{\nu\vee(\tau_\ell\wedge t)} - \int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}}\Big(g(s,Y_s,0)+\widetilde{h}^\ell_s\Big)ds + \int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}}\mathcal{Z}^\ell_s dB_s,$$
$$t\in[0,T] \tag{6.32}$$

also defines an **F**-optional process. Since (6.31), (6.25), (H4), (6.24) and Hölder's inequality imply that

$$\begin{aligned}\big|\widetilde{K}^\ell_t\big| &\le \mathbf{1}_{A_\ell\cap\{t\ge\nu\}}\big(\big|Y_{\tau_\ell\wedge t}\big|+|Y_\nu|\big)+\mathbf{1}_{A_\ell}\int_\nu^{\tau_\ell}\Big(h_t+\kappa|Y_t|+|\widetilde{h}^\ell_t|\Big)dt\\ &\quad+\bigg|\int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}}\mathcal{Z}^\ell_s dB_s\bigg|\\ &\le 3\ell+\kappa\ell T+\bigg(T\int_\nu^{\tau_\ell}|\widetilde{h}^\ell_t|^2dt\bigg)^{1/2}+\sup_{t\in[0,T]}\bigg|\int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}}\mathcal{Z}^\ell_s dB_s\bigg|,\qquad \forall\, t\in[0,T].\end{aligned}$$

Doob's martingale inequality and (1.5) show that

$$\begin{aligned}\mathbb{E}\Big[\big(\widetilde{K}^\ell_*\big)^2\Big] &\le C_0\ell^2+3T\mathbb{E}\int_\nu^{\tau_\ell}|\widetilde{h}^\ell_t|^2dt+3\mathbb{E}\Bigg[\sup_{t\in[0,T]}\bigg|\int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}}\mathcal{Z}^\ell_s dB_s\bigg|^2\Bigg]\\ &\le C_0\ell^2+C_0\mathbb{E}\int_\nu^{\tau_\ell}\Big(\big|\widetilde{h}^\ell_t\big|^2+\big|\mathcal{Z}^\ell_t\big|^2\Big)dt<\infty.\end{aligned} \tag{6.33}$$

We next claim that

$\widetilde{K}^\ell$ satisfies the conditions of Lemma A.3 and is thus an increasing process. (6.34*)

As $\mathbb{E}\Big[\big(\widetilde{K}^\ell_T\big)^2\Big]<\infty$ by (6.33), it holds $\mathbb{P}$-a.s. that $\widetilde{K}^\ell_T<\infty$. Then applying Lemma 2.2 of [44] to (6.32) shows that both process $\widetilde{K}^\ell$ and process $\big\{Y_{\nu\vee(\tau_\ell\wedge t)}\big\}_{t\in[0,T]}$ have $\mathbb{P}$-a.s. càdlàg paths.

(2) By Hölder's inequality and (6.29), $\mathbb{E}\int_\nu^{\tau_\ell}(Y^n_t-L_t)^-dt=\frac{1}{n}\mathbb{E}\big[K^n_{\tau_\ell}\big]\le\frac{1}{n}\big\{\mathbb{E}\big[(K^n_{\tau_\ell})^2\big]\big\}^{1/2}\le\frac{1}{n}C_0\ell$, $\forall\, n\in\mathbb{N}$. Letting $n\to\infty$, we know from the monotone convergence theorem that

$$\mathbb{E}\int_\nu^{\tau_\ell}(Y_t-L_t)^-dt=\lim_{n\to\infty}\downarrow\mathbb{E}\int_\nu^{\tau_\ell}(Y^n_t-L_t)^-dt=0,$$

so it holds $dt\otimes d\mathbb{P}$-a.s. that $\mathbf{1}_{\{\nu<t<\tau_\ell\}}(Y_t-L_t)^-=0$. Since $\big\{Y_{\nu\vee(\tau_\ell\wedge t)}\big\}_{t\in[0,T]}$ has $\mathbb{P}$-a.s. càdlàg paths by part (1) and L has $\mathbb{P}$-a.s. continuous paths, one can deduce that for any $\omega\in A_\ell$ except a $\mathbb{P}$-null set $\widetilde{\mathcal{N}}_\ell$, $Y_t(\omega)\ge L_t(\omega)$ for any $t\in\big[\nu(\omega),\tau_\ell(\omega)\big)$. Given $\omega\in\{\nu<\tau\}\cap\mathcal{N}_1^c\cap\mathcal{N}_2^c\cap\big(\cup_{\ell\in\mathbb{N}}\widetilde{\mathcal{N}}_\ell\big)^c$, there exists an $n_\omega\in\mathbb{N}$ such that $\tau_{n_\omega}(\omega)=\tau(\omega)>\nu(\omega)$. So $\omega\in A_{n_\omega}\cap\widetilde{\mathcal{N}}^c_{n_\omega}=\big\{\omega'\in\Omega:\nu(\omega')<\tau_{n_\omega}(\omega')\big\}\cap\mathcal{N}_2^c\cap\widetilde{\mathcal{N}}^c_{n_\omega}$ and $Y_t(\omega)\ge L_t(\omega)$ holds for any $t\in\big[\nu(\omega),\tau_{n_\omega}(\omega)\big)=\big[\nu(\omega),\tau(\omega)\big)$. In summary, it holds for $\mathbb{P}$-a.s. $\omega\in\{\nu<\tau\}$ that $Y_t(\omega)\ge L_t(\omega)$ for any $t\in\big[\nu(\omega),\tau(\omega)\big)$, which together with $\mathbb{P}\{Y_\tau\ge L_\tau\}=1$ shows that for any $\omega\in\{\nu<\tau\}$ except on a $\mathbb{P}$-null set $\widehat{\mathcal{N}}$

$$Y_t(\omega)\ge L_t(\omega),\qquad \forall\, t\in\big[\nu(\omega),\tau(\omega)\big]. \tag{6.35}$$

Now we freeze the parameter ℓ again and let $\omega\in A_\ell\cap\widehat{\mathcal{N}}^c$. As $A_\ell\subset\{\nu<\tau\}\cap\mathcal{N}_2^c$, we see from (6.35) that $Y_t(\omega)\ge L_t(\omega)$ for any $t\in\big[\nu(\omega),\tau_\ell(\omega)\big]$. Since continuous function

$\big(Y^n_t - L_t\big)^-(\omega)$, $t \in \big[\nu(\omega), \tau_\ell(\omega)\big]$ is decreasing to $(Y_t - L_t)^-(\omega) = 0$, $t \in \big[\nu(\omega), \tau_\ell(\omega)\big]$ when $n \to \infty$, Dini's theorem shows that

$$\lim_{n\to\infty} \downarrow \sup_{t\in[\nu(\omega),\tau_\ell(\omega)]} \big(Y^n_t - L_t\big)^-(\omega) = 0.$$

As $\mathbf{1}_{A_\ell} \sup_{t\in[\nu,\tau_\ell]} \big(Y^n_t - L_t\big)^- \le \mathbf{1}_{A_\ell} \sup_{t\in[\nu,\tau_\ell]} \big(L^+_t + |Y^n_t|\big) \le 2\ell$, $\forall n \in \mathbb{N}$ by (6.24), (6.27) and (6.25), an application of the bounded convergence theorem yields that

$$\lim_{n\to\infty} \downarrow \mathbb{E}\left[\mathbf{1}_{A_\ell} \sup_{t\in[\nu,\tau_\ell]} \big((Y^n_t - L_t)^-\big)^2\right] = 0. \tag{6.36}$$

Similar to the arguments used in [20] (see pages 21–22 therein), we can deduce from (6.36) that

$$\big\{Y^{\ell,n}\big\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } \mathbb{S}^2 \text{ and } \big\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}} Z^n_t\big\}_{t\in[0,T]},\ n \in \mathbb{N}$$
$$\text{is a Cauchy sequence in } \mathbb{H}^{2,2}. \tag{6.37*}$$

Let $\mathcal{Y}^\ell \in \mathbb{S}^2$ and $\widetilde{\mathcal{Z}}^\ell \in \mathbb{H}^{2,2}$ be their limits respectively, i.e.

$$\lim_{n\to\infty} \downarrow \mathbb{E}\left[\sup_{t\in[0,T]} \big|Y^{\ell,n}_t - \mathcal{Y}^\ell_t\big|^2\right] + \lim_{n\to\infty} \mathbb{E}\int_0^T \big|\mathbf{1}_{\{\nu<t\le\tau_\ell\}} Z^n_t - \widetilde{\mathcal{Z}}^\ell_t\big|^2 dt = 0. \tag{6.38}$$

Up to a subsequence of $\big\{Y^{\ell,n}\big\}_{n\in\mathbb{N}}$, one has $\lim_{n\to\infty} \downarrow \sup_{t\in[0,T]} \big|Y^{\ell,n}_t - \mathcal{Y}^\ell_t\big| = 0$, $\mathbb{P}$-a.s. It follows from (6.27) that $\mathbb{P}$-a.s.

$$\mathcal{Y}^\ell_t = \lim_{n\to\infty} \uparrow Y^{\ell,n}_t = \lim_{n\to\infty} \uparrow \mathbf{1}_{A_\ell} Y^n_t = \mathbf{1}_{A_\ell} Y_t, \qquad \forall t \in [\nu, \tau_\ell], \tag{6.39}$$

which together with the continuity of $\mathcal{Y}^\ell$ shows that

$$\big\{\mathbf{1}_{A_\ell} Y_{\nu\vee(\tau_\ell\wedge t)}\big\}_{t\in[0,T]} \text{ is a continuous process.} \tag{6.40}$$

On the other hand, the strong limit $\widetilde{\mathcal{Z}}^\ell$ and the weak limit $\mathcal{Z}^\ell$ of $\big\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}} Z^n_t\big\}_{t\in[0,T]}$, $n \in \mathbb{N}$ must coincide, i.e. $\widetilde{\mathcal{Z}}^\ell_t = \mathcal{Z}^\ell_t$, $dt \otimes d\mathbb{P}$-a.s., which together with (6.38), (6.27) and (6.39) and (6.30) shows that

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_{A_\ell} \sup_{t\in[\nu,\tau_\ell]} \big|Y^n_t - Y_t\big|^2\right] + \lim_{n\to\infty} \mathbb{E}\int_\nu^{\tau_\ell} |Z^n_t - \mathcal{Z}^\ell_t|^2 dt = 0. \tag{6.41}$$

(3) By (6.31) and (6.40), $Y_\nu - Y_{\nu\vee(\tau_\ell\wedge t)} = \mathbf{1}_{A_\ell}\big(Y_\nu - Y_{\nu\vee(\tau_\ell\wedge t)}\big)$, $t \in [0,T]$ is an $\mathbf{F}$-adapted continuous process, then so is

$$\mathcal{K}^\ell_t := Y_\nu - Y_{\nu\vee(\tau_\ell\wedge t)} - \int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}} g(s, Y_s, \mathcal{Z}^\ell_s) ds + \int_0^t \mathbf{1}_{\{\nu<s\le\tau_\ell\}} \mathcal{Z}^\ell_s dB_s,$$
$$t \in [0,T]. \tag{6.42}$$

One can deduce from (6.41) that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[\nu,\tau_\ell]} \big|K^n_t - \mathcal{K}^\ell_t\big|^2\right] = 0. \tag{6.43*}$$

So up to a subsequence of $\{K^n\}_{n\in\mathbb{N}}$, it holds $\mathbb{P}$-a.s. that

$$\lim_{n\to\infty} \sup_{t\in[\nu,\tau_\ell]} \left|K^n_t - \mathcal{K}^\ell_t\right| = 0 \quad \text{and thus} \quad \mathcal{K}^\ell_t = \lim_{n\to\infty} K^n_t, \qquad \forall\, t \in [\nu, \tau_\ell], \tag{6.44}$$

which together with the monotonicity of K^n's show that for $\mathbb{P}$-a.s. $\omega \in \Omega$, the path $\mathcal{K}^\ell_\cdot(\omega)$ is increasing over period $[\nu(\omega), \tau_\ell(\omega)]$. One can also deduce from (6.44) that for $\mathbb{P}$-a.s. $\omega \in \Omega$, the measure $dK^n_t(\omega)$ converges weakly to the measure $d\mathcal{K}^\ell_t(\omega)$ on period $[\nu(\omega), \tau_\ell(\omega)]$. It then follows that $\mathbb{P}$-a.s.

$$\int_t^{\tau_\ell} (Y_s - L_s) d\mathcal{K}^\ell_s = 0, \qquad \forall\, t \in [\nu, \tau_\ell]. \tag{6.45*}$$

(4) *Setting $\tau_0 := \nu$, we next show that process Y together with processes*

$$Z_t := \sum_{\ell\in\mathbb{N}} \mathbf{1}_{\{\tau_{\ell-1}<t\le\tau_\ell\}} \mathcal{Z}^\ell_t \quad \text{and} \quad K_t := \sum_{\ell\in\mathbb{N}} \big(\mathcal{K}^\ell_{\tau_\ell\wedge t} - \mathcal{K}^\ell_{\tau_{\ell-1}\wedge t}\big), \qquad t \in [0, T] \tag{6.46}$$

solves (5.4).

As $\big\{\mathbf{1}_{\{\tau_{\ell-1}<t\le\tau_\ell\}}\big\}_{t\in[0,T]}$ is an $\mathbf{F}$-adapted càglàd process (thus $\mathbf{F}$-predictable) for each $\ell \in \mathbb{N}$, the process Z is $\mathbf{F}$-predictable. On the other hand, it is clear that K is an $\mathbf{F}$-adapted process with $K_0 = 0$.

Let $\mathcal{N}_3$ be the $\mathbb{P}$-null set such that for any $\omega \in \mathcal{N}_3^c$ and $\ell \in \mathbb{N}$,

$$\int_0^T |\mathcal{Z}^\ell_t(\omega)|^2 dt < \infty \text{ and the path } \big\{\mathcal{K}^\ell_t(\omega)\big\}_{t\in[0,T]} \text{ is continuous and increasing over period } [\nu(\omega), \tau_\ell(\omega)].$$

Given $\omega \in (\mathcal{N}_1 \cup \mathcal{N}_3)^c$, both sums in (6.46) are finite sums:

$$\begin{aligned} Z_t(\omega) &:= \sum_{\ell=1}^{N_\omega} \mathbf{1}_{\{\tau_{\ell-1}(\omega)<t\le\tau_\ell(\omega)\}} \mathcal{Z}^\ell_t(\omega) \quad \text{and} \\ K_t(\omega) &:= \sum_{\ell=1}^{N_\omega} \Big(\mathcal{K}^\ell\,(\tau_\ell(\omega) \wedge t, \omega) - \mathcal{K}^\ell\,(\tau_{\ell-1}(\omega) \wedge t, \omega)\Big), \qquad t \in [0, T]. \end{aligned} \tag{6.47}$$

The former implies that $\int_0^T |Z_t(\omega)|^2 dt = \int_0^{\tau(\omega)} |Z_t(\omega)|^2 dt = \sum_{\ell=1}^{N_\omega} \int_{\tau_{\ell-1}(\omega)}^{\tau_\ell(\omega)} |\mathcal{Z}^\ell_t(\omega)|^2 dt \le \sum_{\ell=1}^{N_\omega} \int_0^T |\mathcal{Z}^\ell_t(\omega)|^2 dt < \infty$, so $Z \in \mathbb{H}^{2,0}$. We see from the latter of (6.47) that the path $\{K_t(\omega)\}_{t\in[0,T]}$ is equal to 0 over period $[0, \nu(\omega)]$, is a connection of continuous increasing pieces from $\mathcal{K}^\ell\,(\tau_{\ell-1}(\omega), \omega)$ to $\mathcal{K}^\ell\,(\tau_\ell(\omega), \omega)$, $\ell = 1, \ldots, N_\omega$ over period $[\nu(\omega), \tau(\omega)]$, and then remains constant over period $\big[\tau(\omega), T\big]$. Thus, $\{K_t(\omega)\}_{t\in[0,T]}$ is a continuous increasing path, which shows $K \in \mathbb{K}^0$.

Let $\ell \in \mathbb{N}$. One can deduce that

$$\begin{aligned} K_t &= \sum_{i\in\mathbb{N}} \big(\mathcal{K}^i_{\tau_i\wedge t} - \mathcal{K}^i_{\tau_{i-1}\wedge t}\big) = \sum_{i=1}^{\ell} \big(\mathcal{K}^i_{\tau_i\wedge t} - \mathcal{K}^i_{\tau_{i-1}\wedge t}\big) \\ &= \sum_{i=1}^{\ell} \left(-Y_{\tau_i\wedge t} + Y_{\tau_{i-1}\wedge t} - \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t} g(s, Y_s, \mathcal{Z}^i_s) ds + \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t} \mathcal{Z}^i_s dB_s\right) \\ &= \sum_{i=1}^{\ell} \left(-Y_{\tau_i\wedge t} + Y_{\tau_{i-1}\wedge t} - \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t} g(s, Y_s, Z_s) ds + \int_{\tau_{i-1}\wedge t}^{\tau_i\wedge t} Z_s dB_s\right) \end{aligned}$$

$$= -Y_{\tau_\ell \wedge t} + Y_{\nu \wedge t} - \int_{\nu \wedge t}^{\tau_\ell \wedge t} g(s, Y_s, Z_s) ds + \int_{\nu \wedge t}^{\tau_\ell \wedge t} Z_s dB_s$$

$$= -Y_t + Y_\nu - \int_\nu^t g(s, Y_s, Z_s) ds + \int_\nu^t Z_s dB_s, \quad \forall t \in [\nu, \tau_\ell]. \tag{6.48}$$

It follows that

$$Y_t = Y_{\tau_\ell} + \int_t^{\tau_\ell} g(s, Y_s, Z_s) ds + K_{\tau_\ell} - K_t - \int_t^{\tau_\ell} Z_s dB_s, \quad \forall t \in [\nu, \tau_\ell]. \tag{6.49}$$

Since the increment of K over $[\tau_{i-1}, \tau_i]$ is that of $\mathcal{K}^i$ over $[\tau_{i-1}, \tau_i]$ for any $i \in \mathbb{N}$, (6.45*) implies that

$$\int_\nu^{\tau_\ell} (Y_t - L_t) dK_t = \sum_{i=1}^{\ell} \int_{\tau_{i-1}}^{\tau_i} (Y_t - L_t) dK_t$$

$$= \sum_{i=1}^{\ell} \int_{\tau_{i-1}}^{\tau_i} (Y_t - L_t) d\mathcal{K}_t^i = 0, \quad \mathbb{P}\text{-a.s.} \tag{6.50}$$

Because of $\mathbb{P}\{Y_\tau \geq L_\tau\} = 1$, (5.4) clearly holds $\mathbb{P}$-a.s. on the set $\{\nu = \tau\}$, and $\big\{(Y_{\nu \vee t})(\omega) \equiv (Y_\nu)(\omega)\big\}_{t \in [0,T]}$ is a constant path for any $\omega \in \{\nu = \tau\}$. Let $\mathcal{N}_4$ be the $\mathbb{P}$-null set such that for any $\omega \in \{\nu < \tau\} \cap \mathcal{N}_2^c \cap \mathcal{N}_4^c$ and $\ell \in \mathbb{N}$,

(6.35) and (6.50) hold on scenario ω, and $\big\{(Y_{\nu \vee (\tau_\ell \wedge t)})(\omega)\big\}_{t \in [0,T]}$ is a continuous path (see (6.40)).

For any $\omega \in \{\nu = \tau\} \cap \big(\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_4\big)^c$, we can deduce from (6.49) that (5.4) holds on scenario ω and $\big\{(Y_{\nu \vee t})(\omega) = (Y_{\nu \vee (\tau \wedge t)})(\omega)\big\}_{t \in [0,T]}$ is a continuous path. □

Proof of Proposition 5.3. The flat-off condition of reflected BSDEs implies that $\mathbb{P}$-a.s.

$$0 \leq \int_t^s \mathbf{1}_{\{Y_r^1 > Y_r^2\}} dK_r^1 = \int_t^s \mathbf{1}_{\{L_r^1 = Y_r^1 > Y_r^2\}} dK_r^1$$

$$\leq \int_t^s \mathbf{1}_{\{L_r^1 > L_r^2\}} dK_r^1 = 0, \quad \forall\, 0 \leq t < s \leq T.$$

It follows that $\mathbb{P}$-a.s.

$$\int_t^s \mathbf{1}_{\{Y_r^1 > Y_r^2\}} (dK_r^1 - dK_r^2) = -\int_t^s \mathbf{1}_{\{Y_r^1 > Y_r^2\}} dK_r^2 \leq 0, \quad \forall\, 0 \leq t < s \leq T.$$

Then we can apply Proposition 3.2 over period $[0, T]$ with $V^i = K^i$, $i = 1, 2$ to get the conclusion. □

Proof of Theorem 5.1. (1) (existence) For any $n \in \mathbb{N}$, we define function g_n as in (1.10), which satisfies (H1)–(H5) since $L \in \mathbb{S}_+^1$. In light of Proposition 3.1, the BSDE(ξ, g_n) admits a unique solution $(Y^n, Z^n) \in \cap_{p \in (0,1)} (\mathbb{S}^p \times \mathbb{H}^{2,p})$ such that Y^n is of class (D). Also, Proposition 3.3 shows that for any $\omega \in \Omega$ except on a $\mathbb{P}$-null set $\mathcal{N}$

$$Y_t^n(\omega) \leq Y_t^{n+1}(\omega), \quad \forall t \in [0, T], \ \forall n \in \mathbb{N}. \tag{6.51}$$

We can let (6.51) hold for any $\omega \in \Omega$ by setting $Y_t^n(\omega) := \mathbf{1}_{\{\omega \in \mathcal{N}^c\}} Y_t^n(\omega)$, $(t, \omega) \in [0, T] \times \Omega$, $n \in \mathbb{N}$ (each modified Y^n still belongs to $\cap_{p \in (0,1)} \mathbb{S}^p$, of class (D) and satisfies BSDE(ξ, g_n) with Z^n).

