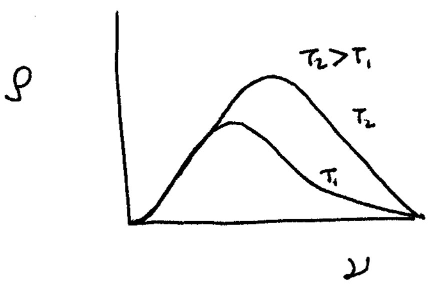
Grade:

Question 1 [50 points]

(a) Consider the Black-body radiation experiment.

(i) Plot the experimental observations. Indicate two ways by which the data changes as temperature increases. Clearly define the variable plotted along the y-axis. [10 points]



ρ = Energy per unit volume . per unit freq.

i.e. $\dfrac{\text{Energy}}{\text{vol. freq.}}$

1) Area under curve increases
2) frequency w/ maximum ρ increases

(ii) State two ways by which the classical predictions fail to explain the experimental observations. [10 points]

1) $\rho \propto \nu^2$, there is no fall off at high frequency
2) Area under curve is ∞ at all temperatures

(b)) State an experiment that shows the particle nature of light. [10 points]

Photoelectric Effect - Light shining on some metals can release electrons

(c) State an experiment that reveals the wave nature of light. [10 points]

Young's 2- slit experiment - Interference Pattern is observed when light is passed through two narrow and closeby slits

(d) Describe an experiment that reveals the wave nature of particles, such as electrons. [5 points]

Eletrons refleted from Nickel crystal show diffraction/interference pattern

(e) State two conditions under which the quantum nature of particles can be easily observed. [5 points]

1) IF $\Delta E \gg RT$
2) IF lengthscales in the experiment are in the order of the wavelength of the particle.

Question 2 [100 points]

(a) Consider a Particle of mass "m" in a one-dimensional box (PIB) of length "a".

(i) Determine $\langle p_x^2 \rangle$ for a 1-D particle in a box (PIB) of length "a". [15 points]

$$\psi_n = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$$

Need $$\langle P_x^2 \rangle = \frac{\int_0^a \psi_n^* \hat{P}_x . \hat{P}_x \psi_n \, dx}{\int_0^a \psi_n^* \psi_n \, dx}$$ But $\int_0^a \psi_n^* \psi_n \, dx = 1$ { class notes

$$\hat{P}_x \hat{P}_x \psi_n = \left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} = -\hbar^2 \frac{\partial}{\partial x}\sqrt{\frac{2}{a}}\left(\frac{n\pi}{a}\right)\cos\frac{n\pi x}{a} = +\hbar^2\left(\frac{n^2\pi^2}{a^2}\right)\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

$$\Rightarrow \langle P_x^2 \rangle = \int_0^a \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} . \hbar^2\left(\frac{n^2\pi^2}{a^2}\right)\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2\hbar^2}{a}\frac{n^2\pi^2}{a^2}\int_0^a \sin^2\frac{n\pi x}{a}\,dx$$ Let $\frac{n\pi x}{a} = y \Rightarrow dx = \frac{a}{n\pi}dy$

$$= \frac{2\hbar^2}{a}\frac{n^2\pi^2}{a^2}\frac{a}{n\pi}\int_0^{n\pi}\sin^2 y\,dy$$

$$= \frac{2\hbar^2}{a}.\frac{n^2\pi^2}{a^2}.\frac{a}{n\pi}\left[\frac{y}{2} - \frac{\sin 2y}{4}\Big|_0^{n\pi}\right]$$

$$= \frac{\cancel{2}\hbar^2 n^2\pi^2 \cancel{a}}{\cancel{a}.a^2.\cancel{n\pi}}.\frac{\cancel{n\pi}}{\cancel{2}} = \frac{n^2\pi^2\hbar^2}{a^2} = \frac{n^2\cancel{\pi^2}h^2}{a^2 4\cancel{\pi^2}} = \frac{n^2h^2}{4a^2} \neq 0$$

(ii) In class we showed that $\langle p_x \rangle = 0$ for this case. Why is $\langle p_x^2 \rangle$ not zero.

