ASSIGNMENT 2

(a)

4.5) $A \cos\frac{n\pi x}{a}$ is NOT acceptable because it does not satisfy the boundary condition that $\psi(0) = 0$

b) $B(x+x^2)$ is NOT acceptable because $\psi(x=a) \neq 0$

c) $Cx^3(x-a) = 0$ when $x=0$ or when $x=a$
It can also be normalized.
It is acceptable

d) $\frac{D}{\sin\frac{n\pi x}{a}} = \infty$ when $x=0$. NOT acceptable

4.B) $$\int_0^a \psi^*\psi \, dx = A\int_0^a x^2\left(1-\frac{x}{a}\right)^2 dx$$

$$= A^2\int_0^a \left(\frac{x^4}{a^2} - \frac{2x^3}{a} + x^2\right)dx$$

$$= A^2\left[\frac{x^5}{5a^2} - \frac{x^4}{2a} + \frac{x^3}{3}\bigg|_0^a\right]$$

$$= A^2\left[\frac{a^3}{5a^2} - \frac{a^4}{2a} + \frac{a^3}{3}\right] = A^2\frac{a^3}{30} = 1$$

$$\Rightarrow A = \sqrt{\frac{30}{a^3}}$$

with $A = \sqrt{\frac{30}{a^3}} \Rightarrow \int_0^a \psi^*\psi \, dx = 1$

$$\langle x\rangle = \int_0^a \psi^* x\psi \, dx = \frac{30}{a^3}\int_0^a x\left(1-\frac{x}{a}\right)x\,x\left(1-\frac{x}{a}\right)dx$$

$$= \frac{30}{a^3}\left[\frac{x^4}{4} - \frac{2x^5}{5a} + \frac{x^6}{6a^2}\bigg|_0^a\right] = \frac{30}{a^3}\left[\frac{a^4}{4} - \frac{2a^4}{5} + \frac{a^4}{4}\right] = \frac{30}{a^3}\cdot\frac{a^4}{60} = \frac{a}{2}$$

$$\langle x^2\rangle = \int_0^a \psi^* x^2\psi \, dx = \frac{30}{a^3}\int_0^a x\left(1-\frac{x}{a}\right)x^2\cdot x\left(1-\frac{x}{a}\right)dx$$

$$= \frac{30}{a^3}\left[\frac{x^5}{5} - \frac{x^6}{3a} + \frac{x^7}{7a^2}\bigg|_0^a\right] = \frac{30}{a^3}\left[\frac{a^5}{5} - \frac{a^5}{3} + \frac{a^5}{7}\right] = \frac{30}{a^3}\cdot\frac{a^5}{105} = \frac{2a^2}{7}$$

(5)

4.16 $\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2}$

$$= \frac{-\hbar^2}{2m}\left[-\frac{n_x^2\pi^2}{a^2} N\sin\frac{n_x\pi x}{a}\sin\frac{n_y\pi y}{a} - \frac{n_y^2\pi^2}{b^2}\cdot N\sin\frac{n_y\pi y}{b}\sin\frac{n_x\pi x}{a}\right]$$

$$= +\frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{b^2}\right)\psi$$

$$\Rightarrow \quad E = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{b^2}\right) = \frac{h^2}{8m}\left(\frac{n_x^2}{a^2}+\frac{n_y^2}{b^2}\right)$$

