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# *A New Look at Galileo's Search for Mathematical Proofs*

P. PALMIERI

*Communicated by* N. M. SWERDLOW

**Contents**



## 1. Introduction

Galileo's processes of mathematical discovery have been the subject of interest of scholars, who have attempted to reconstruct his path towards the theorems published in the celebrated *Two New Sciences* (1638). So far the results have been, in my view, rather inconclusive. Though not denying that light has been cast on specific questions, I think that the reason why little progress has been made concerning Galileo's mathematical discoveries, and no consensus reached, has to do with the historiographic approach that has been the distinctive feature of the relevant literature.

The basis of most studies concerning these issues has been the so-called *Manuscript 72*, a manuscript collection which preserves notes, diagrams, computations, and various drafts on the mathematical treatment of motion written by Galileo in the course of about four decades. No chronology has been established for the various materials incorporated in the manuscript, which are undated and *apparently* haphazardly bound. Thus no wonder that disagreement as to matters of date is writ large in Galileo scholarship. So far the principal historiographic approach has been the re-ordering of the *Manuscript 72* materials according to the order of publication in *Two new sciences* of the more or less corresponding results in their final state. Often this approach has led to the regrouping of the sparse fragments according to the perceived relevance to a particular theorem and/or proof in *Two new sciences*. These are the editing criteria followed by Antonio Favaro, the editor of the so-called *National Edition* of Galileo's works (the foundation of modern Galileo scholarship), who at the turn of the twentieth century published these materials, most for the first time.[1]

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[1] See Galilei 1890–1909. In this paper, I have quoted this edition, the so-called *National Edition*, in 20 volumes, by giving the volume in Roman numerals and the pages in Arabic numerals.

In recent decades Stillman Drake, Winifred Wisan, Paolo Galluzzi, and Jürgen Renn *et al.* analyzed various portions of *Manuscript 72* and furnished tentative reconstructions of Galileo's processes of discovery, often in relation to the mathematics of the laws of falling bodies and projectile motion.[2] These scholars tend to follow the same criteria pioneered by Antonio Favaro. They organize the material of *Manuscript 72* around blocks of theorems and proofs as they appear in *Two new sciences*. Consequently they tend to look at that material as a reservoir of relevant but only partially successful antecedents of the finally published results. However, they often reach incompatible conclusions. Why this puzzling situation? Obviously, because of its fragmentary nature, it is true that *Manuscript 72* lends itself to different interpretations. But I believe that a richer explanation of this historiographic puzzle is to be found in an assumption which has shaped both the strengths and weaknesses of this scholarship.

Undoubtedly the regrouping of the fragments of *Manuscript 72* around blocks of theorems and proofs has proved to be useful in highlighting relations between impervious materials, which at first sight are difficult to interpret out of context. It has, however, one drawback. In re-organizing the fragments in order to form a more coherent context for particular proofs, this methodology unwittingly adulterates the significance of the originals. It does so by mechanically re-assembling them according to the deductive sequences which are read off the proofs published in *Two new sciences*. In other words, this methodology works on the assumption that the final structure of the published proofs is in itself the explanatory principle of Galileo's processes of mathematical discovery. This assumption is too strong an organizing criterion. It needs relaxing if we are to penetrate the processes underlying mathematical creativity.

In this paper, I will challenge this assumption and suggest a new approach to *Manuscript 72*. I will show that the form that two proofs finally take in *Two new sciences* is the result of a modeling process which is (mostly) not driven by the deductive sequence shaping the final proofs. I will describe the reasoning processes underlying Galileo's

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Most of the material of *Manuscript 72* was published as an appendix to *Two new sciences*, in volume 8 of this edition (cf. Galilei 1999, for an electronic facsimile of the manuscript). A thorough attempt at a chronological re-ordering of *Manuscript 72* was made by Stillman Drake (Drake 1979). Drake cut and pasted (so to speak) bits and pieces of the original manuscript and reprinted them in collage-like form according to the probable year of composition. The work was carried out on the basis of watermarks and other internal criteria, such as style of handwriting, but on the whole failed to give a convincing chronology. The reason is that it is possible to distinguish fairly accurately the hands of the young and mature Galileo, but finer differences in style are harder to detect. Even Antonio Favaro, who studied Galileo's manuscripts for fifty years or so, could not establish a detailed chronology beyond the basic partition between juvenile and mature styles (see Favaro's comments, in Galilei 1890–1909, VIII, pp. 35ff.).

