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The Obscurity of the Equimultiples

Clavius' and Galileo's Foundational Studies of Euclid's Theory of Proportions

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Contents

Introduction

In this paper, I analyse a few aspects of the relationship between Galileo's and Clavius' foundational studies of the Euclidean theory of proportions. As is well known, Euclid furnishes two definitions of proportionality:

1) a general definition, valid for any magnitudes whatever, which is based on the notion of equimultiples; let us call it 'equimultiple proportionality';
2) a more restricted definition, valid for numbers, which is based on the notion of 'being the same multiple, or the same part, or the same parts'; let us call it 'rational proportionality'.

Galileo believed that equimultiple proportionality was an obscure notion and therefore proposed to replace it with a new definition of proportionality. In 1641, he began to dictate a tract on proportions to the young mathematician, Evangelista Torricelli, who assisted him in Arcetri during his terminal illness. The task Galileo had set himself was extremely complex. At the beginning of his tract, he recognised that he would either have to demonstrate the entire Fifth Book of Euclid's *Elements* with his new definition or prove that he could deduce Euclid's equimultiple proportionality from his new definition of proportionality. He chose the second approach and devised a set of four theorems in order to show that Euclid's equimultiple proportionality was indeed deducible from his new definition. Let us call Galileo's four-theorem model '*Galileo's model of equivalence*'.

In his massive commentary on Euclid's *Elements*, first published in 1574, Christoph Clavius (1538–1612), one of the most highly respected mathematicians of the late sixteenth century, devoted numerous sections to the study of proportions. Clavius, too, for different reasons, was concerned with the 'obscurity' of equimultiple proportionality.

Why, Clavius asks, did Euclid choose to base the general definition of proportionality on the equimultiples? Because, he answers, rational proportionality applies only to rational ratios. According to Clavius, to define proportionality between irrational ratios, Euclid had no alternative but to find a property that was clearly discernible in proportions between rational ratios and extend it to irrational ratios. The property that, according to Clavius, Euclid found and boldly extended to irrational ratios is the perpetual accord of the equimultiples required by the definition of equimultiple proportionality. But Euclid did not prove that proportional rational ratios satisfy this property, i.e. that ratios proportional in the sense of rational proportionality respect the perpetual accord set forth in the definition of equimultiple proportionality. Neither did he prove the converse of the former proposition, i.e. that if the accord of the equimultiple is respected then rational proportionality holds true. These proofs are furnished by Clavius. In particular, he demonstrates four theorems, by means of which he shows that, in the case of proportional rational ratios, Euclid's two definitions of proportionality are consistent with each other, i.e. that ratios proportional according to *rational proportionality* are also proportional according to *equimultiple proportionality*, and vice versa. Let us call Clavius' four-theorem model '*Clavius' model of equivalence*'.

I will show that the mathematical structure, the purpose, and a few linguistic features of *Galileo's model of equivalence* are similar to those of *Clavius' model of equivalence* and that the second and third theorems of both models have analogous characteristics concerning structure and language. These similar features strongly suggest that Galileo's model of equivalence was influenced by Clavius' model of equivalence. I will also show some of the difficulties encountered by Galileo in applying the equimultiple technique outside of the domain of pure mathematics, and which may eventually have led him to seek an alternative to Euclid's equimultiple proportionality.

Over the past thirty years or so, studies by many scholars have demonstrated the importance of the notion of proportionality for Galileo's 'new' mathematical natural philosophy and, more specifically, the importance of his 'reform' of the Euclidean theory of proportions for the subsequent developments of the Galilean school in the seventeenth century.[1] Thus, the influence that Clavius' foundational approach to proportions exerted on Galileo casts new light on Clavius himself as mathematician. Both his 'rational' reconstruction of Euclid's extension of the accord of the equimultiples to irrational ratios and his insight into the lack of internal 'completeness' in Euclid's general treatment of proportions may have been a source of inspiration for Galileo's re-thinking of equimultiple proportionality. As we shall see in the final section, Galileo first used the equimultiple technique in his early *De Motu* (*ca*. 1592). Then, he apparently abandoned the equimultiples only to resume them in *Two New Sciences*. It is quite possible that Clavius' concern with Euclid's unproven extension of the equimultiple property to irrational ratios en-

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[1] Cf. Drake 1973, 1974a, 1974b, 1987, Giusti 1986 and 1993, Palladino 1991. For a general treatment of the various aspects of the Euclidean theory of proportions I have relied on: Grattan-Guinnes 1996, Sasaki 1985, Saito 1986 and 1993. Rose 1975 is the best survey of Renaissance Italian mathematics from a non-technical point of view. Cf. also Sylla 1984, pp. 11–43.

couraged Galileo to seek a totally new alternative to equimultiple proportionality and that Galileo's approach was subsequently transmitted to the Galilean school.[2]

I now give an outline of the paper.

In Sect. 1, I will discuss Clavius' interpretation of Euclid's two definitions of proportionality and Clavius' model of equivalence. I will subsequently draw on this discussion to show the partial dependence of Galileo's model of equivalence on Clavius' model of equivalence. Section 1 consists of an in-depth analysis of Clavius' model of equivalence. Sections 2 and 3, will be devoted to a general analysis of Galileo's reform of the theory of proportions and to showing the details of the partial dependence of Galileo's model of equivalence on Clavius' model of equivalence. Finally, Sect. 4 will analyse the difficulties that Galileo encountered when he applied the equimultiple technique outside of the domain of pure mathematics and briefly draw some conclusions.

1. Euclidean proportionality and Clavius' model of equivalence

Christoph Clavius published the first edition of his commentary on Euclid's *Elements* in 1574.[3] The book enjoyed wide popularity and was published many a time during Clavius' lifetime, for example, in 1589, 1591, 1603, 1607, and finally as the first volume of his *Opera Mathematica*.[4] There were also a few posthumous editions. In the second edition, in 1589, Clavius added considerable new material to his commentary, especially regarding the theory of proportions. In the second edition, he published his *model of equivalence* for the first time. Given that the contents of the book remained virtually unchanged in all subsequent editions, we can assume that Clavius considered the version of his commentary on Euclid published in 1589 as the definitive one.[5] A brief elucidation of the notions of ratio and sameness of ratios precedes the presentation of the model of equivalence.

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[2] Giusti 1993 expounds various reforms of Euclid's theory of proportions proposed by Galileans, the most important of which are those by Evangelista Torricelli and Giovanni Alfonso Borelli. Palladino 1991 is mainly concerned with Giovanni Alfonso Borelli's theory.

[3] Cf. Clavius 1574a and 1574b. The first volume contains the first nine books and the second the remaining five plus a book by François de Foix (Clavius agreed with those who thought that the fourteenth and the fifteenth books were not Euclid's but Hypsicles'). Cf. Clavius 1999, p. 559.

[4] Clavius 1611–1612. I have quoted Clavius's commentary from Clavius 1999, which is a facsimile edition of the first volume of Clavius' *Opera Mathematica*.

[5] The place of Clavius in the history of the renaissance of mathematics in the late sixteenth century has to be reassessed. I believe that this will not be possible until a thorough investigation of the long and tortuous history (both philological and conceptual) of Euclid's *Elements* in the sixteenth century has been produced. Cf. especially Giusti 1993, pp. 4–12. Contributions devoted to Clavius' mathematics in general are Homann 1980, 1983. Knobloch 1995 and Garibaldi 1995 have investigated the interesting problem of the quadratrix. For biographical data cf. Knobloch 1988 and 1990. Finally, see the three-part article by Naux, 1983a, 1983b, 1983c. As regards specifically the theory of proportions, Clavius' commentary is also mentioned a few times by Giovanni Vailati in his article *Sulla Teoria delle Proporzioni*, in Enriques 1983, II, pp. 143–191. On the fifth postulate, cf. Maierù 1978 and 1982. On the question of the 'angle of contact', cf. Maierù 1990, pp. 115–137. In general, on Clavius, cf. Lattis 1994, Baldini 1992. Baldini also published previously unknown manuscript material on the question of the *Theorica solis*, probably part of an ampler work on the *Theoricae planetarum*. Cf. *ibid.*, pp. 469–564. Finally, cf. the recent collection of essays edited by Ugo Baldini, in Baldini 1995.

First of all, according to Clavius' rendering of Euclid's text:

> Ratio is a certain mutual habitude [*habitudo*] between two magnitudes of the same kind with respect to quantity.[6]

For Clavius, examples of two quantities of the same kind are numbers, lines, surfaces, and solids. The meaning of the expression 'habitude. . .with respect to quantity' is clarified in Clavius' commentary as follows:

> When two quantities [. . .] of the same kind are compared to each other with respect to quantity, i.e. with respect to the fact that one is greater than, less than, equal to, the other, this comparison is called either mutual habitude, or ratio, or, according to others, proportion.[7]

Clavius notes here that terminology may vary and that the use of 'ratio' and 'proportion' is still common in his time to indicate what we call 'ratio'. In Clavius' time, terminology was not yet well established. *Proportio* and *ratio* were commonly used to indicate what we call 'ratio', whereas *proportionalitas* was used to indicate 'proportion'. Clavius normally uses *proportio* for our 'proportion' and *ratio* for our 'ratio'.

A ratio, Clavius says, can be found properly in quantities only, though there are other things that partake of the nature of quantity, such as time, sound, voice, place, motion, weight and power [*potentia*], and they can be said to form ratios if one considers their habitude with respect to quantity. On the other hand, there are other things such as, for example, lines, that may form ratios when their habitudes are taken with respect to quantity, but may not form ratios when their habitudes are taken with respect to quality – e.g. when lines are compared to one another with respect, for example, to colour, or heat.[8]

Having given Euclid's definition of ratio, Clavius goes on to present Euclid's definition of proportion.

> Proportion is similarity between ratios.[9]

Instead of *proportionalitas*, *proportio* is here preferred by Clavius. He comments that exactly in the same way in which a comparison of two quantities with each other is called a ratio, a comparison of two or more ratios with each other is called a proportion. Therefore, according to Clavius,

> [. . .] if the ratio of quantity A to quantity B is similar to the ratio of quantity C to quantity D, then the habitude between these ratios is called proportion.[10]

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[6] "Ratio est duarum magnitudinum eiusdem generis mutua quaedam secundum quantitatem, habitudo". Clavius 1999, p. 167.

[7] "Quando duae quantitates eiusdem generis [. . .] inter se comparantur secundum quantitatem, hoc est secundum quod una maior est, quam altera, vel minor, vel aequalis; appellatur huiusmodi comparatio, seu habitudo mutua, Ratio seu (ut aliis placet) Proportio". Clavius 1999, p. 167.

[8] Clavius 1999, p. 167.

[9] "Proportio vero est rationum similitudo". Clavius 1999, p. 167.

[10] "[. . .] si proportio quantitatis A, ad quantitatem B, similis fuerit proportioni quantitatis C, ad quantitatem D, dicetur habitudo inter has proportiones, proportionalitas". Christoph Clavius 1999, 167. To avoid confusion, I have translated consistently *proportionalitas* and *proportio* with

But what does ‘similarity of ratios’ actually mean? All Clavius says, is that similarity between ratios means a comparison between ratios and that a proportion is a habitude between ratios. A figure shows that similarity may have a visual connotation.[11]

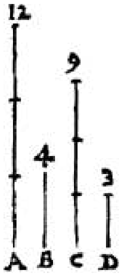

Fig. 1. An example of similarity between lines

In Fig.1, we have four lines A, B, C, D, to which four numbers are attached. Note that the ratio of quantity A to B is similar to the ratio of quantity C to D in a visual sense. The numbers are given for the sole purpose of specifying what the reader’s eye could not appreciate with sufficient precision, namely, that the lines have to one another a very precise *habitudo*.

Euclid’s definition of sameness of ratios is couched by Clavius in the following words:

> Magnitudes are said to be in the same ratio, the first to second, and the third to the fourth, when the equimultiples of the first and the third, both alike equal, alike exceed, or alike fall short of, the equimultiples of the second and the fourth – whatever this multiplication may be – and those equimultiples are being considered [compared] that correspond to each other.[12]

This is the Euclidean definition of equimultiple proportionality. According to Clavius’ comments, Euclid has recourse to the notion of *equimultiples* – i.e. equal *multiples* of two magnitudes – in order to furnish a definition of sameness of ratios that encompasses both rational and irrational ratios. Having proposed his rendering of Euclid’s definition, Clavius goes on to examine the meaning of the technique of equimultiples.

First of all, what are *equimultiples of magnitudes*?

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proportion, even though Clavius here uses both *proportio* and *proportionalitas* with the meaning of proportion.

[11] On the interesting aspects of visualization in the mathematics and natural philosophy of the sixteenth and seventeenth centuries literature is very scant. For a sketchy attempt to understand the role of graphic representation in the mathematization of nature, cf. John J. Roche, “The Semantics of Graphics in Mathematical Natural Philosophy”, in Mazzolini 1993, pp. 197–233.

[12] “In eadem ratione magnitudines dicuntur esse, prima ad secundam, et tertia ad quartam, cum primae et tertiae aeque multiplicia, a secundae et quartae aeque multiplicibus, qualiscunque sit haec multiplicatio, utrumque ab utroque vel una deficiunt, vel una equalia sunt, vel una excedunt; si ea sumantur quae inter se respondent”. Clavius 1999, p. 209.

