# Grade:

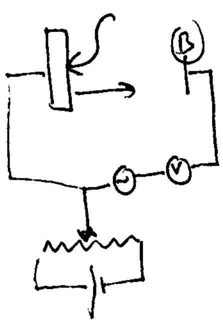
## Question 1 [70 points]

(i) Determine all the statements that are true. Indicate your answer in the space provide. [20 points]

(a) the photoelectric effect experiment shows that radiation is a wave
(b) the photoelectric effect experiment shows that radiation is a particle
(c) the work function of a metal can be measured from the photoelectric effect expt.
(d) the kinetic energy of emitted electrons increases when the intensity of radiation increases in the photoelectric effect experiment.
(e) the blackbody radiation shows that the energy of the oscillators is quantized
(f) blackbody radiation experiment shows that light is a wave
(g) blackbody radiation shows that the energy of the oscillators is quantized
(h) when electrons are reflected off a Ni crystal a diffraction pattern results.

**TRUE STATEMENTS ARE:** (b), (c), (e), (g) (h)

(ii) In this problem you will consider the Photoelectric experiment. In this experiment the maximum Kinetic Energy of electrons emitted from a metal by radiation is measured. Describe how the the maximum Kinetic Energy is measured in the experiment. [10 points]




electrons are emitted upon shining rad$^n$ on the metals.
These electrons are repelled - ~~when~~ the energy required to repel the fastest electron = Max. Kinetic energy. ①
electrons are repelled by applying a negative potential on B. The voltage required to stop the current flow is measured. This voltage, $V_{stop}$, provides the maximum Kinetic energy, $T_{max}$: ②

$$T_{max} = eV_{stop}$$

(iii) A radiation of 300 nm is impinged of the metal which has a work-function of 1 eV. What is the maximum Kinetic energy, and de-broglie wavelength of the emitted electrons? [20 points]

$$T_{max} = h\nu - \Phi$$

$$= \frac{hc}{\lambda} - \Phi \quad (5)$$

$$= \frac{6.626 \times 10^{-34}\ Js.\ 3 \times 10^{8}\ ms^{-1}}{300 \times 10^{-9}\ m} - eV1 \times \frac{1}{6.242 \times 10^{18}}\ J/eV$$

$$= 6.626 \times 10^{-19}\ J - 1.602 \times 10^{-19}\ J \quad (5)$$

$$= 5.024 \times 10^{-19}\ J$$

$$T_{max} = \frac{P_{max}^2}{2m_e} \quad (5) \qquad m_e = 9.1094 \times 10^{-31}\ kg$$

$$\Rightarrow P_{max} = \sqrt{2m_e T_{max}}$$

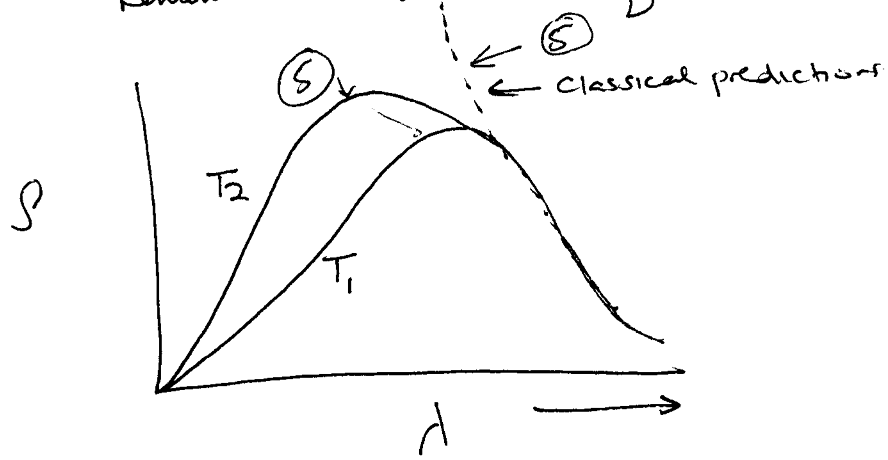
$$= 9.567 \times 10^{-25} \quad (5)$$

$$\lambda = \frac{h}{P_{max}} = \frac{6.626 \times 10^{-34}}{9.567 \times 10^{-25}} = 6.92 \times 10^{-10}\ m$$