Applying Proposition 5.1 with $(Y^n, Z^n, J^n) = (Y^n, Z^n, 0)$, $n \in \mathbb{N}$ shows that the limit process $Y_t := \lim_{n\to\infty} \uparrow Y^n_t$, $t \in [0,T]$ is an $\mathbf{F}$-predictable process of class (D) satisfying $\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^p] < \infty$, $\forall\, p \in (0,1)$. It follows that $\sup_{t\in[0,T]} \big((Y^1_t)^- + Y^+_t\big) \le Y^1_* + Y_* < \infty$, $\mathbb{P}$-a.s. As $Y_T = \lim_{n\to\infty} \uparrow Y^n_T = \xi \ge L_T$, $\mathbb{P}$-a.s., applying Proposition 5.2 with $(\nu,\tau) = (0,T)$ yields that $Y \in \cap_{p\in(0,1)} \mathbb{S}^p$ solves RBSDE(ξ, g, L) with some $(Z, K) \in \mathbb{H}^{2,0} \times \mathbb{K}^0$. Moreover, applying Lemma A.2 with $(\nu,\tau) = (0,T)$ and using Hölder's inequality show that

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^{p/2}\right] + \mathbb{E}\left[K_T^p\right] \le C_p\, \mathbb{E}\left[(Y_*)^p\right] + C_p \left(\mathbb{E}\int_0^T h_t dt\right)^p < \infty,$$
$$\forall\, p \in (0,1).$$

Namely, $(Z, K) \in \cap_{p\in(0,1)} (\mathbb{H}^{2,p} \times \mathbb{K}^p)$.

(2) (uniqueness) Let $(Y^1, Z^1, K^1), (Y^2, Z^2, K^2) \in \cap_{p\in(0,1)} (\mathbb{S}^p \times \mathbb{H}^{2,p} \times \mathbb{K}^p)$ be two solutions of RBSDE(ξ, g, L) such that Y^1, Y^2 is of class (D). We know from Proposition 5.3 that $\mathbb{P}\{Y^1_t = Y^2_t,\ \forall\, t \in [0,T]\} = 1$, so it holds $\mathbb{P}$-a.s. that

$$\int_t^T g(s, Y^1_s, Z^1_s) ds + K^1_T - K^1_t - \int_t^T Z^1_s dB_s$$
$$= \int_t^T g(s, Y^2_s, Z^2_s) ds + K^2_T - K^2_t - \int_t^T Z^2_s dB_s, \quad t \in [0,T].$$

Comparing martingale parts on both side shows that $Z^1_t = Z^2_t$, $dt \otimes d\mathbb{P}$-a.s. Then it follows that $\mathbb{P}$-a.s.

$$K^1_t = Y^1_0 - Y^1_t - \int_0^t g(s, Y^1_s, Z^1_s) ds + \int_0^t Z^1_s dB_s = Y^2_0 - Y^2_t - \int_0^t g(s, Y^2_s, Z^2_s) ds$$
$$+ \int_0^t Z^2_s dB_s = K^2_t, \quad t \in [0,T].$$

(3) (proof of (5.1) and (5.2)) Fix $\nu \in \mathcal{T}$ and $\gamma \in \mathcal{T}_{\nu,T}$. We will simply denote $\tau_\sharp(\nu)$ by $\widehat{\tau}$. The uniform integrability of $\{Y_\gamma\}_{\gamma\in\mathcal{T}}$ implies that $Y_\gamma \in L^1(\mathcal{F}_\gamma)$, so we see from (A.2) that $\mathbb{P}$-a.s.

$$Y^{\gamma,Y_\gamma}_t = Y_\gamma + \int_t^\gamma g\big(s, Y^{\gamma,Y_\gamma}_s, Z^{\gamma,Y_\gamma}_s\big) ds - \int_t^\gamma Z^{\gamma,Y_\gamma}_s dB_s, \quad \forall\, t \in [\nu,\gamma]. \tag{6.52}$$

Since it holds $\mathbb{P}$-a.s. that

$$Y_t = Y_\gamma + \int_t^\gamma g\big(s, Y_s, Z_s\big) ds + K_\gamma - K_t - \int_t^\gamma Z_s dB_s, \quad \forall\, t \in [\nu,\gamma],$$

applying Proposition 3.2 with $(Y^1, Z^1, V^1) = \big(Y^{\gamma,Y_\gamma}, Z^{\gamma,Y_\gamma}, 0\big)$ and $(Y^2, Z^2, V^2) = (Y, Z, K)$ yields that $\mathbb{P}$-a.s., $Y^{\gamma,Y_\gamma}_t \le Y_t$ for any $t \in [\nu,\gamma]$. In particular,

$$\mathcal{E}^g_{\nu,\gamma}[Y_\gamma] = Y^{\gamma,Y_\gamma}_\nu \le Y_\nu, \quad \mathbb{P}\text{-a.s.} \tag{6.53}$$

As $Y_\gamma \ge \mathbf{1}_{\{\gamma<T\}} L_\gamma + \mathbf{1}_{\{\gamma=T\}} \xi = \mathcal{R}_\gamma$, $\mathbb{P}$-a.s., we see from the monotonicity of g-evaluation that

$$Y_\nu \ge \mathcal{E}^g_{\nu,\gamma}[Y_\gamma] \ge \mathcal{E}^g_{\nu,\gamma}[\mathcal{R}_\gamma], \quad \mathbb{P}\text{-a.s.} \tag{6.54}$$

Since it holds $\mathbb{P}$-a.s.that $Y_t > \mathcal{R}_t = L_t$ for any $t \in [\nu, \widehat{\tau})$, the flat-off condition in RBSDE(ξ, g, L) implies that $\mathbb{P}$-a.s. $K_t = K_\nu$ for any $t \in [\nu, \widehat{\tau}]$. Then it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} Y_t &= Y_{\widehat{\tau}\wedge\gamma} + \int_t^{\widehat{\tau}\wedge\gamma} g\big(s, Y_s, Z_s\big)ds + K_{\widehat{\tau}\wedge\gamma} - K_t - \int_t^{\widehat{\tau}\wedge\gamma} Z_s dB_s \\ &= Y_{\widehat{\tau}\wedge\gamma} + \int_t^{\widehat{\tau}\wedge\gamma} g\big(s, Y_s, Z_s\big)ds - \int_t^{\widehat{\tau}\wedge\gamma} Z_s dB_s, \quad \forall\, t \in \big[\nu, \widehat{\tau}\wedge\gamma\big]. \end{aligned}$$

Similar to (6.52), one has that $\mathbb{P}$-a.s.

$$\begin{aligned} Y_t^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}} &= Y_{\widehat{\tau}\wedge\gamma} + \int_t^{\widehat{\tau}\wedge\gamma} g\big(s, Y_s^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}, Z_s^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}\big)ds \\ &\quad - \int_t^{\widehat{\tau}\wedge\gamma} Z_s^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}} dB_s, \quad \forall\, t \in \big[\nu, \widehat{\tau}\wedge\gamma\big]. \end{aligned}$$

Applying Proposition 3.2 again yields that $\mathbb{P}$-a.s., $Y_t = Y_t^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}$ for any $t \in \big[\nu, \widehat{\tau}\wedge\gamma\big]$. It thus follows that

$$Y_\nu = Y_\nu^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}} = \mathcal{E}^g_{\nu, \widehat{\tau}\wedge\gamma}\big[Y_{\widehat{\tau}\wedge\gamma}\big], \quad \mathbb{P}\text{-a.s.}, \tag{6.55}$$

which together with (6.53) proves (5.1).

As $Y_T = \xi = \mathcal{R}_T$, $\mathbb{P}$-a.s., we can deduce from the continuity of process Y and the right-continuity of process $\mathcal{R}$ that $Y_{\widehat{\tau}} = \mathcal{R}_{\widehat{\tau}}$, $\mathbb{P}$-a.s. So taking $\gamma = T$ in (6.55) yields that $Y_\nu = \mathcal{E}^g_{\nu,\widehat{\tau}}[Y_{\widehat{\tau}}] = \mathcal{E}^g_{\nu,\widehat{\tau}}[\mathcal{R}_{\widehat{\tau}}]$, $\mathbb{P}$-a.s., which together with (6.54) implies (5.2). □

6.3. *Proof of Theorem 2.1*

(1) (existence) We shall follow [25]'s approach by pasting local solutions to construct a global solution of DRBSDE (ξ, g, L, U), see our introduction for a synopsis.

(1a) *(increasing penalization scheme)*

For $n \in \mathbb{N}$, we define function g_n as in (1.10) which satisfies (H1)–(H5) since $L \in \mathbb{S}^1_+$. Theorem 5.1 and Remark 5.2 show that the following reflected BSDE with generator g_n and upper obstacle U

$$\begin{cases} U_t \ge Y_t = \xi + \displaystyle\int_t^T g_n(s, Y_s, Z_s)ds - J_T + J_t - \int_t^T Z_s dB_s, \quad t \in [0, T], \\ \displaystyle\int_0^T (U_t - Y_t)dJ_t = 0 \end{cases} \tag{6.56}$$

admits a unique solution $(Y^n, Z^n, J^n) \in \cap_{p\in(0,1)}(\mathbb{S}^p \times \mathbb{H}^{2,p} \times \mathbb{K}^p)$ such that Y^n is of class (D). In light of Proposition 5.3 and Remark 5.2, it holds for any $\omega \in \Omega$ except on a $\mathbb{P}$-null set $\mathcal{N}$ that

$$Y_t^n(\omega) \le Y_t^{n+1}(\omega), \quad \forall\, t \in [0, T], \ \forall\, n \in \mathbb{N}. \tag{6.57}$$

We can let (6.57) hold for any $\omega \in \Omega$ by setting $Y_t^n(\omega) := \mathbf{1}_{\{\omega\in\mathcal{N}^c\}} Y_t^n(\omega)$, $(t, \omega) \in [0, T] \times \Omega$, $n \in \mathbb{N}$ (each modified Y^n still belongs to $\cap_{p\in(0,1)} \mathbb{S}^p$, of class (D) and satisfies (6.56) with (Z^n, J^n)). By Proposition 5.1, the limit process $Y_t := \lim_{n\to\infty} \uparrow Y_t^n$, $t \in [0, T]$ is an $\mathbf{F}$-predictable process of class (D) that satisfies

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]} |Y_t|^p\Bigg] < \infty, \quad \forall\, p \in (0, 1). \tag{6.58}$$

Let $\nu \in \mathcal{T}$. For any $n \in \mathbb{N}$, define a stopping time $\gamma_\nu^n := \inf\{t \in [\nu, T] : Y_t^n = U_t\} \wedge T \in \mathcal{T}$. As it holds $\mathbb{P}$-a.s. that $Y_t^n < U_t$ for any $t \in \big[\nu, \gamma_\nu^n\big)$, we can deduce from the flat-off condition in (6.56) that $\mathbb{P}\big\{J_t^n = J_\nu^n,\ \forall\, t \in [\nu, \gamma_\nu^n]\big\} = 1$. It then follows that $\mathbb{P}$-a.s.

$$0 = J_{\gamma_\nu^n}^n - J_t^n = Y_{\gamma_\nu^n}^n - Y_t^n + \int_t^{\gamma_\nu^n} g_n(s, Y_s^n, Z_s^n)ds - \int_t^{\gamma_\nu^n} Z_s^n dB_s. \tag{6.59}$$

Clearly, γ_ν^n is decreasing in n, and their limit $\gamma_\nu := \lim_{n\to\infty} \uparrow \gamma_\nu^n \ge \nu$ is still a stopping time thanks to the right continuity of filtration $\mathbf{F}$. We claim that

$$Y_{\gamma_\nu} = \mathbf{1}_{\{\gamma_\nu = T\}}\xi + \mathbf{1}_{\{\gamma_\nu < T\}}U_{\gamma_\nu}, \quad \mathbb{P}\text{-a.s.} \tag{6.60}$$

(which will be shown in the Appendix). So $Y_{\gamma_\nu} \ge \mathbf{1}_{\{\gamma_\nu = T\}}L_T + \mathbf{1}_{\{\gamma_\nu < T\}}L_{\gamma_\nu} = L_{\gamma_\nu}$, $\mathbb{P}$-a.s. Since $\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t^1|^p + \sup_{t\in[0,T]}|Y_t|^p\Big] < \infty$, $\forall\, p \in (0, 1)$ and since it holds $\mathbb{P}$-a.s. that

$$Y_t^n = Y_{\gamma_\nu}^n + \int_t^{\gamma_\nu} g_n(s, Y_s^n, Z_s^n)ds - \int_t^{\gamma_\nu} Z_s^n dB_s, \quad \forall\, t \in [\nu, \gamma_\nu] \tag{6.61}$$

for any $n \in \mathbb{N}$ by (6.59), applying Proposition 5.2 to $\big\{(Y^n, Z^n)\big\}_{n\in\mathbb{N}}$ yields that process $\big\{Y_{\nu\vee(\gamma_\nu\wedge t)}\big\}_{t\in[0,T]}$ has $\mathbb{P}$-a.s. continuous paths and there exist $(Z^\nu, K^\nu) \in \mathbb{H}^{2,0} \times \mathbb{K}^0$ such that $\mathbb{P}$-a.s.

$$\begin{cases} L_t \le Y_t = Y_{\gamma_\nu} + \displaystyle\int_t^{\gamma_\nu} g(s, Y_s, Z_s^\nu)ds + K_{\gamma_\nu}^\nu - K_t^\nu - \int_t^{\gamma_\nu} Z_s^\nu dB_s, \\ \quad \forall\, t \in [\nu, \gamma_\nu], \\ \displaystyle\int_\nu^{\gamma_\nu} (Y_t - L_t)dK_t^\nu = 0. \end{cases} \tag{6.62}$$

Since $\mathbb{E}[|Y_\nu|] < \infty$ by the uniform integrability of $\{Y_\zeta\}_{\zeta\in\mathcal{T}}$, Lemma A.2, Hölder's inequality and (6.58) show that

$$\mathbb{E}\left[\left(\int_\nu^\tau |Z_t|^2dt\right)^{p/2}\right] \le C_p\,\mathbb{E}\Big[\sup_{t\in[\nu,\tau]}|Y_t|^p\Big] + C_p\left(\mathbb{E}\int_\nu^\tau h_t dt\right)^p < \infty,$$
$$\forall\, p \in (0, 1). \tag{6.63}$$

(1b) *(decreasing penalization scheme)*

Similar to g_L discussed in Remark 1.3(4), $g_U(t, \omega, y) := (y - U_t(\omega))^+$, $(t, \omega, y) \in [0, T] \times \Omega \times \mathbb{R}$ is clearly a $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable function satisfying (H2)–(H4). For any $n \in \mathbb{N}$, we see from Remark 1.3(3) that

$$\widetilde{g}_n(t, \omega, y, z) := g(t, \omega, y, z) - n\,(y - U_t(\omega))^+, \quad \forall\,(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$$

defines a generator, and Theorem 5.1 shows that RBSDE$\big(\xi, \widetilde{g}_n, L\big)$ admits a unique solution $\big(\widetilde{Y}^n, \widetilde{Z}^n, \widetilde{K}^n\big) \in \cap_{p\in(0,1)}(\mathbb{S}^p \times \mathbb{H}^{2,p} \times \mathbb{K}^p)$ such that $\widetilde{Y}^n$ is of class (D). Since $\widetilde{g}_n$ is decreasing in n, Proposition 5.3 shows that $\mathbb{P}$-a.s.

$$\widetilde{Y}_t^n \ge \widetilde{Y}_t^{n+1}, \quad \forall\, t \in [0, T], \forall\, n \in \mathbb{N}. \tag{6.64}$$

As in (6.57), we can assume that (6.64) holds everywhere on Ω.

Set $\big(\widetilde{L},\widetilde{U}\big):=(-U,-L)\in\mathbb{S}^1_+\times\mathbb{S}^1_-$. For any $n\in\mathbb{N}$, $\big(\widehat{Y}^n,\widehat{Z}^n,\widehat{J}^n\big):=\big(-\widetilde{Y}^n,-\widetilde{Z}^n,-\widetilde{K}^n\big)$ satisfies that $\mathbb{P}$-a.s.

$$\begin{aligned}\widetilde{U}_t &= -L_t\ge \widehat{Y}^n_t=-\xi-\int_t^T g\big(s,\widetilde{Y}^n_s,\widetilde{Z}^n_s\big)ds+n\int_t^T\big(\widetilde{Y}^n_s-U_s\big)^+ds-\widetilde{K}^n_T+\widetilde{K}^n_t\\&\quad+\int_t^T\widetilde{Z}^n_s dB_s\\&=\widehat{Y}^n_T+\int_t^T g_-\big(s,\widehat{Y}^n_s,\widehat{Z}^n_s\big)ds+n\int_t^T\big(\widehat{Y}^n_s-\widetilde{L}_s\big)^-ds+\widehat{J}^n_T-\widehat{J}^n_t\\&\quad-\int_t^T\widehat{Z}^n_s dB_s,\qquad t\in[0,T].\end{aligned}\tag{6.65}$$

Since g_- is a generator by Remark 1.3(1), applying Proposition 5.1 to $\Big\{\big(\widehat{Y}^n,\widehat{Z}^n,\widehat{J}^n\big)\Big\}_{n\in\mathbb{N}}$ yields that $\widehat{Y}_t:=\lim_{n\to\infty}\uparrow\widehat{Y}^n_t$, $t\in[0,T]$ is an **F**-predictable process of class (D) that satisfies $\mathbb{E}[\sup_{t\in[0,T]}\big|\widehat{Y}_t\big|^p]<\infty$, $\forall\,p\in(0,1)$.

Let $\nu\in\mathcal{T}$. The stopping times $\tau^n_\nu:=\inf\big\{t\in[\nu,T]:\widehat{Y}^n_t=\widetilde{U}_t\big\}\wedge T=\inf\big\{t\in[\nu,T]:\widetilde{Y}^n_t=L_t\big\}\wedge T\in\mathcal{T}$ is decreasing in n. Analogous to (6.60), $\tau_\nu:=\lim_{n\to\infty}\downarrow\tau^n_\nu\ge\nu$ is still a stopping time that satisfies

$$\begin{aligned}\widehat{Y}_{\tau_\nu}&=-\mathbf{1}_{\{\tau_\nu=T\}}\xi+\mathbf{1}_{\{\tau_\nu<T\}}\widetilde{U}_{\tau_\nu}\ge-\mathbf{1}_{\{\tau_\nu=T\}}U_T-\mathbf{1}_{\{\tau_\nu<T\}}L_{\tau_\nu}\ge-U_{\tau_\nu}\\&\ge\widetilde{L}_{\tau_\nu},\quad\mathbb{P}\text{-a.s.}\end{aligned}\tag{6.66}$$

For any $n\in\mathbb{N}$, similar to (6.61), we can deduce from (6.65) that $\mathbb{P}$-a.s.

$$\widehat{Y}^n_t=\widehat{Y}^n_{\tau_\nu}+\int_t^{\tau_\nu}g_n\big(s,\widehat{Y}^n_s,\widehat{Z}^n_s\big)ds+n\int_t^{\tau_\nu}\big(\widehat{Y}^n_s-\widetilde{L}_s\big)^-ds-\int_t^{\tau_\nu}\widehat{Z}^n_s dB_s,\qquad\forall\,t\in[\nu,\tau_\nu].$$

As $\mathbb{E}[\sup_{t\in[0,T]}\big|\widehat{Y}^1_t\big|^p+\sup_{t\in[0,T]}\big|\widehat{Y}_t\big|^p]<\infty$, $\forall\,p\in(0,1)$, using (6.66) and applying Proposition 5.2 yield that process $\big\{\widehat{Y}_{\nu\vee(\tau_\nu\wedge t)}\big\}_{t\in[0,T]}$ has $\mathbb{P}$-a.s. continuous paths and there exist $\big(\widehat{Z}^\nu,\widehat{K}^\nu\big)\in\mathbb{H}^{2,0}\times\mathbb{K}^0$ such that $\mathbb{P}$-a.s.

$$\begin{cases}\widetilde{L}_t\le\widehat{Y}_t=\widehat{Y}_{\tau_\nu}+\displaystyle\int_t^{\tau_\nu}g_-(s,\widehat{Y}_s,\widehat{Z}^\nu_s)ds+\widehat{K}^\nu_{\tau_\nu}-\widehat{K}^\nu_t-\int_t^{\tau_\nu}\widehat{Z}^\nu_s dB_s,\\\quad\forall\,t\in[\nu,\tau_\nu],\\\displaystyle\int_\nu^{\tau_\nu}(\widehat{Y}_t-\widetilde{L}_t)d\widehat{K}^\nu_t=0.\end{cases}\tag{6.67}$$

Since g_- satisfies (H4) and (H5) with the same function h as g, an analogy to (6.63) shows that

$$\mathbb{E}\left[\left(\int_\nu^\tau|\widehat{Z}_t|^2dt\right)^{p/2}\right]\le C_p\,\mathbb{E}\Big[\sup_{t\in[\nu,\tau]}\big|\widehat{Y}_t\big|^p\Big]+C_p\left(\mathbb{E}\int_\nu^\tau h_t dt\right)^p<\infty,\quad\forall\,p\in(0,1).\tag{6.68}$$

Set $\big(\widetilde{Y}, \widetilde{Z}^\nu, \widetilde{J}^\nu\big) = \big(-\widehat{Y}, -\widehat{Z}^\nu, -\widehat{K}^\nu\big)$, it follows from (6.67) that $\mathbb{P}$-a.s.

$$\begin{cases} U_t \ge \widetilde{Y}_t = \widetilde{Y}_{\tau_\nu} + \int_t^{\tau_\nu} g(s, \widetilde{Y}_s, \widetilde{Z}^\nu_s)ds - \widetilde{J}^\nu_{\tau_\nu} + \widetilde{J}^\nu_t - \int_t^{\tau_\nu} \widetilde{Z}^\nu_s dB_s, \quad \forall t \in [\nu, \tau_\nu], \\ \int_\nu^{\tau_\nu} (U_t - \widetilde{Y}_t) d\widetilde{J}^\nu_t = 0. \end{cases} \tag{6.69}$$

(1c) *Next, we show that except on a $\mathbb{P}$-null set $\mathcal{N}_1$*

$$L_t \le \widetilde{Y}_t = Y_t \le U_t, \quad t \in [0, T]. \tag{6.70}$$

Given $n \in \mathbb{N}$, we set $V^n_t := n\int_0^t (Y^n_s - L_s)^- ds - J^n_t$ and $\widetilde{V}^n_t := -n\int_0^t (\widetilde{Y}^n_s - U_s)^+ ds + K^n_t$, $t \in [0, T]$. As (Y^n, Z^n, J^n) solves (6.56) and $\big(\widetilde{Y}^n, \widetilde{Z}^n, \widetilde{K}^n\big)$ solves RBSDE$\big(\xi, \widetilde{g}_n, L\big)$, it holds $\mathbb{P}$-a.s. that

$$Y^n_t \le U_t \quad \text{and} \quad \widetilde{Y}^n_t \ge L_t, \quad \forall t \in [0, T]. \tag{6.71}$$