[2] See Wisan 1974, Drake 1970, 1973, 1974, 1979, 1987, 1995 (all reprinted in Drake 1999), Galluzzi 1979, Renn 1988, Damerow et al. 1992, 2004 (basically a 2nd edition of idem 1992, but with some reworking), Damerow et al. 1996. An exception to the general tendency is Renn, Damerow, & Rieger 2000, who, in the case of the parabolic trajectory of projectiles, propose to do away with the notion of "discovery" conceived of as a circumscribable event occurring at a precise point in time. In relation to the law of falling bodies, I myself have proposed a radical reconsideration of traditional historiography in "Galileo's construction of idealized fall in the void", forthcoming in *History of Science*.

search for new proofs, and try to shed light on the power of his practice. I will also show how blocks of deductive sequences gradually are formed out of the modeling process. I will describe the rich processes, based on diagrams, underlying Galileo's search for new proofs. I will also try to illuminate the power of his original approach to discovering and solving problems.

Given the complexity of the material in *Manuscript 72* I will not be able to illuminate the processes that led Galileo to all the proofs published in *Two new sciences*. I will restrict my analysis mostly to situations for which, in my view, sufficient evidence is extant in *Manuscript 72*. In particular, I will explore Galileo's path to Problems 13 and 14 in finer detail.[3]. They basically consist in finding the length of a vertical (Problem 13) or the length of a portion of an incline (Problem 14), such that certain conditions on the times of descent along the vertical and the inclines are satisfied.

Winifred Wisan reconstructed Galileo's process of discovery of the solution to one of these problems (Problem 13).[4] Wisan believes that Galileo is here working according to the Greek method of *analysis*.[5] Though not explaining what she means by "Greek method of analysis" it seems that Wisan intends to refer to a process by which mathematical proofs are first discovered in a certain deductive sequence, the *analysis*, which is subsequently reversed in the *synthesis*, the standard form of presentation of theorems in classical Greek geometry. This meaning is what, I think, underlies her work. The difficulty with this approach is that Wisan reads off Galileo's published proofs a deductive sequence which she then mechanically reverses, looking for antecedents in the manuscripts. In fact entire blocks of deductions which she inserts in her reconstructed analysis of Problem 13 are nowhere to be found in the manuscripts.[6]

Further, and most interestingly, Wisan excludes Problem 14 from her considerations, the wealth of relevant materials in the manuscripts notwithstanding. Why? Because, in *Manuscript 72*, a partial *analysis* (in Wisan's terminology) of Problem 14 appears to have been written by Galileo in a handwriting style which she is convinced to be later than the style of an autograph version of the corresponding *synthetic* proof, eventually published in *Two new sciences*. How could the analysis possibly have been thought up and/or written *later than* the synthesis? The exclusion, however, is simply the artifact of two assumptions and a problematic piece of handwriting evidence.[7] The first assumption is that Galileo is following the Greek method of analysis; the second assumption is that the latter consists in devising proofs whose deductive structure is subsequently reversed

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[3] Galilei 1974, pp. 207–210, and Galilei 1890–1909, VIII, pp. 255–259

[4] Wisan 1974, pp. 249–258.

[5] Wisan 1974, p. 249.

[6] See Wisan 1974, p. 256, deductive steps labeled g-h-i-j-k-l-m, and q-r-s. Of course it is also possible that lost folios might have contained more relevant material than has actually been preserved in *Manuscript 72*, though at present this remains an unsolvable problem. Cf. the editors' introduction to the history of the Galileo codex, in Galilei 1999. The fact that not only are the above-mentioned blocks of deductions not in the surviving manuscripts, but no clues to those blocks of deductions are to be found in the extant folios suggests to me that they never existed.