(iii) Consider the Black-body radiation experiment conducted at two temperatures, $T_1$ and $T_2$ (with $T_2 > T_1$). The spectral density of emitted radiation is measured as a function of **wavelength**. [20 points]

(a) Plot the experimental observations at the two temperatures – the x axis must be **wavelength** and y axis is spectral density. On the plot draw the predictions from classical theory. [10 points]

Remember $\nu$: $\lambda = \frac{c}{\nu}$




{else 5}

(b) Write down an equation that relates the spectral density to frequency. What aspect of this was modified in light of the results of the blackbody ~~radiation~~ experiment.

(5)

$$\rho = \frac{8\pi\nu^2 \bar{E}_{osc}}{c^3}$$

Classically $\bar{E}_{osc} = kT$. The blackbody experiment showed that the Energy of each oscillators: $E_n = nh\nu$

$$\bar{E}_{osc} = \frac{h\nu}{e^{h\nu/kT} - 1}$$ (5)

## Question 2 [80 points]

(i) Complete the following sentences [20 points]:

(a) The probability of finding a particle in a region of space dx around x is:

$\psi^* \psi \, dx$

(b) The operator for Kinetic energy in one-dimension in quantum mechanics is:

$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

(c) The only values of Kinetic energy that can be measured in a single experiment are:

its eigenvalues

(d) If kinetic energy of a system which is in a state $\psi(x)$, is measured once in each of many identically prepared systems the average value of Kinetic Energy is given by:

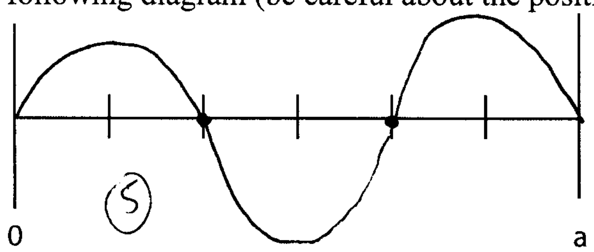
$$\langle T \rangle = \frac{\int_{-\infty}^{\infty} \psi^* \hat{T} \psi \, dx}{\int_{-\infty}^{\infty} \psi^* \psi \, dx} = \frac{-\int_{-\infty}^{\infty} \psi^* \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \, dx}{\int_{-\infty}^{\infty} \psi^* \psi \, dx}$$

(e) The evolution in time of $\psi$ is given by:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \hat{H} \Psi(x,t) = E \Psi(x,t)$$

(a) Consider a Particle of mass "m" in a one-dimensional box (PIB) of length "a". The potential is: V=0 for 0<x <a and the potential is infinity at the walls and outside the box. Imagine that the particle is in the n = 3 state.

(i) Write down the wavefunction in this state. Clearly draw the wavefunction on the following diagram (be careful about the positions of the nodes). [10 points]




$n = 3 \Rightarrow$ 2 nodes between 0 & a.

$\psi = \sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a}$

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(ii) What value of energy is measured in a single experiment in this state? [10 points]

$E_n = \frac{n^2 h^2}{8ma^2} = \frac{9h^2}{8ma^2}$

(iii) The unconstrained free particle (ie quantum mechanical free particle) has a continuous energy spectrum. Using key results from the unconstrained free particle and your diagram of part (i) show that the value of Energy in the n=3 state of a PIB is reasonable. [20 points]

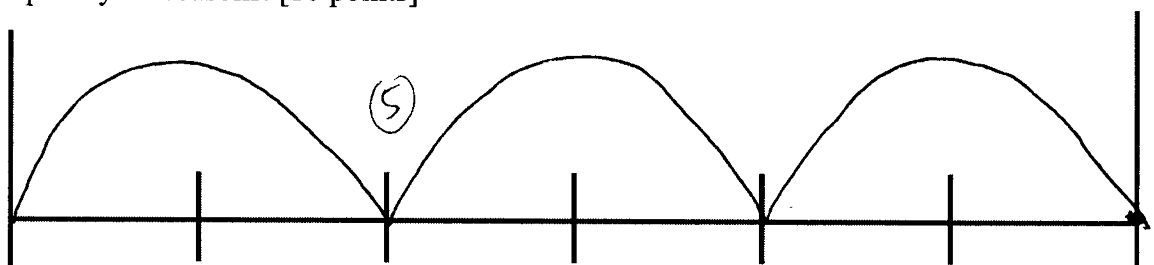
For a free particle $k = \frac{2\pi}{\lambda}$ & $k = \sqrt{\frac{2mE}{\hbar^2}} \Rightarrow E = \frac{k^2\hbar^2}{2m}$ (5)