We can then deduce that $\mathbb{P}$-a.s.

$$\begin{aligned} \int_t^s \mathbf{1}_{\{Y^n_r > \widetilde{Y}^n_r\}} \big(dV^n_r - d\widetilde{V}^n_r\big) &\le n \int_t^s \mathbf{1}_{\{Y^n_r > \widetilde{Y}^n_r\}} \big((Y^n_r - L_r)^- + (\widetilde{Y}^n_r - U_r)^+\big) dr \\ &= n \int_t^s \Big(\mathbf{1}_{\{L_r \ge Y^n_r > \widetilde{Y}^n_r\}} (Y^n_r - L_r)^- + \mathbf{1}_{\{Y^n_r > \widetilde{Y}^n_r \ge U_r\}} (\widetilde{Y}^n_r - U_r)^+\Big) dr \\ &\le n \int_t^s \Big(\mathbf{1}_{\{L_r > \widetilde{Y}^n_r\}} (Y^n_r - L_r)^- + \mathbf{1}_{\{Y^n_r > U_r\}} (\widetilde{Y}^n_r - U_r)^+\Big) dr = 0, \quad \forall 0 \le t < s \le T. \end{aligned}$$

Since $Y^n_T = \widetilde{Y}^n_T = \xi$, $\mathbb{P}$-a.s., applying Proposition 3.2 over period $[0, T]$ with $g^1 = g^2 = g$, $(Y^1, Z^1, V^1) = (Y^n, Z^n, V^n)$ and $(Y^2, Z^2, V^2) = \big(\widetilde{Y}^n, \widetilde{Z}^n, \widetilde{V}^n\big)$ yields that $\mathbb{P}\big\{Y^n_t \le \widetilde{Y}^n_t, \ \forall t \in [0, T]\big\} = 1$. It follows that $\mathbb{P}$-a.s.

$$Y_t = \lim_{n\to\infty} \uparrow Y^n_t \le \lim_{n\to\infty} \downarrow \widetilde{Y}^n_t = \widetilde{Y}_t, \quad t \in [0, T]. \tag{6.72}$$

On the other hand, let $\nu \in \mathcal{T}$. By (6.66),

$$\begin{aligned} \widetilde{Y}_{\tau_\nu \wedge \gamma_\nu} &= \mathbf{1}_{\{\tau_\nu > \gamma_\nu\}} \widetilde{Y}_{\gamma_\nu} + \mathbf{1}_{\{\tau_\nu \le \gamma_\nu\}} \widetilde{Y}_{\tau_\nu} = \mathbf{1}_{\{\tau_\nu > \gamma_\nu\}} \widetilde{Y}_{\gamma_\nu} + \mathbf{1}_{\{\tau_\nu \le \gamma_\nu, \tau_\nu < T\}} L_{\tau_\nu} + \mathbf{1}_{\{\tau_\nu = \gamma_\nu = T\}} \xi \\ &\le \mathbf{1}_{\{\tau_\nu > \gamma_\nu\}} U_{\gamma_\nu} + \mathbf{1}_{\{\tau_\nu \le \gamma_\nu, \tau_\nu < T\}} Y_{\tau_\nu} + \mathbf{1}_{\{\tau_\nu = \gamma_\nu = T\}} \xi = \mathbf{1}_{\{\tau_\nu > \gamma_\nu\}} Y_{\gamma_\nu} + \mathbf{1}_{\{\tau_\nu \le \gamma_\nu\}} Y_{\tau_\nu} \\ &= Y_{\tau_\nu \wedge \gamma_\nu}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Also, we see from (6.69)and (6.62) that $\mathbb{P}$-a.s.

$$\widetilde{Y}_t = \widetilde{Y}_{\tau_\nu \wedge \gamma_\nu} + \int_t^{\tau_\nu \wedge \gamma_\nu} g(s, \widetilde{Y}_s, \widetilde{Z}^\nu_s) ds - \widetilde{J}^\nu_{\tau_\nu \wedge \gamma_\nu} + \widetilde{J}^\nu_t - \int_t^{\tau_\nu \wedge \gamma_\nu} \widetilde{Z}^\nu_s dB_s,$$

and

$$Y_t = Y_{\tau_\nu \wedge \gamma_\nu} + \int_t^{\tau_\nu \wedge \gamma_\nu} g(s, Y_s, Z^\nu_s) ds + K^\nu_{\tau_\nu \wedge \gamma_\nu} - K^\nu_t - \int_t^{\tau_\nu \wedge \gamma_\nu} Z^\nu_s dB_s, \\ \forall t \in [\nu, \tau_\nu \wedge \gamma_\nu].$$

Since both Y and $\widetilde{Y}$ are of class (D), using (6.63), (6.68) and applying Proposition 3.2 over stochastic interval $[\![\nu, \tau_\nu \wedge \gamma_\nu]\!]$ with $(Y^1, Z^1, V^1) = \big(\widetilde{Y}, \widetilde{Z}^\nu, -\widetilde{J}^\nu\big)$ and $(Y^2, Z^2, V^2) =$

(Y, Z^ν, K^ν) yield that $\mathbb{P}$-a.s., $\widetilde{Y}_t \le Y_t$, $\forall\, t \in [\nu, \tau_\nu \wedge \gamma_\nu]$. In particular, one has $\widetilde{Y}_\nu \le Y_\nu$, $\mathbb{P}$-a.s. As ν varies over $\mathcal{T}$, the cross-section theorem (see Theorem IV.86 of [16]) and (6.71) imply that $\mathbb{P}$-a.s.

$$L_t \le \lim_{n\to\infty} \downarrow \widetilde{Y}^n_t = \widetilde{Y}_t \le Y_t = \lim_{n\to\infty} \uparrow Y^n_t \le U_t, \quad t \in [0, T],$$

which together with (6.72) proves (6.70). In particular, we see from (6.62) and (6.69) that $(Y, Z^\nu, K^\nu, 0)$ locally solves the doubly reflected BSDE over the stochastic interval $[\![\nu, \gamma_\nu]\!]$ and $(Y, \widetilde{Z}^\nu, 0, \widetilde{J}^\nu) = (\widetilde{Y}, \widetilde{Z}^\nu, 0, \widetilde{J}^\nu)$ locally solves the doubly reflected BSDE over the stochastic interval $[\![\nu, \tau_\nu]\!]$.

(1d) *(construction of a solution via pasting)*

For any $n \in \mathbb{N}$ and $t \in [0, T]$, set $\mathcal{I}^n_t := [(t - 2^{-n}) \vee 0, (t + 2^{-n}) \wedge T]$. Similar to (A.19), we can deduce from the continuity of Y^n's, $\widetilde{Y}^n$'s and (6.70) that $\mathbb{P}$-a.s.

$$\begin{aligned}
\lim_{n\to\infty} \uparrow \inf_{s\in\mathcal{I}^n_t} Y_s &= \lim_{n\to\infty} \uparrow \inf_{s\in\mathcal{I}^n_t} \lim_{m\to\infty} \uparrow Y^m_s \ge \lim_{m\to\infty} \uparrow \lim_{n\to\infty} \uparrow \inf_{s\in\mathcal{I}^n_t} Y^m_s \\
&= \lim_{m\to\infty} \uparrow Y^m_t = Y_t = \widetilde{Y}_t = \lim_{m\to\infty} \downarrow \widetilde{Y}^m_t = \lim_{m\to\infty} \downarrow \lim_{n\to\infty} \downarrow \sup_{s\in\mathcal{I}^n_t} \widetilde{Y}^m_s \\
&\ge \lim_{n\to\infty} \downarrow \sup_{s\in\mathcal{I}^n_t} \lim_{m\to\infty} \downarrow \widetilde{Y}^m_s = \lim_{n\to\infty} \downarrow \sup_{s\in\mathcal{I}^n_t} \widetilde{Y}_s = \lim_{n\to\infty} \downarrow \sup_{s\in\mathcal{I}^n_t} Y_s \\
&\ge \lim_{n\to\infty} \uparrow \inf_{s\in\mathcal{I}^n_t} Y_s, \quad \forall\, t \in [0, T],
\end{aligned}$$

which shows that Y is a continuous process. So $Y \in \cap_{p\in(0,1)} \mathbb{S}^p$ by (6.58).

Let $\nu_1 := 0$, we recursively set stopping times $\nu'_\ell := \gamma_{\nu_\ell}$, $\nu_{\ell+1} := \tau_{\nu'_\ell}$, $\ell \in \mathbb{N}$, and define processes

$$\begin{aligned}
&Z_t := \sum_{\ell\in\mathbb{N}} \mathbf{1}_{\{\nu_\ell < t \le \nu'_\ell\}} Z^{\nu_\ell}_t + \mathbf{1}_{\{\nu'_\ell < t \le \nu_{\ell+1}\}} \widetilde{Z}^{\,\nu'_\ell}_t, \qquad K_t := \sum_{\ell\in\mathbb{N}} \Big(K^{\nu_\ell}_{\nu'_\ell \wedge t} - K^{\nu_\ell}_{\nu_\ell \wedge t} \Big), \\
&J_t := \sum_{\ell\in\mathbb{N}} \Big(\widetilde{J}^{\nu'_\ell}_{\nu_{\ell+1}\wedge t} - \widetilde{J}^{\nu'_\ell}_{\nu'_\ell \wedge t} \Big), \quad t \in [0, T].
\end{aligned} \tag{6.73}$$

Since $\big\{\mathbf{1}_{\{\nu_\ell < t \le \nu'_\ell\}}\big\}_{t\in[0,T]}$ and $\big\{\mathbf{1}_{\{\nu'_\ell < t \le \nu_{\ell+1}\}}\big\}_{t\in[0,T]}$ are $\mathbf{F}$-adapted càglàd processes (thus $\mathbf{F}$-predictable) for each $\ell \in \mathbb{N}$, the process Z is $\mathbf{F}$-predictable. Also, it is clear that K and J are $\mathbf{F}$-adapted processes with $K_0 = J_0 = 0$.

Let $\mathcal{N}_2$ be the $\mathbb{P}$-null set such that for any $\omega \in \mathcal{N}^c_2$, the paths $L_\cdot(\omega)$, $U_\cdot(\omega) Y_\cdot(\omega)$ are continuous and $L_t(\omega) < U_t(\omega)$ for any $t \in [0, T]$. By (6.60) and (6.66), it holds except on a $\mathbb{P}$-null set $\mathcal{N}_3$ that

$$\mathbf{1}_{\{\nu'_\ell < T\}} Y_{\nu'_\ell} = \mathbf{1}_{\{\nu'_\ell < T\}} U_{\nu'_\ell} \quad \text{and} \quad \mathbf{1}_{\{\nu_{\ell+1} < T\}} \widetilde{Y}_{\nu_{\ell+1}} = \mathbf{1}_{\{\nu_{\ell+1} < T\}} L_{\nu_{\ell+1}}, \qquad \forall\, \ell \in \mathbb{N}. \tag{6.74}$$

We claim that $\{\nu_n\}_{n\in\mathbb{N}}$ is stationary: more precisely, for any $\omega \in (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)^c$

$$T = \nu_{N_\omega}(\omega) \quad \text{for some } N_\omega \in \mathbb{N}. \tag{6.75}$$

Assume not, then it holds for some $\omega \in (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)^c$ that $\nu_n(\omega) < T$ for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, as $\nu_n(\omega) \le \nu'_n(\omega) \le \nu_{n+1}(\omega) < T$, (6.74) shows that

$$\big(Y_{\nu'_n}\big)(\omega) = \big(U_{\nu'_n}\big)(\omega) \quad \text{and} \quad \big(Y_{\nu_{n+1}}\big)(\omega) = \big(\widetilde{Y}_{\nu_{n+1}}\big)(\omega) = \big(L_{\nu_{n+1}}\big)(\omega). \tag{6.76}$$

Let $t_* = t_*(\omega) = \lim_{n\to\infty} \uparrow \nu_n(\omega) = \lim_{n\to\infty} \uparrow \nu'_n(\omega) \in [0, T]$. As $n \to \infty$ in (6.76), we see from the continuity of paths $L_\cdot(\omega)$, $U_\cdot(\omega)$ and $Y_\cdot(\omega)$ that

$$L_{t_*}(\omega) = \lim_{n\to\infty} \big(L_{\nu_{n+1}}\big)(\omega) = \lim_{n\to\infty} \big(\widetilde{Y}_{\nu_{n+1}}\big)(\omega) = \widetilde{Y}_{t_*}(\omega) = Y_{t_*}(\omega) = \lim_{n\to\infty} \big(Y_{\nu'_n}\big)(\omega)$$
$$= \lim_{n\to\infty} \big(U_{\nu'_n}\big)(\omega) = U_{t_*}(\omega).$$

A contradiction appears, so (6.75) holds. Then the three sums in (6.73) are finite sums. An analogous discussion to the one below (6.47) shows that $Z \in \mathbb{H}^{2,0}$ and $K, J \in \mathbb{K}^0$.

Let $\ell \in \mathbb{N}$ with $\ell \ge 2$. Similar to (6.48), we can deduce from (6.69), (6.62) and (6.70) that $\mathbb{P}$-a.s.

$$\begin{aligned} K_t - J_t &= \sum_{i=1}^{\ell-1} \Big(K^{\nu_i}_{\nu'_i\wedge t} - K^{\nu_i}_{\nu_i\wedge t}\Big) - \sum_{i=1}^{\ell-1} \Big(\widetilde{J}^{\nu'_i}_{\nu_{i+1}\wedge t} - \widetilde{J}^{\nu'_i}_{\nu'_i\wedge t}\Big) \\ &= \sum_{i=1}^{\ell-1} \Bigg(-Y_{\nu'_i\wedge t} + Y_{\nu_i\wedge t} - \int_{\nu_i\wedge t}^{\nu'_i\wedge t} g\big(s, Y_s, Z^{\nu_i}_s\big)ds \\ &\quad + \int_{\nu_i\wedge t}^{\nu'_i\wedge t} Z^{\nu_i}_s dB_s - \widetilde{Y}_{\nu_{i+1}\wedge t} + \widetilde{Y}_{\nu'_i\wedge t} \\ &\quad - \int_{\nu'_i\wedge t}^{\nu_{i+1}\wedge t} g\big(s, \widetilde{Y}_s, \widetilde{Z}^{\nu'_i}_s\big)ds + \int_{\nu'_i\wedge t}^{\nu_{i+1}\wedge t} \widetilde{Z}^{\nu'_i}_s dB_s\Bigg) \\ &= \sum_{i=1}^{\ell-1} \Big(-Y_{\nu_{i+1}\wedge t} + Y_{\nu_i\wedge t} - \int_{\nu_i\wedge t}^{\nu_{i+1}\wedge t} g\big(s, Y_s, Z_s\big)ds + \int_{\nu_i\wedge t}^{\nu_{i+1}\wedge t} Z_s dB_s\Big) \\ &= -Y_t + Y_0 - \int_0^t g\big(s, Y_s, Z_s\big)ds + \int_0^t Z_s dB_s, \quad \forall\, t \in [0, \nu_\ell]. \end{aligned}$$

It follows that $\mathbb{P}$-a.s.

$$Y_t = Y_{\nu_\ell} + \int_t^{\nu_\ell} g\big(s, Y_s, Z_s\big)ds + K_{\nu_\ell} - K_t - J_{\nu_\ell} + J_t - \int_t^{\nu_\ell} Z_s dB_s,$$
$$\forall\, t \in [0, \nu_\ell]. \tag{6.77}$$

Since the increment of K over $[\nu_i, \nu'_i]$ is that of K^{ν_i} over $[\nu_i, \nu'_i]$ (K is constant over $[\nu'_i, \nu_{i+1}]$) and since the increment of J over $[\nu'_i, \nu_{i+1}]$ is that of $J^{\nu'_i}$ over $[\nu'_i, \nu_{i+1}]$ (J is constant over $[\nu_i, \nu'_i]$), (6.69), (6.62) and (6.70) again imply that

$$\int_0^{\nu_\ell} (Y_t - L_t)dK_t = \sum_{i=1}^{\ell-1} \int_{\nu_i}^{\nu'_i} (Y_t - L_t)dK_t = \sum_{i=1}^{\ell-1} \int_{\nu_i}^{\nu'_i} (Y_t - L_t)dK^{\nu_i}_t = 0, \tag{6.78}$$

and

$$\begin{aligned} \int_0^{\nu_\ell} (U_t - Y_t)dJ_t &= \int_0^{\nu_\ell} \big(U_t - \widetilde{Y}_t\big)dJ_t = \sum_{i=1}^{\ell-1} \int_{\nu'_i}^{\nu_{i+1}} \big(U_t - \widetilde{Y}_t\big)dJ_t \\ &= \sum_{i=1}^{\ell-1} \int_{\nu'_i}^{\nu_{i+1}} \big(U_t - \widetilde{Y}_t\big)dJ^{\nu'_i}_t = 0, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{6.79}$$

Clearly, $Y_T = \lim_{n\to\infty} \uparrow Y^n_T = \xi$, $\mathbb{P}$-a.s. Letting $\ell \to \infty$ in (6.77), (6.78) and (6.79), we see from (6.75) and (6.70) that (Y, Z, K, J) solves DRBSDE(ξ, g, L, U).

(2) (Proof of (2.1)–(2.3)). Fix $\nu \in \mathcal{T}$. We will simply denote τ_ν^* by $\widehat{\tau}$ and γ_ν^* by $\widehat{\gamma}$. Since it holds $\mathbb{P}$-a.s. that

$$Y_t > L_t, \quad \forall\, t \in \big[\nu, \widehat{\tau}\big) \quad \text{and} \quad Y_t < U_t, \quad \forall\, t \in \big[\nu, \widehat{\gamma}\big),$$

the flat-off conditions in DRBSDE(ξ, g, L, U) implies that $\mathbb{P}$-a.s.

$$K_t = K_\nu, \quad \forall\, t \in [\nu, \widehat{\tau}] \quad \text{and} \quad J_t = J_\nu, \quad \forall\, t \in \big[\nu, \widehat{\gamma}\big]. \tag{6.80}$$

Let $\tau, \gamma \in \mathcal{T}_{\nu,T}$, we see from (6.80) that $\mathbb{P}$-a.s.

$$Y_t = Y_{\widehat{\tau}\wedge\gamma} + \int_t^{\widehat{\tau}\wedge\gamma} g(s, Y_s, Z_s)ds - J_{\widehat{\tau}\wedge\gamma} + J_t - \int_t^{\widehat{\tau}\wedge\gamma} Z_s dB_s, \quad \forall\, t \in \big[\nu, \widehat{\tau}\wedge\gamma\big].$$

As $Y_{\widehat{\tau}\wedge\gamma} \in L^1(\mathcal{F}_{\widehat{\tau}\wedge\gamma})$ by the uniform integrability of $\{Y_{\gamma'}\}_{\gamma'\in\mathcal{T}}$, (A.2) shows that $\mathbb{P}$-a.s.

$$Y_t^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}} = Y_{\widehat{\tau}\wedge\gamma} + \int_t^{\widehat{\tau}\wedge\gamma} g\Big(s, Y_s^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}, Z_s^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}\Big)ds - \int_t^{\widehat{\tau}\wedge\gamma} Z_s^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}} dB_s,$$
$$\forall\, t \in \big[\nu, \widehat{\tau}\wedge\gamma\big]. \tag{6.81}$$

Applying Proposition 3.2 with $(Y^1, Z^1, V^1) = (Y, Z, -J)$ and $(Y^2, Z^2, V^2) = \Big(Y^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}, Z^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}, 0\Big)$ yields that $\mathbb{P}$-a.s., $Y_t \le Y_t^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}}$ for any $t \in \big[\nu, \widehat{\tau}\wedge\gamma\big]$. It follows that

$$Y_\nu \le Y_\nu^{\widehat{\tau}\wedge\gamma, Y_{\widehat{\tau}\wedge\gamma}} = \mathcal{E}^g_{\nu,\widehat{\tau}\wedge\gamma}\big[Y_{\widehat{\tau}\wedge\gamma}\big], \quad \mathbb{P}\text{-a.s.} \tag{6.82}$$

Similarly, we can deduce that

$$Y_\nu \ge \mathcal{E}^g_{\nu,\tau\wedge\widehat{\gamma}}\big[Y_{\tau\wedge\widehat{\gamma}}\big], \quad \mathbb{P}\text{-a.s.}, \tag{6.83}$$

proving (2.1).