$P_x = mV_x$: Since particle can move ~~right~~ to the right or the left roughly half time a positive momentum is measured while the other half the same but negative momentum is measured - Hence the average momentum = 0

To measure the averge of momentum-squared i.e. $\langle P_x^2 \rangle$ you square each measurement - thus each value is positive. Hence the average is not zero

(iii) Determine the result when $\hat{x}\hat{p}_x - \hat{p}_x\hat{x}$ operates on the wavefunction of a PIB. [20 points]

$$(\hat{x}\hat{p}_x - \hat{p}_x\hat{x})\psi_n = \left[-x i\hbar\frac{\partial}{\partial x} + i\hbar\frac{\partial}{\partial x}x\right]\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} = -x i\hbar\frac{\partial}{\partial x}\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} + i\hbar\frac{\partial}{\partial x}x\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

$$= -i\hbar x\sqrt{\frac{2}{a}}\cdot\left(\frac{n\pi}{a}\right)\cos\frac{n\pi x}{a} + i\hbar\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} + i\hbar x\sqrt{\frac{2}{a}}\left(\frac{n\pi}{a}\right)\cos\frac{n\pi x}{a}$$

$$= i\hbar\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} = i\hbar\psi_n$$

(b) The state of particle of mass "m" in a 1-D box of length "a" is given by:

$$\psi(x) = \sqrt{\frac{2}{a}}\left(\frac{1}{\sqrt{4}}\sin\left(\frac{2\pi x}{a}\right) + \frac{1}{\sqrt{4}}\sin\left(\frac{5\pi x}{a}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{3\pi x}{a}\right)\right) \quad [1]$$

Terms in brackets are ψ_2, ψ_5, ψ_3, where $\psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

These have an energy $\frac{n^2h^2}{8ma^2}$

i.e. $E_2 = \frac{4h^2}{8ma^2}$

$E_5 = \frac{25h^2}{8ma^2}$

$E_3 = \frac{9h^2}{8ma^2}$

(i) A measurement of energy is made on the particle. What values of energy can possibly be observed on this system. [4 points]

The possible energies are E_2, E_5, E_3

or $\frac{4h^2}{8ma^2}, \frac{25h^2}{8ma^2}, \frac{9h^2}{8ma^2}$

(ii) Many copies of the particle are made, all in the same state, ψ(x), given in Equation 1. The value of energy is measured in each box and the average energy <E> determined. Determine an equation for <E>. [4 points]

$\cancel{\langle E\rangle = \frac{1}{4}E_2 + \frac{1}{4}E_5 + \frac{1}{\sqrt{2}}E_3 =}$

$$\langle E\rangle = \frac{1}{4}E_2 + \frac{1}{4}E_5 + \frac{1}{2}E_3 = \frac{1}{4}\cdot\frac{4h^2}{8ma^2} + \frac{1}{4}\cdot\frac{25h^2}{8ma^2} + \frac{9h^2}{2\cdot 8ma^2} = \frac{h^2}{8ma^2}\left(1 + \frac{25}{4} + \frac{9}{2}\right)$$

$$= \frac{h^2}{8ma^2}\left(\frac{4+25+18}{4}\right) = \frac{47h^2}{8ma^2}$$

(iii) What is the probability that an energy of $\frac{9h^2}{8ma^2}$ is measured in a single experiment? [4 points]

$\frac{1}{2}$

(iv) What is the probability that an energy of $\frac{h^2}{8ma^2}$ is measured in a single experiment? [4 points]

0

(v) What values of p_x can possibly be observed on this system. [4 points]

There is no information on possible values of P_x

(c) Consider a particle in a 3-D box of lengths, a, b, and c along the x, y and z directions respectively. The values of $n_x = 2$, $n_y = 2$, $n_z = 3$. $\psi_n = \sqrt{\frac{8}{abc}} \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{b} \sin \frac{n_y \pi z}{c}$

(i) Write out the wavefunction for the particle in this state. [4 points]

$$\psi_n = \sqrt{\frac{8}{abc}} \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b} \sin \frac{3\pi z}{c}$$

(ii) Write an expression for the energy of this level? [4 points]

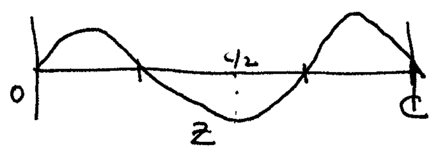
$$E_{n_x n_y n_z} = \frac{n_x^2 h^2}{8ma^2} + \frac{n_y^2 h^2}{8mb^2} + \frac{n_z^2 h^2}{8mc^2} = \frac{h^2}{8m}\left(\frac{4}{a} + \frac{4}{b} + \frac{9}{c}\right)$$

(iii) What is the degeneracy of this level if a=b=c? [4 points]

IF $a=b=c$, then states with $n_x=2$, $n_y=2$, $n_z=3$ has the same energy as

$$\left.\begin{array}{l} n_x=3,\ n_y=2,\ n_z=2 \\ n_x=2,\ n_y=3,\ n_z=3 \end{array}\right\} \text{degeneracy} = 3$$

(iv) Draw a plot of $\psi_3(z)$ versus z. [4 points]



(v) What average value of y-component of momentum, i.e. $<p_y>$, do you anticipate (no calculations are required but provide your reason)? [4 points]

0 ⇒ a negative momentum is equally likely as a positive momentum.