[7] Drake 1979, pp. 107–108, does not comment on Wisan's claims but implicitly disagrees with her, attributing both the manuscript analysis and the manuscript synthesis to 1609, with the analysis preceding the synthesis.

in the syntheses. Both assumptions are unwarranted. As for the first I will show why in the following sections. As for second let us note that no consensus whatsoever has been reached in the literature about what the so-called "Greek method of analysis" consists of.[8] In a word, nobody knows what it is, let alone if it ever was practiced, by whom and for what purposes. Thus, in my view, having recourse to the Greek method of analysis in order to explain Galileo's mathematical practice is tantamount to explaining *ignotum per ignotius*.

Indeed, in my view, Problems 13 and 14 are the most complex examples of Galileo's creativity in mathematics. First of all, Galileo had to excogitate them and then solve them. The two processes, however, as we shall see, can hardly be distinguished, either chronologically or logically. Let us examine the first emergence of the two problems, the form they finally took, and the discovery of their solutions. We shall try to reconstruct Galileo's mathematical practice on the basis of Galileo's manuscripts.

## 2. Diagrams at work

Is there a fundamental activity allowing Galileo to start shaping mathematical objects? Before answering this question one thing to keep in mind is that at the turn of the sixteenth century the mathematical search space is not an empty recipient to fill with new proofs. On the contrary, as we shall see, it is exceedingly rich. It is more like a junkyard of past mathematical results, though. And for Galileo the past results that mattered were those of classical Greek geometers. Euclid and Archimedes figured most prominently, but also Apollonius and Pappus, the latter freshly coming onto the printing-press scene at the time when Galileo was just about to start his academic career at Pisa.[9] Unfortunately,

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[8] My ideas about analysis have been influenced by highly stimulating discussions with Kenneth Manders. A valuable summary paper of the questions raised by scholars in connection with analysis (not only Greek analysis) is Beaney 2003. I have relied also on: Euclid 1956 (various commentaries by Thomas Heath), Mahoney 1968, Hintikka and Remes 1974, Knorr 1993, Behboud 1994, Netz 2000 (with further bibliography on the debate). I believe that Heath's comment to the effect that what Greek analysis consisted of "...must remain apparently an insoluble mystery" still holds true today (Euclid 1956, III, p. 246). This transpires from the strong disagreements in the literature. I am referring especially to Netz 2000, whose shocking conclusions are as follows. Contrary to received wisdom, according to which analysis somehow provides a form of heuristics, "...Greek mathematical analysis should be understood as a tool for the presentations of results, rather than as a tool for their discovery", in ibid., p. 156. The view that Greek analysis affords heuristic power to the mathematician is stressed, for instance, in Hintikka and Remes 1974, p. xiii, and Behboud 1994, pp. 72ff.

[9] The first printed edition of Pappus' *Mathematical collections* was published in 1588 (Pappus 1588). As for the Renaissance Archimedes, see, for instance, Archimedes 1543 and 1544. On Galileo's mathematics, and more specifically on his adoption of Euclidean proportion theory, a

we have lost the original activities that shaped the lives of Greek geometers. All we have at our disposal, and all Galileo had at his disposal, are presentations in a deductive style, all too often corrupted by centuries of copying and editing. The Greek scrapheap, so to speak, was luridly rich, but the wrecks rusty and out of order.[10] Fortunately some of Galileo's original activities, or at least vestiges thereof still survive. They afford us the rare opportunity of a catching a glimpse of a mathematician in the flesh busy at work.

Upon surveying *Manuscript 72* I recognize two most important morphological characterisitics, *diagram* and *text*, the latter often written in the spaces around diagrams. In many cases texts neatly wrap around the diagrams they refer to. This suggests that diagrams are drawn first. They tend to receive attention before the texts.[11] Diagramming must have been the breaking of the ice in the process of giving shape to a mathematical object. There is at least another important feature which would deserve sustained scrutiny, *computing*.[12] However, since I do not find it relevant to Problems 13 and 14, the focus of this paper, I will leave it aside.