From the plot for $n = 3$; $\lambda = \frac{2}{3}a$ (10)

$$\Rightarrow k = \frac{2\pi \cdot 3}{2a} = \frac{3\pi}{a} \quad (5)$$

$$\Rightarrow E = \frac{k^2\hbar^2}{2m} = \frac{9\pi^2}{a^2}\cdot\frac{h^2}{4\pi^2}\cdot\frac{1}{2m} = \frac{9h^2}{8ma^2}$$ as required.

(iv) Draw a plot that shows the probability of finding the particle in the region 0<x<a. Without any calculations state the most probable location of the particle – clearly explain your reasons. [10 points]



(5)

$\langle x \rangle = \frac{a}{2}$ (5)

{$\Psi^*\Psi$ is symmetric around $\frac{a}{2}$} thus the average of the distribution function must be $\frac{a}{2}$

0

a

(v) What value of $p_x$ is measured for a single measurement in the n=3 state. You can just state the answer with your reasoning [10 points]

$\langle P_x \rangle = 0$ (5) ; Since the $\Psi^*\Psi$ is symmetric half the time the particle will be found moving to the right with some velocity, the other half to the left with the same velocity. The average is thus zero (5)

(vi) Determine $<p_x^2>$ for the particle in the n= 3 state. Explain why the answer is not the same as $<p_x>$ that was derived in class. [20 points]

$$\hat{P}_x = -i\hbar \frac{\partial}{\partial x}$$

$$\text{New } \langle P_x^2 \rangle = \frac{\int_{-\infty}^{\infty} \psi_3^* \hat{P}_x^2 \psi_3 \, dx}{\int_{-\infty}^{\infty} \psi_3^* \psi_3 \, dx} \quad \text{(5)}$$

$$\hat{P}_x^2 \psi_3 = -i\hbar \frac{\partial}{\partial x} \cdot \left[ -i\hbar \frac{\partial}{\partial x} \sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a} \right]$$

$$= -i\hbar \frac{\partial}{\partial x} \left[ -i\hbar \cdot \frac{3\pi}{a} \cdot \sqrt{\frac{2}{a}} \cos \frac{3\pi x}{a} \right]$$

$$= -i\hbar \left[ -i\hbar \cdot \left(\frac{3\pi}{a}\right)\left(\frac{3\pi}{a}\right)\left(-\sqrt{\frac{2}{a}} \sin \frac{3\pi x}{a}\right) \right]$$

$$= \hbar^2 \cdot \frac{9\pi^2}{a^2} \psi_3 \quad \text{(5)}$$

$$\langle \hat{P}_x^2 \rangle = \int_{-\infty}^{\infty} \psi_3^* \hbar^2 \frac{9\pi^2}{a^2} \psi_3 \, dx$$

$$= \frac{9\pi^2 \hbar^2}{a^2} \int_{-\infty}^{\infty} \psi_3^* \psi_3 \, dx$$

$$= \frac{9\pi^2}{a^2} \cdot \frac{h^2}{4\pi^2} \quad \text{(5)}$$

$$= \frac{9h^2}{a^2} \quad \text{(5)}$$

Momentum-squared is always positive. ~~Thus~~ Since the lowest energy is not zero, momentum is never zero. Thus momentum-squared must be non-zero and greater than zero.