The continuity of processes Y, L and U implies that $\mathbf{1}_{\{\widehat{\tau}<T\}}Y_{\widehat{\tau}} = \mathbf{1}_{\{\widehat{\tau}<T\}}L_{\widehat{\tau}}$ and $\mathbf{1}_{\{\widehat{\gamma}<T\}}Y_{\widehat{\gamma}} = \mathbf{1}_{\{\widehat{\gamma}<T\}}U_{\widehat{\gamma}}$, $\mathbb{P}$-a.s. It follows that $\mathbb{P}$-a.s.

$$R(\widehat{\tau}, \gamma) = \mathbf{1}_{\{\widehat{\tau}<\gamma\}}L_{\widehat{\tau}} + \mathbf{1}_{\{\gamma\le\widehat{\tau}\}\cap\{\gamma<T\}}U_\gamma + \mathbf{1}_{\{\widehat{\tau}=\gamma=T\}}\xi \ge \mathbf{1}_{\{\widehat{\tau}<\gamma\}}Y_{\widehat{\tau}} + \mathbf{1}_{\{\gamma\le\widehat{\tau}\}\cap\{\gamma<T\}}Y_\gamma$$
$$+ \mathbf{1}_{\{\widehat{\tau}=\gamma=T\}}Y_T = Y_{\widehat{\tau}\wedge\gamma}, \tag{6.84}$$

and

$$R(\tau, \widehat{\gamma}) = \mathbf{1}_{\{\tau<\widehat{\gamma}\}}L_\tau + \mathbf{1}_{\{\widehat{\gamma}\le\tau\}\cap\{\widehat{\gamma}<T\}}U_{\widehat{\gamma}} + \mathbf{1}_{\{\tau=\widehat{\gamma}=T\}}\xi \le \mathbf{1}_{\{\tau<\widehat{\gamma}\}}Y_\tau + \mathbf{1}_{\{\widehat{\gamma}\le\tau\}\cap\{\widehat{\gamma}<T\}}Y_{\widehat{\gamma}}$$
$$+ \mathbf{1}_{\{\tau=\widehat{\gamma}=T\}}Y_T = Y_{\tau\wedge\widehat{\gamma}}. \tag{6.85}$$

Then (6.82), (6.83) and the monotonicity of g-evaluation show that

$$\mathcal{E}^g_{\nu,\tau\wedge\widehat{\gamma}}\big[R(\tau, \widehat{\gamma})\big] \le \mathcal{E}^g_{\nu,\tau\wedge\widehat{\gamma}}\big[Y_{\tau\wedge\widehat{\gamma}}\big] \le Y_\nu \le \mathcal{E}^g_{\nu,\widehat{\tau}\wedge\gamma}\big[Y_{\widehat{\tau}\wedge\gamma}\big] \le \mathcal{E}^g_{\nu,\widehat{\tau}\wedge\gamma}\big[R(\widehat{\tau}, \gamma)\big], \quad \mathbb{P}\text{-a.s.}$$

Taking essential supremum over $\tau \in \mathcal{T}_{\nu,T}$ and essential infimum over $\gamma \in \mathcal{T}_{\nu,T}$ respectively yields that

$$\underset{\tau\in\mathcal{T}_{\nu,T}}{\text{esssup}}\ \underset{\gamma\in\mathcal{T}_{\nu,T}}{\text{essinf}}\ \mathcal{E}^g_{\nu,\tau\wedge\gamma}\big[R(\tau, \gamma)\big] \le \underset{\gamma\in\mathcal{T}_{\nu,T}}{\text{essinf}}\ \underset{\tau\in\mathcal{T}_{\nu,T}}{\text{esssup}}\ \mathcal{E}^g_{\nu,\tau\wedge\gamma}\big[R(\tau, \gamma)\big]$$
$$\le \underset{\tau\in\mathcal{T}_{\nu,T}}{\text{esssup}}\ \mathcal{E}^g_{\nu,\tau\wedge\widehat{\gamma}}\big[R(\tau, \widehat{\gamma})\big]$$

$$\le Y_\nu \le \underset{\gamma\in\mathcal{T}_{\nu,T}}{\operatorname{essinf}}\ \mathcal{E}^g_{\nu,\widehat{\tau}\wedge\gamma}\big[R(\widehat{\tau},\gamma)\big]$$
$$\le \underset{\tau\in\mathcal{T}_{\nu,T}}{\operatorname{esssup}}\ \underset{\gamma\in\mathcal{T}_{\nu,T}}{\operatorname{essinf}}\ \mathcal{E}^g_{\nu,\tau\wedge\gamma}\big[R(\tau,\gamma)\big], \quad \mathbb{P}\text{-a.s.} \tag{6.86}$$

By (6.80) again, it holds $\mathbb{P}$-a.s. that

$$Y_t = Y_{\widehat{\tau}\wedge\widehat{\gamma}} + \int_t^{\widehat{\tau}\wedge\widehat{\gamma}} g(s,Y_s,Z_s)ds - \int_t^{\widehat{\tau}\wedge\widehat{\gamma}} Z_s dB_s, \quad \forall\, t\in\big[\nu,\widehat{\tau}\wedge\widehat{\gamma}\big].$$

Comparing it to (6.81) with $\gamma=\widehat{\gamma}$, we can deduce from applying Proposition 3.2 that $\mathbb{P}$-a.s., $Y_t = Y_t^{\widehat{\tau}\wedge\widehat{\gamma},Y_{\widehat{\tau}\wedge\widehat{\gamma}}}$ for any $t\in\big[\nu,\widehat{\tau}\wedge\widehat{\gamma}\big]$. Taking $\gamma=\widehat{\gamma}$ in (6.84) and $\tau=\widehat{\tau}$ in (6.85) yields that $Y_\nu = Y_\nu^{\widehat{\tau}\wedge\widehat{\gamma},Y_{\widehat{\tau}\wedge\widehat{\gamma}}} = \mathcal{E}^g_{\nu,\widehat{\tau}\wedge\widehat{\gamma}}\big[Y_{\widehat{\tau}\wedge\widehat{\gamma}}\big] = \mathcal{E}^g_{\nu,\widehat{\tau}\wedge\widehat{\gamma}}\big[R(\widehat{\tau},\widehat{\gamma})\big]$, $\mathbb{P}$-a.s., which together with (6.86) proves (2.2) and (2.3).

(3) (uniqueness) Let $(\mathscr{Y},\mathscr{Z},\mathscr{K},\mathscr{J})\in\big(\cap_{p\in(0,1)}\mathbb{S}^p\big)\times\mathbb{H}^{2,0}\times\mathbb{K}^0\times\mathbb{K}^0$ be another solution of DRBSDE (ξ,g,L,U) such that $\mathscr{Y}$ is of class (D). Since $\mathscr{Y}$ also satisfies (2.3), it holds for any $t\in[0,T]$ that

$$\mathscr{Y}_t = \underset{\tau\in\mathcal{T}_{t,T}}{\operatorname{esssup}}\ \underset{\gamma\in\mathcal{T}_{t,T}}{\operatorname{essinf}}\ \mathcal{E}^g_{t,\tau\wedge\gamma}\big[R(\tau,\gamma)\big] = \underset{\gamma\in\mathcal{T}_{t,T}}{\operatorname{essinf}}\ \underset{\tau\in\mathcal{T}_{t,T}}{\operatorname{esssup}}\ \mathcal{E}^g_{t,\tau\wedge\gamma}\big[R(\tau,\gamma)\big]$$
$$= Y_t, \quad \mathbb{P}\text{-a.s.} \tag{6.87}$$

The continuity of Y and $\mathscr{Y}$ then shows that $\mathbb{P}$-a.s.

$$\xi + \int_t^T g(s,Y_s,Z_s)\,ds + K_T - K_t - J_T + J_t - \int_t^T Z_s dB_s = Y_t = \mathscr{Y}_t$$
$$= \xi + \int_t^T g(s,\mathscr{Y}_s,\mathscr{Z}_s)\,ds + \mathscr{K}_T - \mathscr{K}_t - \mathscr{J}_T + \mathscr{J}_t - \int_t^T \mathscr{Z}_s dB_s, \quad t\in[0,T].$$

Comparing the martingale parts on both sides shows that $Z_t=\mathscr{Z}_t$, $dt\otimes d\mathbb{P}$-a.s., and it follows that $\mathbb{P}$-a.s.

$$K_t - J_t = \mathscr{K}_t - \mathscr{J}_t, \quad t\in[0,T]. \tag{6.88}$$

The flat-off conditions in DRBSDE(ξ,g,L,U) implies that $\mathbb{P}$-a.s.

$$K_t = \int_0^t \mathbf{1}_{\{Y_s=L_s\}}dK_s, \qquad \mathscr{K}_t = \int_0^t \mathbf{1}_{\{\mathscr{Y}_s=L_s\}}d\mathscr{K}_s, \qquad J_t = \int_0^t \mathbf{1}_{\{Y_s=U_s\}}dJ_s,$$
$$\mathscr{J}_t = \int_0^t \mathbf{1}_{\{\mathscr{Y}_s=U_s\}}d\mathscr{J}_s, \quad t\in[0,T]. \tag{6.89}$$

As $\mathbb{P}\{L_t<U_t,\ \forall\, t\in[0,T]\}=1$, we can deduce that $\mathbb{P}$-a.s.

$$\int_0^t \mathbf{1}_{\{Y_s=U_s\}}dK_s = \int_0^t \mathbf{1}_{\{Y_s=U_s\}}\mathbf{1}_{\{Y_s=L_s\}}dK_s = 0 \quad \text{and}$$
$$\int_0^t \mathbf{1}_{\{\mathscr{Y}_s=U_s\}}d\mathscr{K}_s = \int_0^t \mathbf{1}_{\{\mathscr{Y}_s=U_s\}}\mathbf{1}_{\{\mathscr{Y}_s=L_s\}}d\mathscr{K}_s = 0, \quad t\in[0,T],$$

which together with (6.89), (6.87) and (6.88) leads to that $\mathbb{P}$-a.s.

$$\begin{aligned} J_t &= \int_0^t \mathbf{1}_{\{Y_s=U_s\}} dJ_s + \int_0^t \mathbf{1}_{\{\mathscr{Y}_s=U_s\}} d\mathscr{K}_s = \int_0^t \mathbf{1}_{\{Y_s=U_s\}} dJ_s + \int_0^t \mathbf{1}_{\{Y_s=U_s\}} d\mathscr{K}_s \\ &= \int_0^t \mathbf{1}_{\{Y_s=U_s\}} d\mathscr{J}_s + \int_0^t \mathbf{1}_{\{Y_s=U_s\}} dK_s = \int_0^t \mathbf{1}_{\{\mathscr{Y}_s=U_s\}} d\mathscr{J}_s + \int_0^t \mathbf{1}_{\{Y_s=U_s\}} dK_s \\ &= \mathscr{J}_t, \quad t \in [0, T]. \end{aligned}$$

Then it easily follows from (6.88) that $\mathbb{P}$-a.s., $K_t = \mathscr{K}_t$, $\forall\, t \in [0, T]$. □

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Appendix

Lemma A.1. *Given $\xi \in L^1(\mathcal{F}_\tau)$, let $\xi \in L^1(\mathcal{F}_\tau)$ and let $g(t, \omega, y, z) = \mathbf{1}_{\{t \le \tau(\omega)\}} g(t, \omega, y, z)$, $(t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ be a generator. Then one has*

$$\mathbb{P}\big\{Y_t^{\tau,\xi} = Y_{\tau\wedge t}^{\tau,\xi},\ \forall\, t \in [0, T]\big\} = 1 \quad \text{and} \quad Z_t^{\tau,\xi} = \mathbf{1}_{\{t \le \tau\}} Z_t^{\tau,\xi}, \quad dt \otimes d\mathbb{P}\text{-a.s.} \tag{A.1}$$

(see (4.1) for the notation $(Y^{\tau,\xi}, Z^{\tau,\xi})$). In particular, it holds $\mathbb{P}$-a.s. that

$$Y_t^{\tau,\xi} = \xi + \int_t^\tau g\big(s, Y_s^{\tau,\xi}, Z_s^{\tau,\xi}\big) ds - \int_t^\tau Z_s^{\tau,\xi} dB_s, \qquad \forall\, t \in [0, \tau]. \tag{A.2}$$

Proof. Given $n \in \mathbb{N}$, we define a stopping time

$$\gamma_n := \inf\left\{t \in [0, T] : \int_0^t \big|Z_s^{\tau,\xi}\big|^2 ds > n\right\} \wedge T \in \mathcal{T}. \tag{A.3}$$

Since $Y_{\tau\wedge\gamma_n}^{\tau,\xi} = Y_{\gamma_n}^{\tau,\xi} + \int_{\tau\wedge\gamma_n}^{\gamma_n} \mathbf{1}_{\{s \le \tau\}} g\big(s, Y_s^{\tau,\xi}, Z_s^{\tau,\xi}\big) ds - \int_{\tau\wedge\gamma_n}^{\gamma_n} Z_s^{\tau,\xi} dB_s = Y_{\gamma_n}^{\tau,\xi} - \int_{\tau\wedge\gamma_n}^{\gamma_n} Z_s^{\tau,\xi} dB_s$, $\mathbb{P}$-a.s., taking conditional expectation $\mathbb{E}\big[\cdot|\mathcal{F}_{\tau\wedge\gamma_n}\big]$ yields that $\mathbb{P}$-a.s.

$$\begin{aligned} Y_{\tau\wedge\gamma_n}^{\tau,\xi} &= \mathbb{E}\big[Y_{\gamma_n}^{\tau,\xi}\big|\mathcal{F}_{\tau\wedge\gamma_n}\big] = \mathbf{1}_{\{\tau \le \gamma_n\}} \mathbb{E}\big[Y_{\gamma_n}^{\tau,\xi}\big|\mathcal{F}_\tau\big] + \mathbf{1}_{\{\tau > \gamma_n\}} \mathbb{E}\big[Y_{\gamma_n}^{\tau,\xi}\big|\mathcal{F}_{\gamma_n}\big] \\ &= \mathbf{1}_{\{\tau \le \gamma_n\}} \mathbb{E}\big[Y_{\gamma_n}^{\tau,\xi}\big|\mathcal{F}_\tau\big] + \mathbf{1}_{\{\tau > \gamma_n\}} Y_{\gamma_n}^{\tau,\xi}. \end{aligned} \tag{A.4}$$

As $Z^{\tau,\xi} \in \cap_{p \in (0,1)} \mathbb{H}^{2,p} \subset \mathbb{H}^{2,0}$, $\{\gamma_n\}_{n\in\mathbb{N}}$ is stationary. Letting $n \to \infty$, we can deduce from the uniform integrability of $\big\{Y_\gamma^{\tau,\xi}\big\}_{\gamma \in \mathcal{T}}$ that

$$Y_\tau^{\tau,\xi} = \mathbf{1}_{\{\tau \le T\}} \mathbb{E}\big[Y_T^{\tau,\xi}\big|\mathcal{F}_\tau\big] + \mathbf{1}_{\{\tau > T\}} Y_T^{\tau,\xi} = \mathbb{E}\big[Y_T^{\tau,\xi}\big|\mathcal{F}_\tau\big] = \mathbb{E}\big[\xi\big|\mathcal{F}_\tau\big] = \xi, \quad \mathbb{P}\text{-a.s.}$$

Then it follows that $\mathbb{P}$-a.s.

$$\begin{aligned} Y^{\tau,\xi}_{\tau\wedge t} &= Y^{\tau,\xi}_{\tau} + \int_{\tau\wedge t}^{\tau} \mathbf{1}_{\{s\le\tau\}} g\big(s, Y^{\tau,\xi}_s, Z^{\tau,\xi}_s\big)ds - \int_{\tau\wedge t}^{\tau} Z^{\tau,\xi}_s dB_s \\ &= \xi + \int_t^T \mathbf{1}_{\{s\le\tau\}} g\big(s, Y^{\tau,\xi}_{\tau\wedge s}, \mathbf{1}_{\{s\le\tau\}} Z^{\tau,\xi}_s\big)ds - \int_t^T \mathbf{1}_{\{s\le\tau\}} Z^{\tau,\xi}_s dB_s, \\ & t\in[0,T] \end{aligned} \tag{A.5}$$

which shows that $\big\{\big(Y^{\tau,\xi}_{\tau\wedge t}, \mathbf{1}_{\{t\le\tau\}} Z^{\tau,\xi}_t\big)\big\}_{t\in[0,T]}$ also solves BSDE(ξ, g_τ). Clearly, $\big\{Y^{\tau,\xi}_{\tau\wedge t}\big\}_{t\in[0,T]}$ is an $\mathbf{F}$-adapted continuous process such that $\mathbb{E}[\sup_{t\in[0,T]} \big|Y^{\tau,\xi}_{\tau\wedge t}\big|^p] \le \mathbb{E}[\sup_{t\in[0,T]} \big|Y^{\tau,\xi}_t\big|^p] < \infty$ for any $p \in (0,1)$ and that $\big\{Y^{\tau,\xi}_\gamma\big\}_{\gamma\in\mathcal{T}_{0,\tau}}$ is uniformly integrable. As $\big\{\mathbf{1}_{\{t\le\tau\}}\big\}_{t\in[0,T]}$ is an $\mathbf{F}$-adapted càglàd process (and thus $\mathbf{F}$-predictable), we see that $\big\{\mathbf{1}_{\{t\le\tau\}} Z^{\tau,\xi}_t\big\}_{t\in[0,T]}$ is an $\mathbf{F}$-predictable process satisfying $\mathbb{E}\Big[\Big(\int_0^T \mathbf{1}_{\{t\le\tau\}}|Z^{\tau,\xi}_t|^2\Big)^{p/2}\Big] \le \mathbb{E}\Big[\Big(\int_0^T |Z^{\tau,\xi}_t|^2\Big)^{p/2}\Big] < \infty$ for any $p \in (0,1)$. Hence, by the uniqueness of solution of BSDE(ξ, g_τ), (A.1) holds.

Moreover, (A.5) can be alternatively expressed as: $\mathbb{P}$-a.s.

$$Y^{\tau,\xi}_{\tau\wedge t} = \xi + \int_{\tau\wedge t}^{\tau} g\big(s, Y^{\tau,\xi}_s, Z^{\tau,\xi}_s\big)ds - \int_{\tau\wedge t}^{\tau} Z^{\tau,\xi}_s dB_s, \quad t\in[0,T],$$

which leads to (A.2). □

Lemma A.2. *Let* $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ *be a* $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$*-measurable function satisfying* (H1) *and* (H4). *Given* $\nu, \tau \in \mathcal{T}$ *with* $\nu\le\tau$, *let* $(Y, Z, K) \in \mathbb{S}^0 \times \widetilde{\mathbb{H}}^{2,0} \times \mathbb{K}^0$ *satisfy that* $\mathbb{P}$*-a.s.*

$$Y_t = Y_\tau + \int_t^\tau g(s, Y_s, Z_s)ds + K_\tau - K_t - \int_t^\tau Z_s dB_s, \quad \forall\, t\in[\nu,\tau]. \tag{A.6}$$

If $\mathbb{E}\big[|Y_\nu|\big] < \infty$, *then for any* $p \in (0,\infty)$, $\mathbb{E}\Big[\big(\int_\nu^\tau |Z_t|^2 dt\big)^{p/2}\Big] + \mathbb{E}\big[(K_\tau - K_\nu)^p\big] \le C_p\,\mathbb{E}\Big[\sup_{t\in[\nu,\tau]} |Y_t|^p\Big] + C_p\mathbb{E}\Big[\big(\int_\nu^\tau h_t dt\big)^p\Big]$.

Proof. Let $\mathbb{E}\big[|Y_\nu|\big] < \infty$ and fix $p \in (0,\infty)$. By the Burkholder–Davis–Gundy inequality, there exists $\mathfrak{c}_p > 0$ such that for any continuous local martingale M

$$\mathbb{E}\big[(M_*)^p\big] \le \mathfrak{c}_p\mathbb{E}\Big[\langle M\rangle_T^{p/2}\Big] \quad \text{and} \quad \mathbb{E}\Big[(M_*)^{p/2}\Big] \le \mathfrak{c}_p\mathbb{E}\Big[\langle M\rangle_T^{p/4}\Big]. \tag{A.7}$$

Set $\Psi := \sup_{t\in[\nu,\tau]} |Y_t|$ and suppose $\mathbb{E}\big[\Psi^p\big] < \infty$, otherwise the result trivially holds. We let $n\in\mathbb{N}$ and define a stopping time $\tau_n := \inf\{t\in[\nu,\tau] : \int_\nu^t |Z_s|^2 ds > n\} \wedge \tau \in \mathcal{T}$. It is clear that $\nu \le \tau_n \le \tau$. Since (H1), (H4) and Hölder's inequality imply that

$$\begin{aligned} K_{\tau_n} - K_\nu &= Y_\nu - Y_{\tau_n} - \int_\nu^{\tau_n} g(t, Y_t, Z_t)dt + \int_\nu^{\tau_n} Z_t dB_t \\ &\le 2\Psi + \int_\nu^{\tau_n} (h_t + \kappa|Y_t| + \kappa|Z_t|)\, dt + \left|\int_0^T \mathbf{1}_{\{\nu\le s\le\tau_n\}} Z_t dB_t\right| \end{aligned}$$

$$\le (2+\kappa T)\Psi + \int_\nu^\tau h_t dt + \kappa\sqrt{T}\bigg(\int_\nu^{\tau_n} |Z_t|^2 dt\bigg)^{1/2} + \sup_{t\in[0,T]}\bigg|\int_0^t \mathbf{1}_{\{\nu\le s\le\tau_n\}} Z_s dB_s\bigg|, \quad \mathbb{P}\text{-a.s.},$$

taking the expectation of pth power, we can deduce from (1.5) and (A.7) that

$$\mathbb{E}\Big[\big(K_{\tau_n}-K_\nu\big)^p\Big] \le (1\vee 4^{p-1})\bigg\{(2+\kappa T)^p\,\mathbb{E}\big[\Psi^p\big] + \mathbb{E}\bigg[\bigg(\int_\nu^\tau h_t dt\bigg)^p\bigg] + \Big(\kappa^p T^{p/2} + \mathfrak{c}_p\Big)\,\mathbb{E}\bigg[\bigg(\int_\nu^{\tau_n}|Z_t|^2 dt\bigg)^{p/2}\bigg]\bigg\}. \tag{A.8}$$

As $\mathbb{E}\big[|Y_\nu|\big] < \infty$, Corollary 3.1 implies that there exists a unique $\widetilde{Z} \in \cap_{p\in(0,1)} \mathbb{H}^{2,p}$ such that $\mathbb{P}\big\{\mathbb{E}[Y_\nu|\mathcal{F}_t] = \mathbb{E}[Y_\nu] + \int_0^t \widetilde{Z}_s dB_s,\ \forall\, t\in[0,T]\big\} = 1$. Similar to (6.3), (A.6) shows that $\mathbb{P}$-a.s.

$$\widetilde{Y}_t := \mathbb{E}[Y_\nu|\mathcal{F}_{\nu\wedge t}] + Y_{\nu\vee(\tau\wedge t)} - Y_\nu = \mathbb{E}[Y_\nu] - \int_0^t \mathbf{1}_{\{\nu<s\le\tau\}} g(s,Y_s,Z_s)ds - \int_0^t \mathbf{1}_{\{\nu<s\le\tau\}} dK_s + \int_0^t \Big(\mathbf{1}_{\{s\le\nu\}}\widetilde{Z}_s + \mathbf{1}_{\{\nu<s\le\tau\}} Z_s\Big) dB_s, \quad t\in[0,T]. \tag{A.9}$$

So $\widetilde{Y}$ is an $\mathbf{F}$-adapted continuous process, i.e. $\widetilde{Y}\in\mathbb{S}^0$.