First of all, diagrams in *Manuscript 72* are not static entities. They express an intrinsic dynamics that the mathematician's eye may spot. This seems to have happened with the set of four diagrams laid out by Galileo on folio 96 verso (Fig. 1). There is also a text on the folio, which wraps around two of the diagrams. When Galileo ran out of paper space at the bottom he continued the text at the top following a recall sign. Obviously the text was written last. Thus a process of modeling developed around the diagrams before the writing of the text. Galileo drew the diagrams according to the sequence 1, 2, 3, 4 (see Fig. 1 and Table 1).

considerable body of literature is now available, which allows us to understand most of its technical aspects better. Cf. Armijo 2001, Drake 1973, 1974, 1987, Frajese 1964, Giusti 1981, 1986, 1990, 1992, 1993, 1994, and 1995, Maracchia 2001, Palladino 1991, Palmieri 2001 and 2002. For a general treatment of relevant aspects of the Euclidean theory of proportions I have relied on: Grattan-Guinnes 1996, Sasaki 1985, Saito 1986 and 1993. Rose 1975 is an extensive, immensely erudite survey of Renaissance mathematics in Italy from a non-technical point of view. Cf. also Sylla 1984 and 1986.

[10] Giusti 1993 and Palmieri 2001 document the struggle of mathematicians to come to terms with Euclid's theory of proportionality, in the late sixteenth- and early seventeenth centuries. A comprehensive study of the influence of Archimedes on Galileo's work is badly needed (see Frajese 1964).

[11] See, for instance, Galilei 1999, folios 86 recto and 54 verso, where texts wrap around two diagrams concerning parabolic trajectories and a circle, respectively.

[12] Cf. especially Galilei 1890–1909, VIII, pp. 419ff. Here Galileo calculates the times of descent through chords of arcs of a circle in order to approximate the time of descent through the arc. This example constitutes the background of the conjecture put forward in *Two new sciences* that the arc of a circle is the brachistochrone. It is a fascinating computational approach that, as far as I know, has never been considered in the Galileo literature.

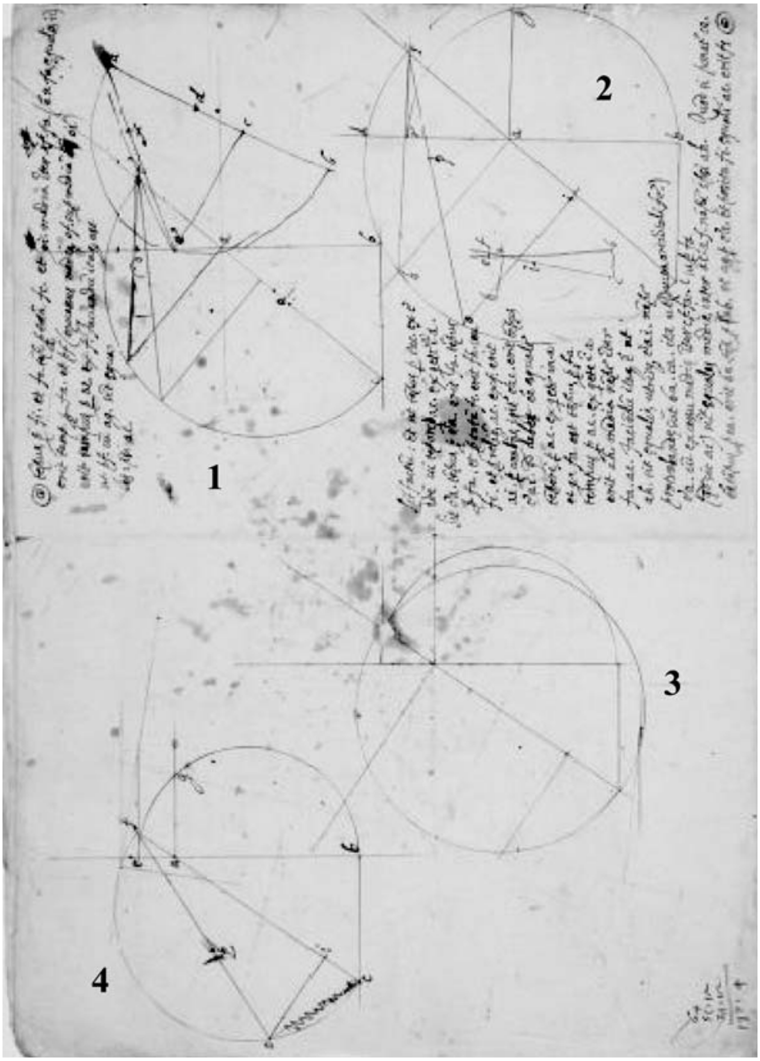



**Fig. 1.** Folio 96 verso, *Manuscript 72*. The four diagrams have been drawn according to the sequence 1, 2, 3, 4. Notice also the text wrapping around diagrams 2 and 1

So the starting diagram was number 1, which also appears to be more tentative (a few lines are drawn in free hand). The remaining three are wholly drawn with ruler and pair of compasses. Superficially all the diagrams look the same. They are not.