The notion of *equimultiples of magnitudes* is not defined by Euclid. Euclid simply defines the notion of *multiple*, in Definition 2, Book V. Clavius' interprets it as follows:

> The greater part [of a magnitude] is multiple of the less part [of the magnitude] when the less part measures the greater part.[13]

Commenting on this definition, Clavius points out that 'part' of a magnitude (defined by Euclid, in Definition 1, Book V, as a magnitude that measures another magnitude) is to be conceived of as a part that perfectly [*perfecte*] measures the magnitude of which it is part. This rules out the possibility, Clavius says, that part of a magnitude may be *any* part of it, which would have the awkward consequence that 7 would be multiple of 6, since 6, being part of 7, would measure it. Thus, in Clavius' view, 6 cannot be considered part of 7 in the sense of *part* defined by Euclid. Concluding his remarks on Euclid's Definition 2, Clavius' proposes his own definition of *equimultiples of magnitudes*.

> When two lesser magnitudes equally measure two greater magnitudes, i.e. one lesser [magnitude] is contained in one greater [magnitude] as many times as the other smaller [magnitude] is contained in the second greater [magnitude], these two greater [magnitudes] are called equimultiples of the two less [magnitudes].[14]

Clavius now goes on to clarify the workings of the equimultiples. Let us consider four magnitudes A, B, C, D, (Fig. 2).

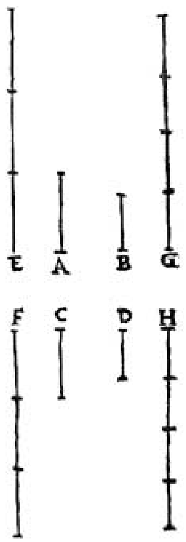

Fig. 2. Equimultiples E, F, G, H, of lines A, C, B, D

[13] "Multiplex autem est maior minoris, cum minor metitur maiorem". Clavius 1999, p. 166.

[14] "[. . .] quando duae magnitudines minores duas alias maiores aeque metiuntur, hoc est, una minor in una maiore toties continetur, quoties altera minor in altera maiore; dicuntur duae hae maiores duarum illarum minorum aequemultiplices". Clavius 1999, p. 167.

Let E and F be any equimultiples whatever of the first, A, and third, C, respectively. In the same manner, let G and H be any equimultiples whatever of the second magnitude B, and the fourth, D, respectively. These may be either the same equimultiples as the former ones or different. Now, if one compares the equimultiples corresponding to each other, that is, (E, F), and (G, H), and finds that they are always such that:

a) when E, multiple of A, is less than G, multiple of B, then F, multiple of C, is less than H, multiple of D;
b) when E, multiple of A, is equal to G, multiple of B, then F, multiple of C, is equal to H, multiple of D;
c) when E, multiple of A, is greater than G, multiple of B, then F, multiple of C, is greater than H, multiple of D;

then, if it is not possible to find some equimultiple that does not satisfy the above three conditions, one may conclude that the ratio of the first magnitude, A, to the second magnitude, B, is the same as the ratio of the third magnitude, C, to the fourth magnitude, D.[15] A proof may directly use this definition of sameness of ratios in order to prove that four magnitudes have the same ratio. To do so, it must show that any equimultiples whatever of the first and third magnitudes are always in accordance with conditions a, b, c, when they are compared with any equimultiples whatever of the second and fourth magnitudes.

We now need to examine the definition of what we have called 'rational proportionality'. According to Clavius' extension of Euclid's definition 20, Book VII, to ratios of rational quantities, (not necessarily numbers),

> four magnitudes are proportional when the first is the same multiple of the second, or the same part, or the same parts that the third is of the fourth; or certainly, as we have added to the definition, when the first contains the second, or the second plus a part of it, or parts of it, in a way exactly equal to that in which the third contains the fourth, or the fourth plus a part of it, or parts of it.[16]

This is the formal definition of 'rational proportionality'. So, Clavius comments, Euclid might have chosen to define proportional magnitudes in this way, had he been concerned with rational ratios only, since all rational ratios can be 'exhibited' in numbers. For a ratio between numbers is a habitude, a certain *habitudo* of a number to another number. But incommensurable magnitudes form irrational ratios, and therefore there cannot be a part common to the two magnitudes.[17] Therefore, according to Clavius, Euclid had to excogitate a mechanism whereby he could investigate the comparison between ratios of incommensurable magnitudes. He turned to ratios of commensurable magnitudes, that is, to ratios of numbers – since every rational ratio can be reduced to a ratio between numbers, as is proven in proposition 5 of Book X. In other words, according to Clavius,

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[15] Clavius 1999, p. 209.

[16] "Magnitudines proportionales sunt, cuma prima secundae, et tertia quartae, aequimultiplex est, vel eadem pars, vel eaedem partes; vel certe (ut nos ad eam defi. Addidimus) cum prima secundam, et tertia quartam, aequaliter, continet, eademque insuper illius partem, vel easdem partes." Clavius 1999, p. 211.

[17] Clavius 1999, p. 211.

Euclid went back to the case of commensurability because he had to investigate a property that was clearly applicable to both numbers and commensurable magnitudes forming proportions, in which case, according to Clavius, similarity is clearly discernible.

If Euclid had found that a property applicable to the case of proportional commensurable magnitudes could somehow be extended to incommensurable magnitudes, then he would have had the right to conclude that that property was the the criterion of proportionality even in the case of incommensurability. According to Clavius, the perpetual accord of the equimultiple technique is the property that Euclid eventually found.[18] Here Clavius tries to make sense of the fact that ratios of rational quantities show openly their similarity, or dissimilarity, and consequently, he tries to justifies the extension of this property to irrational ratios. But he could find no proof in Euclid of the validity of this extension.

In other words, there is no proof in the text of the *Elements* that proportional numbers (according to Book VII, definition 20) are in accordance with the general definition of sameness of ratios based on the equimultiple technique given in Book V.[19] There is no *Euclidean* model of equivalence.

We summarise the structure of Clavius' model of equivalence in Table 1, a schema of the set of four theorems that Clavius proves.[20] Let

Eq=	be the equimultiple definition,
Eq>	be the equimultiple definition of 'greater ratio' according to Euclid,[21]
D20VII=	be the definition of proportional numbers according to definition 20, Book VII,
D20VII>	be the definition of greater ratio according to Clavius completion of definition 20, Book VII,[22]
➔	be the implication symbol (i.e. *then*).

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[18] Clavius 1999, 211–212.

[19] Thomas Heath notes that 'Euclid has nowhere proved (though the fact cannot have escaped him) that the proportion of numbers is included in the proportion of magnitudes as a special case. This is proved by [Robert] Simson as being necessary to the 5th and 6th propositions of Book X'. Heath then gives the proof furnished by Simson without mentioning Clavius' commentary. It is noteworthy that Clavius' search for an explanation of the perpetual accord of the equimultiples had to a large extent anticipated Simson, more than one hundred and fifty years earlier. See Heath's comments in Euclid 1956, II, pp. 126–128.

[20] Clavius 1999, p. 212ff.

[21] In Clavius' version this definition is as follows: 'When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second [magnitude], but the multiple of the third does not exceed the multiple of the fourth, then the first [magnitude] is said to have to the second [magnitude] a greater ratio than the third [magnitude] has to the fourth [magnitude]'. In Latin: "Cum vero aeque multiplicium, multiplex primae magnitudinis excesserit multiplicem secundae; at multiplex tertiae non excesserit multiplicem quartae; tunc prima ad secundam maiorem rationem habere dicetur, quam tertia ad quartam". Clavius 1999, p. 210.

[22] Commenting definition 20, Book VII, Clavius says that: 'the ratio of a greater number to the same [number] is greater than [the ratio of] a smaller number [to the same number]. And contrariwise [the ratio of] the same [number] to a smaller [number] is greater than [the ratio of] of a greater [number] [to the same number] [. . .]. This is clear if one understands correctly this definition [i.e. Def. 20, Book VII]'. In Latin: "maioris numeri ad eundem, maiorem proportionem esse, quam

Thus, for example, '**D20VII= ➔**' should be read as 'definition of proportional numbers according to definition 20, Book VII, implies ...'. Here is Clavius' model of equivalence.

Table 1. The set of four theorems forming Clavius' model of equivalence

Theorem I	four proportional numbers according to **D20VII= ➔** four proportional numbers according to **Eq=**
Theorem II	greater ratio for numbers according to **D20VII> ➔** greater ratio for numbers according to **Eq>**
Theorem III	if four numbers obey **Eq= ➔** they obey **D20VII=**
Theorem IV	greater ratio for numbers according to **Eq> ➔** greater ratio for numbers according to **D20VII>**

Clavius' next step is to prove that proportional numbers in the sense of rational proportionality – and proportional commensurable magnitudes in the sense of rational proportionality – satisfy conditions a, b, c, as set forth in the equimultiple definition. This important proof shows that *rational proportionality* is in perfect accordance with *equimultiple proportionality*. We now examine Clavius' Theorem I.

Clavius' statement of Theorem I.
Given four proportional numbers, let equimultiples of the first and third be taken according to whatever multiplication, and, again, let equimultiples of the second and fourth be taken according to whatever multiplication; if the multiple of the first is greater than the multiple of the second, then the multiple of the third will be greater than the multiple of the fourth; if the multiple of the first is equal to the multiple of the second, then the multiple of the third will be equal to the multiple of the fourth; and finally, if the multiple of the first is less than the multiple of the second, then the multiple of the third will be less than the multiple of the fourth.[23]

Clavius' proof of the theorem depends upon a few propositions regarding numbers that are given in Book VII, but, as Clavius notes, since these propositions are independent of both any propositions proved in Book V and of the general definition of proportionality

minoris. Et contra, eiusdem ad minorem, maiorem esse proportionem, quam ad maiorem. [...] Quae omnia perspicuae sunt, si recte haec definitio intelligatur". Clavius 1999, p. 311.

23 "Propositis quatuor numeris proportionalibus, sumptisque primi ac tertii aeque multiplicibus iuxta quamvis multiplicationem, item secundi et quarti aeque multiplicibus iuxta quamcunque etiam multiplicationem; si multiplex primi maior sit multiplice secundi, erit quoque multiplex tertii maior multiplice quarti; et si multiplex primi aequalis sit multiplici secundi, erit quoque multiplex tertii aequalis multiplici quarti; si denique multiplex primi sit multiplice secundi minor, erit et multiplex tertii multiplice quarti minor". Clavius 1999, p. 212.

based on the equimultiple technique, they can be assumed as given since they could be easily proved before Book V.[24] The proof proceeds as follows:

> Let a number, A, have to a second number, B, the same ratio that a third number, C, has to a fourth number, D. Let E, F be equimultiples of the first, A, and third, C, and G, H equimultiples of the second, B, and fourth, D, according to whatever multiplication. I say that if E, multiple of the first, A, is greater than G, multiple of the second, B, then F, multiple of the third, C, will be greater than H, multiple of the fourth, D; if E is equal to G, then F will be equal to H; and finally, if E is less than G, then F will be less than H. For, since A is to B as C is to D, by alternating, A is to C as B is to D. Also, as A is to C so E is to F, since the same number multiplying A, C has produced E, F, inasmuch as E, F have been taken as equimultiples of A, C. For the same reason, G is to H as B is to D. Therefore, by the lemma given as a premise to proposition 14, in Book 7,[25] G is to H as E is to F. By alternating, F is to H as E is to G. Thus, if E is greater than G, then F will be greater than H, as in the first example. If E is equal to G, then F will be equal to H, as in the second example. Finally, if E is less than G, then F will be less than H, as in the third example. Q.E.D.[26]

The examples referred to in the final part of Clavius' text are given in the following table (Table 2).

Table 2. Clavius' examples

E 9		A 3	C 6		F 18
G 4		B 2	D 4		H 8
E 18		A 3	C 6		F 36
G 18		B 2	D 4		H 36
E 12		A 3	C 6		F 24
G 14		B 2	D 4		H 28

[24] *Ibid.*

[25] The lemma here referred to asserts that 'It can be proved that two ratios of numbers that are the same to a third ratio of numbers are also the same to each other', in other words, two ratios equal to a third one are equal to each other. The Latin has: 'Quod autem duae proportiones numerorum, quae eidem proportioni eaedem sunt inter se quoque sint eaedem [. . .] ita demonstrabitur'. [Clavius 1999, 324].

[26] "Habeat numerus primus A, ad secundum B, eandem proportionem, quam tertius C, ad quartum D; sumanturque primi A, et tertii C, aequemultiplices E, F: Item secundi B, et quarti D, aequemultiplices G, H, qualisqunque haec multiplicatio sit. Dico si E, multiplex primi A, maior est quam G, multiplex secundi B, maiorem quoque esse F, multiplicem tertii C, quam H, multiplicem quarti D. Et si E, aequalis sit ipsi G, aequalem quoque esse F, ipsi H. Si denique E, minor sit quam G, minorem quoque esse F, quam H. Quoniam enim est, ut A, ad B, ita C ad D; erit permutando etiam, ut A, ad C, ita B, ad D: ut autem A, ad C, ita est E, ad F; quod idem numerus ipsos A, C, multiplicans produxerit ipsos E, F, quippe cum E, et F, ipsorum A, et C, sunti sint

Clavius has now demonstrated that *rational proportionality* – i.e. proportionality in numbers and commensurable quantities – is in accordance with the property set forth by conditions a, b, c, in the equimultiple technique.

Clavius' proof that numbers proportional according to definition 20, Book VII, are also proportional according to the more general definition based on the equimultiple technique is a brilliant piece of foundational work. Clavius' model of equivalence is complete in the sense that he is also able to prove the inverse of the proposition given above.

We now focus on Theorem II and Theorem III. As will be clear later on, when we discuss Galileo, Theorem IV is not relevant for our purposes.