(d) The lowest energy of an unconfined quantum mechanical free particle can be zero. However the lowest energy of a Particle in a box is not zero. Explain the difference using results derived for the unconfined free particle [20 points]

For unconfined Q.M. particle $k = \frac{2\pi}{\lambda} = \sqrt{\frac{2mE}{\hbar^2}}$

Since λ can take any value, E can take any value.

For PIB, (length of box = a) $\lambda_{max} = 2a$

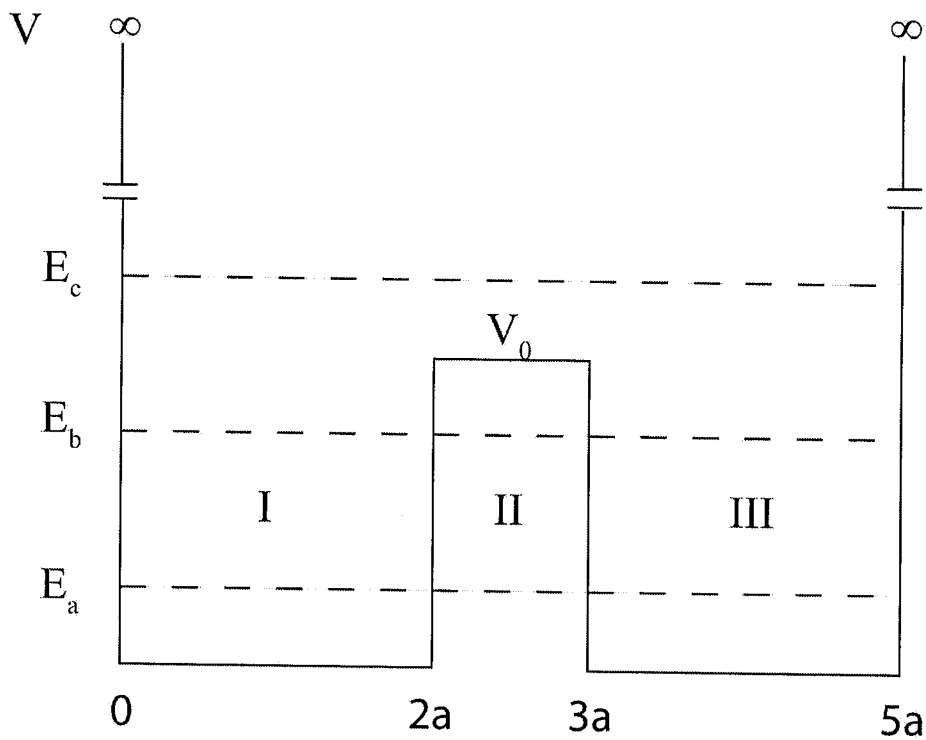
$$\Rightarrow \frac{2\pi}{2a} = \sqrt{\frac{2m E_{min}}{\hbar^2}}$$

$$\Rightarrow E_{min} = \frac{\hbar^2 \pi^2}{a^2 2m} = \frac{h^2 \pi^2}{8\pi^2 m a^2} = \frac{h^2}{8ma^2}$$

Since wavelength is restricted by boundary conditions only certain Energy values can exist. Since max wavelength is 2a $\left(E \propto \frac{1}{\lambda^2}\right)$ minimum Energy is $\frac{h^2}{8ma^2}$

Question 3 [50 points]

Consider the problem of a particle in a box of length 5a and potential as sketched below. In this question you will consider two particles - the first with a mass of "m" and the second with a mass of "4m" - that occupy this box.