First of all, what is represented in the diagrams? In Table 1, I have emphasized the important elements of the four diagrams so as to make them more legible. Galileo is interested in time-space relations among bodies falling along the inclined planes and the verticals. The spaces are represented by lines, inclined or vertical. Galileo also knows that the diagrams automatically, so to say, embed the times of descent. He has learned this peculiar fact well before drawing these diagrams. We shall see how later on towards

**Table 1.** Diagrams 1,2,3,4. The grey triangles represent inclined planes. The black, thick vertical segments represent heights from the topmost point of which bodies start falling before deviating along the inclines. I have also added in black circles representing falling bodies along the inclines. Note that *the vertical segment also represents the time of descent along the vertical segment itself*

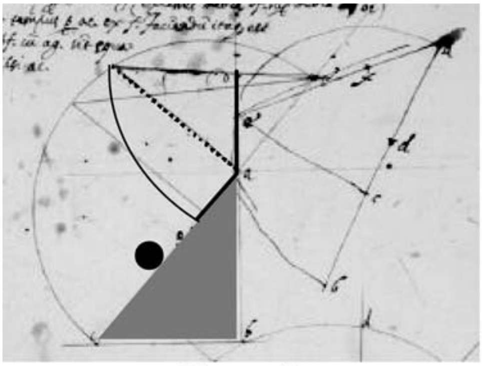


Diagram 1

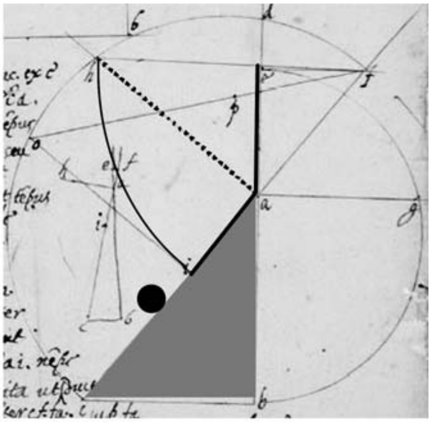


Diagram 2

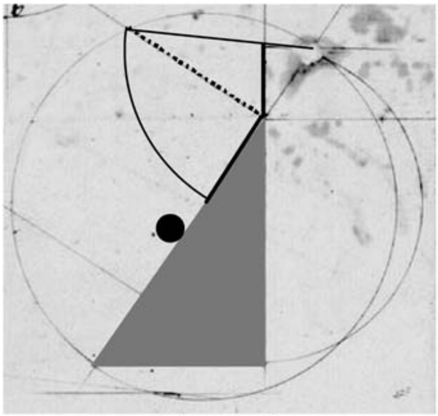


Diagram 3

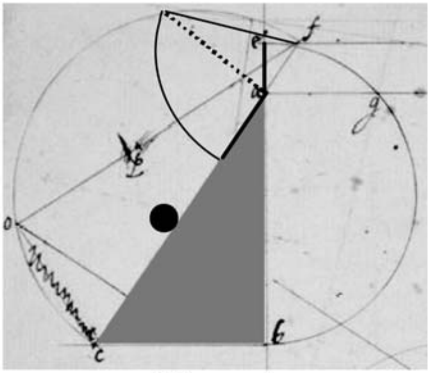


Diagram 4

the end of this section. For the time being it will suffice to note the following. First Galileo assumes that the vertical segment is also representing the time of descent along that vertical segment (this is equivalent to establishing an arbitrary unit for the times of descent). The time of descent of a body starting from rest at the *topmost point of the inclined plane* is represented by the segment perpendicular to the plane at the point of departure and intersecting the circle whose diameter is the inclined plane prolonged as far as the horizontal through the topmost point of the vertical segment (dashed lines in the diagrams; the line in diagram 4 is not drawn by Galileo, I have added it). The time of descent of a body starting from rest at the *topmost point of the vertical segment* and deviating along the incline is a little more complex. The vertical segment represents the time along the segment itself. Then Galileo adds the time along the incline when the body deviates from the vertical, continuing downwards. The latter is given by the