The complexity of Clavius' proof of Theorem II depends mainly on the fact that, since he works with integers, he has to manage to prove some properties of the ratios he considers without the simplification offered by modern algebraic symbolism. A second difficulty lies in the fact that any arithmetical concept he uses has to be based on Euclid's so-called arithmetical books, i.e. Books VII, VIII, and IX. It must be borne in mind that operations like addition, subtraction, multiplication, and division were normally explained in texts devoted to practical arithmetic, but they were not justified in terms of rigorous proofs. Theoretical, i.e. rigorous, elementary arithmetic, for the most part amounted to what Euclid had furnished in his arithmetical books. For example, in order to prove that a number was greater than another number one had to prove that the ratio of the former number to a third one was greater than the ratio of the latter number to the same third one. As we shall see, by replacing integers with continuous magnitudes, Galileo was able to simplify Clavius' proof significantly. But this he achieved at the cost of mathematical constructibility.

We divide Clavius' proof into six parts in order both to understand better its workings and to show how Galileo modelled the proof of his own Theorem II after Clavius. We first furnish the complete text and then analyse it. Table 3 contains two numerical examples that Clavius proposes, and which accompany his proof. Clavius' table is important because it shows a spatial disposition of numbers that may have had a role in suggesting to Galileo how to derive from it the diagram and the general pattern of his own proof. Even more importantly, this disposition shows the basic fact of his innovative proposal consisting in the idea that from a magnitude *greater than it need be* to form a ratio equal to another ratio a part may be cut in order to make it *exactly what it must be* for that ratio to be the same as the other one (cf. section 3). As we shall see in a moment, number A, forming ratio A to B, is decomposed by Clavius in numbers F, G, in such a way that F is exactly *what it must be* for the ratio of F to E to be the same as the ratio of C to D. In other words, in such a way that F is to E as C is to D.

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aequemultiplices. Et eadem de causa, ut B, ad D, ita est G, ad H. Igitur ex lemmate propos. 14, lib. 7, erit quoque, ut E, ad F, ita G, ad H. Et permutando, ut E, ad G, ita F, ad H. Quocirca si E, maior est quam G, erit quoque F, maior quam H, ut in primo exemplo.Si vero E [the text has F, which is a misprint for E], aequalis est ipsi G, erit quoque F, ipsi H, aequalis, ut in secundo exemplo: si denique E, minor est quam G, erit quoque F, minor quam H, ut in tertio exemplo. Quod erat demonstrandum". Clavius 1999, pp. 212–213.

Table 3. Clavius' two numerical examples

		Ex. 1		
K 56	H 7		O 21	I 28
F 8	G 1		**A 3**	**C 4**
	E 6		**B 2**	**D 3**
M 54	N 6		P 20	L 30
		Ex. 2		
K 160	H 440		O 60	I 20
F 8	G 22		**A 3**	**C 1**
	E 80		**B 8**	**D 10**
M 160	N 80		P 24	L 30

Clavius' statement of Theorem II.
Given four non-proportional numbers [*according to Clavius' completion of the definition of proportional numbers, i.e. Definition 7, Book VII*], in such a way that the ratio of the first to the second is greater than [the ratio] of the third to the fourth, if equimultiples of the first and third are taken, and equimultiples of the second and fourth [are also taken], it can happen that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth.

Proof. Part I.[27]
Let the first number, A, have to the second number, B, a greater ratio than the third, C, has to the fourth, D. I say that taking equimultiples of the first, A, and third, C, and equimultiples of the second, B, and fourth, D, it can happen that the multiple of the first, A, is greater than the multiple of the second, B, while the multiple of the third, C, is not greater than the multiple of the fourth, D. Let the multiplication of numbers B, D, yield E. Since D measures E, by axiom 7, Book VII, and E measures [the number] generated by C multiplied into E, by the same axiom, then D too

[27] "Habeat primus numerus A, ad secundum B, maiorem proportionem, quam tertius C, ad quartum D. Dico fieri posse, ut sumptis aequemultiplicibus primi A, ut tertii C, item aequemultiplicibus secundi B, et quarti D, multiplex ipsius A, primi sit maior quam multiplex ipsius B, secundi, at multiplex ipsius C, tertii maior non sit, quam multiplex ipsius D, quarti, multiplicantes enim se mutuo numeri B, D, faciant E. Et quoniam D, metitur E, ex pronunciato 7 lib. 7 et E, metitur genitum ex C, in E, ex eodem pronunciato; metietur quoque D, eundem genitum ex C, in E, ex pronunciato 11 lib. 7. Metiatur D, genitum ex C, in E, per F; fietque propterea ex D, in F, numerus, quem D, per F, metitur, ex pronunciato 9, libr. 7, hoc est numerus idem, qui fit ex C, in E. Quia igitur idem numerus gignitur ex D, primo in F, quartum, qui ex C, secundo fit in E, tertium, [19 septimi] erit ud D, primus ad C, secundum, ita E, tertius ad F, quartum: Et convertendo, ut C, ad D, ita F ad E." Clavius 1999, p. 213.

measures the same [number] generated by C multiplied into E, by axiom 11, Book VII. Let D measure [the number] generated by C multiplied into E by [number] F; consequently, a number is generated by D multiplied into F which will measure D by F, according to axiom 9, Book VII, that is, the same number generated by C multiplied into E. Now, since the same number is generated by D, the first, multiplied into F, the fourth, as [the number that is generated by] C, the second, multiplied into E, the third, then, by proposition 19, Book VII, as D, the first, is to C, the second, so E, the third, is to F, the fourth: and by inversion, as C to D so F to E.[28]

Proof. Part II.[29]
Again, since by axiom 7, Book VII, B measures E, and E measures [the number] generated by A multiplied into E, by the same axiom, B too will measure the same [number] generated by A multiplied into E, by the same axiom 7, Book VII. Let B measure [the number] generated by A multiplied into E by [number] GF, so that by B in FG there will be generated the number that B measures by FG, by axiom 9, Book VII, that is the same number that is generated by A multiplied into E. Now, since the same number is generated by B, the first, multiplied into FG, the fourth, and by A, the second, multiplied into E, the third, then, by proposition 19, Book VII, as B, the first, is to A, the second, so E, the third, is to FG, the fourth. And, by

[28] Clavius here needs axioms 7 and 9, Book VII, and propostition 19, Book VII. Axiom 7 is as follows: “If a number produces a number by multiplication with another number, then by the multiplied [number] the multiplying [number] will measure the product, and by the multiplying [number] the multiplied [number] will measure the same [product].” In Latin: “Si numerus numerum multiplicans, aliquem produxerit, metietur multiplicans productum per multiplicatum, multiplicatus autem eundem per multiplicantem.” Clavius 1999, p. 313. Axiom 9 is as follows: “If a number masuring a number multiplies the number by which it measures or it is multiplied by it, it will produce the [number] that it measures.” In Latin: “Si numerus numerum metiens, multiplicet eum, per quem metitur, vel ab eo multiplicetur, illum quem metitur, producet.”. Clavius 1999, p. 314. Proposition 19 is as follows: “If four numbers are proportional, the number that is formed by [multiplication of] the first anf fourth [numbers] will be equal to [the number] formed [by multiplication] of the second and third [numbers]. And if the number formed by the first and fourth is equal to that formed by the second and third, then these [numbers] are proportional.” In Latin: “Si quatuor numeri proportionales fuerint, qui ex primo, et quarto fit, numerus, aequalis erit ei, qui ex secundo et quarto fit, numero. Et si, qui ex primo, et quarto fit, numerus, aequalis fuerit ei, qui ex secundo, et tertio fit, numero; ipsi quatuor numeri proportionales erunt.” Clavius 1999, p. 326.

[29] “Rursus quia B, metitur E, ex pronunciato 7 libr. 7 et E, metitur procreatum ex A, in E, ex eodem pronunciato; metietur quoque B, eundem procreatum ex A, in E, ex pronunciato 11 libr. 7. Metiatur B, procreatum ex A, in E, per FG, fietque idcirco ex B, in FG, numerus, quem B, metitur per FG, ex pronunciato 9 lib.7, hoc est numerus idem qui fit ex A, in E. Quia igitur idem numerus ex B, primo in FG, quartum, et ex A, secundo in E, tertium gignitur, [19 septimi] erit ut B, primus ad A, secundum, ita E, tertius ad FG, quartum: et convertendo ut A, ad B, ita FG, ad E. Quoniam ergo est ut FG, ad E, ita A, ad B. Est autem proportio A, ad B, posita maior quam C, ad D; erit quoque proportio FG, ad E, maior quam C, ad D: ostensum autem est, esse ut C, ad D, ita F, ad E. Igitur proportio FG, ad E, maior quoque erit proportionem F ad E, ac proinde numerus FG, maior erit numero F: quae omnia ex iis quae in defin. 20 lib. 7 scripsimus perspique consecuuntur. Superet igitur FG, ipsum F, numero G.” Clavius 1999, p. 213.

inversion, as A is to B, so FG is to E. Since as FG is to E so A is to B, and the ratio of A to B has been taken greater than [the ratio] of C to D, then the ratio of FG to E too will be greater than [the ratio] of C to D. But since it has been proven that as C is to D so F is to E, then the ratio of FG to E will be greater than the ratio of F to E, and therefore number FG will be greater than the number F, which is clearly a consequence of all that we have written in [our comment on] definition 20, Book VII. Therefore number FG exceeds number F by number G.[30]

Proof. Part III.[31]
Let equimultiples K, H, I, of F, G, C, be taken according to such a law that H be greater than E, but I not less than D. Then, by the scholium to proposition 5, Book VII, the whole KH will be multiple of the whole FG as K is of F or I of C. Again, let equimultiples L, MN of D, E, be taken according to such a law that L is a multiple of D, immediately-greater [*proxime maior*] than I, that is let the equimultiples be such that by subtracting number D from [number] L the resulting number is not greater than I, but is equal [to I], as in the second example [Table 3], or less [than I], as in the first example [Table 3]. Since it has been shown that as D is to C so E is to F, and, by construction, as F is to K so C is to I, because I, K, have been taken as equimultiples of C, F, then, by proposition 14, Book VII, by equidistance of ratios, as D is to I so E is to K.[32]

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30 It should be noted here that Clavius assumes that FG represents the sum of F and G, i.e. in modern symbols F + G. Addition and subtraction are operations that are not explicitly defined in the arithmetical books of the *Elements*. Cf. the discussion below for the meaning of number FG.

31 "Sumantur iam ipsorum F, G, C, aequemultiplices K, H, I, ea lege, ut H, sit quidem maior quam E, at I non minor quam D, eritque ex scholio propos. 5 lib. 7 totus KH, ita multiplex totius FG, ut est multiplex K, ipsius F, vel I, ipsius C. Sumantur rursus ipsorum D, E, aequemultiplices L, MN, ea lege, ut L, sit multiplex ipsius D, proxime maior quam I, hoc est, tales aequemultiplices, ut subtracto numero D, ex L, reliquus numerus maior non sit, quam I, sed vel aequalis, ut in secundo exemplo, vel minor, ut in primo exemplo contingit. Et quoniam ostensum est ita esse D, ad C, ut E, ad F; et ita est C, ad I, ut F, ad K, ex constructione quod I, K, sumpti sint ipsorum C, F, aequemultiplices; [14 septimi] erit quoque ex aequo, ut D, ad I, ita E, ad K." Clavius 1999, p. 213.

32 As to this part of the proof, Clavius needs a result proven in the scholium to proposition 5, Book VII, and Proposition 14, Book VII. The result proven in the scholium is as follows: "If any numbers are given that are equimultiples of other numbers, one to one, then as many times a number is multiple of another number, so many times all numbers taken together will be multiples of all the others". In Latin: "Si sint quotcunque numeri quotcunque numerorum aequalium numero, siguli singulorum, aeque multiplices: quam multiplex est unius unus numerus, tam multiplices erunt omnes omnium". (Clavius 1999, p. 318). Proposition 14 is as follows: "If any set of numbers are given and another equal set of numbers are given such that if taken two by two [one from each set] they are in the same ratio, then they are in the same ratio even by equality." In Latin: "Si sint quotcunque numeri, et alii illis aequales in multitudine, qui bini sumantur, et in eadem ratione: Etiam ex aequalitate in eadem ratione erunt." Clavius 1999, p. 323. The meaning of this proposition only becomes clear upon reading the beginning of the proof. Therefore we also give it in part in order to let the reader understand what Clavius means by the expression 'by equality' [*ex aequalitate*]. Clavius goes

Proof. Part IV.[33]

Since it has been shown that D is to C as E is to F, then, by alternating, D is to E as C is to F. And as D is to E so L is to MN, because the same number that multiplies D, E, has produced L, MN since L, MN have been taken equimultiples of D, E. For the same reason, as C is to F so I is to K. Therefore, by the lemma to proposition 14, Book VII, as L is to MN so I is to K. Thus, by alternating, as L is to I so MN is K.[34]

Proof. Part V.[35]

Now, since as the whole L is to I so the whole MN is to K, and as D, [the part] taken away from L, is to the same I, so E, [the part] taken away from MN, is to the same K, then, by theorem 6 in the scholium to proposition 22, Book VII, as the remaining [part] from L is to I, so the remaining [part] from MN is to K. And the remaining [part] from L is not greater than I, but it is either less [than I], as in the first example [Table 3], or equal [to I], as in the second example [Table 3], because the remaining [part] from L, together with D, produces L, the multiple of D immediately-greater [*proxime maiorem*] than I, by construction. So, the remaining [part] of MN too will not be greater than K, so that if N, which is equal to E, is subtracted from MN, the remaining [part] M will be less than K, as in the first example [Table 3], or equal [to K] as in the second [example] [Table 3]. And since H, multiple of G, is greater than E, or N, by construction, the whole KH will be greater than the whole MN.