## Question 3 (50 points)

(b) The state of particle of mass "m" in a 1-D box of length "a" is given by:

$$\psi(x) = c\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) + d\sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right) \qquad [1]$$

(i) A measurement of energy is made on many copies of the particle prepared in this state ψ, and it is found that either $\frac{4h^2}{8ma^2}$ or $\frac{25h^2}{8ma^2}$ is measured. Determine the values of "n" and "m". [10 points]

$$E_n = \frac{n^2h^2}{8ma^2}$$

$$\Rightarrow \quad n = 4, \quad m = 5$$

(ii) What is the probability that an energy of $\frac{4h^2}{8ma^2}$ is measured in a single experiment? [10 points]

$$c^2$$

(iii) Many copies of the particle are made, all in the same state, ψ(x), given by Equation 1. The value of energy is measured in each box and the average energy <E> is found to be $\frac{h^2}{8ma^2} + \frac{75h^2}{32ma^2}$. Determine the values of "c" and 'd" in Equation 1. [20 points]

$$\frac{h^2}{8mc^2} = \frac{1}{4}\cdot\frac{4h^2}{8mc^2} \Rightarrow c^2 = \frac{1}{4} \text{ or } c = \frac{1}{2} \quad 10$$

$$\frac{75h^2}{32mc^2} = \frac{3}{4}\cdot\frac{25h^2}{8mc^2} \Rightarrow d^2 = \frac{3}{4} \text{ or } d = \sqrt{\frac{3}{4}} \quad 10$$

$$\Rightarrow \quad c^2 + d^2 = \frac{1}{4} + \frac{3}{4} = 1$$

(iv) Show that ψ(x) is normalized when $c^2+d^2 = 1$. [10 points] 5

$$\int_0^a \psi^*\psi\,dx = \int_0^a \left(c\sqrt{\tfrac{2}{a}}\sin\tfrac{4\pi x}{a} + d\sqrt{\tfrac{2}{a}}\sin\tfrac{5\pi x}{a}\right)\left(c\sqrt{\tfrac{2}{a}}\sin\tfrac{4\pi x}{a} + d\sqrt{\tfrac{2}{a}}\sin\tfrac{5\pi y}{a}\right)dx$$

$$= c^2\int_0^a \sqrt{\tfrac{2}{a}}\sin\tfrac{4\pi x}{a}\sqrt{\tfrac{2}{a}}\sin\tfrac{4\pi x}{a}\,dx + cd\int_0^a \sqrt{\tfrac{2}{a}}\sin\tfrac{4\pi x}{a}\sqrt{\tfrac{2}{a}}\sin\tfrac{5\pi x}{a}\,dx + dc\int_0^a \sqrt{\tfrac{2}{a}}\sin\tfrac{5\pi y}{a}\sin\tfrac{4\pi x}{a}\,dx + d^2\int_0^a \sqrt{\tfrac{2}{a}}\sin\tfrac{4\pi x}{a}\sin\tfrac{5\pi x}{a}\,dx\Big]$$

$$= c^2 + d^2 = 1$$

5

## Question 4 [50 points]

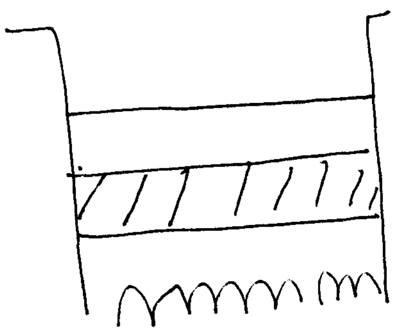
In this problem we will consider applications of the PIB and quantum mechanical tunneling through a finite barrier.

(a) Using a diagram briefly explain how the PIB model is used to rationalize the conductivity of metal such as sodium.

The valence 3s elettron is delocalized. In a 1 cm long wire of sodium there exist ~ $2\times10^7$ valence electrons. These can be treated as a Particle in a Box. Each "n" level has 2 electrons ⇒ Thus $n \sim 10^7$ fill all the delocalized electrons.

But $E_{n+1} - E_n = \frac{h^2}{8mL^2}(2n+1) \sim 10^{-27}$ J

Thus the energy levele overlap and form a band of possible energetic states. In "Na" this band is half filled. Thus when the wire is connected to a battery electron from the negative end of the battery can occupy unfilled levels and travel to the positive end.

