Proofs of Theorems

We will make use of the following lemma, which we state without proof (Puterman 1994):

**Lemma 1.** Suppose \( \{v_i\} (i=1,\ldots,N) \) is a sequence of numbers, and \( \{p_i\} \) and \( \{q_i\} (i=1,\ldots,N) \) are two discrete probability distributions such that
\[
\sum_{i=k}^{N} q_i \geq \sum_{i=k}^{N} p_i \quad \text{for all } k \in \{1,\ldots,N\}.
\]
Then, if \( \{v_i\} \) is nondecreasing (nonincreasing),
\[
\sum_{i=1}^{N} q_i v_i \geq \left( \leq \right) \sum_{i=1}^{N} p_i v_i.
\]

**Proof of Theorem 1**

We prove this by induction. Let \( i \in \{0,1,\ldots\} \) represent iteration \( i \) of the value iteration algorithm, and let \( v^i \) be the resulting value vector of that iteration. We suppose that \( v^i(s) \) is nonnegative and nondecreasing in \( s \) (note that \( v^0(s) = 0 \) for all \( s \) satisfies this property for \( i = 0 \)). By this assumption and assumption \( \text{As2} \) (i.e., \( P \) is IFR), we apply Lemma 1 to obtain:
\[
\sum_{j \in S} p(j|s+1)v^i(j) \geq \sum_{j \in S} p(j|s)v^i(j) \geq 0 \quad \text{for all } s \in S.
\]
Combining this with assumption \( \text{As1} \) (i.e., \( r(s) \) is nonnegative and nondecreasing in \( s \) for all \( s \)), we have:
\[
 r(s+1) + \sum_{j \in S} p(j|s+1)v^i(j) \geq r(s) + \sum_{j \in S} p(j|s)v^i(j) \geq 0 \quad \text{for all } s \in S. \quad (A1)
\]
We also know by assumption \( \text{As1} \) that:
\[
 R(s+1) \geq R(s) \geq 0 \quad \text{for all } s \in S. \quad (A2)
\]
Combining (A1) and (A2) yields:
\[
 \max \left\{ r(s+1) + \sum_{j} p(j|s+1)v^i(j), \quad R(s+1) \right\} \\
 \geq \max \left\{ r(s) + \sum_{j} p(j|s)v^i(j), \quad R(s) \right\} \geq 0 \quad \text{for all } s \in S. \quad (A3)
\]
According to the value iteration algorithm,
\[
v^{i+1}(k) = \max \left\{ r(k) + \sum_{j} p(j|k)v^i(j), \quad R(k) \right\} \quad \text{for all } k \in S. \quad (A4)
\]
Therefore, (A3) and (A4) imply:

\[ v^{i+1}(s + 1) \geq v^{i+1}(s) \geq 0 \quad \text{for all } s \in S. \]

Thus, \( v^{i+1}(s) \) is nonnegative and nondecreasing in \( s \). Then, because \( \lim_{n \to \infty} v^n(s) = v^*(s) \), it follows that \( v^*(s) \) is nonnegative and nondecreasing in \( s \). \( \square \)

**Proof of Theorem 2**

If \( a^*(s) = I \) for all \( s \), then \( v^*(s) = R(s) \) for all \( s \). Furthermore, \( v^* \) must satisfy Bellman’s equations:

\[
 v(s) = \max \left\{ r(s) + \sum_j p(j|s)v(j), \; R(s) \right\} \quad \text{for all } s \in \{1, 2, \ldots, N\}, \text{ and} \\
 v(0) = 0,
\]

which implies:

\[
 R(s) \geq r(s) + \sum_j p(j|s)R(j) \quad \text{for all } s \in S.
\]

Now suppose

\[
 R(s) \geq r(s) + \sum_j p(j|s)R(j) \quad \text{for all } s \in S.
\]

Letting \( v(s) = R(s) \), we see that \( v \) satisfies Bellman’s equations given in (1). Therefore, \( v(s) = R(s) \) is an optimal value function which is achieved by the policy \( a^*(s) = I \) for all \( s \). \( \square \)

**Proof of Corollary 1**

Clearly \( v^*(s) \geq R(s) \) for all \( s \in S \). Therefore, if \( R(s') < r(s') + \sum_j p(j|s')R(j) \) it follows that \( R(s') < r(s') + \sum_j p(j|s')v^*(j) \) which implies that \( a^*(s') = W \), uniquely. \( \square \)

**Proof of Theorem 3**

As a consequence of Corollary 1, it is uniquely optimal to wait in state \( s \) if

\[
 mL(s) < 1 + \sum_j p(j|s)mL(j).
\]

This is equivalent to

\[
 L(s) < 1/m + \sum_j p(j|s)L(j),
\]
which is equivalent to

\[ \frac{1}{m} > L(s) - \sum_j p(j|s) L(j). \]

By the definition of \( \Delta_s \), the above is equivalent to:

\[ \frac{1}{m} > 1 + \Delta_s. \]

\( \square \)

**Proof of Corollary 2**

The sufficiency part of the proof follows directly from Theorem 3.

To prove necessity, suppose \( a^*(s_{\text{min}}) = W \), uniquely. Also, suppose (towards a contradiction) that \( 1/m \leq 1 + \Delta_{\text{min}} \). Then it follows that \( 1/m \leq 1 + \Delta_s \) for all \( s \). By the definition of \( \Delta_s \), we have:

\[ 1/m \leq L(s) - \sum_j p(j|s) L(j) \quad \text{for all } s \in S. \]

Multiplying through by \( m \) and rearranging terms yields:

\[ mL(s) \geq 1 + \sum_j p(j|s) m L(j) \quad \text{for all } s \in S. \]

By Theorem 2, this implies that \( a^*(s) = I \) for all \( s \), which contradicts our first assumption that \( a^*(s_{\text{min}}) = W \), uniquely. Therefore, \( 1/m > 1 + \Delta_{\text{min}} \). \( \square \)

**Proof of Theorem 4**

Let \( m_1 \leq m_2 \). We prove this by performing parallel iterations of the value iteration algorithm to solve for \( v_a(s|m_1) \) and \( v_a(s|m_2) \) for all \( s \). Suppose for some iteration, \( i \), of the algorithm \( v_a^i(s|m_1) \leq v_a^i(s|m_2) \) for all \( s \) (note that starting each problem with a vector of zeroes, satisfies this). Then for each \( s \geq 1 \) (the value associated with state \( s = 0 \) is always 0),

\[

d_{a}^{i+1}(s|m_1) = \max \left\{ 1 + \sum_j p(j|s) v_a^i(j|m_1), \ m_1 L(s) \right\}, \quad \text{and} \\
n_{a}^{i+1}(s|m_2) = \max \left\{ 1 + \sum_j p(j|s) v_a^i(j|m_2), \ m_2 L(s) \right\}.
\]

By the inductive assumption, \( 1 + \sum_j p(j|s) v_a^i(j|m_1) \leq 1 + \sum_j p(j|s) v_a^i(j|m_2) \) and by the assumption that \( m_1 \leq m_2, m_1 L(s) \leq m_2 L(s) \). Therefore, \( v_a^{i+1}(s|m_1) \leq v_a^{i+1}(s|m_2) \). Taking the limit of the value iterates proves the result. \( \square \)