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on as follows: "Let any set of numbers A, B, C, be taken, and another equal set, D, E, F, [be taken], and as A is to B so let D be to E, and as B is to C, so let E be to F. I say by equality that as A is to C so D is to F." In Latin: "Sint quotqunque numeri A, B, C, et alii totidem D, E, F, sitque ut A, ad B; ita D, ad E; et ut B, ad C; ita E, ad F. Dico ex aequalitate quoque esse ut A, ad C, ita D, ad F." Clavius 1999, p. 323.

33 "Praeterea quoniam ostensum est esse D, ad C, ut E, ad F; erit permutando quoque D, ad E, ut C, ad F. Ut autem D, ad E, ita est L, ad MN; quod idem numerus ipsos D, E, multiplicans fecerit L, MN, quippe cum L, MN, sumpti sint ipsorum D, E, aequemultiplices: Et eadem de causa, ut C, ad F, ita est I, ad K. Igitur ex lemmate proposit. 14 libr. 7 erit quoque, ut L, ad MN, ita I, ad K, et permutando, ut L, ad I, ita MN, ad K." Clavius 1999, p. 213.

34 The lemma to proposition 14, Book VII, is as follows: " That two ratios of numbers that are equal to same ratio are also equal to each other – as in the [preceding] proof the ratios of A to D and of C to F, which were shown to be the same as the ratio of B to E – both if the numbers are integers and if they are fractions, can be proven as follows." In Latin: "Quod autem duae proportiones numerorum, quae eidem proportioni eaedem sunt inter se quoque sint eaedem, quales sunt in demonstratione proportiones A, ad D, et C, ad F, quae eaedem ostensae sunt proportioni, B, ad E, sive numeri sint integri, sive fracti, ita demonstrabitur." Clavius 1999, p. 324.

35 "Quia ergo est, ut totus L, ad I, ita totus MN, ad K: et ut D, ex L, ablatus ad eundem I, ita E, ex MN, ablatus ad eundem K; erit quoque ex theor. 6 scholii propositio 22 lib. 7: ut reliquus ex L, ad I, ita reliquus ex MN, ad K: Est autem reliquus ex L, non maior quam I, sed vel minor, ut in priori exemplo, vel aequalis, ut in posteriori; propterea quod reliquus ex L, cum D, facit ipsum L, multiplicem ipsius D, proxime maiorem ipso I, ex constructione. Igitur et reliquus ex MN, maior non erit quam K; atque idcirco si ex MN, detrahantur N, ipsi E, aequalis, erit reliquus M, vel minor ipso K, ut in priori exemplo, vel aequalis, ut in posteriori. Cum ergo H, multiplex ipsius G, sit maior quam E, vel N, ex constructione; erit totus KH, toto MN, maior." Clavius 1999, pp. 213–214.

Proof. Part VI.[36]

Finally, let O be taken equal multiple of A as KH is multiple of FG, or as I [is multiple] of C, and let P [be taken] equal multiple of B as MN is multiple of E, or L [is multiple] of D. Since it has been shown that as FG is to E so A is to B, and equimultiples KH, O, of FG, A, the first and the third, have been taken, and equimultiples MN, P, of E, B, the second and the fourth, have also been taken, it follows from the preceding proposition that if KH is greater than MN, then O is greater than P. Since KH has been proven to be greater than MN, O will then be greater than P. Thus, since O, I, are equimultiples of A, C, the first and the third, and P, L, are equimultiples of B, D, the second and the fourth, and since it has been proven that O is greater than P, and [since] I is less than L, by construction, then it can happen that O, the multiple of the first, is greater than P, the multiple of the second, while I, the multiple of the third, is not greater than L, the multiple of the fourth, if the ratio of A, first, to B, second, be greater than the ratio of C, third, to D, fourth, and the equimultiples be taken as explained. Q.E.D.

Now, if the ratio of the first to the second is less than [the ratio of] the third to the fourth, and equimultiples of the first, second, third, and fourth are taken, it can happen that the multiple of the first is less than the multiple of the second while the multiple of the third is not less than the multiple of the fourth. For in the same example, the ratio of C [taken as the] first [number] to D [taken as the] second [number] is less than [the ratio] of A [taken as the] third [number] to B [taken as the] fourth [number], and it has been proven that I, multiple of the first [number], C, is less than L, multiple of the second [number], D, but O, multiple of A, the third [number], is greater than P, multiple of B, the fourth [number].[37]

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[36] "Denique sumpto O, ita multiplici ipsius A, ut KH, multiplex est ipsius FG, vel I, ipsius C: item P, ita multiplici ipsius B, ut MN, multiplex est ipsius E, vel L, ipsius D: quoniam ostensum est, ita esse FG, ad E, ut A, ad B, sumptique sunt ipsorum FG, et A, primi ac tertii, aequemultiplices KH, et O; item ipsorum E, et B, secundi ac quarti aequemultiplices MN, et P, sequitur ex antecedente propositione si KH, maior est quam MN, ipsum quoque O, maiorem esse quam P. Cum ergo KH, ostensus sit maior quam MN, erit quoque O maior quam P. Quocirca cum O, I, sint aequemultiplices ipsorum A, C, primi ac tertii; et P, L, aequemultiplices ipsorum B, D, secundi ac quarti, demonstratumque sit, maiorem esse O, quam P, sit autem I, minor quam L, ex constructione; fieri potest, ut existente maiore proportione primi A, ad secundum B, quam tertii C, ad D, quartum, sumptisque aequemultiplicibus, ut dictum est, multiplex primi, nimirum O, maior sit quam P, multiplex secundi, multiplex autem tertii, nimirum I, maior non sit quam L, multiplex quarti. Quod demonstrandum erat.

Quod si minor sit proportio primi ad secundum, quam tertii ad quartum, sumanturque aequemultiplices primi ac tertii, item aequemultiplices secundi ac quarti, fieri quoque potest, ut nonnunquam multiplex primi sit minor multiplice secundi, multiplex vero tertii non sit multiplice quarti minor. In eodem enim exemplo, minor est proportio C, primi ad D, secundum, quam A, tertii ad B, quartum, demonstratumque est, I, multiplicem primi C, minorem esse, quam L, multiplicem secundi D, at O, multiplicem tertii A, maiorem esse quam P, multiplicem quarti B." Clavius 1999, p. 214.

[37] This sixth part of Clavius' proof, after the concluding formula Q.E.D., finishes up with a coda that simply shows that by inverting the order of the numbers A, B, C, D, forming the two starting ratios the proof reaches the same conclusion even if the first ratio is less than the second one.

Given the sheer complexity of Clavius' proof, we need to summarise the partial results of the six parts using some form of short-hand representation. The reader should also be cautioned not to over-interpret this quasi-algebraic symbolism in order not to superimpose inadvertently any modern meaning on what Clavius is doing. To facilitate further the understanding of this beautiful proof, we will also replicate Table 3 as many times as necessary, progressively filling it with the data that are required by our step-by-step analysis. I can promise that after this Clavian tour de force, the analysis of Galileo's proof of his own Theorem II will turn out to be a pleasant promenade!

Part I. We represent the product of two numbers by means of the symbol '*', the relation of proportionality (according to Definition 20, Book VII) by '::', the ratio between two numbers by ':', and the sum of two numbers by '+'. Finally, '>' and '<' indicate greater than and less than, respectively. Clavius begins by assuming two ratios formed by four numbers, A, B, C, D, respectively, such that the ratio of A to B is greater than the ratio of C to D. Thus A:B > C:D. Then, he assumes that F = C * B and E = B * D. Clavius' aim is to decompose ratio A:B in such a way that a part of it (ratio F:E) turns out to be equal to ratio C:D. From the previous relations, it follows that C * E = F * D and therefore C:D :: F:E. This is reached by Clavius not by means of algebraic manipulations – as we did – but by proposition 19, Book VII.

Table 3 for Part I

		Ex. 1		
F 8			**A 3**	**C 4**
	E 6		**B 2**	**D 3**
		Ex. 2		
F 8			**A 3**	**C 1**
	E 80		**B 8**	**D 10**

Part II. Since E = B * D, by construction, then A * E = A * B * D, and A * E = B * (F + G). Here Clavius looks for a number G such that the previous relation is true, i.e., in his language, such that number B measures the number generated by A and E by number FG. Thus, we must have B:A :: E:(F + G). Again, note that this is not obtained by Clavius by means of algebra. Here we have to point out that at this juncture Clavius tacitly uses an unproven property. Assuming that A * E = B * (F + G) implies that since G can only be positive, it must be A * D > F, and therefore A * D > C * B. This is equivalent to the following property: the product of the first and fourth terms of a proportion between numbers is greater than the product of the second and third terms if the first ratio is greater than the second ratio. Of course this is true in our case since the hypothesis of the theorem precisely assumes that A:B > C:D. Nevertheless, it is interesting to note that this result is neither proven by Euclid nor by Clavius. By inverting B:A :: E:(F + G), Clavius obtains A:B :: (F + G):E. Note that Clavius has reached his goal, namely, to decompose A:B in such a way that C:D :: F:E (cf. Part I, and below, Table 3, Part III, in which grey areas represent the decomposition of ratio A:B). Then, by simple considerations of inequality of ratios, Clavius proves that F + G > F. Surprised? Again, there is no proof in Euclid that given any two numbers, then the sum of them is greater than one of the two numbers. In fact, there is no definition of the 'simple' notion of addition of two numbers at all.

Part III. Here we enter into the real core of the proof. Up to this point, Clavius has simply prepared the way for the application of the equimultiple technique. The reason why this has proven to be so complicated is that numbers are non-continuous quantities and ratios of numbers are not fractions in the modern sense. This means that manipulations of ratios can only be carried out within the framework of properties available in the arithmetical books of the *Elements*. Now Clavius proceeds to choose the particular equimultiples he needs in order to show that since A:B > C:D, then a violation of the three conditions set forth by the equimultiple definition can indeed occur. It must be stressed that Clavius is free to choose any way of forming the equimultiples whatever, since he only has to prove that under a particular circumstance a violation can occur.

Table 3 for Part III

		Ex. 1		
K 56	H 7			I 28
F 8	G 1		**A 3**	**C 4**
	E 6		**B 2**	**D 3**
M 54	N 6			L 30
		Ex. 2		
K 160	H 440			I 20
F 8	G 22		**A 3**	**C 1**
	E 80		**B 8**	**D 10**
M 160	N 80			L 30

Equimultiples K, H, I, of F, G, C, and L, MN, of D, E are taken according to these rules: a) H must greater than E, but I not less than D (this applies to the second example, it is irrelevant in the first example where C > D); b) L must be immediately-greater (according to my translation of the Latin *proxime maior*) than I, i.e. if D is subtracted from L then the remaining number is not greater than I. In symbols: L – D <= I (where <= means less than or equal to). In the first example, L – D < I and in the second one L – D = I. As we shall see later on, this condition will be used by Galileo who will render Clavius' Latin expression *proxime maior* as the Italian *prossimamente maggiore*. As for the rest, Clavius arrives at this proportion D:I :: E:K. He is going to use this relation in the final stage of the proof.

Part IV. This part is essentially needed in order to prove one proportion: L:I :: (M + N):K.

Part V. By putting together the relations previously obtained in Part III and IV, Clavius shows that:

$$(L - D):I :: (M + N - E):K.$$

Since by construction (L – D) <= I (remember L has been chosen immediately-greater [*proxime maior*] than I) then M + N – E <= K and since N = E it follows that M <= K. Again by construction H > E and, finally, K + H > M + N. This is the first inequality between equimultiples that Clavius has reached. K + H is equimultiple of F + G and M + N is equimultiple of E. In other words, Clavius has constructed the equimultiples of ratio (F + G):E, which represents the decomposition of ratio A:B. Now he simply needs to complete the construction of the equimultiples by adding the equimultiples of C and D.

Table 3 for Part VI

		Ex. 1		
K 56	H 7		O 21	I 28
F 8	G 1		**A 3**	**C 4**
	E 6		**B 2**	**D 3**
M 54	N 6		P 20	L 30
		Ex. 2		
K 160	H 440		O 60	I 20
F 8	G 22		**A 3**	**C 1**
	E 80		**B 8**	**D 10**
M 160	N 80		P 24	L 30

Part VI. In Table 3 Part VI we have added O, P, equimultiples of A, B according to the same multiplicity factor as that of equimultiples K, H, I, and L, MN, respectively. Now, since it has been proved in Part II that (F + G):E :: A:B and, in Part V, that

$$K + H \text{ (equimultiple of } F + G) > M + N \text{ (equimultiple of } E),$$

then, by Theorem I, it follows that O > P. Let us recall that Theorem I states that for proportional numbers according to Definition 20, Book VII, the three conditions of the equimultiple definition are satisfied, i.e. those numbers are also proportional according to the equimultiple definition. But by construction, I < L, since L has been chosen immediately-greater [*proxime maior*] than I. Therefore it can happen that if

A:B > C:D then O > P, that is the equimultiple of A is greater than the equimultiple of B, while I < L, that is the equimultiple of C is less than the equimultiple of D. To sum up, it must be underlined that the strategy pursued by Clavius in his proof hinges on two main devices (which we shall find in Galileo as well):

a) the decomposition of first term A of greater ratio A:B into the two terms F and G of ratio (F + G):E in such a way that F:E :: C:D;
b) the clever choice of L, the immediately-greater [*proxime maior*] equimultiple of D, which allows Clavius to arrive at the first inequality K + H > M + N between the equimultiples of F + G and E.