Set $a := 2(\kappa+\kappa^2)$ and $\delta := \big[3(1\vee 4^{p/2-1})(1\vee 4^{p-1})\big(\kappa^p T^{p/2} + \mathfrak{c}_p\big)\big]^{-2/p}$. Applying Itô's formula to process $\big\{e^{at}|\widetilde{Y}_t|^2\big\}_{t\in[0,T]}$, we can deduce from (A.9) that $\mathbb{P}$-a.s.

$$\begin{aligned} e^{at}\big|\widetilde{Y}_t\big|^2 &= \big(\mathbb{E}[Y_\nu]\big)^2 + a\int_0^t e^{as}\big|\widetilde{Y}_s\big|^2 ds - 2\int_0^t \mathbf{1}_{\{\nu<s\le\tau\}} e^{as}\widetilde{Y}_s g(s,Y_s,Z_s)ds \\ &\quad + \int_0^t e^{as}\Big(\mathbf{1}_{\{s\le\nu\}}|\widetilde{Z}_s|^2 + \mathbf{1}_{\{\nu<s\le\tau\}}|Z_s|^2\Big) ds \\ &\quad - 2\int_0^t \mathbf{1}_{\{\nu<s\le\tau\}} e^{as}\widetilde{Y}_s dK_s + 2\int_0^t e^{as}\widetilde{Y}_s\Big(\mathbf{1}_{\{s\le\nu\}}\widetilde{Z}_s + \mathbf{1}_{\{\nu<s\le\tau\}} Z_s\Big) dB_s, \\ &\quad t\in[0,T]. \end{aligned}$$

An analogy to (6.6) shows that $\widetilde{Y}_t = Y_t,\ \forall\, t\in[\nu,\tau]$. Hence, it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} e^{a\tau}|Y_\tau|^2 &= e^{a\tau}\big|\widetilde{Y}_\tau\big|^2 = e^{at}\big|\widetilde{Y}_t\big|^2 + a\int_t^\tau e^{as}\big|\widetilde{Y}_s\big|^2 ds - 2\int_t^\tau \mathbf{1}_{\{\nu<s\le\tau\}} e^{as}\widetilde{Y}_s g(s,Y_s,Z_s)ds \\ &\quad - 2\int_t^\tau \mathbf{1}_{\{\nu<s\le\tau\}} e^{as}\widetilde{Y}_s dK_s + 2\int_t^\tau e^{as}\widetilde{Y}_s\Big(\mathbf{1}_{\{s\le\nu\}}\widetilde{Z}_s + \mathbf{1}_{\{\nu<s\le\tau\}} Z_s\Big) dB_s \\ &\quad + \int_t^\tau e^{as}\Big(\mathbf{1}_{\{s\le\nu\}}|\widetilde{Z}_s|^2 + \mathbf{1}_{\{\nu<s\le\tau\}}|Z_s|^2\Big) ds \\ &= e^{at}|Y_t|^2 + \int_t^\tau e^{as}\Big(a|Y_s|^2 + |Z_s|^2 - 2Y_s g(s,Y_s,Z_s)\Big) ds \\ &\quad - 2\int_t^\tau e^{as} Y_s dK_s + 2\int_t^\tau e^{as} Y_s Z_s dB_s, \ \ \forall\, t\in[\nu,\tau]. \end{aligned} \tag{A.10}$$

Then (H1) and (H4) imply that $\mathbb{P}$-a.s.

$$\begin{aligned}
&e^{a\nu}\big|Y_\nu\big|^2+\int_\nu^{\tau_n}e^{as}\left(a|Y_s|^2+|Z_s|^2\right)ds\\
&=e^{a\tau_n}\big|Y_{\tau_n}\big|^2+2\int_\nu^{\tau_n}e^{as}Y_s g(s,Y_s,Z_s)ds+2\int_\nu^{\tau_n}e^{as}Y_s dK_s-2\int_\nu^{\tau_n}e^{as}Y_sZ_sdB_s\\
&\le e^{aT}\Psi^2+2\int_\nu^{\tau_n}e^{as}\left(|Y_s|h_s+\kappa|Y_s|^2+\kappa|Y_s||Z_s|\right)ds+2e^{aT}\Psi(K_{\tau_n}-K_\nu)\\
&\quad+2\left|\int_0^T\mathbf{1}_{\{\nu\le s\le\tau_n\}}e^{as}Y_sZ_sdB_s\right|\\
&\le\Big(1+\frac{2}{\delta}\Big)e^{2aT}\Psi^2+2e^{aT}\Psi\cdot\int_\nu^\tau h_sds+2(\kappa+\kappa^2)\int_\nu^{\tau_n}e^{as}|Y_s|^2ds\\
&\quad+\frac12\int_\nu^{\tau_n}e^{as}|Z_s|^2ds+\frac{\delta}{2}(K_{\tau_n}-K_\nu)^2+2\left|\int_0^T\mathbf{1}_{\{\nu\le s\le\tau_n\}}e^{as}Y_sZ_sdB_s\right|.
\end{aligned}$$

It follows that $\mathbb{P}$-a.s.

$$\begin{aligned}
\int_\nu^{\tau_n}|Z_t|^2dt\le\int_\nu^{\tau_n}e^{as}|Z_t|^2dt&\le\Big(4+\frac{4}{\delta}\Big)e^{2aT}\Psi^2+2\left(\int_\nu^\tau h_tdt\right)^2\\
&\quad+\delta(K_{\tau_n}-K_\nu)^2+4\sup_{t\in[0,T]}\left|\int_0^t\mathbf{1}_{\{\nu\le s\le\tau_n\}}e^{as}Y_sZ_sdB_s\right|.
\end{aligned}$$

Taking the expectation of $p/2$th power, we can deduce from (1.5) and (A.8) that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_\nu^{\tau_n}|Z_t|^2dt\right)^{p/2}\right]\\
&\le(1\vee 4^{p/2-1})\Bigg\{\Big(4+\frac{4}{\delta}\Big)^{p/2}e^{apT}\mathbb{E}\left[\Psi^p\right]+2^{p/2}\mathbb{E}\left[\left(\int_\nu^\tau h_tdt\right)^p\right]\\
&\quad+\delta^{p/2}\mathbb{E}\left[(K_{\tau_n}-K_\nu)^p\right]+4^{p/2}\mathfrak{c}_p\mathbb{E}\left[\left(\int_\nu^{\tau_n}e^{2at}|Y_t|^2|Z_t|^2dt\right)^{p/4}\right]\Bigg\}\\
&\le C_p\mathbb{E}\left[\Psi^p\right]+C_p\mathbb{E}\left[\left(\int_\nu^\tau h_tdt\right)^p\right]+\frac13\mathbb{E}\left[\left(\int_\nu^{\tau_n}|Z_t|^2dt\right)^{p/2}\right]\\
&\quad+C_p\mathbb{E}\left[(\Psi)^{p/2}\left(\int_\nu^{\tau_n}|Z_t|^2dt\right)^{p/4}\right]\\
&\le C_p\mathbb{E}\left[\Psi^p\right]+C_p\mathbb{E}\left[\left(\int_\nu^\tau h_tdt\right)^p\right]+\frac23\mathbb{E}\left[\left(\int_\nu^{\tau_n}|Z_t|^2dt\right)^{p/2}\right].
\end{aligned}$$

So $\mathbb{E}\left[\left(\int_\nu^{\tau_n}|Z_t|^2dt\right)^{p/2}\right]\le C_p\mathbb{E}\left[\Psi^p\right]+C_p\mathbb{E}\left[\left(\int_\nu^\tau h_tdt\right)^p\right]$, which together with (A.8) shows that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_\nu^{\tau_n}|Z_t|^2dt\right)^{p/2}\right]+\mathbb{E}\Big[\big(K_{\tau_n}-K_\nu\big)^p\Big]\\
&\le C_p\mathbb{E}\left[\Psi^p\right]+C_p\mathbb{E}\left[\left(\int_\nu^\tau h_tdt\right)^p\right].
\end{aligned}\tag{A.11}$$

As $Z \in \widetilde{\mathbb{H}}^{2,0}$, it holds for $\mathbb{P}$-a.s. $\omega \in \Omega$ that $\tau(\omega) = \tau_{N_\omega}(\omega)$ for some $N_\omega \in \mathbb{N}$. Then letting $n \to \infty$ in (A.11), we can apply the monotone convergence theorem to obtain the conclusion. □

Lemma A.3. *Let X be an $\mathbf{F}$-optional process with $\mathbb{P}$-a.s. right upper semi-continuous paths (i.e., for any $\omega \in \Omega$ except a $\mathbb{P}$-null set $\mathcal{N}_X$, $X_t \ge \overline{\lim}_{s \searrow t} X_s$, $\forall\, t \in [0,T)$). If $X_\nu \le X_{\widetilde{\nu}}$, $\mathbb{P}$-a.s. for any $\nu, \widetilde{\nu} \in \mathcal{T}$ with $\nu \le \widetilde{\nu}$, $\mathbb{P}$-a.s., then X is an increasing process.*

Proof. Set $\mathcal{D}_k := \big\{t_i^k := \frac{i}{2^k} \wedge T\big\}_{i=0}^{\lceil 2^k T \rceil}$, $\forall\, k \in \mathbb{N}$ and $\mathcal{D} := \cup_{k\in\mathbb{N}} \mathcal{D}_k$. Given $t \in [0,T)$, we define $\underline{X}_t := \lim_{n\to\infty} \uparrow \inf_{s \in \Theta_t^n} X_s$, where $\Theta_t^n := \mathcal{D} \cap \big(t, (t+2^{-n}) \wedge T\big]$. Clearly,

$$\Theta_t^n = \underset{k>n}{\cup}\, \Theta_t^{n,k}, \quad \text{where } \Theta_t^{n,k} := \mathcal{D}_k \cap \big(t, (t+2^{-n}) \wedge T\big]. \tag{A.12}$$

For any $m, n \in \mathbb{N}$ with $m < n$, since Θ_t^n is a countable subset of $\big(t, (t+2^{-n}) \wedge T\big]$, the random variable $\inf_{s\in\Theta_t^n} X_s$ is clearly $\mathcal{F}_{(t+2^{-n})\wedge T}$-measurable. So $\underline{X}_t = \lim_{\substack{n\to\infty \\ n>m}} \uparrow \inf_{s\in\Theta_t^n} X_s \in \mathcal{F}_{(t+2^{-m})\wedge T}$. As $m \to \infty$, the right-continuity of the filtration $\mathbf{F}$ shows that

$$\underline{X}_t \in \underset{m\in\mathbb{N}}{\cap} \mathcal{F}_{(t+2^{-m})\wedge T} = \mathcal{F}_{t+} = \mathcal{F}_t. \tag{A.13}$$

(1) *Additionally setting $\underline{X}_T := X_T \in \mathcal{F}_T$, we first show the process $\underline{X}$ is $\mathbf{F}$-progressively measurable.*

For any $t \in [0,T)$, $c \in \mathbb{R}$ and $n, k \in \mathbb{N}$ with $k > n$, since it holds for $i = 0, \dots, \lfloor 2^k t \rfloor$ and any $s \in [t_i^k, t_{i+1}^k) \cap [0,t]$ that

$$\begin{aligned}\Theta_i^{n,k} &:= \Theta_{t_i^k}^{n,k} = \{t_j^k : j = i+1, \cdots, i+2^{k-n}\} = \Theta_s^{n,k} \subset \big(s, (s+2^{-n}) \wedge T\big] \\ &\subset \big(0, (t+2^{-n}) \wedge T\big],\end{aligned}$$

we can deduce that

$$\begin{aligned}&\Big\{(s,\omega) \in [0,t] \times \Omega : \min_{r\in\Theta_s^{n,k}} X_r(\omega) \ge c\Big\} \\ &= \bigcup_{i=0}^{\lfloor 2^k t \rfloor} \Big\{(s,\omega) \in \Big([t_i^k, t_{i+1}^k) \cap [0,t]\Big) \times \Omega : \min_{r\in\Theta_s^{n,k}} X_r(\omega) \ge c\Big\} \\ &\bigcup_{i=0}^{\lfloor 2^k t \rfloor} \Big\{(s,\omega) \in \Big([t_i^k, t_{i+1}^k) \cap [0,t]\Big) \times \Omega : \min_{r\in\Theta_i^{n,k}} X_r(\omega) \ge c\Big\} \\ &= \bigcup_{i=0}^{\lfloor 2^k t \rfloor} \bigcap_{r\in\Theta_i^{n,k}} \Big\{(s,\omega) \in \Big([t_i^k, t_{i+1}^k) \cap [0,t]\Big) \times \Omega : X_r(\omega) \ge c\Big\} \\ &= \bigcup_{i=0}^{\lfloor 2^k t \rfloor} \bigcap_{r\in\Theta_i^{n,k}} \Big([t_i^k, t_{i+1}^k) \cap [0,t]\Big) \times \{X_r \ge c\} \in \mathscr{B}([0,t]) \otimes \mathcal{F}_{(t+2^{-n})\wedge T}.\end{aligned} \tag{A.14}$$

Now, let $\widetilde{t} \in [0,T]$ and $\widetilde{c} \in \mathbb{R}$. If $\widetilde{t} = 0$, then (A.13) shows that $\big\{(s,\omega) \in \big[0,\widetilde{t}\,\big] \times \Omega : \underline{X}_s(\omega) > \widetilde{c}\big\} = \{0\} \times \{\underline{X}_0 > \widetilde{c}\} \in \mathscr{B}(\{0\}) \otimes \mathcal{F}_0$; if $\widetilde{t} > 0$, for any $m > m_0 := \Big\lceil -\frac{\ln \widetilde{t}}{\ln 2} \Big\rceil$, we can deduce from (A.14) and (A.12) that

$$\begin{aligned}&\big\{(s,\omega) \in \big[0, \widetilde{t} - 2^{-m}\big] \times \Omega : \underline{X}_s(\omega) > \widetilde{c}\big\} \\ &= \Big\{(s,\omega) \in \big[0, \widetilde{t} - 2^{-m}\big] \times \Omega : \lim_{\substack{n\to\infty \\ n>m}} \uparrow \inf_{r\in\Theta_s^n} X_r(\omega) > \widetilde{c}\Big\}\end{aligned}$$

$$= \bigcup_{n>m} \Big\{(s,\omega) \in \big[0, \widetilde{t} - 2^{-m}\big] \times \Omega : \inf_{r \in \Theta_s^n} X_r(\omega) > \widetilde{c}\Big\}$$

$$= \bigcup_{n>m} \bigcup_{\ell \in \mathbb{N}} \Big\{(s,\omega) \in \big[0, \widetilde{t} - 2^{-m}\big] \times \Omega : \inf_{r \in \Theta_s^n} X_r(\omega) \ge \widetilde{c} + 1/\ell\Big\}$$

$$= \bigcup_{n>m} \bigcup_{\ell \in \mathbb{N}} \bigcap_{k>n} \Big\{(s,\omega) \in \big[0, \widetilde{t} - 2^{-m}\big] \times \Omega : \min_{r \in \Theta_s^{n,k}} X_r(\omega) \ge \widetilde{c} + 1/\ell\Big\} \in \mathscr{B}\big(\big[0, \widetilde{t} - 2^{-m}\big]\big) \otimes \mathcal{F}_{\widetilde{t}},$$

which together with (A.13) shows that

$$\big\{(s,\omega) \in \big[0, \widetilde{t}\,\big] \times \Omega : \underline{X}_s(\omega) > \widetilde{c}\big\}$$

$$= \Big\{(s,\omega) \in \Big(\bigcup_{m>m_0} \big[0, \widetilde{t} - 2^{-m}\big]\Big) \times \Omega : \underline{X}_s(\omega) > \widetilde{c}\Big\} \cup \big\{(s,\omega) \in \big\{\widetilde{t}\big\} \times \Omega : \underline{X}_s(\omega) > \widetilde{c}\big\}$$

$$= \Big(\bigcup_{m>m_0} \big\{(s,\omega) \in \big[0, \widetilde{t} - 2^{-m}\big] \times \Omega : \underline{X}_s(\omega) > \widetilde{c}\big\}\Big) \cup \big(\big\{\widetilde{t}\big\} \times \big\{\underline{X}_{\widetilde{t}} > \widetilde{c}\big\}\big) \in \mathscr{B}\big(\big[0, \widetilde{t}\,\big]\big) \otimes \mathcal{F}_{\widetilde{t}}.$$

So $\Lambda := \Big\{\mathcal{E} \subset \mathbb{R} : \big\{(s,\omega) \in \big[0, \widetilde{t}\,\big] \times \Omega : \underline{X}_s(\omega) \in \mathcal{E}\big\} \in \mathscr{B}\big(\big[0, \widetilde{t}\,\big]\big) \otimes \mathcal{F}_{\widetilde{t}}\Big\}$ contains all open sets of form $(\widetilde{c}, \infty)$, which generates $\mathscr{B}(\mathbb{R})$. Clearly, Λ is a σ-field of $\mathbb{R}$. It follows that $\mathscr{B}(\mathbb{R}) \subset \Lambda$, i.e. $\big\{(s,\omega) \in \big[0, \widetilde{t}\,\big] \times \Omega : \underline{X}_s(\omega) \in \mathcal{E}\big\} \in \mathscr{B}\big(\big[0, \widetilde{t}\,\big]\big) \otimes \mathcal{F}_{\widetilde{t}}$ for any $\mathcal{E} \in \mathscr{B}(\mathbb{R})$. Hence, $\underline{X}$ is $\mathbf{F}$-progressively measurable.

(2) Fix $\ell \in \mathbb{N}$. Since both X and $\underline{X}$ are $\mathbf{F}$-progressively measurable, the Debut theorem shows that

$$\tau_\ell := \inf\{t \in [0,T] : \underline{X}_t \le X_t - 1/\ell\} \wedge T$$

defines a stopping time, i.e. $\tau_\ell \in \mathcal{T}$. We claim that $A_\ell := \{\tau_\ell < T\} \in \mathcal{F}_T$ is a $\mathbb{P}$-null set: *Assume not*, so $A_\ell \backslash \mathcal{N}_X$ is not empty. Let $\omega \in A_\ell \backslash \mathcal{N}_X$ and set $s := \tau_\ell(\omega)$. there exists $\{s_i\}_{i \in \mathbb{N}} \subset [s, T)$ with $\lim_{i\to\infty} \downarrow s_i = s$ such that

$$\underline{X}_{s_i}(\omega) \le X_{s_i}(\omega) - 1/\ell, \qquad \forall\, i \in \mathbb{N}. \tag{A.15}$$

Given $m \in \mathbb{N}$, we can find some $\widehat{i} = \widehat{i}(m) \in \mathbb{N}$ and $\widehat{n} = \widehat{n}(m) \ge m$ such that for any $i \ge \widehat{i}$ and $n \ge \widehat{n}$, $\big(s_i, (s_i + 2^{-n}) \wedge T\big] \subset \big(s, (s + 2^{-m}) \wedge T\big]$ and thus

$$\Theta_{s_i}^n = \Big(\bigcup_{k>n} \mathcal{D}_k\Big) \cap \big(s_i, (s_i + 2^{-n}) \wedge T\big] \subset \Big(\bigcup_{k>m} \mathcal{D}_k\Big) \cap \big(s, (s + 2^{-m}) \wedge T\big] = \Theta_s^m.$$

It follows that $\inf_{r \in \Theta_s^m} X_r(\omega) \le \inf_{r \in \Theta_{s_i}^n} X_r(\omega)$. Letting $n \to \infty$, we see that $\inf_{r \in \Theta_s^m} X_r(\omega) \le \underline{X}_{s_i}(\omega)$. As $i \to \infty$, (A.15) and the right upper semi-continuity of $X_\cdot(\omega)$ imply that

$$\inf_{r \in \Theta_s^m} X_r(\omega) \le \varliminf_{i\to\infty} \underline{X}_{s_i}(\omega) \le \varlimsup_{i\to\infty} X_{s_i}(\omega) - 1/\ell \le \varlimsup_{r \searrow s} X_r(\omega) - 1/\ell \le X_s(\omega) - 1/\ell.$$

Now, letting $m \to \infty$ yields that $\underline{X}_s(\omega) \le X_s(\omega) - 1/\ell$, which shows that

$$\underline{X}_{\tau_\ell} \le X_{\tau_\ell} - 1/\ell \quad \text{on } A_\ell \backslash \mathcal{N}_X. \tag{A.16}$$

The **F**-optional measurability of X implies that of the stopped process $\{X_{\tau_\ell \wedge t}\}_{t\in[0,T]}$ (see e.g. Corollary 3.24 of [33]), so $\mathcal{X}^\ell_t := \mathbf{1}_{\{X_{\tau_\ell\wedge t}\le X_t\}}$, $t \in [0,T]$ is also an **F**-optional process. Since $\mathcal{X}^\ell_\nu = \mathbf{1}_{\{X_{\tau_\ell\wedge \nu}\le X_\nu\}} = 1$, $\mathbb{P}$-a.s. for any $\nu \in \mathcal{T}$, the cross-section theorem (see Theorem IV.86 of [16]) shows that for any $\omega \in \Omega$ except on a $\mathbb{P}$-null set $\mathcal{N}_\ell$,

$$\mathcal{X}^\ell_t(\omega) = 1 \quad \text{or} \quad \big(X_{\tau_\ell\wedge t}\big)(\omega) \le X_t(\omega), \qquad \forall\, t \in [0,T]. \tag{A.17}$$

Let $\omega \in A_\ell \backslash (\mathcal{N}_X \cup \mathcal{N}_\ell)$. As $X(\tau_\ell(\omega),\omega) \le X(t,\omega)$, $\forall\, t \in [\tau_\ell(\omega), T]$ by (A.17), we can deduce from (A.16) that

$$X(\tau_\ell(\omega),\omega) \le \underline{X}(\tau_\ell(\omega),\omega) \le X(\tau_\ell(\omega),\omega) - 1/\ell.$$

An contradiction appears, so $0 = \mathbb{P}(A_\ell) = \mathbb{P}\{\underline{X}_t \le X_t - 1/\ell \text{ for some } t \in [0,T)\}$. Letting $\ell \to \infty$ yields that $\mathbb{P}\{\underline{X}_t < X_t, \text{ for some } t \in [0,T)\} = \lim_{\ell\to\infty} \uparrow \mathbb{P}\{\underline{X}_t \le X_t - 1/\ell$ for some $t \in [0,T)\} = 0$, which together with the right upper semi-continuity of X shows that except on a $\mathbb{P}$-null set $\mathcal{N}$

$$\begin{aligned}\underline{X}_t &\ge X_t \ge \varlimsup_{s\searrow t} X_s = \lim_{n\to\infty} \downarrow \sup_{s\in(t,(t+2^{-n})\wedge T]} X_s \ge \lim_{n\to\infty}\downarrow \sup_{s\in\Theta^n_t} X_s \\ &\ge \lim_{n\to\infty}\uparrow \inf_{s\in\Theta^n_t} X_s = \underline{X}_t, \qquad \forall\, t\in[0,T).\end{aligned}$$

To wit, it holds for any $\omega \in \mathcal{N}^c$ that

$$X_t(\omega) = \lim_{\substack{s\searrow t\\ s\in\mathcal{D}\cap(t,T]}} X_s(\omega), \qquad \forall\, t\in[0,T). \tag{A.18}$$

Set $\widetilde{\mathcal{N}} := \mathcal{N} \cup \big(\cup_{s,s'\in\mathcal{D},s<s'}\{X_s > X_{s'}\}\big)$, which is also a $\mathbb{P}$-null set. Given $\omega \in \widetilde{\mathcal{N}}^c$ and $t, t' \in [0,T]$ with $t < t'$, let $\{s_n\}_{n\in\mathbb{N}} \subset \mathcal{D}\cap(t,t')$ with $\lim_{n\to\infty}\downarrow s_n = t$ and let $\{s'_n\}_{n\in\mathbb{N}} \subset \mathcal{D}\cap \big((t',T)\cup\{T\}\big)$ with $\lim_{n\to\infty}\downarrow s'_n = t'$. We can deduce from (A.18) that $X_t(\omega) = \lim_{n\to\infty} X_{s_n}(\omega) \le \lim_{n\to\infty} X_{s'_n}(\omega) = X_{t'}(\omega)$. Therefore, X is an increasing process. $\square$

Proof of (6.34*). (1) The continuity of Y^n's implies that for $\mathbb{P}$-a.s. $\omega\in\Omega$

$$\begin{aligned}\varliminf_{s\searrow t} Y_s(\omega) &= \lim_{n\to\infty}\uparrow \inf_{s\in(t,(t+2^{-n})\wedge T]} Y_s(\omega) = \lim_{n\to\infty}\uparrow \inf_{s\in(t,(t+2^{-n})\wedge T]} \lim_{m\to\infty}\uparrow Y^m_s(\omega) \\ &\ge \lim_{m\to\infty}\uparrow \lim_{n\to\infty}\uparrow \inf_{s\in(t,(t+2^{-n})\wedge T]} Y^m_s(\omega) \\ &= \lim_{m\to\infty}\uparrow \varliminf_{s\searrow t} Y^m_s(\omega) = \lim_{m\to\infty}\uparrow Y^m_t(\omega) \\ &= Y_t(\omega), \qquad \forall\, t\in\big[\nu(\omega),\tau(\omega)\big),\end{aligned} \tag{A.19}$$

which shows that the process $\{Y_{\nu\vee(\tau_\ell\wedge t)}\}_{t\in[0,T]}$ has $\mathbb{P}$-a.s. right lower semi-continuous paths. It then follows from (6.32) that $\widetilde{K}^\ell$ has $\mathbb{P}$-a.s. right upper semi-continuous paths.