(a) For the lighter particle state and explain how frequency of oscillation changes in region I, II and III if the energy is E_c. **[5 points]**

FREQ of oscillation is same in Region of I & III
FREQ of oscillation is smaller in REGION II →

FREQ $k \propto \sqrt{\frac{2mE_c}{\hbar^2}}$

$k' \propto \sqrt{\frac{2m(E_c - V_0)}{\hbar^2}}$

$k' \overset{\text{sm}}{<} k$

(b) For the lighter particle state the boundary condition at x=0, x=2a, x=3a and x= 5a. **[5 points]**

$$\psi_I(x=0) = 0; \quad \psi_I(x=2a) = \psi_{II}(x=2a)$$
$$\psi_{II}(x=3a) = \psi_{III}(x=3a); \text{ and } \psi_{III}(x=5a) = 0$$

(c) For the lighter particle state and explain how frequency of oscillation changes as the Energy is raised from E_a to E_b to E_c in region I. **[5 points]**

FREQ. increases as Energy increase → FREQ $k \propto \sqrt{\frac{2mE}{\hbar^2}}$

(d) How is the frequency of the heavier particle related to the frequency of the lighter particle in Region I? **[5 points]**

GIVEN FREQ $k \propto \sqrt{\frac{2mE}{\hbar^2}}$; mass increase by a factor of 4
⇒ frequency increases by a factor of 2

(g) Describe how the wavefunctions in region II change as the height of the barrier (i.e. V_0) increases. **[10 points]**

Since decay goes as $K \propto \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

when V_0 increase K increases

$\Rightarrow$ decay into Region II is faster,

The particle tunnels a shorter distance in Region II

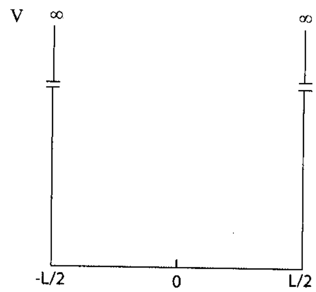
Question 4

Consider a particle of mass m in box of length L. The potential energy diagram is sketched on the right. Note that the potential is given by:

$V(x) = \infty$ for $x \geq L/2$, $x \leq -L/2$ and

$V(x) = 0$ for $-L/2 < x < L/2$

Which of the following are acceptable wavefunctions for this particle.



(a) $\Psi(x) = N\sin(\frac{n\pi x}{L})$

(b) $\Psi(x) = N\sin(\frac{2n\pi x}{L})$

(c) $\Psi(x) = N\cos(\frac{n\pi x}{L})$

(d) $\Psi(x) = N\cos\left[\frac{(2n+1)\pi x}{L}\right]$

(e) $\Psi(x) = N\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\sin(\frac{2n\pi x}{L})$

Clearly explain your reasoning **[30 points]**

(a) $N\sin\left(\frac{n\pi x}{L}\right) \Rightarrow \psi(-\frac{L}{2}) = N\sin\left(-\frac{n\pi}{2}\right) = -N\sin(\frac{n\pi}{2}) = \pm N \neq 0$ if n is odd unless $N=0$, in this

if $N=0$ ~~case~~ $\psi(x)=0$ everywhere

NOT acceptable for n odd or 0.

Wavefunction is acceptable for $n = 2, 4, 8, 12, 6, 10 \ldots$

b) $\psi(\frac{-L}{2}) = N\sin(-n\pi) = -N\sin(n\pi) = 0$ } acceptable

Similarly $\psi(\frac{L}{2}) = N\sin(n\pi) = 0$

c) $\psi(-\frac{L}{2}) = N\cos\left(-\frac{n\pi}{2}\right) = N\cos\left(\frac{n\pi}{2}\right)$ careful here

$\psi(\frac{L}{2}) = N\cos(\frac{n\pi}{2}) = N\cos(\frac{n\pi}{2})$

this is 0 for $n = 1, 3, 5$

But is Not zero IF $n = 2, 4, 6, 8 \ldots$

wavefunction is acceptable only if n is odd

d) $\psi(-\frac{L}{2}) = N\cos\left[-\frac{(2n+1)\pi}{2}\right] = N\cos\left[\frac{(2n+1)\pi}{2}\right]$

$\psi(\frac{L}{2}) = N\cos\left[\frac{(2n+1)\pi}{2}\right] = N\cos\left[\frac{(2n+1)\pi}{2}\right]$

} $= 0$ for $n = 1, 2, 3, 4 \ldots$

acceptable

(e) $\psi(x) = Ne^{i\theta}\sin\left(\frac{2n\pi x}{L}\right)$ → will obey boundary conditions, (see above) and yield identical values as $\sin\left(\frac{2n\pi x}{L}\right)$

$\theta = 45°$

acceptable.