b) For $a = b \Rightarrow E = \frac{h^2}{8ma^2}(n_x^2+n_y^2)$

a) $n_x = 1, \quad n_y = 1$

b) $n_x = 2; \quad n_y = 3$

c) $n_x = 3, \quad n_y = 1$

d) $n_x = 2, \quad n_y = 2$

e) $n_x = 1; \quad n_y = 5$

f) $n_x = 2, \quad n_y = 1$

4.21 Now $E = \frac{h^2}{8ma^2}(n_x^2+n_y^2)$

a) $n_x^2 + n_y^2 = 5$

$\Rightarrow$ Either $n_x = 1, \; n_y = 2$

or $n_x = 2, \; n_y = 1$

$\Big\}$ degeneracy = 2

For 3D $E = \frac{h^2}{8ma^2}(n_x^2+n_y^2+n_z^2)$, $a=b=c$ $\Rightarrow$

b) $n_z^2 + n_x^2 + n_y^2 = 9$

$n_x = 1, \; n_y = 2, \; n_z = 2$

or

$n_x = 2, \; n_y = 1, \; n_z = 2$

or

$n_x = 2, \; n_y = 2, \; n_z = 1$

$\Big\}$ degeneracy = 3

4.25) $\int \phi_i^* \phi_j \, dx = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq 0 \end{cases}$ (c)

a) $\int \Psi^* \Psi \, dx = \int \left[\frac{1}{2}\phi_1^* + \frac{1}{4}\phi_2^* + \frac{3-\sqrt{2}i}{4}\phi_3^*\right)\left(\frac{1}{2}\phi_1 + \frac{1}{4}\phi_2 + \frac{3+\sqrt{2}}{4}i\,\phi_3\right) dx$

Only $i=j$ terms survive $\int$.

$$= \frac{1}{4}\int \phi_1^* \phi_1 \, dx + \frac{1}{16}\int \phi_2^* \phi_2 \, dx + \frac{(3-\sqrt{2}i)(3+\sqrt{2}i)}{16}\int \phi_3^* \phi_3 \, dx$$

$$= \frac{1}{4} + \frac{1}{16} + \frac{9+2}{16} = \frac{4+1+11}{16} = \underline{1}$$

(b) Either E_1 or $3E_1$ or $7E_1$

(c) Prob of measuring E_1 is $\frac{1}{4}$ $\left(\frac{1}{2^2}\right)$

" " " $3E_1$ is $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

" " " $7E_1$ is $\frac{(3+\sqrt{2}i)(3-\sqrt{2}i)}{4 \cdot 4} = \frac{11}{16}$

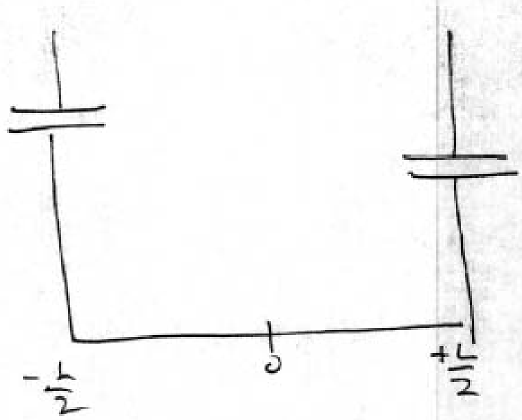
(d) $\langle E \rangle = \frac{1}{4}E_1 + \frac{1}{16} \cdot 3E_1 + \frac{11}{16} \cdot 7E_1$

$$= \frac{4}{16}E_1 + \frac{3E_1}{16} + \frac{77E_1}{16}$$

$$= \frac{84E_1}{16}$$

$$= \frac{21}{4}E_1$$

3) (a)



$\psi = A \sin kx + B \cos kx$

$\psi(x = -\frac{L}{2}) = 0 \Rightarrow \psi = A \sin(-\frac{kL}{2}) + B \cos(-\frac{kL}{2})$

$\left\{ \begin{array}{l} \sin(-\theta) = -\sin\theta \\ \cos(-\theta) = \cos\theta \end{array} \right\}$

$\Rightarrow \psi = -A \sin \frac{kL}{2} + B \cos \frac{kL}{2} = 0 \quad -①$

Similarly $\psi(x = \frac{L}{2}) = 0 = A \sin \frac{kL}{2} + B \cos \frac{kL}{2} = 0 \quad -②$