(2) *We next show that $\widetilde{K}^\ell_\gamma$ is a weak limit of $\{K^n_{\tau_\ell\wedge\gamma}\}_{n\in\mathbb{N}}$ in $L^2(\mathcal{F}_T)$ for any $\gamma\in\mathcal{T}$.*

Let $\chi \in L^2(\mathcal{F}_T)$. In virtue of martingale representation theorem, there exists a unique $Z^\chi \in \mathbb{H}^{2,2}$ such that $\mathbb{P}$-a.s.

$$M^\chi_t := \mathbb{E}[\chi|\mathcal{F}_t] = \mathbb{E}[\chi] + \int_0^t Z^\chi_s\, dB_s, \qquad \forall\, t\in[0,T].$$

Set $\zeta = \zeta^\ell := \nu\vee(\tau_\ell\wedge\gamma) \in \mathcal{T}$ and let $n\in\mathbb{N}$. We define $\Upsilon^{\ell,n}_t := K^n_{\nu\vee(\zeta\wedge t)} + Y^{\ell,n}_{\nu\vee(\zeta\wedge t)} - Y^{\ell,n}_\nu - \Big(\widetilde{K}^\ell_{\nu\vee(\zeta\wedge t)} + Y_{\nu\vee(\zeta\wedge t)} - Y_\nu\Big)$, $t\in[0,T]$. As $K^n_\nu = 0$ by (6.22), one can deduce from (6.28) that

$\mathbb{P}$-a.s.

$$\begin{aligned}\Upsilon^{\ell,n}_t &= -\int_\nu^{\nu\vee(\zeta\wedge t)} \Big(g(s, Y^{\ell,n}_s, Z^n_s) - g(s, Y_s, 0) - \widetilde{h}^\ell_s\Big)\,ds + \int_\nu^{\nu\vee(\zeta\wedge t)} \Big(Z^n_s - \mathcal{Z}^\ell_s\Big)\,dB_s \\ &= -\int_0^t \mathbf{1}_{\{\nu<s\le\zeta\}} \Big(g(s, Y^{\ell,n}_s, Z^n_s) - g(s, Y_s, 0) - \widetilde{h}^\ell_s\Big)\,ds \\ &\quad + \int_0^t \mathbf{1}_{\{\nu<s\le\zeta\}} \Big(Z^n_s - \mathcal{Z}^\ell_s\Big)\,dB_s, \quad t\in[0,T],\end{aligned}$$

thus $\Upsilon^{\ell,n}$ is an $\mathbf{F}$-adapted continuous process. Since (6.27), (6.31) and (6.25) shows that $|\Upsilon^{\ell,n}_t| \le 4\ell + K^n_{\nu\vee(\zeta\wedge t)} + \big|\widetilde{K}^\ell_{\nu\vee(\zeta\wedge t)}\big|$, $\forall\, t\in[0,T]$, (1.5), (6.29) and (6.33) imply that

$$\begin{aligned}\mathbb{E}\Big[\Big(\Upsilon^{\ell,n}_*\Big)^2\Big] &\le 3\mathbb{E}\Big[16\ell^2 + \big(K^n_{\tau_\ell}\big)^2 + \big(\widetilde{K}^\ell_*\big)^2\Big] \\ &\le C_0\ell^2 + C_0\mathbb{E}\int_\nu^{\tau_\ell} \Big(|\widetilde{h}^\ell_t|^2 + |\mathcal{Z}^\ell_t|^2\Big)\,dt < \infty,\end{aligned} \tag{A.20}$$

which shows that $\Upsilon^{\ell,n}\in\mathbb{S}^2$. Like (6.22), one has

$$\widetilde{K}^\ell_t = 0, \qquad \forall\, t\in[0,\nu]. \tag{A.21}$$

So $\Upsilon^{\ell,n}_\nu = K^n_\nu - \widetilde{K}^\ell_\nu = 0$. Integrating by parts the process $M^\chi\Upsilon^{\ell,n}$ yields that $\mathbb{P}$-a.s.

$$\begin{aligned}\chi\,\Upsilon^{\ell,n}_T &= M^\chi_T\Upsilon^{\ell,n}_T = M^\chi_t\Upsilon^{\ell,n}_t + \int_t^T M^\chi_s\,d\Upsilon^{\ell,n}_s + \int_t^T \Upsilon^{\ell,n}_s\,dM^\chi_s \\ &\quad + \langle M^\chi, \Upsilon\rangle_T - \langle M^\chi, \Upsilon\rangle_t \\ &= -\int_t^T \mathbf{1}_{\{\nu<s\le\zeta\}} M^\chi_s \Big(g(s, Y^{\ell,n}_s, Z^n_s) - g(s, Y_s, 0) - \widetilde{h}^\ell_s\Big)\,ds \\ &\quad + \int_t^T \mathbf{1}_{\{\nu<s\le\zeta\}} M^\chi_s \Big(Z^n_s - \mathcal{Z}^\ell_s\Big)\,dB_s \\ &\quad + \int_t^T \Upsilon^{\ell,n}_s Z^\chi_s\,dB_s + \int_t^T \mathbf{1}_{\{\nu<s\le\zeta\}} Z^\chi_s \Big(Z^n_s - \mathcal{Z}^\ell_s\Big)\,ds, \qquad t\in[0,T].\end{aligned} \tag{A.22}$$

Since Doob's martingale inequality shows that $\mathbb{E}\Big[\big(M^\chi_*\big)^2\Big] \le 4\mathbb{E}\big[|M^\chi_T|^2\big] = 4\mathbb{E}\big[|\chi|^2\big] < \infty$ (i.e. $M^\chi\in\mathbb{S}^2\subset\mathbb{H}^{2,2}$), applying the Burkholder–Davis–Gundy inequality and Hölder's inequality, we see from (A.20) that

$$\begin{aligned}&\mathbb{E}\Bigg[\sup_{t\in[0,T]}\bigg|\int_0^t \mathbf{1}_{\{\nu<s\le\zeta\}} M^\chi_s \Big(Z^n_s - \mathcal{Z}^\ell_s\Big)\,dB_s\bigg| + \sup_{t\in[0,T]}\bigg|\int_0^t \Upsilon^{\ell,n}_s Z^\chi_s\,dB_s\bigg|\Bigg] \\ &\le C_0\mathbb{E}\Bigg[M^\chi_*\bigg(\int_\nu^\zeta |Z^n_s - \mathcal{Z}^\ell_s|^2 ds\bigg)^{1/2}\Bigg] \\ &\quad + C_0\mathbb{E}\Bigg[\Upsilon^{\ell,n}_*\bigg(\int_0^T |Z^\chi_s|^2 ds\bigg)^{1/2}\Bigg] \le C_0\bigg\{\mathbb{E}\Big[\big(M^\chi_*\big)^2\Big]\cdot\mathbb{E}\int_\nu^\zeta \big|Z^n_s - \mathcal{Z}^\ell_s\big|^2 ds\bigg\}^{1/2} \\ &\quad + C_0\bigg\{\mathbb{E}\Big[\Big(\Upsilon^{\ell,n}_*\Big)^2\Big]\cdot\mathbb{E}\int_0^T |Z^\chi_s|^2 ds\bigg\}^{1/2} < \infty.\end{aligned}$$

Namely, $\big\{\int_0^t \mathbf{1}_{\{\nu<s\le\zeta\}} M_s^\chi \big(Z_s^n - \mathcal{Z}_s^\ell\big)\, dB_s\big\}_{t\in[0,T]}$ and $\big\{\int_0^t \mathbf{1}_{\{s\ge\nu\}} \Upsilon_s^{\ell,n} Z_s^\chi\, dB_s\big\}_{t\in[0,T]}$ are uniformly integrable martingales. Then taking expectation in (A.22) for $t=0$ yields that

$$
\begin{aligned}
\mathbb{E}\Big[\chi\Big(K_\zeta^n - \widetilde{K}_\zeta^\ell\Big)\Big] &= \mathbb{E}\Big[\chi\Big(-Y_\zeta^{\ell,n} + Y_\zeta + Y_\nu^{\ell,n} - Y_\nu\Big)\Big] + \mathbb{E}[\chi\,\Upsilon_T^{\ell,n}] \\
&= \mathbb{E}\Big[\chi\Big(-Y_\zeta^{\ell,n} + Y_\zeta + Y_\nu^{\ell,n} - Y_\nu\Big)\Big] \\
&\quad - \mathbb{E}\int_0^T \mathbf{1}_{\{\nu<s\le\zeta\}} M_s^\chi\Big(g(s,Y_s^{\ell,n},Z_s^n) - g(s,Y_s^{\ell,n},0) - \widetilde{h}_s^\ell\Big)ds \\
&\quad - \mathbb{E}\int_0^T \mathbf{1}_{\{\nu<s\le\zeta\}} M_s^\chi\Big(g(s,Y_s^{\ell,n},0) - g(s,Y_s,0)\Big)\,ds \\
&\quad + \mathbb{E}\int_0^T \mathbf{1}_{\{\nu<s\le\zeta\}} Z_s^\chi\big(Z_s^n - \mathcal{Z}_s^\ell\big)ds := I_1^n - I_2^n - I_3^n + I_4^n.
\end{aligned}
$$

As $M^\chi, Z^\chi \in \mathbb{H}^{2,2}$, the weak convergence of $\big\{\mathbf{1}_{\{\nu<s\le\tau_\ell\}}(g(s,Y_s^{\ell,n},Z_s^n) - g(s,Y_s^{\ell,n},0))\big\}_{s\in[0,T]}$, $n\in\mathbb{N}$ to $\big\{\mathbf{1}_{\{\nu<s\le\tau_\ell\}}\widetilde{h}_s^\ell\big\}_{s\in[0,T]}$ and that of $\big\{\mathbf{1}_{\{\nu<s\le\tau_\ell\}}Z_s^n\big\}_{s\in[0,T]}$, $n\in\mathbb{N}$ to $\big\{\mathbf{1}_{\{\nu<s\le\tau_\ell\}}\mathcal{Z}^\ell\big\}_{s\in[0,T]}$ by (6.30) show that $\lim_{n\to\infty} I_2^n = \lim_{n\to\infty} I_4^n = 0$. Since $\Big|\chi\big(-Y_\zeta^{\ell,n} + Y_\zeta + Y_\nu^{\ell,n} - Y_\nu\big)\Big| \le 4\ell|\chi|$ by (6.27), (6.31), (6.25) and since $\mathbb{E}[|\chi|] \le 1+\mathbb{E}[|\chi|^2] < \infty$ by (1.6), the dominated convergence theorem imply that $\lim_{n\to\infty} I_1^n = 0$. Moreover, (H3) shows that

$$
\begin{aligned}
\lim_{n\to\infty} \mathbf{1}_{\{\nu<s\le\zeta\}}\big(g(s,Y_s^{\ell,n},0) - g(s,Y_s,0)\big) &= \lim_{n\to\infty} \mathbf{1}_{A_\ell\cap\{\nu<s\le\zeta\}}\big(g(s,Y_s^n,0) - g(s,Y_s,0)\big) \\
&= 0, \quad ds\otimes d\mathbb{P}\text{-a.s.},
\end{aligned}
$$

while (H4), (6.27) and (6.25) imply that $ds\otimes d\mathbb{P}$-a.s.

$$
\begin{aligned}
\Big|\mathbf{1}_{\{\nu<s\le\zeta\}} M_s^\chi\big(g(s,Y_s^{\ell,n},0) - g(s,Y_s,0)\big)\Big| &\le \mathbf{1}_{\{\nu<s\le\zeta\}}|M_s^\chi|\big(2h_s + \kappa\big|Y_s^{\ell,n}\big| + \kappa|Y_s|\big) \\
&\le \mathbf{1}_{\{\nu<s\le\zeta\}}|M_s^\chi|\big(2h_s + 2\kappa\ell\big).
\end{aligned}
$$

As (6.24) and Hölder's inequality show that

$$
\begin{aligned}
\mathbb{E}\int_0^T \mathbf{1}_{\{\nu<s\le\zeta\}}|M_s^\chi|\,(2h_s + 2\kappa\ell)\,ds &\le 2\ell(1+\kappa T)\,\mathbb{E}\big[M_*^\chi\big] \\
&\le 2\ell(1+\kappa T)\Big\{\mathbb{E}\Big[\big(M_*^\chi\big)^2\Big]\Big\}^{1/2} < \infty,
\end{aligned}
$$

we can apply the dominated convergence theorem again to obtain $\lim_{n\to\infty} I_3^n = 0$. Hence $\lim_{n\to\infty}\mathbb{E}\Big[\chi\Big(K_\zeta^n - \widetilde{K}_\zeta^\ell\Big)\Big] = 0$.

Since (6.22) and (A.21) imply that for any $n\in\mathbb{N}$

$$
\begin{aligned}
K_{\tau_\ell\wedge\gamma}^n - \widetilde{K}_\gamma^\ell &= K_{\tau_\ell\wedge\gamma}^n - \widetilde{K}_{\tau_\ell\wedge\gamma}^\ell = K_{\tau_\ell\wedge\gamma}^n - K_{\nu\wedge(\tau_\ell\wedge\gamma)}^n - \Big(\widetilde{K}_{\tau_\ell\wedge\gamma}^\ell - \widetilde{K}_{\nu\wedge(\tau_\ell\wedge\gamma)}^\ell\Big) \\
&= K_{\nu\vee(\tau_\ell\wedge\gamma)}^n - K_\nu^n - \Big(\widetilde{K}_{\nu\vee(\tau_\ell\wedge\gamma)}^\ell - \widetilde{K}_\nu^\ell\Big) = K_\zeta^n - \widetilde{K}_\zeta^\ell,
\end{aligned}
$$

one gets $\lim_{n\to\infty}\mathbb{E}\Big[\chi\big(K_{\tau_\ell\wedge\gamma}^n - \widetilde{K}_\gamma^\ell\big)\Big] = 0$, which shows that $\big\{K_{\tau_\ell\wedge\gamma}^n\big\}_{n\in\mathbb{N}}$ converges weakly to $\widetilde{K}_\gamma^\ell$ in $L^2(\mathcal{F}_T)$.

(3) Now, let $\gamma, \widetilde{\gamma} \in \mathcal{T}$ such that $\gamma \le \widetilde{\gamma}$, $\mathbb{P}$-a.s. For any $n \in \mathbb{N}$, since K^n is an increasing process, it holds $\mathbb{P}$-a.s. that

$$K^n_{\tau_\ell \wedge \gamma} \le K^n_{\tau_\ell \wedge \widetilde{\gamma}} . \tag{A.23}$$

Then we must have $\widetilde{K}^\ell_\gamma \le \widetilde{K}^\ell_{\widetilde{\gamma}}$, $\mathbb{P}$-a.s.: *Assume not*, i.e. the $\mathbb{P}$-measure of set $A := \left\{\widetilde{K}^\ell_\gamma > \widetilde{K}^\ell_{\widetilde{\gamma}}\right\} \in \mathcal{F}_T$ is strictly larger than 0, it would follow that $\mathbb{E}\left[\mathbf{1}_A \widetilde{K}^\ell_\gamma\right] > \mathbb{E}\left[\mathbf{1}_A \widetilde{K}^\ell_{\widetilde{\gamma}}\right]$. However, we know from part (2) and (A.23) that

$$\mathbb{E}\left[\mathbf{1}_A \widetilde{K}^\ell_\gamma\right] = \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_A K^n_{\tau_\ell \wedge \gamma}\right] \le \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_A K^n_{\tau_\ell \wedge \widetilde{\gamma}}\right] = \mathbb{E}\left[\mathbf{1}_A \widetilde{K}^\ell_{\widetilde{\gamma}}\right].$$

An contradiction appears. Therefore, $\widetilde{K}^\ell_\gamma \le \widetilde{K}^\ell_{\widetilde{\gamma}}$, $\mathbb{P}$-a.s. Then Lemma A.3 shows that $\widetilde{K}^\ell$ is an increasing process. □

Proof of (6.37*). Set $\mathfrak{a} := 2(\lambda^+ + \kappa^2)$ and Fix $m, n \in \mathbb{N}$ with $m > n$. We define processes $\Xi^{m,n}_t := \Xi^m_t - \Xi^n_t$, $t \in [0,T]$ for $\Xi = Y, Y^\ell, Z$. Similar to (A.10), we can deduce from (6.26) that $\mathbb{P}$-a.s.

$$\begin{aligned} & e^{\mathfrak{a}t}\left|Y^{\ell,m,n}_t\right|^2 + \int_t^{\tau_\ell} e^{\mathfrak{a}s}\left(\mathfrak{a}|Y^{\ell,m,n}_s|^2 + |Z^{m,n}_s|^2\right) ds \\ &\quad = e^{\mathfrak{a}\tau_\ell}\left|Y^{\ell,m,n}_{\tau_\ell}\right|^2 + 2\int_t^{\tau_\ell} e^{\mathfrak{a}s} Y^{\ell,m,n}_s \left(g(s, Y^{\ell,m}_s, Z^m_s) - g(s, Y^{\ell,n}_s, Z^n_s)\right) ds \\ &\qquad + 2\int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y^{\ell,m,n}_s dK^m_s - 2\int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y^{\ell,m,n}_s dK^n_s - 2\int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y^{\ell,m,n}_s Z^{m,n}_s dB_s, \\ &\quad \forall\, t \in [\nu, \tau_\ell]. \end{aligned} \tag{A.24}$$

By (H1) and (H2), it holds $ds \otimes d\mathbb{P}$-a.s. that

$$\begin{aligned} & Y^{\ell,m,n}_s \left(g(s, Y^{\ell,m}_s, Z^m_s) - g(s, Y^{\ell,n}_s, Z^n_s)\right) \\ &\quad = Y^{\ell,m,n}_s \left(g(s, Y^{\ell,m}_s, Z^m_s) - g(s, Y^{\ell,n}_s, Z^m_s)\right) \\ &\qquad + Y^{\ell,m,n}_s \left(g(s, Y^{\ell,n}_s, Z^m_s) - g(s, Y^{\ell,n}_s, Z^n_s)\right) \\ &\quad \le \lambda |Y^{\ell,m,n}_s|^2 + \kappa |Y^{\ell,m,n}_s||Z^{m,n}_s| \le (\lambda^+ + \kappa^2)|Y^{\ell,m,n}_s|^2 + \frac{1}{4}|Z^{m,n}_s|^2. \end{aligned} \tag{A.25}$$

Also, one can deduce from the definition of process K^m that

$$\begin{aligned} \int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y^{\ell,m,n}_s dK^m_s &= \mathbf{1}_{A_\ell} \int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y^{m,n}_s dK^m_s = \mathbf{1}_{A_\ell} \int_t^{\tau_\ell} \mathbf{1}_{\{Y^m_s < L_s\}} e^{\mathfrak{a}s}\, Y^{m,n}_s dK^m_s \\ &\le \mathbf{1}_{A_\ell} \int_t^{\tau_\ell} \mathbf{1}_{\{Y^m_s < L_s\}} e^{\mathfrak{a}s}\, (L_s - Y^n_s) dK^m_s \\ &\le e^{\mathfrak{a}T} \mathbf{1}_{A_\ell} \int_\nu^{\tau_\ell} (Y^n_s - L_s)^- dK^m_s \\ &\le e^{\mathfrak{a}T} \mathbf{1}_{A_\ell} \left(\sup_{s\in[\nu,\tau_\ell]} (Y^n_s - L_s)^-\right) K^m_{\tau_\ell}, \qquad \forall\, t \in [\nu, \tau_\ell]. \end{aligned} \tag{A.26}$$

Similarly,

$$
\begin{aligned}
-\int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y_s^{\ell,m,n} dK_s^n &\le \mathbf{1}_{A_\ell} \int_t^{\tau_\ell} \mathbf{1}_{\{Y_s^n < L_s\}} e^{\mathfrak{a}s}\, (L_s - Y_s^m)\, dK_s^n \\
&\le e^{\mathfrak{a}T} \mathbf{1}_{A_\ell} \left(\sup_{s\in[\nu,\tau_\ell]} (Y_s^m - L_s)^- \right) K_{\tau_\ell}^n \\
&\le e^{\mathfrak{a}T} \mathbf{1}_{A_\ell} \left(\sup_{s\in[\nu,\tau_\ell]} (Y_s^n - L_s)^- \right) K_{\tau_\ell}^n, \quad \forall\, t \in [\nu, \tau_\ell].
\end{aligned}
$$

Plugging this and (A.25), (A.26) back into (A.24) shows that $\mathbb{P}$-a.s.

$$
e^{\mathfrak{a}t} \big| Y_t^{\ell,m,n} \big|^2 + \frac{1}{2} \int_t^{\tau_\ell} e^{\mathfrak{a}s} |Z_s^{m,n}|^2 ds \le \eta - 2 \int_t^{\tau_\ell} e^{\mathfrak{a}s}\, Y_s^{\ell,m,n} Z_s^{m,n} dB_s, \quad \forall\, t \in [\nu, \tau_\ell],
$$

where $\eta := e^{\mathfrak{a}\tau_\ell} \big| Y_{\tau_\ell}^{\ell,m,n} \big|^2 + 2e^{\mathfrak{a}T} \mathbf{1}_{A_\ell} \big(\sup_{s\in[\nu,\tau_\ell]} (Y_s^n - L_s)^- \big) \big(K_{\tau_\ell}^m + K_{\tau_\ell}^n \big)$.