As to the body of the proof, the difficulties principally lie in the fact that integers are non-continuous quantities and can only be manipulated according to the specific and restricted number of rules set forth by Euclid in the arithmetical books of the *Elements*. That this mathematical world is far removed from our algebra-dominated mindset admits of no question.

Now we briefly turn to Theorem III. First, we furnish Clavius' statement of Theorem III, secondly, we give the proof, and, thirdly, we discuss the structure and the meaning of the proof.

Clavius' statement of Theorem III.
Given four numbers and given equimultiples of the first and third, according to any multiplication whatever, and equimultiples of the second and fourth, according to any multiplication whatever, if, the equimultiple of the first being greater than the equimultiple of the second, the equimultiple of the third is necessarily greater than the equimultiple of the fourth, and, if the former being equal, the latter will always be equal too, and, if the former being less, the latter will perpetually be less too, then the ratio of the first to the second will be the same as [the ratio of] the third to the fourth [*according to the definition of proportional numbers, Def. 20 Book VII*].[38]

Clavius' proof of Theorem III.
Let four numbers A, B, C, D, be given and let any equimultiples whatever E, F, of the first A, and third C, be taken. By the same token, let equimultiples G, H, of the second, B, and fourth, D, be taken according to any multiplication. I say that if E, F, multiples of the first and third, are always greater than, equal to, less than G, H, multiples of the second and fourth, then A, first, is to B, second, as C, third, is to D, fourth. For, if the ratio of A to B were greater or less than the ratio of C to D, it could happen that – as has been proven in the preceding proposition [i.e. Theorem II] – E, multiple of the first, is sometimes greater or less than G, multiple of the

[38] "Propositis quatuor numeris, sumptisque aequemultiplicibus primi ac tertii iuxta quamvis multiplicationem, item aequemultiplicibus secundi ac quarti iuxta quamvis etiam multiplicationem, si multiplici primi existente maiore, quam multiplex secundi, multiplex tertii maior quoque sit necessario, quam multiplex quartii: et illo existente aequali, hic quoque semper sit aequalis; illo denique existente minore, hic quoque perpetuo minor sit: erit eadem proportio primi ad secundum, quae tertii ad quartum". Clavius 1999, p. 214.

> second, while F, multiple of the third, is not greater or less than H, multiple of the fourth, which is contrary to the hypothesis. Thus, A is to B as C is to D. Q.E.D.[39]

The proof is based on a *reductio ad absurdum*. Here we must note a few interesting features of this proof. As should be clear by now, one of the most important themes of Clavius' work on proportions is his interest in exploring the meaning of Euclid's equimultiple definition. The model of the four theorems that connect equimultiple proportionality with rational proportionality, can be interpreted as an attempt to capture the elusive meaning of the perpetual accord of the equimultiples. The hypothesis of Theorem III assumes that the conditions of the equimultiple definition are satisfied by numbers A, B, C, D. This involves an infinity of cases. If we are able to satisfy ourself that for all infinite cases the equimultiples are in perpetual accordance with one another, we can assert that A, B, C, D are in effect proportional according to Definition 20, Book VII. So, Clavius has recourse to the last resort of *reductio ad absurdum*. If it were not true that our four numbers are proportional according to Definition 20, Book VII, i.e. if it were the case that, for example, the ratio of A to B is greater than the ratio of C to D, then (by Theorem II) we would find some cases in which the accordance of the equimultiples is violated, contrary to the hypothesis.

On the one hand, as Theorem I proves, if rational proportionality obtains, then the perpetual accordance of the equimultiples obtains too (the triple condition of the equimultiple definition). And this undoubtedly confers a clear meaning on this perpetual accordance in all cases in which rational proportionality obtains. On the other hand, if the perpetual accordance of the equimultiples obtains, can we say that a *certain* other form of proportionality obtains too? And is this *certain* proportionality *rational proportionality*, i.e. proportionality according to Definition 20, Book VII? By means of his *reductio* proof Clavius demonstrates that this is indeed the case. Clavius succeeds in coercing equimultiple proportionality into the more restricted form of rational proportionality. He is able to map the territory ruled by this bizarre perpetual accord of the equimultiples.[40]

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[39] "Sint quatuor numeri A, B, C, D, sumanturque primi A, et tertii C, aequemultiplices qualescunque E, F. Item secundi B, et quarti D, aequemultiplices G, H, qualiscunque etiam sit haec multiplicatio. Dico si E, F, multiplices primi ac tertii semper sint vel maiores, quam G, H, multiplices secundi et quarti, vel aequales, vel minores, ita esse A, primum ad B, secundum, ut C, tertium, ad D, quartum. Si namque foret maior proportio A, ad B, vel minor, quam C, ad D, fieri posset, veluti in antecedente propos. demonstratum est, ut E, multiplex primi esset aliquando maior, aut minor, quam G, multiplex secundi, at F, multiplex tertii non maior, aut minor quam H, multiplex quarti. Quod est contra hypotesin. Est ergo A, ad B, ut C, ad D. Quod erat ostendendum." Clavius 1999, p. 214.

[40] Professor Curtis Wilson had the insight of recognising in the structure of Clavius' model of equivalence the following logical pattern and the question it poses. Let A and B be the two definitions of proportionality of the first and second theorems of Clavius' model of equivalence. Then, according to the sentential calculus, since Clavius has proven in the first and second theorems that A ➔ B and notA ➔ notB, respectively, it is clear that Theorems III and IV immediately follow. This raises the very interesting question of the meaning of the presence of the third and fourth theorems. I believe that Clavius simply did not 'see'

To sum up, Clavius had proven that:

a) proportional numbers according to rational proportionality [Definition 20, Book VII], are also proportional according to equimultiple proportionality;
b) if four numbers satisfy the equimultiple condition, then they can be said to be proportional according to rational proportionality [Definition 20, Book VII].

Looking at Clavius' foundational work on proportions, the equivalence model, one realizes that the driving forces behind such a programme must have been a constant concern about the possibility that either absurd results could be derived from the equimultiple definition or the commonly accepted results concerning proportions of numbers could not be derived from this definition.

2. Galileo's *Dialogue on Proportions* and Clavius' model of equivalence

This section is divided into two parts. First, I discuss Galileo's new definition of proportionality; secondly, I propose the statements of the four theorems of both Clavius' and Galileo's model of equivalence in a synoptic form, in order to show their similarity in terms of purpose and general structure.

Galileo's model has come down to us indirectly.[41] In 1641, Galileo dictated to Torricelli what he intended to be a new dialogue to be added to *Two New Sciences*. Once again, Galileo summoned his three famous literary characters, Salviati, Sagredo, and Simplicio, to present his reflections on the difficulties of Euclid's Fifth Book. After a brief introduction, in which Sagredo notes that the theory of motion in *Two New Sciences* is founded on the definition of *equable motion* and this in turn is founded on Euclid's ambiguous equimultiple definition, Salviati confesses that he too was 'shrouded in the same fogs' as Sagredo for sometime after studying Euclid's Fifth Book.[42] For Galileo-Salviati:

> [...] in order to give a definition of the assumed proportional magnitudes suitable to produce in the mind of a reader some concept of the nature of these proportional

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this logical pattern. A clue to a possible answer to this question is this. Unlike Theorem III, Theorem IV is not proven by *reductio ad absurdum* as, I think, would likely have been the case if Clavius had recognised the pattern. Cf. Clavius' proof of theorem IV in Clavius 1999, p. 214. On the other hand, since Galileo's Theorem IV is a *reductio* proof, the previous conclusion suggests that Galileo might have recognised the pattern and proceeded accordingly. Cf. Galileo's proof in Drake 1995, p. 431.

[41] Enrico Giusti has furnished a new edition of the text in 1993, pp. 279–298. Whereas Antonio Favaro, the editor of the *National Edition*, had based his edition on the first printed version of Galileo's tract, published by Vincenzo Viviani, Giusti has tried to reconstruct a version nearer to Torricelli's original manuscript. Apart from a specific point regarding a definition of proportion for integers and/or rational quantities that Galileo seems to use only for didactic purposes before abandoning it altogether, and which has been differently phrased in Giusti's version, the remaining few differences are mainly stylistic. With the exception of a few points, which will be indicated below, we have therefore followed Stillman Drake's translation, which is based on the text of the *National Edition*. Stillman Drake's translation of the dialogue on proportions was published in Drake 1995, pp. 422 436. The original Italian can be seen in Galilei 1890–1909, VIII, pp. 349–362.

[42] See Salviati's comments in Drake 1995, p. 423.

> magnitudes, we must select one of their properties. Now, the simplest [property] of all is precisely that which is deemed most intelligible even by the average man who has not been introduced to mathematics; Euclid himself has proceeded thus in many places. Remember that he does not say [for example] that the circle is a plane figure within which two intersecting straight lines will produce rectangles such that that which is made of the parts of one line will equal that which is made with the parts of the other, or [that it is a plane figure] within which all quadrilaterals have their opposite angles equal to two right angles. These would have been good definitions, had he spoken thus; but since he knew another property of the circle more intelligible than the preceding, and easier to form a concept of, he did much better to set forth that clearer and more evident property [equidistance from a point] as a definition [. . .].[43]

Sagredo fully agrees, and adds that

> [. . .] very few indeed are the minds that would be completely satisfied by this definition, when I say with Euclid:
>
> Four magnitudes are proportional when equal multiples of the first and third, taken according to any multiplication, always alike exceed, fall short of, or equal, equal multiples of the second and fourth.
>
> Who is there so fortunate of mind as to be able to be certain that when four magnitudes are proportional, these equal multiples will always [thus] agree ? Indeed, who knows that such equal multiples will not always agree, even when the magnitudes are not proportional?[44]

It is the ‘obscurity’ of the equimultiple definition that does not satisfy Galileo. At this point, there is a sudden change of tone in the dialogue. After a few didactic exchanges in which Galileo has the Aristotelian Simplicio agree that, at least in principle, there can be a more intuitive manner of defining proportionality, for example by having recourse to simple rational ratios of integers, Salviati abruptly introduces Galileo’s new proposals:

> I shall add this other mode in which four magnitudes are to be understood as proportional:
>
> When, in order to have the same ratio to the second that the third has to the fourth, the first is neither greater nor less than it need be, then the first is understood to have to the second the same ratio as the third [has] to the fourth.
>
> At this point I shall also define “greater ratio,” saying
>
> When the first magnitude is greater than it need be in order to have to the second the same ratio which the third has to the fourth, then we must agree to say that the first has a “greater ratio” to the second than that which the third has to the fourth.[45]

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[43] Drake 1995, p. 424.
[44] Drake 1995, p. 424.
[45] Drake 1995, p. 426.

Enrico Giusti thinks that this definition is not circular, if we admit as a postulate, as Galileo has explicitly done a few lines before Salviati's above passage, that proportional magnitudes exist, i.e. that there are such things in nature as *four proportional magnitudes*.[46] Still, I believe that here we have a scarcely more intuitive definition than Euclid's. Although it is very difficult to penetrate the secrets of the so-called *context of discovery*, I would argue that Galileo may have worked backwards, starting from Clavius' model and then trying to adapt a new definition of proportionality to what he interpreted as being formally the perfect sequence of theorems that was needed to establish a sort of isomorphism between his new definition of proportionality and Euclid's. After all, Clavius' model accomplishes exactly this: the establishment of a sort of isomorphism between Euclid's definition of proportionality of Book VII and the equimultiple definition of Book V. In other words, Clavius' model offered to Galileo the formal shell in which a new general definition of proportionality might be inserted in the place of Definition 20, Book VII. If this is the case, Galileo may have been somehow forced by the cogency of the mathematical framework of Clavius' model to consider his own new definition as being quite acceptable. This isomorphism is precisely what Sagredo requires to be proven by Salviati.

For Sagredo, the new definition is all very well but, he comments,

> It seems to me that you have placed yourself [i.e. Salviati] under the obligation to add one of two things, that is, either to demonstrate from these principles of yours the entire fifth Book of Euclid, or else to deduce, from the two definitions you have set forth, the two other [definitions] that Euclid puts for the fifth and seventh among his definitions, on which he bases the whole structure of the fifth Book.[47]

Though I believe that it is plausible that Galileo's excogitation of his new definition was the result of his pondering Clavius' model, I must acknowledge that it is only a hypothesis (cf. the discussion at the beginning of the next section). Moreover, there are no extant documents that allow us to date with accuracy Galileo's thoughts on the foundations of proportionality. We only know that he dictated his brief tract in form of dialogue to Torricelli at the end of 1641.

We turn now to presenting Clavius' and Galileo's models of equivalence in synoptic form.

In order to make it easier for the reader to understand the two sets of theorems, we first furnish a summary table (Table 4), in which the long statements of the theorems have been contracted by using the symbols given above, and secondly, in Table 5, we quote the two sets of theorems by Clavius and Galileo in full. In column

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[46] Giusti 1993, pp. 76–77.

[47] Drake 1995, p. 427. The fifth and seventh definitions mentioned by Sagredo are the equimultiple definition and the equimultiple definition of 'greater ratio', respectively. There were a few differences in the numbers associated with Euclid's definitions and propositions, according to the different editions of Euclid. For example, in Clavius' edition the equimultiple definition is the sixth, while in Commandino's edition it is the fifth. Commandino had shifted the position of the definition of proportionality as similarity of ratios, so that, as a result, the equimultiple definition had become the fifth. Cf. Euclid 1572, pp. 57 verso – 58 recto.