Add EQNS ① + ② to get

$2B \cos \frac{kL}{2} = 0$ $\Big\}$ either $B = 0$ OR $\frac{kL}{2} = \frac{(2n+1)\pi}{2}$

Subtract EQ1 ① from ② to get

$2A \sin \frac{kL}{2} = 0$ $\Big\}$ either $A = 0$ or $\frac{kL}{2} = n\pi$

Thus 2 soln are possible

① $B = 0$ & $\frac{kL}{2} = n\pi \Rightarrow k = 2n\pi/L$

$\psi(x) = A \sin \frac{2n\pi}{L} x$

$n = 1, 2, 3, 4 \ldots$

$\left\{ \begin{array}{l} \text{when } x = \frac{L}{2} \\ \psi(+\frac{L}{2}) = 0 \\ \text{when } x = -L/2 \\ \psi(-\frac{L}{2}) = 0 \end{array} \right.$

OR

② $A = 0$; $\frac{kL}{2} = \frac{(2n+1)\pi}{2} \Rightarrow k = \frac{(2n+1)\pi}{L}$

$\psi(x) = B \cos \frac{(2n+1)\pi x}{2}$

$n = 0, 1, 2, 3, \ldots$

$\left\{ \begin{array}{l} \text{when } x = L/2 \\ \psi(\frac{L}{2}) = 0 \\ \text{when } x = -L/2 \\ \psi(-\frac{L}{2}) = 0 \end{array} \right.$

4 $\psi = Ae^{-Kx} + Be^{+Kx}$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left[(-K)Ae^{-K(x)} + KBe^{Kx}\right]$$

$$= -\frac{\hbar^2}{2m}\left[K^2Ae^{-Kx} + K^2Be^{Kx}\right]$$

$$= -\frac{\hbar^2}{2m}K^2\left[Ae^{-Kx} + Be^{Kx}\right]$$

$$= -\frac{\hbar^2}{2m}K^2\psi$$

Now $K = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{-\cancel{\hbar^2}}{\cancel{2m}}\cdot\frac{\cancel{2m}(V_0-E)}{\cancel{\hbar^2}}\psi$$

$$= (E-V_0)\psi$$

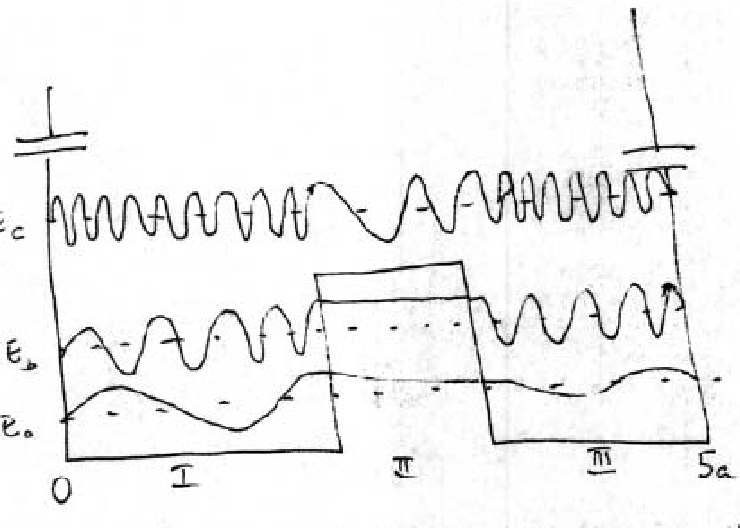
Rewrite to get

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\psi = E\psi$$

{this is Sch Eqn. for the region outside the box}

(7)

5)



or Relative features $\psi(x=0)=0$; $\psi(x=5a)=0$

Notes

1) Region I/III ⇒ freq. of oscillation increases from E_a to E_c.
$\psi(x=0)=0=\psi(x=5a)$

2) Region II
For E_a/E_b wavefunc decays (No oscillation) ⇒ Faster decay for E_a

For E_c ⇒ wavefunction oscillates @ lower freq. than in I or III