Taking expectation for $t = \nu$, we see from Hölder's inequality, (6.27) and (6.29) that

$$
\begin{aligned}
&\mathbb{E} \int_0^T \mathbf{1}_{\{\nu < t \le \tau_\ell\}} |Z_t^{m,n}|^2 dt \le \mathbb{E} \int_\nu^{\tau_\ell} e^{\mathfrak{a}s} |Z_s^{m,n}|^2 ds \le 2\mathbb{E}[\eta] \le 2\mathbb{E} \Big[e^{\mathfrak{a}\tau_\ell} \big| Y_{\tau_\ell}^{\ell,m,n} \big|^2 \Big] \\
&\quad + 4e^{\mathfrak{a}T} \left\{ \mathbb{E} \left[\mathbf{1}_{A_\ell} \sup_{s\in[\nu,\tau_\ell]} \big((Y_s^n - L_s)^- \big)^2 \right] \times \mathbb{E} \Big[\big(K_{\tau_\ell}^m + K_{\tau_\ell}^n \big)^2 \Big] \right\}^{1/2} \\
&\le C_0 \mathbb{E} \Big[\mathbf{1}_{A_\ell} \big| Y_{\tau_\ell} - Y_{\tau_\ell}^n \big|^2 \Big] + C_0 \ell \left\{ \mathbb{E} \left[\mathbf{1}_{A_\ell} \sup_{s\in[\nu,\tau_\ell]} \big((Y_s^n - L_s)^- \big)^2 \right] \right\}^{1/2}. \qquad \text{(A.27)}
\end{aligned}
$$

On the other hand, the Burkholder–Davis–Gundy inequality implies that

$$
\begin{aligned}
\mathbb{E} \left[\sup_{t\in[\nu,\tau_\ell]} \big| Y_t^{\ell,m,n} \big|^2 \right] &\le \mathbb{E} \left[\sup_{t\in[\nu,\tau_\ell]} e^{\mathfrak{a}t} \Big| Y_t^{\ell,m,n} \Big|^2 \right] \le \mathbb{E}[\eta] \\
&\quad + 2\mathbb{E} \left[\sup_{t\in[0,T]} \left| \int_t^T \mathbf{1}_{\{\nu < s \le \tau_\ell\}} e^{\mathfrak{a}s}\, Y_s^{\ell,m,n} Z_s^{m,n} dB_s \right| \right] \\
&\le \mathbb{E}[\eta] + C_0 \mathbb{E} \left[\left(\sup_{t\in[\nu,\tau_\ell]} \big| Y_t^{\ell,m,n} \big| \right) \cdot \left(\int_\nu^{\tau_\ell} e^{\mathfrak{a}t}\, |Z_t^{m,n}|^2 dt \right)^{1/2} \right] \\
&\le \mathbb{E}[\eta] + \frac{1}{2} \mathbb{E} \left[\sup_{t\in[\nu,\tau_\ell]} \big| Y_t^{\ell,m,n} \big|^2 \right] + C_0 \mathbb{E} \int_\nu^{\tau_\ell} e^{\mathfrak{a}t}\, |Z_t^{m,n}|^2 dt.
\end{aligned}
$$

As $\mathbb{E}[\sup_{t\in[\nu,\tau_\ell]} \big| Y_t^{\ell,m,n} \big|^2] \le 4\ell^2$ by (6.27) and (6.25), it follows from (A.27) that

$$
\mathbb{E} \left[\sup_{t\in[\nu,\tau_\ell]} \big| Y_t^{\ell,m,n} \big|^2 \right] \le 2\mathbb{E}[\eta] + C_0 \mathbb{E} \int_\nu^{\tau_\ell} e^{\mathfrak{a}t}\, |Z_t^{m,n}|^2 dt \le C_0 \mathbb{E}[\eta]. \qquad \text{(A.28)}
$$

Since Doob's martingale inequality and (6.27) show that

$$\mathbb{E}\left[\sup_{t\in[0,\nu]}\left|Y_t^{\ell,m,n}\right|^2\right] \le \mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbb{E}\big[\mathbf{1}_{A_\ell}Y_\nu^{m,n}\big|\mathcal{F}_t\big]\right|^2\right] \le 4\mathbb{E}\left[\left|\mathbf{1}_{A_\ell}Y_\nu^{m,n}\right|^2\right]$$
$$\le 4\mathbb{E}\left[\sup_{t\in[\nu,\tau_\ell]}\left|Y_t^{\ell,m,n}\right|^2\right],$$

we see from (6.27) and (A.28) that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_t^{\ell,m,n}\right|^2\right] \le \mathbb{E}\left[\sup_{t\in[0,\nu]}\left|Y_t^{\ell,m,n}\right|^2 + \sup_{t\in[\nu,\tau_\ell]}\left|Y_t^{\ell,m,n}\right|^2\right]$$
$$\le 5\mathbb{E}\left[\sup_{t\in[\nu,\tau_\ell]}\left|Y_t^{\ell,m,n}\right|^2\right] \le C_0\mathbb{E}[\eta].$$

This together with (A.27) leads to that

$$\sup_{m>n}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_t^{\ell,m,n}\right|^2\right] + \mathbb{E}\int_0^T \mathbf{1}_{\{\nu<t\le\tau_\ell\}}|Z_t^{m,n}|^2dt\right\}$$
$$\le C_0\mathbb{E}[\eta] \le C_0\mathbb{E}\left[\mathbf{1}_{A_\ell}\left|Y_{\tau_\ell} - Y_{\tau_\ell}^n\right|^2\right] + C_0\ell\left\{\mathbb{E}\left[\mathbf{1}_{A_\ell}\sup_{t\in[\nu,\tau_\ell]}\left((Y_t^n - L_t)^-\right)^2\right]\right\}^{1/2}.$$

As $\mathbf{1}_{A_\ell}\left|Y_{\tau_\ell} - Y_{\tau_\ell}^n\right| \le 2\ell$, $\forall n \in \mathbb{N}$ by (6.25), letting $n \to \infty$, we see from bounded convergence theorem and (6.36) that

$$\lim_{n\to\infty}\sup_{m>n}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}\left|Y_t^{\ell,m,n}\right|^2\right] + \mathbb{E}\int_0^T \mathbf{1}_{\{\nu<t\le\tau_\ell\}}|Z_t^{m,n}|^2dt\right\} = 0.$$

Hence, $\left\{Y^{\ell,n}\right\}_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{S}^2$ and $\left\{\mathbf{1}_{\{\nu<t\le\tau_\ell\}}Z_t^n\right\}_{t\in[0,T]}$, $n \in \mathbb{N}$ is Cauchy sequence in $\mathbb{H}^{2,2}$. □

Proof of (6.43*). As $K_\nu^n = \mathcal{K}_\nu^\ell = 0$ by (6.22) and (6.42), one can deduce from (6.28) that $\mathbb{P}$-a.s.

$$K_t^n - \mathcal{K}_t^\ell = (K_t^n - K_\nu^n) - (\mathcal{K}_t^\ell - \mathcal{K}_\nu^\ell) = Y_\nu^{\ell,n} - Y_t^{\ell,n} - \mathbf{1}_{A_\ell}(Y_\nu - Y_t)$$
$$- \int_\nu^t \Big(g(s, Y_s^{\ell,n}, Z_s^n) - g(s, Y_s, \mathcal{Z}_s^\ell)\Big)ds$$
$$+ \int_\nu^t (Z_s^n - \mathcal{Z}_s^\ell)dB_s, \quad \forall t \in [\nu, \tau_\ell].$$

Then (6.27) and (H1) show that $\mathbb{P}$-a.s.

$$\left|K_t^n - \mathcal{K}_t^\ell\right| \le \mathbf{1}_{A_\ell}|Y_\nu^n - Y_\nu| + \mathbf{1}_{A_\ell}\left|Y_t^n - Y_t\right|$$
$$+ \int_\nu^t \Big(\kappa|Z_s^n - \mathcal{Z}_s^\ell| + |g(s, Y_s^{\ell,n}, \mathcal{Z}_s^\ell) - g(s, Y_s, \mathcal{Z}_s^\ell)|\Big)ds$$
$$+ \left|\int_\nu^t \left(Z_s^n - \mathcal{Z}_s^\ell\right)dB_s\right|, \ \forall t \in [\nu, \tau_\ell].$$

Since Hölder's inequality and (1.5) imply that

$$\big|K_t^n - \mathcal{K}_t^\ell\big|^2 \le C_0 \mathbf{1}_{A_\ell}|Y_\nu^n - Y_\nu|^2 + C_0 \mathbf{1}_{A_\ell}\big|Y_t^n - Y_t\big|^2 + C_0 \int_\nu^t |Z_s^n - \mathcal{Z}_s^\ell|^2 ds$$
$$+ C_0 \left(\mathbf{1}_{A_\ell} \int_\nu^t |g(s, Y_s^{\ell,n}, \mathcal{Z}_s^\ell) - g(s, Y_s, \mathcal{Z}_s^\ell)| ds\right)^2$$
$$+ C_0 \sup_{\widetilde{t}\in[0,T]} \left| \int_0^{\widetilde{t}} \mathbf{1}_{\{\nu < s \le \tau_\ell\}} \left(Z_s^n - \mathcal{Z}_s^\ell\right) dB_s\right|^2, \quad \forall\, t \in [\nu, \tau_\ell],$$

we can deduce from Doob's martingale inequality that

$$\mathbb{E}\left[\sup_{t\in[\nu,\tau_\ell]} \big|K_t^n - \mathcal{K}_t^\ell\big|^2\right] \le C_0 \mathbb{E}\left[\mathbf{1}_{A_\ell}|Y_\nu^n - Y_\nu|^2\right]$$
$$+ C_0 \mathbb{E}\left[\mathbf{1}_{A_\ell} \sup_{t\in[\nu,\tau_\ell]} \big|Y_t^n - Y_t\big|^2\right] + C_0 \mathbb{E}\int_\nu^{\tau_\ell} |Z_t^n - \mathcal{Z}_t^\ell|^2 dt$$
$$+ C_0 \mathbb{E}\left[\left(\int_\nu^{\tau_\ell} \mathbf{1}_{A_\ell}\big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t, \mathcal{Z}_t^\ell)\big| dt\right)^2\right].$$

The bounded convergence theorem and (6.25) imply that $\lim_{n\to\infty} \downarrow \mathbb{E}\left[\mathbf{1}_{A_\ell}|Y_\nu^n - Y_\nu|^2\right] = 0$. Thanks to (6.41), it remains to show that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_\nu^{\tau_\ell} \mathbf{1}_{A_\ell}\left|g\left(t, Y_t^n, \mathcal{Z}_t^\ell\right) - g\left(t, Y_t, \mathcal{Z}_t^\ell\right)\right| dt\right)^2\right] = 0. \tag{A.29}$$

By (H3), it holds $dt \otimes d\mathbb{P}$-a.s. that $\lim_{n\to\infty} \mathbf{1}_{A_\ell \cap \{\nu < t \le \tau_\ell\}}\big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t, \mathcal{Z}_t^\ell)\big| = 0$. Also, (H1), (H4) and (6.25) imply that for any $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbf{1}_{A_\ell \cap \{\nu < t \le \tau_\ell\}}\big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t, \mathcal{Z}_t^\ell)\big| \\
&\le \mathbf{1}_{A_\ell \cap \{\nu < t \le \tau_\ell\}}\Big(\big|g(t, Y_t^n, 0)\big| + \big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t^n, 0)\big| + \big|g(t, Y_t, 0)\big| \\
&\quad + \big|g(t, Y_t, \mathcal{Z}_t^\ell) - g(t, Y_t, 0)\big|\Big) \le \mathbf{1}_{A_\ell \cap \{\nu < t \le \tau_\ell\}} \left(2h_t + 2\kappa\ell + 2\kappa|\mathcal{Z}_t^\ell|\right) \\
&:= \mathfrak{h}_t^\ell, \quad dt \otimes d\mathbb{P}\text{-a.s.}
\end{aligned} \tag{A.30}$$

As $\mathbb{E}\int_0^T \mathfrak{h}_t^\ell\, dt \le 2\ell + 2\kappa\ell T + 2\kappa T^{1/2}\left(\mathbb{E}\int_\nu^{\tau_\ell} |\mathcal{Z}_t^\ell|^2 dt\right)^{1/2} < \infty$ by (6.24) and Hölder's inequality, applying the dominated convergence theorem yields that

$$\lim_{n\to\infty} \mathbb{E}\int_\nu^{\tau_\ell} \mathbf{1}_{A_\ell}\big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t, \mathcal{Z}_t^\ell)\big| dt = 0.$$

So up to a subsequence of $\{Y^n\}_{n\in\mathbb{N}}$, it holds $\mathbb{P}$-a.s. that $\lim_{n\to\infty} \int_\nu^{\tau_\ell} \mathbf{1}_{A_\ell}\big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t, \mathcal{Z}_t^\ell)\big| dt = 0$. Since (A.30) shows that for any $n \in \mathbb{N}$, $\left(\int_\nu^{\tau_\ell} \mathbf{1}_{A_\ell}\big|g(t, Y_t^n, \mathcal{Z}_t^\ell) - g(t, Y_t, \mathcal{Z}_t^\ell)\big| dt\right)^2 \le \left(\int_0^T \mathfrak{h}_t^\ell dt\right)^2$, $\mathbb{P}$-a.s. and since Hölder's inequality implies that $\mathbb{E}\left[\left(\int_0^T \mathfrak{h}_t^\ell dt\right)^2\right] \le \mathbb{E}\left[\left(2\ell + 2\kappa\ell T + 2\kappa \int_\nu^{\tau_\ell} |\mathcal{Z}_t^\ell| dt\right)^2\right] \le C_0\ell^2 + C_0\mathbb{E}\int_\nu^{\tau_\ell} |\mathcal{Z}_t^\ell|^2 dt < \infty$, applying the dominated convergence theorem again yields (A.29). □

Proof of (6.45*). For any $n \in \mathbb{N}$, Hölder's inequality and (6.29) imply that

$$\mathbb{E}\int_\nu^{\tau_\ell} \big|Y_t^n - Y_t\big| dK_t^n = \mathbb{E}\left[\mathbf{1}_{A_\ell}\int_\nu^{\tau_\ell} \big|Y_t^n - Y_t\big| dK_t^n\right] \le \mathbb{E}\left[\mathbf{1}_{A_\ell}\left(\sup_{t\in[\nu,\tau_\ell]}\big|Y_t^n - Y_t\big|\right)K_{\tau_\ell}^n\right]$$
$$\le \left\{\mathbb{E}\left[\left(K_{\tau_\ell}^n\right)^2\right]\mathbb{E}\left[\mathbf{1}_{A_\ell}\sup_{t\in[\nu,\tau_\ell]}\big|Y_t^n - Y_t\big|^2\right]\right\}^{1/2}$$
$$\le C_0\ell\left\{\mathbb{E}\left[\mathbf{1}_{A_\ell}\sup_{t\in[\nu,\tau_\ell]}\big|Y_t^n - Y_t\big|^2\right]\right\}^{1/2}.$$

As $n \to \infty$, (6.41) shows that $\lim_{n\to\infty}\mathbb{E}\left[\left|\int_\nu^{\tau_\ell}\big(Y_t^n - Y_t\big)dK_t^n\right|\right] = 0$. So up to a subsequence of $\{Y^n\}_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty}\int_\nu^{\tau_\ell}\big(Y_t^n - Y_t\big)dK_t^n = 0, \quad \mathbb{P}\text{-a.s.} \tag{A.31}$$

For $\mathbb{P}$-a.s. $\omega \in \Omega$ such that (6.44) holds and that path $Y_t(\omega) - L_t(\omega)$ is continuous from $t = \nu(\omega)$ to $t = \tau_\ell(\omega)$ by (6.40), one can deduce from (6.44) that measure $dK_t^n(\omega)$ converges weakly to measure $d\mathcal{K}_t^\ell(\omega)$ on period $[\nu(\omega), \tau_\ell(\omega)]$, so

$$\lim_{n\to\infty}\int_{\nu(\omega)}^{\tau_\ell(\omega)}\left(Y_t(\omega) - L_t(\omega)\right) dK_t^n(\omega) = \int_{\nu(\omega)}^{\tau_\ell(\omega)}\left(Y_t(\omega) - L_t(\omega)\right) d\mathcal{K}_t^\ell(\omega).$$

Adding this to (A.31), we see from (6.35) that $\mathbb{P}$-a.s.

$$0 \le \mathbf{1}_{\{\nu<\tau_\ell\}}\int_t^{\tau_\ell}(Y_s - L_s)d\mathcal{K}_s^\ell \le \mathbf{1}_{\{\nu<\tau_\ell\}}\int_\nu^{\tau_\ell}(Y_s - L_s)d\mathcal{K}_s^\ell = \int_\nu^{\tau_\ell}(Y_s - L_s)d\mathcal{K}_s^\ell$$
$$= \lim_{n\to\infty}\int_\nu^{\tau_\ell}\big(Y_s^n - L_s\big)dK_s^n$$
$$= \lim_{n\to\infty}\int_\nu^{\tau_\ell}\mathbf{1}_{\{Y_s^n<L_s\}}(Y_s^n - L_s)dK_s^n \le 0, \quad \forall\, t \in [\nu, \tau_\ell],$$

proving (6.45*). □

Proof of Claim (6.60). It is clear that $Y_{\gamma_\nu} = \mathbf{1}_{\{\gamma_\nu=T\}}Y_T + \mathbf{1}_{\{\gamma_\nu<T\}}Y_{\gamma_\nu} \le \mathbf{1}_{\{\gamma_\nu=T\}}\xi + \mathbf{1}_{\{\gamma_\nu<T\}}U_{\gamma_\nu}$, $\mathbb{P}$-a.s., so we only need to show the converse inequality.