Table 4. A sketch of the general structure of Clavius' and Galileo's models

	Clavius	**Galileo**
Theorem I	four proportional numbers according to **D20VII=** ➔ four proportional numbers according to **Eq=**	four proportional magnitudes according to **G=** ➔ four proportional magnitudes according to **Eq=**
Theorem II	Greater ratio for numbers according to **D20VII>** ➔ greater ratio for numbers according to **Eq>**	greater ratio for magnitudes according to **G>** ➔ greater ratio for magnitudes according to **Eq>**
Theorem III	if four numbers obey **Eq=** ➔ they obey **D20VII=**	if four magnitudes obey **Eq=** ➔ they obey **G=**
Theorem IV	greater ratio for numbers according to **Eq>** ➔ greater ratio for numbers according to **D20VII>**	greater ratio for magnitudes according to **Eq>** ➔ greater ratio for magnitudes according to **G>**

one, there are the names of theorems, here abbreviated simply as theorem I, II, III, IV, in column two, Clavius' wordings, and, in column three, Galileo's wordings. Let

G= be Galileo's new definition of proportionality and

G> be Galileo's new definition of 'greater ratio'.

(The other symbols have been defined above.)

Of Galileo's four theorems, the first is the only one whose statement is not formally given by Salviati at the beginning of the proof. Instead, the theorem's thesis develops step by step during the dialogue with Simplicio. It is one of the many ways in which Galileo's extraordinary talent for shaping the Italian language is used to improve accessibility to complex concepts. I have therefore collated the introduction by Salviati and the conclusion by Sagredo omitting the intermediate passages in which the complete proof is derived by Salviati. Later on, we shall note some striking similarities of terminology between Clavius' Latin and Galileo's Italian.

[48] Clavius 1999, p. 213. The Latin text has been given in note 23.
[49] Drake 1995, pp. 427–429. Cf. the original in Galilei1890–1909, VIII, pp. 355–356.
[50] "Propositis quator numeris non proportionalibus, ita ut maior sit proportio primi ad secundum, quam tertii ad quartum, si sumantur aequemultiplices primi ac tertii, item aequemultiplices secundi ac quarti, fieri potest, ut multiplex primi maior sit, quam multiplex secundi, multiplex autem tertii non maior, quam multiplex quarti." Clavius 1999, p. 213.
[51] Drake 1995, p. 429. Cf. the original in Galilei 1890–1909, VIII, pp. 356–357.
[52] The Latin text has been given in note 38.
[53] Drake 1995, p. 431. Cf. The original in Galilei 1890–1909, VIII, pp. 357–358.
[54] "Propositis quator numeris, sumptisque primi ac tertii aequemultiplicibus, item secundi et quarti aequemultiplicibus; si quando contingat, multiplicem primi maiorem esse multiplice secundi, multiplicem vero tertii non maiore multiplice quarti; maior erit proportio primi ad secundum, quam tertii ad quartum." Clavius 1999, p. 214.
[55] Drake 1995, p. 431. Cf. The original in Galilei 1890–1909, VIII, p. 358.

Table 5. The wording of all the statements of the theorems of Clavius' and Galileo's models

	Clavius	**Galileo**
Theorem I	Given four proportional numbers [*according to Definition 20, Book VII*], let equimultiples of the first and third be taken according to whatever multiplication, and, again, let equimultiples of the second and fourth be taken according to whatever multiplication; if the multiple of the first is greater than the multiple of the second, then the multiple of the third will be greater than the multiple of the fourth; if the multiple of the first is equal to the multiple of the second, then the multiple of the third will be equal to the multiple of the fourth; and finally, if the multiple of the first is less than the multiple of the second, then the multiple of the third will be less than the multiple of the fourth.[48]	*Salviati*. Supposing that the four magnitudes A, B, and C, D, are proportional [in that order] [*according to Galileo's definition of proportionality*] – that is, A.B.C.D, that the first, A, has to second, B, the same ratio that the third, C, has to the fourth, D, [...]. *Sagredo*. [...] I now understand quite well the necessity by which equal multiples of four proportional magnitudes agree eternally in being greater, or less, or equal etc.[49]
Theorem II	Given four non-proportional numbers [*according to Clavius' completion of the definition of proportional numbers, i.e. Definition 7, Book VII*], in such a way that the ratio of the first to the second is greater than [the ratio] of the third to the fourth, if equimultiples of the first and third are taken, and equimultiples of the second and fourth [are also taken], it can happen that the multiple of the first is greater than the multiple of the second, while the multiple of the third is not greater than the multiple of the fourth.[50]	Let there be the four magnitudes AB, C, D, and E, and let the first, AB, be somewhat greater than it need be in order to have to second, C, the same ratio which the third, D, has to the fourth, E [*i.e. let the first ratio be greater than the second one according to Galileo's definition of greater ratio*]. I shall show that equal multiples of the first and third [AB and D] being taken in a particular manner, and other equal multiples of the second and fourth [C and E] being taken, the multiple of the first [AB] will be found to be greater than that of the second [C], while that of the third [D] will in no way be greater than that of the fourth, and indeed I shall prove it to be less [than the like multiple of E].[51]

Table 5. (Cont.)

Theorem III	Given four numbers and given equimultiples of the first and third, according to any multiplication whatever, and equimultiples of the second and fourth, according to any multiplication whatever, if, the equimultiple of the first being greater than the equimultiple of the second, the equimultiple of the third is necessarily greater than the equimultiple of the fourth, and, if the former being equal, the latter will always be equal too, and, if the former being less, the latter will perpetually be less too, then the ratio of the first to the second will be the same as [the ratio of] the third to the fourth [*according to the definition of proportional numbers, Def. 20 Book VII*].[52]	If of the four magnitudes, A, B, C, D, equal multiples of the first and third, taken according to any multiplier, shall always agree in equalling, or falling short of, or exceeding, equal multiples of the second and fourth, jointly, I say the four magnitudes are proportional [*according to Galileo's definition of proportionality*].[53]
Theorem IV	Given four numbers and given the equimultiples of the first and third and equimultiples of the second and fourth, if it happens that, when the multiple of the first is greater than the multiple of the second, the multiple of the third is not greater than the multiple of the fourth, then the ratio of the first to the second will be greater than the ratio of the third to the fourth [*according to Clavius' completion of the definition of proportional numbers, Def. 20 Book VII*].[54]	Let the four magnitudes be A, B, C, and D, and suppose that taking equal multiples of the two antecedents, first and third, in some particular way, as [likewise some] equal multiples of the two consequents, second and fourth, a case is found in which the multiple of A is greater than the multiple of B, while the multiple of C is not greater than the multiple of D. Then I say that A will have a greater ratio to B than C has to D, that is, A will be somewhat greater than it must be in order to have to B the same ratio that C has to D [*which is Galileo's definition of greater ratio*].[55]

By quickly glancing at both tables, one notes that Galileo-Salviati simply needed to put **G=** and **G>** in the place of **D20VII=** and **D20VII>** to satisfy the obligation that Sagredo had pointed out after listening to Salviati's new proposal concerning proportional magnitudes. Of course it was not all that simple. After all, Galileo had first to invent **G=** and **G>** before going on to demonstrate the equivalence of his two definitions with Euclid's.

The proofs of the first and the fourth propositions could not be modelled after Clavius' corresponding ones because Clavius had dealt with numbers and since numbers are homogeneous quantities, couples of numbers lend themselves to being manipulated by operations that are not admissible for couples of non-homogeneous magnitudes. Clavius had taken advantage of this special property and used it in his proofs of the first and fourth theorems by having recourse to the technique of alternating the terms of the two ratios of a proportion. Galileo faced a more general problem. Since his new definitions were not restricted to numbers, he had to furnish proofs that could only assume the homogeneity of the two couples of quantities forming ratios. Therefore he could not resort to the alternating technique, otherwise he would have ended up with mixed ratios of non-homogeneous quantities. Thus, Clavius' and Galileo's first and fourth proofs are not similar, as we have already noted at the beginning of this discussion. But the structures of the second and third ones are clearly isomorphic. Therefore, we leave aside Galileo's first and fourth proofs and focus on both authors' second and third ones.

3. Galileo's visualization of Clavius' model of equivalence

As the reader will have noticed, more than one half of the proof devised by Clavius for Theorem II has to do with resolving the difficulty of decomposing a numerical ratio of a proportion in such a way that its first term forms a ratio with the second term equal to the ratio formed by the third and fourth terms. If one tries to visualize this kind of operation – for example by representing numbers with lines – one notes that since lines are continuous quantities they can be thought of as being easily divisible into parts. This simple fact in connection with Clavius' way of decomposing a numerical ratio *greater than it need be in order for it to be equal to a second ratio* might have suggested to Galileo his new definition of proportional quantities. In a sense, Galileo might have been lured into his extension by the almost ready-to-use formal shell he found in Clavius' treatment of equimultiple proportionality and rational proportionality. This becomes clearer if we look at how Galileo proved his Theorem II.

In what follows, we adopt a schema similar to that we used in discussing Clavius. Firstly, we furnish Galileo's text of the statements and proofs of his Theorems II, III, and, secondly, we briefly comment on their structure comparing Clavius' proofs with Galileo's, trying to show the similarities between them.

Galileo's statement of Theorem II.
Let there be the four magnitudes AB, C, D, and E, and let the first, AB, be somewhat greater than it need be in order to have to second, C, the same ratio which the third, D, has to the fourth, E [*i.e. let the first ratio be greater than the second one according to Galileo's definition of greater ratio*]. I shall show that equal multiples of the first and third [AB and D] being taken in a particular manner, and other equal multiples of the second and fourth [C and E] being taken, the multiple of the first [AB] will be found to be greater than that of the second [C], while that of the third [D] will in no way be greater than that of the fourth, and indeed I shall prove it to be less [than the like multiple of E].

Galileo's proof of Theorem II.
Therefore it is assumed that from the first magnitude AB there shall be removed that excess which makes it greater than it should be for precise proportionality. Let that

excess be FB. The four magnitudes will now be proportional, that is, the remainder AF will have to C the same ratio that D has to E. Let FB be multiplied sufficiently to become greater than C, and let this product be designated HI. Then take HL as many times a multiple of AF, and M likewise of D, as HI is a multiple of FB. This done, no doubt the composite LI will be as many times a multiple of the composite AB as HI is of FB, or M of D. Now take the multiple of C, N, according to such a law that N be immediately-greater [*prossimamente maggiore*] than LH; finally, make O a multiple of E as many times as N is a multiple of C. Now, N being immediately-greater [*prossimamente maggiore*] than LH, if we suppose removed from N one of the magnitudes composing it, which we made equal to C, then the remainder will not be greater than LH. Therefore when we restore to N the magnitude equal to C that we supposed removed, if to LH (which is not less than the said remainder) we add HI, which is greater than the restoration to N, LI will be greater than N. So, this is one case in which the multiple of the first [AB] exceeds the multiple [N] of the second. But the four magnitudes AF, C, D, and E were made proportional by us; and taking the equal multiples LH and M (of the first and third) and N and O (of the second and fourth), the latter will, by what we said before, always agree as to being greater, or less, or equal; hence the multiple LH of the first magnitude being less than the multiple N of the second, by construction the multiple M of the third will necessarily be less than the multiple O of the fourth. Thus it is proved that whenever the first magnitude is somewhat larger than it need be in order to have the same ratio to the second that the third has to the fourth, there is a way of taking equal multiples of first and third, and other [equal multiples] of second and fourth, and showing that this multiple of the first exceeds the multiple of the second, while the [equivalent] multiple of the third does not exceed that of the fourth.[56]

Galileo's statement of Theorem III.
If of the four magnitudes, A, B, C, D, equal multiples of the first and third, taken according to any multiplier, shall always agree in equalling, or falling short of, or exceeding, equal multiples of the second and fourth, jointly, I say the four magnitudes are proportional [*according to Galileo's definition of proportionality*].

Galileo's proof of Theorem III.
For if possible, let them be not proportional; then one of the antecedents will be greater than it need be in order to have to its consequent the same ratio that the other antecedent has to its consequent. Let this be, for instance, the one designated A. Then by what was already shown, taking in such a way multiples equally of A and C, and multiples equally of B and D in the manner already shown, the multiple of A will be shown to be greater than the multiple of B, while the multiple of C will not be greater (but rather, less) than the multiple of D; which is counter to our assumption [of their agreement].[57]

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[56] Drake 1995, pp. 429–430. I have slightly modified Drake's translation, which is not very faithful in rendering the passage in which Galileo uses the expression '*prossimamente maggiore*'. Galileo's sentence in Italian is: "Prendasi ora la N multiplice della C con tal legge, che la stessa N sia prossimamemente maggiore della LH [...]." Cf. Galilei 1890–1909, VIII, p. 356.

[57] Drake 1995, p. 431.

Let us now consider the figure accompanying Galileo's proof of Theorem II (Fig. 3).

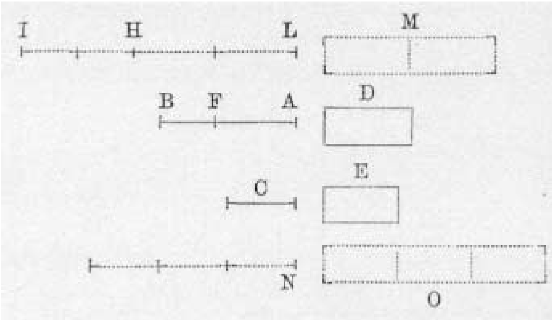

Fig. 3. The figure accompanying Galileo's proof of Theorem II

First of all, Galileo invites us to assume that the ratio of line AB to line C is greater than the ratio of rectangle D to rectangle E (according to his definition of *greater ratio*, cf. above). Secondly, he wants us to imagine cutting from line AB the excess that makes it greater than it need be for precise proportionality. How long is this excess? Or, if you like, how long must the remainder be in order that such remainder be to line C as the third quantity, D, is to the fourth quantity, E? Note that there is no criterion to establish how long this remainder need be, since Galileo's definition of proportionality does not furnish any criterion that may be applicable to the construction of proportional magnitudes. So, all Galileo can do is conjure up before our eyes a diagram in which he assumes that, somewhere in between point A and point B, a third point, F, exists such that AF is the remainder he is seeking. As has already been noted above, he is taking as a postulate that proportional magnitudes exist (as Euclid himself had done, according to Galileo's claim at the beginning of his dialogue on proportions).

