Parameter Sensitivity in an Iterated Tikhonov Regularization Scheme

with Applications to Inverse problems

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Ill-posed problems

Solve for x: $\mathbf{Ax} = \bar{\mathbf{y}}$

- X, Y - Hilbert spaces
- $A : X \to Y$
- Problem data:

$$y = \bar{y} + \epsilon$$

$\bar{y} \in \text{Range}(A)$, $\epsilon = \text{noise}$

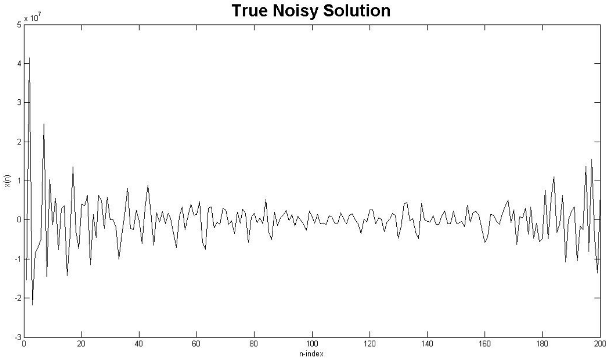
Ill-posed if ...

- NO solution
- Solution NOT unique
- Solution does not NOT depend continuously on data

Ill-posed problems

Does $x_\epsilon := A^{-1}(\bar{y} + \epsilon) \approx x_{true} := A^{-1}\bar{y}$???

noise $\approx 10^{-10}\%$



NO!

Prior Developments:

Motivation for our Research Efforts:

- (1948, 1963) A.N. Tikhonov
  Tikhonov Regularization

- (1979) J.T. King, D. Chillingworth
  Iterated Tikhonov (model formulation, error bounds)

- (1981, 1987) H. W. Engl
  Iterated Tikhonov (consistency results)

- Current Applications: Turbulence Modeling, Coupled Problems, etc.

Iterated Tikhonov Regularization

Definition

- *Fix* $\alpha > 0$, *number of update steps* J
- *Approximate* x_{true} *by* x_j, $j = 1, 2, \ldots, J$

$$(A^*A + \alpha I)x_0 = A^*y$$
$$(A^*A + \alpha I)x_j = A^*y + \alpha x_{j-1}$$

Descent properties of Iterated Tikhonov

Theorem

- x_j: *minimizing sequence of noisy functional*

$$J_\epsilon(x) := \frac{1}{2}(A^*Ax, x)_X - (A^*(\bar{y} + \epsilon), x)_X$$

- *Suppose* $||\epsilon||_X \leq \epsilon_0$ *and*

$$\alpha \geq \frac{\epsilon_0}{||x_{j+1} - x_j||_X},$$

x_j: *"minimizing sequence" of noise-free functional*

$$J_0(x) := \frac{1}{2}(A^*Ax, x)_X - (A^*\bar{y}, x)_X$$

Parameter Sensitivity

Definition

The sensitivity of the iterated Tikhonov updates is

$$s_j(\alpha) = \frac{dx_j(\alpha)}{d\alpha}$$

Applications: Uncertainty assessment, optimization, stopping criterion, etc.

Parameter Sensitivity - Shaw problem

Example - Fredholm Integeral Equation of the 1st Kind
(ref. Shaw, Hansen)

$$\int_{-\pi/2}^{\pi/2} \kappa(t,\tau) x(\tau) dt = \bar{y}(\tau), \quad -\pi/2 \leq \tau \leq \pi/2,$$

$$\kappa(t,\tau) = (cos(t) + cos(\tau))^2 \left(\frac{sin(\lambda)}{\lambda} \right)^2$$

$$\lambda = \pi(sin(t) + sin(\tau))$$

Parameter Sensitivity - Shaw problem

Problem Specifications

- Discretize with 200 nodes: $A \in \mathbb{R}^{200\times 200}$
- $\bar{y} \in \mathbb{R}^{200}$ chosen so that solution x_{true} is sum of 2 Gaussian functions
- Added noise $\epsilon \approx 0.1\%$
- Discrete problem:

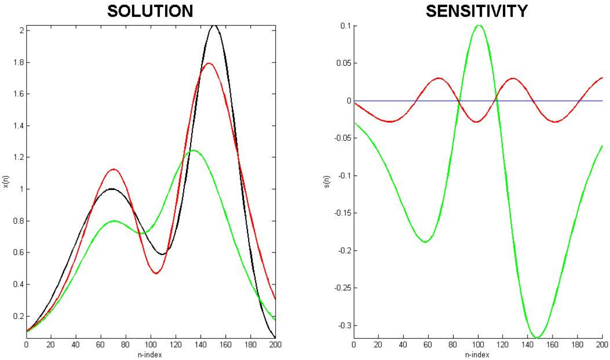
$$Ax = y = \bar{y} + \epsilon$$

condition number $\approx 5 \times 10^{19}$

Parameter Sensitivity - Numerical Results

$\alpha = 1.00$, $J = 80$

— $= x_{true}$, — $= x_0$, — $= x_J$

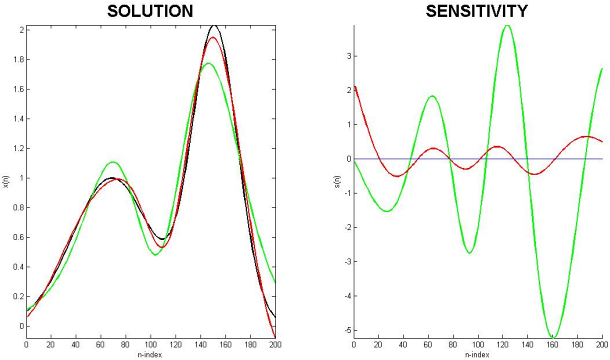


$s_0(min) \approx -0.3167$, $s_0(max) \approx 0.1008$
$s_J(min) \approx -0.0289$, $s_J(max) \approx 0.0301$

Parameter Sensitivity - Numerical Results

$\alpha = 0.01$, $J = 80$

— $= x_{true}$, — $= x_0$, — $= x_J$



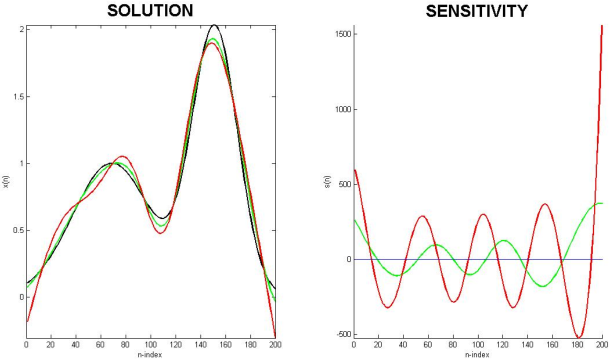
$s_0(min) \approx -5.2353$, $\quad s_0(max) \approx 3.9044$

$s_J(min) \approx -0.5184$, $\quad s_J(max) \approx 2.1215$

Parameter Sensitivity - Numerical Results

$\alpha = 0.00001$, $J = 80$

— $= x_{true}$, — $= x_0$, — $= x_J$



$s_0(min) \approx -1.822E + 2, \quad s_0(max) \approx 3.725E + 2$

$s_J(min) \approx -5.252E + 2, \quad s_J(max) \approx 1.562E + 3$

Parameter Sensitivity - Algorithm

Theorem

Equivalent sensitivity-based algorithm

- *Solve for* x_0, s_0

$$(A^*A + \alpha I)x_0 = A^*y$$
$$(A^*A + \alpha I)s_0 = -x_0$$
$$\Rightarrow \textit{Update, } x_1 = x_0 - \alpha s_0$$

- *For* $j = 1, 2, \ldots, J$

  Solve for s_j

$$(A^*A + \alpha I)s_j = -(x_j - x_{j-1}) + \alpha s_{j-1}$$

  If $j \neq J$

$$x_{j+1} = x_j - \alpha s_j + \alpha^2 (A^*A + \alpha I)^{-1} s_{j-1}$$

Parameter Sensitivity

Definition

The k-th order Taylor expansion of $x_0(\alpha)$ is

$$T_k(x_0(\alpha)) := x_0(\alpha) - \alpha x_0'(\alpha) + \ldots + \frac{(-\alpha)^k}{k!} x_0^{(k)}(\alpha)$$

Theorem

- *For noise-free problem*

$$x_{true} := x_0(\alpha = 0) = T_k(x_0(\alpha)) + O(\alpha^{k+1})$$

- *For noisy problem,*

$$x_J(\alpha) = T_J(x_0(\alpha))$$

Conclusion

Current Research:

- Parameter optimization in terms of s_j's
- Stopping criterion in terms of s_j's
- Applications to "real-world" inverse problems (imaging, fluids, etc.)