Fix $n \in \mathbb{N}$. Clearly, $K_s^n := n\int_0^s(Y_r^n - L_r)^- dr$, $s \in [0, T]$ is a process of $\mathbb{K}^0$ satisfying that $\mathbb{P}$-a.s.

$$Y_t^n = Y_{\gamma_\nu^n}^n + \int_t^{\gamma_\nu^n} g(s, Y_s^n, Z_s^n)ds + K_{\gamma_\nu^n}^n - K_\nu^n - \int_t^{\gamma_\nu^n} Z_s^n dB_s, \quad \forall\, t \in [\nu, \gamma_\nu^n]$$

by (6.59). Since $\mathbb{E}[|Y_\nu^n|] < \infty$ by the uniform integrability of $\{Y_\zeta^n\}_{\zeta\in\mathcal{T}}$, applying Lemma A.2 with $(Y, Z, K) = (Y^n, Z^n, K^n)$ and $\tau = \gamma_\nu^n$, we see from (1.6) that for any $p \in (0, 1)$

$$\mathbb{E}\left[\left(\int_\nu^{\gamma_\nu^n}|Z_t^n|^2dt\right)^{p/2}\right] \le C_p\mathbb{E}\left[\sup_{t\in[\nu,\gamma_\nu^n]}|Y_t^n|^p + \left(\int_\nu^{\gamma_\nu^n}h_tdt\right)^p\right]$$
$$\le C_p\mathbb{E}\left[1 + \sup_{s\in[0,T]}|Y_s^1|^p + \sup_{s\in[0,T]}|Y_s|^p + \int_0^T h_tdt\right]. \tag{A.32}$$

Let $j \in \mathbb{N}$ and define a stopping time $\zeta_j^n := \inf\big\{t \in [0,T] : \int_0^t |Z_s^n|^2 ds > j\big\} \wedge T \in \mathcal{T}$. Since (6.59) shows that

$$\begin{aligned} Y_{\gamma_\nu \wedge \zeta_j^n} &\ge Y^n_{\gamma_\nu \wedge \zeta_j^n} = Y^n_{\gamma_\nu^n \wedge \zeta_j^n} + \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} g(s, Y_s^n, Z_s^n) ds + K^n_{\gamma_\nu^n \wedge \zeta_j^n} - K^n_{\gamma_\nu \wedge \zeta_j^n} \\ &\quad - \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} Z_s^n dB_s \\ &\ge Y^n_{\gamma_\nu^n \wedge \zeta_j^n} + \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} g(s, Y_s^n, Z_s^n) ds - \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} Z_s^n dB_s, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

taking conditional expectation $\mathbb{E}\Big[\ \cdot\ \Big|\mathcal{F}_{\gamma_\nu \wedge \zeta_j^n}\Big]$ yields that $\mathbb{P}$-a.s.

$$\begin{aligned} Y_{\gamma_\nu \wedge \zeta_j^n} &\ge \mathbb{E}\left[Y^n_{\gamma_\nu^n \wedge \zeta_j^n} + \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} g(t, Y_t^n, Z_t^n) dt \Big| \mathcal{F}_{\gamma_\nu \wedge \zeta_j^n} \right] \\ &= \mathbf{1}_{\{\gamma_\nu \ge \zeta_j^n\}} \mathbb{E}\left[Y^n_{\gamma_\nu^n \wedge \zeta_j^n} + \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} g(t, Y_t^n, Z_t^n) dt \Big| \mathcal{F}_{\zeta_j^n} \right] \\ &\quad + \mathbf{1}_{\{\gamma_\nu < \zeta_j^n\}} \mathbb{E}\left[Y^n_{\gamma_\nu^n \wedge \zeta_j^n} + \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} g(t, Y_t^n, Z_t^n) dt \Big| \mathcal{F}_{\gamma_\nu} \right] := I_1^{n,j} + I_2^{n,j}. \qquad (A.33) \end{aligned}$$

As $\{\gamma_\nu \ge \zeta_j^n\} \subset \{\gamma_\nu^n \ge \zeta_j^n\}$, it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} I_1^{n,j} &= \mathbb{E}\left[\mathbf{1}_{\{\gamma_\nu \ge \zeta_j^n\}} Y^n_{\gamma_\nu^n \wedge \zeta_j^n} + \mathbf{1}_{\{\gamma_\nu \ge \zeta_j^n\}} \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} g(t, Y_t^n, Z_t^n) dt \Big| \mathcal{F}_{\zeta_j^n} \right] \\ &= \mathbb{E}\Big[\mathbf{1}_{\{\gamma_\nu \ge \zeta_j^n\}} Y^n_{\zeta_j^n} \Big| \mathcal{F}_{\zeta_j^n} \Big] = \mathbf{1}_{\{\gamma_\nu \ge \zeta_j^n\}} Y^n_{\zeta_j^n}. \qquad (A.34) \end{aligned}$$

Similar to (6.17), (H4), (H5), (1.5) and (1.6) imply that $\big|g(t, Y_t^n, Z_t^n)\big| \le \kappa + (1+\kappa)h_t + 2\kappa|Y_t^n| + \kappa|Z_t^n|^\alpha$, $dt \otimes d\mathbb{P}$-a.s. It then follows from Hölder's inequality that $\mathbb{P}$-a.s.

$$\begin{aligned} \int_{\gamma_\nu \wedge \zeta_j^n}^{\gamma_\nu^n \wedge \zeta_j^n} \big|g(s, Y_s^n, Z_s^n)\big| ds &\le \int_{\gamma_\nu}^{\gamma_\nu^n} \big|g(s, Y_s^n, Z_s^n)\big| ds \le C_0 \int_{\gamma_\nu}^{\gamma_\nu^n} \big(1 + h_s + |Y_s^n|\big) ds \\ &\quad + \kappa(\gamma_\nu^n - \gamma_\nu)^{1-\alpha/2} \left(\int_{\gamma_\nu}^{\gamma_\nu^n} |Z_s^n|^2 ds \right)^{\alpha/2} \qquad (A.35) \\ &\le C_0 \int_0^T \big(1 + h_s + |Y_s^n|\big) ds + C_\alpha \left(\int_0^T |Z_s^n|^2 ds \right)^{\alpha/2}. \qquad (A.36) \end{aligned}$$

By Fubini's Theorem and the uniform integrability of $\{Y_\zeta^n\}_{\zeta \in \mathcal{T}}$, $\mathbb{E}\int_0^T |Y_s^n| ds = \int_0^T \mathbb{E}\big[|Y_s^n|\big] ds \le T \sup_{s \in [0,T]} \mathbb{E}\big[|Y_s^n|\big] < \infty$, which together with $Z^n \in \mathbb{H}^{2,\alpha}$ shows that the last term in (A.36) is integrable. As $Z^n \in \cap_{p \in (0,1)} \mathbb{H}^{2,p} \subset \mathbb{H}^{2,0}$ shows that $\big\{\zeta_j^n\big\}_{j \in \mathbb{N}}$ is stationary, it holds $\mathbb{P}$-a.s. that $\lim_{j \to \infty} Y_{\gamma_\nu \wedge \zeta_j^n} = Y_{\gamma_\nu}$ though we have not yet shown whether Y is a continuous process. Letting $j \to \infty$ in (A.33) and (A.34), we can deduce from the uniform integrability of $\{Y_\zeta^n\}_{\zeta \in \mathcal{T}}$ and the conditional-expectation version of dominated convergence theorem that

$$Y_{\gamma_\nu} \ge \mathbf{1}_{\{\gamma_\nu=T\}}Y^n_T + \lim_{j\to\infty} I_2^{n,j} = \mathbf{1}_{\{\gamma_\nu=T\}}\xi + \mathbf{1}_{\{\gamma_\nu<T\}}\mathbb{E}\left[Y^n_{\gamma^n_\nu} + \int_{\gamma_\nu}^{\gamma^n_\nu} g(t, Y^n_t, Z^n_t)dt\Big|\mathcal{F}_{\gamma_\nu}\right]$$
$$= \mathbf{1}_{\{\gamma_\nu=T\}}\xi + \mathbf{1}_{\{\gamma_\nu<T\}}$$
$$\times\,\mathbb{E}\left[\mathbf{1}_{\{\gamma^n_\nu=T\}}\xi + \mathbf{1}_{\{\gamma^n_\nu<T\}}U_{\gamma^n_\nu} + \int_{\gamma_\nu}^{\gamma^n_\nu} g(t, Y^n_t, Z^n_t)dt\Big|\mathcal{F}_{\gamma_\nu}\right], \quad \mathbb{P}\text{-a.s.}, \tag{A.37}$$

where we used in the last equality the fact that $Y^n_{\gamma^n_\nu} = U_{\gamma^n_\nu}$, $\mathbb{P}$-a.s. on $\{\gamma^n_\nu < T\}$ by the continuity of Y^n and U.

Since $\lim_{n\to\infty}\downarrow \mathbf{1}_{\{\gamma^n_\nu=T\}} = \mathbf{1}_{\{\gamma_\nu=T\}}$ and since $\xi \in L^1(\mathcal{F}_T)$, applying the conditional-expectation version of dominated convergence theorem yields that

$$\lim_{n\to\infty}\mathbf{1}_{\{\gamma_\nu<T\}}\mathbb{E}\left[\mathbf{1}_{\{\gamma^n_\nu=T\}}\xi\big|\mathcal{F}_{\gamma_\nu}\right] = \lim_{n\to\infty}\mathbb{E}\left[\mathbf{1}_{\{\gamma_\nu<T\}}\mathbf{1}_{\{\gamma^n_\nu=T\}}\xi\big|\mathcal{F}_{\gamma_\nu}\right] = 0, \quad \mathbb{P}\text{-a.s.} \tag{A.38}$$

As $\mathbf{1}_{\{\gamma^n_\nu<T\}}\big|U_{\gamma^n_\nu}\big| = \mathbf{1}_{\{\gamma^n_\nu<T\}}\big|Y^n_{\gamma^n_\nu}\big| \le \big|Y^1_{\gamma^n_\nu}\big| + \big|Y_{\gamma^n_\nu}\big|$, $\mathbb{P}$-a.s., the uniform integrability of $\{Y^1_\zeta\}_{\zeta\in\mathcal{T}}$ and $\{Y_\zeta\}_{\zeta\in\mathcal{T}}$ implies that of $\big\{\mathbf{1}_{\{\gamma^n_\nu<T\}}U_{\gamma^n_\nu}\big\}_{n\in\mathbb{N}}$, and it then follows from the continuity of U that

$$\lim_{n\to\infty}\mathbb{E}\left[\mathbf{1}_{\{\gamma^n_\nu<T\}}U_{\gamma^n_\nu}\big|\mathcal{F}_{\gamma_\nu}\right] = \mathbb{E}\left[\mathbf{1}_{\{\gamma_\nu<T\}}U_{\gamma_\nu}\big|\mathcal{F}_{\gamma_\nu}\right] = \mathbf{1}_{\{\gamma_\nu<T\}}U_{\gamma_\nu}, \quad \mathbb{P}\text{-a.s.} \tag{A.39}$$

Set $\widetilde{\alpha} := \frac{1}{2}(1+\alpha) \in (0,1)$. Given $\varepsilon > 0$, with $A^n_\varepsilon := \left\{\mathbb{E}\Big[\int_{\gamma_\nu}^{\gamma^n_\nu}\big|g(s, Y^n_s, Z^n_s)\big|ds\Big|\mathcal{F}_{\gamma_\nu}\Big] > \varepsilon\right\} \in \mathcal{F}_{\gamma_\nu}$, (A.35), Hölder's inequality and (A.32) imply that

$$\mathbb{P}(A^n_\varepsilon) \le \frac{1}{\varepsilon}\mathbb{E}\left[\mathbf{1}_{A^n_\varepsilon}\mathbb{E}\Big[\int_{\gamma_\nu}^{\gamma^n_\nu}\big|g(t, Y^n_t, Z^n_t)\big|dt\Big|\mathcal{F}_{\gamma_\nu}\Big]\right] = \frac{1}{\varepsilon}\mathbb{E}\left[\mathbf{1}_{A^n_\varepsilon}\int_{\gamma_\nu}^{\gamma^n_\nu}\big|g(t, Y^n_t, Z^n_t)\big|dt\right]$$
$$\le \frac{C_0}{\varepsilon}\mathbb{E}\int_{\gamma_\nu}^{\gamma^n_\nu}\big(1+h_t+|Y^n_t|\big)\,dt + \frac{\kappa}{\varepsilon}\mathbb{E}\left[(\gamma^n_\nu-\gamma_\nu)^{1-\alpha/2}\left(\int_{\gamma_\nu}^{\gamma^n_\nu}|Z^n_t|^2dt\right)^{\alpha/2}\right]$$
$$\le \frac{C_0}{\varepsilon}\mathbb{E}\int_{\gamma_\nu}^{\gamma^n_\nu}\Big(1+h_t+|Y^1_t|+|Y_t|\Big)\,dt + \frac{\kappa}{\varepsilon}\left\{\mathbb{E}\left[(\gamma^n_\nu-\gamma_\nu)^{\frac{(2-\alpha)\widetilde{\alpha}}{2(\widetilde{\alpha}-\alpha)}}\right]\right\}^{1-\alpha/\widetilde{\alpha}}$$
$$\times\left\{\mathbb{E}\left[\left(\int_{\gamma_\nu}^{\gamma^n_\nu}|Z^n_t|^2dt\right)^{\widetilde{\alpha}/2}\right]\right\}^{\alpha/\widetilde{\alpha}}$$
$$\le \frac{C_0}{\varepsilon}\mathbb{E}\int_{\gamma_\nu}^{\gamma^n_\nu}\Big(1+h_t+|Y^1_t|+|Y_t|\Big)\,dt + \frac{C_\alpha}{\varepsilon}\left\{\mathbb{E}\left[(\gamma^n_\nu-\gamma_\nu)^{\frac{(2-\alpha)\widetilde{\alpha}}{2(\widetilde{\alpha}-\alpha)}}\right]\right\}^{1-\alpha/\widetilde{\alpha}}$$
$$\times\left\{\mathbb{E}\left[1+\sup_{s\in[0,T]}|Y^1_t|^{\widetilde{\alpha}} + \sup_{s\in[0,T]}|Y_t|^{\widetilde{\alpha}} + \int_0^T h_t dt\right]\right\}^{\alpha/\widetilde{\alpha}}.$$

Since Fubini's Theorem and the uniform integrability of $\{Y^1_\zeta\}_{\zeta\in\mathcal{T}}$, $\{Y_\zeta\}_{\zeta\in\mathcal{T}}$ show that

$$\mathbb{E}\int_0^T\Big(1+h_t+|Y^1_t|+|Y_t|\Big)\,dt \le \int_0^T(1+h_t)dt + \int_0^T\mathbb{E}\Big[|Y^1_t|+|Y_t|\Big]dt$$
$$\le \int_0^T(1+h_t)dt + T\sup_{t\in[0,T]}\mathbb{E}[|Y^1_t|]$$
$$+\,T\sup_{t\in[0,T]}\mathbb{E}[|Y_t|] < \infty,$$

letting $n \to \infty$, we can deduce from the dominated convergence theorem and the bounded convergence theorem that

$$\lim_{n\to\infty} \mathbb{P}\left\{ \mathbb{E}\left[\int_{\gamma_\nu}^{\gamma_\nu^n} \left| g(s, Y_s^n, Z_s^n) \right| ds \middle| \mathcal{F}_{\gamma_\nu} \right] > \varepsilon \right\} = 0, \quad \mathbb{P}\text{-a.s.}$$

Thus, $\mathbb{E}\left[\int_{\gamma_\nu}^{\gamma_\nu^n} g(s, Y_s^n, Z_s^n) ds \middle| \mathcal{F}_{\gamma_\nu} \right]$ converges to 0 in probability $\mathbb{P}$. Up to a subsequence of $\{(Y^n, Z^n)\}_{n\in\mathbb{N}}$, one has

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{\gamma_\nu}^{\gamma_\nu^n} g(s, Y_s^n, Z_s^n) ds \middle| \mathcal{F}_{\gamma_\nu} \right] = 0, \quad \mathbb{P}\text{-a.s.},$$

which together with (A.37)–(A.39) leads to that $Y_{\gamma_\nu} \geq \mathbf{1}_{\{\gamma_\nu = T\}} \xi + \mathbf{1}_{\{\gamma_\nu < T\}} U_{\gamma_\nu}$, $\mathbb{P}$-a.s. □

References

[1] K. Bahlali, S. Hamadène, B. Mezerdi, Backward stochastic differential equations with two reflecting barriers and continuous with quadratic growth coefficient, Stochastic Process. Appl. 115 (2005) 1107–1129.
[2] E. Bayraktar, Y.-J. Huang, On the multidimensional controller-and-stopper games, SIAM J. Control Optim. 51 (2013) 1263–1297.
[3] E. Bayraktar, I. Karatzas, S. Yao, Optimal stopping for dynamic convex risk measures, Illinois J. Math. 54 (2010) 1025–1067.
[4] E. Bayraktar, M. Sîrbu, Stochastic Perron's method and verification without smoothness using viscosity comparison: obstacle problems and Dynkin games, Proc. Amer. Math. Soc. 142 (2014) 1399–1412.
[5] E. Bayraktar, S. Yao, Optimal stopping for non-linear expectations—Part I, Stochastic Process. Appl. 121 (2011) 185–211.
[6] E. Bayraktar, S. Yao, Optimal stopping for non-linear expectations—Part II, Stochastic Process. Appl. 121 (2011) 212–264.
[7] E. Bayraktar, S. Yao, Quadratic reflected BSDEs with unbounded obstacles, Stochastic Process. Appl. 122 (2012) 1155–1203.
[8] E. Bayraktar, S. Yao, On the robust optimal stopping problem, SIAM J. Control Optim. 52 (2014) 3135–3175.
[9] J.-M. Bismut, Conjugate convex functions in optimal stochastic control, J. Math. Anal. Appl. 44 (1973) 384–404.
[10] P. Briand, R. Carmona, BSDEs with polynomial growth generators, J. Appl. Math. Stoch. Anal. 13 (2000) 207–238.
[11] P. Briand, B. Delyon, Y. Hu, E. Pardoux, L. Stoica, L^p solutions of backward stochastic differential equations, Stochastic Process. Appl. 108 (2003) 109–129.
[12] R. Buckdahn, J. Li, Probabilistic interpretation for systems of Isaacs equations with two reflecting barriers, NoDEA Nonlinear Differential Equations Appl. 16 (2009) 381–420.
[13] Z. Chen, W. Tian, G. Zhao, Optimal stopping rule meets ambiguity, in: Real Options, Ambiguity, Risk and Insurance, in: Studies in Probability, Optimization and Statistics, vol. 5, IOS Press, 2013, pp. 97–125.
[14] J. Cvitanić, I. Karatzas, Backward stochastic differential equations with reflection and Dynkin games, Ann. Probab. 24 (1996) 2024–2056.
[15] J. Cvitanić, I. Karatzas, H.M. Soner, Backward stochastic differential equations with constraints on the gains-process, Ann. Probab. 26 (1998) 1522–1551.
[16] C. Dellacherie, P.-A. Meyer, Probabilités et Potentiel, Hermann, Paris, 1975, Chapitres I à IV, Édition entièrement refondue, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. XV, Actualités Scientifiques et Industrielles, No. 1372.
[17] R. Dumitrescu, M.-C. Quenez, A. Sulem, Generalized Dynkin Games and Doubly Reflected BSDEs with Jumps, 2013, Available at http://arxiv.org/abs/1310.2764.
[18] I. Ekren, N. Touzi, J. Zhang, Optimal stopping under nonlinear expectation, Stochastic Process. Appl. 124 (2014) 3277–3311.
[19] B. El Asri, S. Hamadène, H. Wang, L^p-solutions for doubly reflected backward stochastic differential equations, Stoch. Anal. Appl. 29 (2011) 907–932.
[20] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, M.C. Quenez, Reflected solutions of backward SDE's, and related obstacle problems for PDE's, Ann. Probab. 25 (1997) 702–737.
[21] N. El Karoui, S. Peng, M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance 7 (1997) 1–71.

[22] E.H. Essaky, M. Hassani, Generalized BSDE with 2-Reflecting Barriers and Stochastic Quadratic Growth, 2013, Available at http://arxiv.org/abs/0805.2979v3.
[23] E.H. Essaky, M. Hassani, Y. Ouknine, Stochastic Quadratic BSDE with Two RCLL Obstacles, 2011, Available at http://arxiv.org/abs/1103.5373.
[24] S. Hamadène, Mixed zero-sum stochastic differential game and American game options, SIAM J. Control Optim. 45 (2006) 496–518.
[25] S. Hamadène, M. Hassani, BSDEs with two reflecting barriers: the general result, Probab. Theory Related Fields 132 (2005) 237–264.
[26] S. Hamadène, I. Hdhiri, Backward stochastic differential equations with two distinct reflecting barriers and quadratic growth generator, J. Appl. Math. Stoch. Anal. (2006) Article ID 95818, 28.
[27] S. Hamadène, J.-P. Lepeltier, Reflected BSDEs and mixed game problem, Stochastic Process. Appl. 85 (2000) 177–188.
[28] S. Hamadène, J.-P. Lepeltier, A. Matoussi, Double barrier backward SDEs with continuous coefficient, in: Backward Stochastic Differential Equations (Paris, 1995–1996), in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 1997, pp. 161–175.
[29] S. Hamadène, J.-P. Lepeltier, Z. Wu, Infinite horizon reflected backward stochastic differential equations and applications in mixed control and game problems, Probab. Math. Statist. 19 (1999) 211–234.
[30] S. Hamadène, A. Popier, L^p-solutions for reflected backward stochastic differential equations, Stoch. Dyn. 12 (2012) 1150016, 35.
[31] S. Hamadène, E. Rotenstein, A. Zălinescu, A generalized mixed zero-sum stochastic differential game and double barrier reflected BSDEs with quadratic growth coefficient, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 55 (2009) 419–444.
[32] S. Hamadène, J. Zhang, The continuous time nonzero-sum Dynkin game problem and application in game options, SIAM J. Control Optim. 48 (2009/10) 3659–3669.
[33] S.W. He, J.G. Wang, J.A. Yan, Semimartingale Theory and Stochastic Calculus, Kexue Chubanshe (Science Press), Beijing; CRC Press, Boca Raton, FL, 1992.
[34] Y. Hu, J. Ma, S. Peng, S. Yao, Representation theorems for quadratic $\mathcal{F}$-consistent nonlinear expectations, Stochastic Process. Appl. 118 (2008) 1518–1551.
[35] I. Karatzas, W.D. Sudderth, The controller-and-stopper game for a linear diffusion, Ann. Probab. 29 (2001) 1111–1127.
[36] I. Karatzas, H. Wang, Connections between bounded variation control and Dynkin games, in: J.L. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations (Volume in honor of A. Bensoussan), IOS Press, Amsterdam, 2001, pp. 363–373.
[37] I. Karatzas, I.-M. Zamfirescu, Martingale approach to stochastic differential games of control and stopping, Ann. Probab. 36 (2008) 1495–1527.
[38] T. Klimsiak, BSDEs with monotone generator and two irregular reflecting barriers, Bull. Sci. Math. 137 (2013) 268–321.
[39] J.-P. Lepeltier, J. San Martín, Backward SDEs with two barriers and continuous coefficient: an existence result, J. Appl. Probab. 41 (2004) 162–175.
[40] J. Ma, S. Yao, On quadratic g-evaluations/expectations and related analysis, Stoch. Anal. Appl. 28 (2010) 711–734.
[41] M. Nutz, J. Zhang, Optimal stopping under adverse nonlinear expectation and related games, Ann. Appl. Probab. (2014) in press. Available at http://arxiv.org/abs/1212.2140.
[42] É Pardoux, S.G. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1990) 55–61.
[43] S. Peng, Backward SDE and related g-expectation, in: Backward Stochastic Differential Equations (Paris, 1995–1996), in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow, 1997, pp. 141–159.
[44] S. Peng, Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type, Probab. Theory Related Fields 113 (1999) 473–499.
[45] S. Peng, Dynamical evaluations, C. R. Math. Acad. Sci. Paris 339 (2004) 585–589.
[46] S. Peng, Nonlinear Expectations, Nonlinear Evaluations and Risk Measures, in: Lecture Notes in Math, vol. 1856, Springer, Berlin, 2004.
[47] E. Rosazza Gianin, Risk measures via g-expectations, Insurance Math. Econom. 39 (2006) 19–34.
[48] N. Touzi, N. Vieille, Continuous-time Dynkin games with mixed strategies, SIAM J. Control Optim. 41 (2002) 1073–1088. electronic.
[49] M. Xu, Reflected backward SDEs with two barriers under monotonicity and general increasing conditions, J. Theoret. Probab. 20 (2007) 1005–1039.
[50] K. Yosida, Functional Analysis, sixth ed., in: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 123, Springer-Verlag, Berlin, 1980.