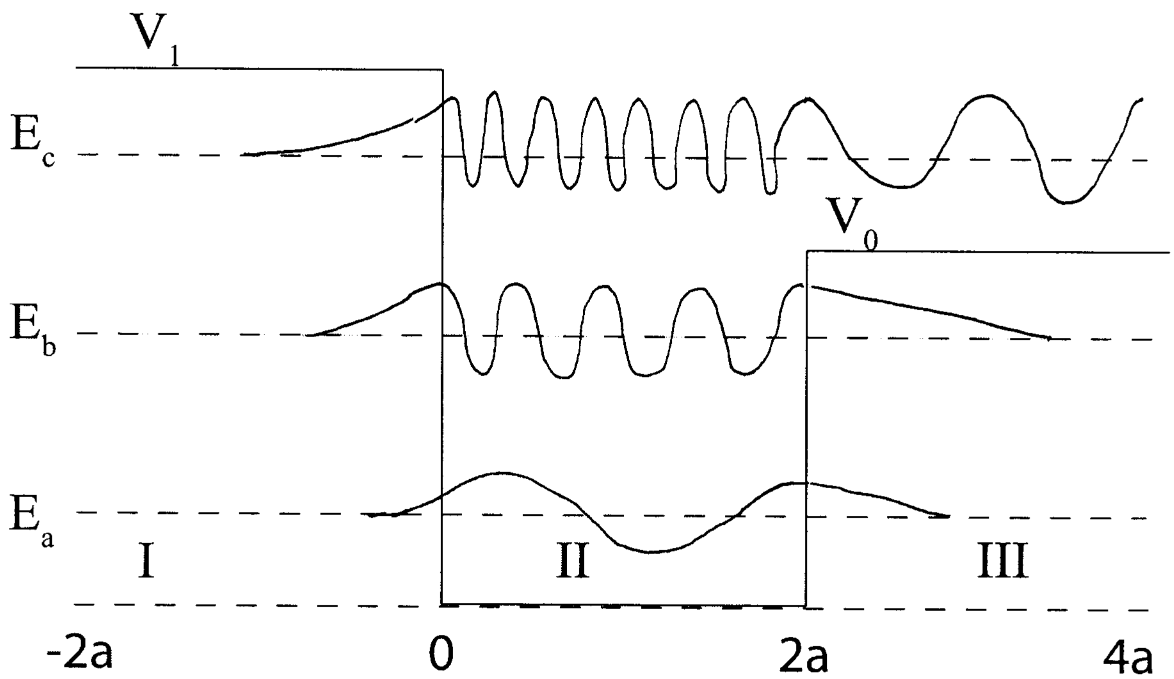


(b) Consider the quantum mechanical particle placed in a box of length 2a and potential as sketched below. Three regions are defined in the sketch and three possible energy levels are also sketched. Briefly the potential, V is:

$V = V_1$ for $-\infty < x < 0$,
$= 0$ for $0 < x < 2a$
$= V_0$ for $2a < x < \infty$ where $V_1$ and $V_0$ are finite constants and $V_1 > V_0$.

(i) Write down all the possible boundary conditions for the wavefunction and its first derivative.

(ii) For each of the three energy levels sketch out the wavefunction of the particle in region I, II, and III. Clearly show relative features (the changes in the amplitude, frequency, and extent of tunneling must be clearly evident as you go from region I, II, and III or $E_a$ to $E_c$.) State the trend in words if your plot is unclear.




{5 each}

(i) $\psi_I(0) = \psi_{II}(0)$ ; $\psi_{II}(2a) = \psi_{III}(2a)$

$\left.\frac{\partial \psi_I}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_{II}}{\partial x}\right|_{x=0}$ ; $\left.\frac{\partial \psi_{II}}{\partial x}\right|_{x=2a} = \left.\frac{\partial \psi_{III}}{\partial x}\right|_{x=2a}$

(ii) Region I ⇒ Tunneling length increases from $E_a$ to $E_c$ (5)

Region II ⇒ Freq. of oscillation increases from $E_a$ to $E_c$ (5)

Region III ⇒ Tunneling length increases from $E_a$ to $E_b$. Note that is more tunneling on the right (III) than the left (I) since $V_0 < V_1$ (5)

For $E_c$ freq. of oscillation changes from II to III (5)