It is interesting to note that by making this assumption on the existence of proportional magnitudes, Galileo is in reality doing something more than hypothesize the existence of such things as proportional magnitudes. He is sweeping away the difficult task of actually having to construct magnitudes that are proportional according to his new definition and replacing this task with a visual aid. If one asks what the meaning of Galileo's new definition really is, one can answer only by looking at Fig. 3. Since AF, C, D, E, must be proportional according to Galileo's new definition, they must reveal what this proportionality is. But all that they reveal is simply that *visually* there seems to be a certain accordance in the way in which C is contained in AF and E is contained in D. In other words, Galileo seems to be transferring the idea of proportional numbers – in the sense of Euclidean arithmetical proportionality – into the domain of continuous magnitudes, which have the added bonus of being easier to manipulate visually than integers (integers are neither lines nor sequences of dots, though such representations were normally used in many editions of Euclid's *Elements* including Clavius').

The dependence of Galileo's new definition on some form of visualization (or, if you prefer, intuition) allows him to get rid of the intricate way in which

Clavius decomposes the greater ratio A:B into one that in turn has its first term composed by two numbers, F, G, such that proportion A:B::(F+G):E holds true. All Galileo needs to do is place point F somewhere in between points A and B in order to have AB:C::(AF+FB):C (the reader should note that here we use the same pseudo-algebraic notation as we have done referring to Clavius, except that the symbol '::', when used in a Galilean proportion, must be interpreted as *Galilean proportionality*, according to Galileo's new definition). But this apparent simplification comes at a significant cost. Let us see how.

The important difference between Clavius and Galileo is that Clavius is actually able to construct the numerical proportion A:B::(F+G):E (Clavius' numerical examples show that it is actually possible to construct this proportion), while Galileo is unable to construct his analogous relation for continuous quantities, AB:C::AF+FB:C (we do not know where point F is really placed and consequently there can be no examples). Thus, although very similar in their formal structure, the two proofs have a very different flavour: Clavius' leads to the actual construction of the mathematical object that is being sought (i.e. proportion A:B::(F+G):E), while Galileo's is simply based on an existence assumption (point F must be located somewhere between A and B).

The strategy Galileo pursues in the course of his proof is clearly modelled after Clavius. The two main technical devices are the same as Clavius':

a) the decomposition of line AB into line AF, the remainder, and line BF, the excess;
b) the clever choice of the law according to which N, the multiple of line C, is formed, i.e. the idea that N be immediately-greater [*prossimamente maggiore* is a literal translation into Italian of Clavius' Latin *proxime maior*] than LH.

K 56	H 7	O 21	I 28
F 8	G 1	**A 3**	**C 4**
	E 6	**B 2**	**D 3**
M 54	N 6	P 20	L 30

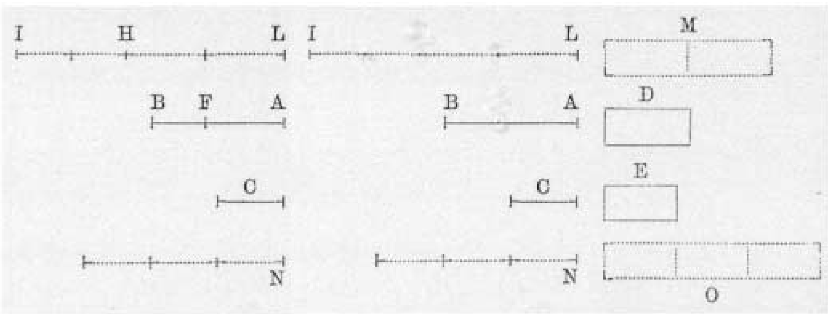

Fig. 4. Theorem II: the analogy between Clavius' decomposition and Galileo's decomposition

Referring to Fig. 4, in which, for the sake of clarity, I have elaborated Galileo's original figure duplicating the left-hand part in order to show better the step-by-step construction of Theorem II, the reader can appreciate that:

a) Galileo's diagram reproduces the spatial disposition of numbers that he found in Clavius' table;
b) Clavius' ratios A:B and C:D correspond to Galileo's ratios AB:C and D:E;
c) Clavius' equimultiples O, P, of A, B, correspond to Galileo's equimultiples IL, N, of AB, C;
d) Clavius' equimultiples I, L, of C, D, correspond to Galileo's equimultiples M, O, of D, E;
e) Clavius' decomposition of ratio A:B into ratio (D+G):E corresponds to Galileo's decomposition of ratio AB:C into ratio (AF+FB):C;
f) Clavius' equimultiples K, H, of F, G, and MN of E, correspond to Galileo's equimultiples LH, HI, and N, of AF, FB, and C, respectively.

As to the rest, it easy to see that the sequence of steps of Galileo's proof follows closely Clavius' sequence. Referring to the choice of N, multiple of C, Galileo also uses another Italian expression, 'con tal legge' [with such a law], which is a literal translation of the Latin *ea lege*, the expression Clavius' used referring to the specific choice of the equimultiples needed to construct the proof. Moreover, Galileo's proof needs the result obtained in his Theorem I, precisely as Clavius needs the result proven in his own Theorem I. Galileo refers to his previous result with the expression 'per le cose già stabilite sopra' [according to the things already established above].[58]

As regards the proof of Theorem III, the reader will already have noticed that it is a proof based on a *reductio ad absurdum*, as Clavius' one is. Again, Galileo refers to his previous theorems with the expression 'per le cose già dimostrate' [according to the things already proven].[59]

I believe that the analysis I have furnished is sufficient proof that Galileo borrowed from Clavius important elements of the formal structure of his foundational research on Euclid's theory of proportions.

4. Conclusion. The equimultiples outside of the domain of pure mathematics

Galileo must have had mixed sentiments regarding the equimultiples. He first used this technique in his early *De Motu* (*ca*. 1590) to prove that weights of different volumes of bodies having the same specific weight are in the same ratio as their volumes.[60] Then he apparently abandoned it, when, in 1612, he wrote the *Discourse* on *Bodies That Stay atop Water, or Move in It*.[61] Finally, he resumed the equimultiples in *Two New Sciences*. Ironically, in the tract on proportions he dictated

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[58] Galilei 1890–1909, VIII, p. 357.

[59] Galilei 1890–1909, VIII, p. 357.

[60] By means of the equimultiples, Galileo proved that the *gravitates* of different bodies of equal [specific] weight are proportional to their volumes ('Gravitates inaequalium molium corporum aeque gravium eam inter se habent proportionem, quam ipsae moles'). Galilei 1890–1909, I, pp. 348–349. This interesting passage is not in the English version published by Drake and Drabkin 1960.

[61] To conclude a proof, Galileo, referring to bodies C and B, asserted 'since C and B are of the same specific weight, then as is the absolute weight of C to the absolute weight of B,

to Torricelli, he strongly objected to their *obscurity*, but in practice, with his model of equivalence, he implicitly preserved them, adopting a substantial portion of the formal structure of Clavius' model. Clavius' attempt at capturing the elusive meaning of the equimultiples set an important example for Galileo and other mathematicians in more ways than one. Indeed, the attraction, or the repulsion, for the equimultiple definition and the search for a general meaning of the perpetual accord of the equimultiples was to fascinate mathematicians up until the nineteenth century.[62]

In the above mentioned passage of his early *De Motu*, Galileo came face to face with the problem of transferring the notion of equimultiple proportionality from the domain of pure mathematics to that of natural philosophy. Let us refer to Fig. 5.

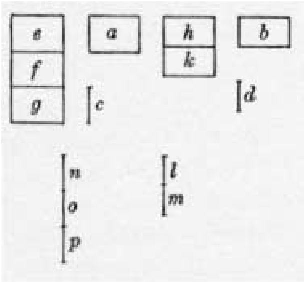

Fig. 5. The mathematisation of weight according to equimultiple proportionality

Let **a** and **b** be two unequal volumes [*moles*]. Let **c** and **d** be their weights [*gravitates*]. In *De Motu*, Galileo sets out to prove that **c** has to **d** the same ratio that **a** has to **b**. Recalling the example discussed above (associated to Fig. 2), one can see that in order to show that two weights of different volumes of bodies having the same specific weight are in the same ratio as their volumes, all that is needed is construct equimultiples of the two weights and two volumes and prove that they satisfy the perpetual accord of the equimultiple definition. Let **efg** and **hk** be the multiples of volumes **a**, **b**, respectively. Let **nop** an **lm** be the multiples of weights **c** and **d**, respectively. Here, by way of example, Galileo has chosen to represent triple and double multiples. According to Galileo, if, for any choice whatever of the multiples,

so will be the volume of C to the volume B' ('perchè C e B sono della medesima gravità in ispecie, sarà come il peso assoluto di C al peso assoluto di B, così la mole C [. . .] alla mole B'). But he did not bother to justify this assertion. The translation is taken from Drake 1981, p. 44. Cf. the original in Galilei 1890–1909, IV, p. 74.

[62] Giusti 1993 is partly devoted to studying the concerns that many Galileans (Torricelli, Borelli, Viviani, and others) showed with the obscurity of Euclid's equimultiple proportionality. For the nineteenth century, cf. De Morgan 1836 and, in general, Heath's discussion in Euclid 1956, II, pp. 120–129.

when **efg** $\gtreqless$ **hk** is true then **nop** $\gtreqless$ **lm** is true, then one has proved that the two weights are in the same ratio as their volumes, in the sense of equimultiple proportionality.[63]

The difficult point is that when Galileo assembles the equimultiples by constructing weight **nop** (three times weight **c**) and weight **lm** (two times weight **d**), he assumes that these assembled weights represent the weight of the assembled volumes, **efg**, **hk**, respectively. Whereas it is intuitive that volumes and weights may be added to form multiples, it must be borne in mind that volumes are geometrical quantities, while weights are physical quantities (and one cannot assume that the simple geometrical *relationships between the lines* chosen by Galileo to 'geometrise' weights isomorphically represent the same physical *relationships between weights*). Thus, who can guarantee that **nop**, three times weight **c**, equals the weight of volume **efg**?[64] This difficulty does not arise in the case of geometrical proofs, such as the first proposition of Book VI of the *Elements*, in which Euclid demonstrates that parallelograms of equal heights are in the same ratios as their bases. The assembling of the equimultiples proceeds immediately from the geometrical properties of the figures that are being considered and their internal relationships.[65]

Was Galileo aware of the difficulty intrinsic to the assembling of the equimultiples of physical quantities, when, in 1612, in the *Discourse* on buoyancy, he decided to assume without proof that weights of different volumes of bodies having the same specific weight are in the same ratio as their volumes? We do not know. But later on, in *Two New Sciences*, he once again resumed the technique of the equimultiples.

At the beginning of the Third Day of *Two New Sciences*, in order to mathematise the relationships governing uniform motion, Galileo proved six theorems, the first three of which depend on the notion of equimultiple proportionality.[66] Moreover, as is clear from a glance at the entire mathematical structure of *Two New Sciences*, the mathematised notion of uniform motion is not the technical embellishment of a virtuoso. It is subsequently used by Galileo to prove other theorems and is an important element of the axiomatic organization of the book.[67] We now turn to the Galilean treatment of equable motion according to equimultiple proportionality in Propositions I and II of the section entitled *On Equable Motion*. First we present the

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[63] Galilei 1890–1909, I, p. 349.

[64] For the sake of clarity, let us use functional symbolism and let W be weight. Who can guarantee that W(efg) = nop? Since efg = 3a, nop = 3c, and c = W(a), assuming that W(3a) = 3c is tantamount to assuming that W(3a) = 3W(a). Therefore W(efg) = nop ➔ W(3a) = 3W(a). But the latter relation is true if and only if W is proportional to volume (if, for example, W were proportional to the square of the volume then it would be false), i.e. if it is a linear function. In other words, in modern functional language, assuming W(efg) = nop, which is the step necessary in order to conclude the proof based on the equimultiples, is tantamount to assuming what has to be proved. Of course, this is clear because we have anachronistically translated the reasoning followed by Galileo into modern functional symbolism.

[65] Cf. the proof in Euclid 1956, II, pp. 191–194.

[66] Giusti 1986 and Drake 1973. Cf. also Giusti's *Introduction* in Galilei 1990.

[67] For example, in the section *On Naturally Accelerated Motion*, in the Third Day, Theorem II, VI (2nd proof), and XI use results proven for uniform motion. Cf. Galilei 1974, pp. 167, 179, 184.

proof of Proposition I. We then proceed to analyse the difficulty connected with the use of the equimultiples in Proposition II, whose text Galileo significantly did not furnish in its entirety.

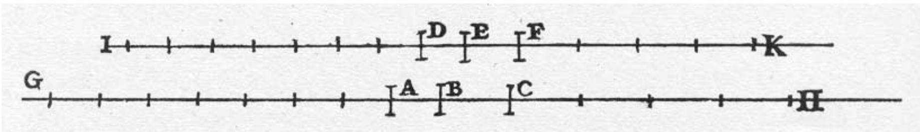

Fig. 6. Galileo's construction for the application of equimultiple proportionality to uniform motion

Referring to Fig. 6, let line IK represent time and let a movable move along line GH. Let AB be the space traversed in time DE and BC the space traversed in time EF. Galileo sets out to prove the following

> **Proposition I.** If a moveable equably carried [*latum*] with the same speed passes through two spaces, the times of motion will be to one another as the spaces passed through.[68]

His proof proceeds as follows. Let a number **m** of spaces, equal to space AB, and a number **n** of spaces, equal to space BC, be taken, respectively, on the left- and right-hand side of AB. Let the same number **m** of times, equal to time DE, and the same number **n** of times, equal to time EF, be taken, respectively, on the left- and right-hand side of DE. By recalling Galileo's definition of *equable motion*,[69] one can assert that: a) there being in EI as many equal times as there are equal spaces in BG, the whole EI is the time necessary to travel the whole distance BG, and the same goes for the whole time KE and the whole distance BH; b) if space GB were equal to space BH, then time IE would be equal to time EK. In addition, by virtue of Galileo's Axiom I,[70] if space GB were greater/smaller than space BH, then time IE would be longer/shorter than time EK. It is therefore true that if $\mathbf{m}AB \gtreqless \mathbf{n}BC$ then $\mathbf{m}\,DE \gtreqless \mathbf{n}EF$. Thus AB is to BC as DE is to EF.[71]

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[68] Galilei 1974, p. 149.

[69] 'Equable or uniform motion I understand to be that of which the parts run through by the moveable in any equal times whatever are equal to one another', Galilei 1974, p. 148.

[70] 'During the same equable motion, the space completed in a longer time is greater than the space completed in shorter time', Galilei 1974, p. 148.

[71] The structure of this proof is similar to that given as first Proposition in Archimedes' *De Lineis Spiralibus*. Cf. Archimedes 1544, p. 101. As is well known, Galileo used to refer to Archimedes as *the divine Archimedes*. He made notes on the *De Sphaera et Cylindro*. And we can assume that he would have known Archimedes' proof very well. At the beginning of his tract on proportions, Galileo said the 'fogs' which his mind had remained shrouded in after studying the Fifth Book Euclid's Elements began to disappear when he encountered the first proposition of Archimedes' *On Spirals*. On that occasion, he says, he began to think whether there was another, better way of going about proportions (Drake 1995, pp. 424–425). Cf. Galileo's notes on *De Sphaera et Cylindro* in Galilei 1890–1909, I, pp. 229–242. See also

We now need Galileo's text of the proof of this Proposition. It is necessary to quote the entire text of the proof because in a moment we shall need to reconstruct from this passage the incomplete text of Proposition II, which Galileo does not furnish in detail, probably assuming that it should be evident from the procedure already applied in Proposition I. With reference to Fig. 6, Galileo asserts:

> Let the moveable equably carried with the same speed pass through two spaces, AB and BC; and let the time of motion through AB be DE, while the time of motion through BC is EF; I say that space AB is to space BC as time DE is to time EF. Extend the spaces toward G and H, and the times toward I and K. In AG take any number of spaces [each] equal to AB, and in DI likewise as many times [each] equal to DE. Further, let there be taken in CH any multitude of spaces [each] equal to CB, and in FK that multitude of times [each] equal to EF. Space BG and time El will now be equimultiples of space BA and time ED [respectively], according to whatever multiplication was taken. Similarly, space HB and time KE will be equimultiples of space CB and time EF in such multiplication. And since DE is the time of movement through AB, the whole of EI will be the time of the whole [space] BG, since this motion is assumed equable, and in EI there are as many equal times DE as there are equal spaces BA in BG; and similarly it is concluded that KE is the time of movement through HB. But since the motion is assumed equable, if the space GB is equal to BH, the time IE will be equal to time EK, while if GB is greater than BH, so will IE be greater than EK; and if less, less. Thus there are four magnitudes, AB first, BC second, DE third, and EF fourth; and of the first and third (that is, of space AB and time DE), equimultiples are taken according to any multiplication, [i.e.] the time IE and the space GB; and it has been demonstrated that these either both equal, both fall short of, or both exceed the time EK and the space BH, which are equimultiples of the second and fourth. Therefore the first has to the second (that is, space AB has to space BC) the same ratio as the third to the fourth (that is, time DE to time EF); Q.E.D.[72]

Let us now turn our attention to Proposition II. The diagram remains the same as that given in Fig. 6, but Galileo assumes that the segments of line IK represent the speeds with which a movable traverses certain spaces along line GH in equal times. Thus, he goes on to assert his

> **Proposition II.** If a moveable passes through two spaces in equal times, these spaces will be to one another as the speeds. And if the spaces are as the speeds, the times will be equal.[73]

Giusti 1993, pp. 106–114, in which Giusti shows that in a note on *De Aequeponderantibus* that was written in the margin of a page of the same edition of Archimedes (Archimedes 1544) from which Antonio Favaro took Galileo's notes on the *De Sphaera et Cylindro* – and which now Giusti attributes to the Galilean school, possibly to Galileo himself, though it was not published by Favaro in the National Edition – there is a correction of the printed text in accord with the wording of Galileo's alternative definition of proportionality. Cf. also Di Girolamo 1999, on the Archimedean influence on Galileo's early theorems on centres of gravity.

[72] Galilei 1974, pp. 149–150.

[73] Galilei 1974, p.150.

At first glance, Galileo's proof of Proposition II would appear to be very straightforward, almost a banal repetition of the preceding proof. Here is his sketchy proof.

> Taking the previous diagram [i.e. that of Fig. 6], let there be two spaces, AB and BC, completed in equal times, space AB with speed DE and space BC with speed EF; I say that space AB is to space BC as speed DE is to speed EF. Again, as above, taking *equimultiples both of spaces and speeds* according to any multiplication – that is, GB and IE [equimultiples] of AB and DE, and likewise HB and KE [equimultiples] of BC and EF – it is concluded in the same way as above that multiples GB and IE either both fall short of, or equal, or exceed equimultiples BH and EK. Therefore the proposition is manifest.[74]

In the proof of Proposition I, we had equimultiples of spaces and times. Here we have again equimultiples of space, but instead of equimultiples of time, Galileo now introduces equimultiples of speed. He does not seem to be aware that, whereas space and time may certainly be conceived of as being quantities somehow intuitively summable – i.e. such that any parts of them may be added to one other so as to obtain a greater amount of the same quantities – the sum of speeds turns out to be more problematic. In order to see exactly why and how, we need to re-construct the complete proof of Proposition II, which Galileo did not furnish. To do so, we re-write the text of the proof of Proposition I, carefully substituting for every occurrence of the term 'time' the term 'speed' and making minor linguistic adjustments. Let us call our exercise 'pseudo-Galilean proof of Proposition II'. We write it in a different indented format.

> Extend the spaces toward G and H, and the ***speeds*** toward I and K. In AG take any number of spaces [each] equal to AB, and in DI likewise as many ***speeds*** [each] equal to DE. Further, let there be taken in CH any multitude of spaces [each] equal to CB, and in FK that multitude of ***speeds*** [each] equal to EF. Space BG and ***speed*** EI will now be equimultiples of space BA and ***speed*** ED [respectively], according to whatever multiplication was taken. Similarly, space HB and ***speed*** KE will be equimultiples of space CB and ***speed*** EF in such multiplication. And since DE is the ***speed*** of movement through AB, the whole of EI will be the ***speed*** of the whole [space] BG, since this motion is assumed equable, and in EI there are as many equal ***speeds*** DE as there are equal spaces BA in BG; and similarly it is concluded that KE is the ***speed*** of movement through HB. But since the motion is assumed equable, if the space GB is equal to BH, the ***speed*** IE will be equal to ***speed*** EK, while if GB is greater than BH, so will IE be greater than EK; and if less, less. Thus there are four magnitudes, AB first, BC second, DE third, and EF fourth; and of the first and third

[74] *Ibid.*

> (that is, of space AB and ***speed*** DE), equimultiples are taken according to any multiplication, [i.e.] the ***speed*** IE and the space GB; and it has been demonstrated that these either both equal, both fall short of, or both exceed the ***speed*** EK and the space BH, which are equimultiples of the second and fourth. Therefore the first has to the second (that is, space AB has to space BC) the same ratio as the third to the fourth (that is, ***speed*** DE to ***speed*** EF); Q.E.D.

If we now dissect the body of the proof in order to examine its internal structure, we encounter one major problem. In what sense can pseudo-Galileo assert that

> since DE is the ***speed*** of movement through AB, the whole of EI will be the ***speed*** of the whole [space] BG?

Again, as in the case of weights and volumes, is Galileo not taking for granted that the elementary geometrical relationships between the lines he chooses to represent physical quantities can be transferred into the domain of physical quantities? Awkwardly, the assembled speed, EI, cannot be the speed of the moveable passing through space GB, because it is assumed that its motion is uniform and its speed is constant, i.e. its speed remains the same as speed DE. It is the assembling of the equimultiples outside of the domain of pure mathematics that turns out to be problematic. The relationships between simple geometrical quantities, like line segments, do not necessarily reflect the relationships between the physical quantities that Galileo associates to them.

Galileo's concerns with the obscurity of the equimultiple definition were probably not only the result of a purely intellectual dissatisfaction with the lack of simplicity that he attributed to this definition,[75] but also the consequence of the awareness of the difficulties he had encountered in applying it to the physical realm. This awareness is clearly signalled by his hesitation over the equimultiples both in the case of weights and volumes (because in the *Discourse* on buoyancy he abandoned his earlier *De Motu* proof) and of uniform motion (because he did not furnish a complete text of Proposition II).

Since Galileo died before publishing a second edition of *Two New Sciences*, we cannot know whether he would have gone so far as to re-formulate some of his ideas on uniform motion taking into account his model of equivalence. He might have replaced the theorems on uniform motion dependent on Euclid's 'obscure' equimultiple proportionality with new ones based on his more 'intuitive' alternative. It is a fascinating possibility that cannot be ruled out. In his *Discourse* on buoyancy, he had already abandoned an earlier proof based on the equimultiples (cf. also the discussion in footnote 71).

To sum up, strong opposition to the 'obscurity' of the equimultiple definition informs Galileo's re-thinking of the foundations of Euclid's theory of proportions. In essence, Galileo's attempt to clarify these foundations is based on a definition

[75] 'I say that, in order to give a definition of the assumed proportional magnitudes suitable to produce in the mind of the reader some concept of the nature of these proportional magnitudes, we must select one of their properties. Now the simplest [property] of all...'. Drake 1995, 424.

of proportionality profoundly different from Euclid's. He rejects the equimultiple definition as obscure and replaces it with a new one that, according to him, should be more intuitive. Then, he proves that the equimultiple definition can be derived from his new definition of proportionality, and *vice versa*. He also gives two other theorems that show that Euclid's definition of greater ratio follows from his own definition of greater ratio, and *vice versa*. That Galileo's model of equivalence was inspired by Clavius' model can be gathered from their following characteristics:

a) the general structure of both models is identical, i.e. they are both based on a set of four theorems that have the same purpose and the statements of which have similar formal structure;
b) the second and the third theorems even share the same pattern of proof.

In particular, the similarity of the complex pattern of proof of the second theorem of both Galileo's and Clavius' models of equivalence is an indication that Galileo must have had Clavius' commentary at hand when he first worked out the details of his demonstration. Indeed, it would have been quite an extraordinary coincidence if they had come up with such complicated proofs independently of each other.

Euclid might seem to be somehow redundant since he gives a general definition of proportionality in Book V and another one, restricted to numbers, in Book VII. Perhaps ancient mathematicians were aware of the difficulty of conferring a clear meaning to the equimultiple technique. Definition 20, Book VII, could well serve the purpose of pointing to a possible way of interpreting the perpetual agreement of the equimultiples. Certainly Clavius tried to shed light on the *obscurity* of that perpetual agreement by means of rational proportionality. And this inspired and encouraged Galileo's foundational study of the theory of proportions.

In his book devoted to the mathematical way in the scientific revolution, Peter Dear, claims that

> recent research has shown that Galileo aimed at developing scientific knowledge, whether of moving bodies or of the motion of the earth, according to the Aristotelian (or Archimedean) deductive formal structure of the mixed mathematical sciences.[76]

Aristotelian traditions (especially that of the pseudo-Aristotle's *Mechanical Questions*) and the Archimedean tradition were certainly important in Galileo's approach to the mathematization of natural philosophy. But the Euclidean theory of proportions with its attendant foundational implications may have had a more direct impact on the formation of Galileo's mathematical natural philosophy. The influence of a 'Eudoxan proportional reasoning' on Galileo's notion of *momento* has been recently recognised.[77]

Clavius' and Galileo's discussions on the foundations of Euclidean proportionality suggest that they saw the concept of proportionality as open to different interpretations. Moreover, as we have seen, the meaning of equimultiple proportionality becomes problematic when it is applied outside of the domain of pure mathematics, because there are no

[76] Dear 1995, p. 126.
[77] De Groot 2000, p. 646.

simple geometrical relationships that can guarantee the principles whereby equimultiples can be associated to one another. We need to know much more about the mathematics of Galileo's time in order to assess the import that proportionality theory had on the 'proportional reasoning' that led to the mathematization of natural philosophy. Further research on the impact that the late sixteenth- and early seventeenth-century debate on the foundations of proportionality had on the natural philosophers who embraced the study of the mathematical language of nature could improve our understanding of why they came to equate *mathematical* enquiry about nature with *rational* inquiry about nature.

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