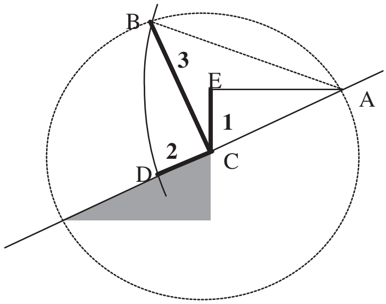



**Fig. 2.** The diagrammatic construction. You start with the inclined plane (grey triangle) and the vertical segment (labeled **1**)

difference between the segment joining two intersections, the intersection between the horizontal already mentioned and the circle's diameter, and the intersection between the circle and the perpendicular to the incline (added black lines on the inclined planes). In words this looks tremendously complicated but it is much easier to read these relations off the diagrams. I summarize the process as follows (Fig 2).

The grey triangle is the inclined plane. Prolong the incline upwards as far as the horizontal passing through the highest point of the vertical. Draw the circle, center C. From the uppermost point of the vertical E draw a horizontal that intersects the circle at A. Next, draw a perpendicular to the inclined plane from its uppermost point that also ends on the circle at B. Draw the chord AB connecting these latter two endpoints. With this chord as radius, and with the endpoint A as center, draw an arc from endpoint B that intesects the inclined plane at D, thereby constructing the segment CD. The thick lines in Fig. 2, i.e., segments CE, CB and CD, represent times. So CB (or **3**) is the time of descent of a body starting from rest at the topmost point of the inclined plane, while EC+CD (or **1+2**) is the time of descent of a body starting from rest at the topmost point of the vertical segment and deviating along the incline.

Thus diagrams embed complex relations between times. At some point Galileo is struck by one fact. It could be possible that time **3** equals time **1+2**. In other words, it could be possible that the time of descent of a body starting from rest at the topmost point of the inclined plane equals the time of descent of a body starting from rest at the topmost point of the vertical segment and deviating along the incline. An interest is quickly developing around this exciting possibility. Note that no commitment need be made at this stage about the final form of the problem. Simply an interest emerges which we might call the *embedding of times of descent in diagrams*. Here we have possibly captured an irreducible element of Galileo's mathematical practice. A powerful (but also powerfully constraining) means of representation, the *diagram*, provides a mathematical object and at the same time the manipulation rules of the object. It all comes in one package, so to say, take it or leave it.

The very possibility that time **3** equals time **1+2** can be further explored by trials and errors, i.e., by redrawing the diagram. This is exactly what the sequence of diagrams 1, 2, 3, 4 accomplishes. In table 2, I have summarized the progressive approach to the equality of the times of descent suggested to Galileo by the exciting possibility that time

**Table 2.** The progressive approach to the equality of the times of descent suggested by the possibility that time **3** might equal time **1+2**. I myself easily read the lengths off Galileo's diagrams with a ruler graduated in millimeters

| | Segment 1 | Segment 2 | Segment 3 | % error |
|---|---|---|---|---|
| Diagram 1 | 26 | 24 | 30 | 66 |
| Diagram 2 | 25 | 20 | 41.5 | 8 |
| Diagram 3 | 13 | 18 | 29 | 7 |
| Diagram 4 | 9 | 15 | 24 | 0 |

**3** might equal time **1+2** (within the degree of precision of my reading lengths off the diagrams).

Within the limit of precision of my reading the lengths off the diagrams, I have found that diagram 4 is "correct", i.e., the relation time **3** = time **1** + time **2** is satisfied (that is the error is equal to zero). Of course this does not mean that an exact solution must exist, since in fact the equality is satisfied only within the precision of the reading of lengths off the diagrams. However, Galileo now feels that an exact solution must exist. He repeatedly and explicitly tells himself that this must be the case. We can see his statements on folio 61 verso (Fig. 3). A diagram and two blocks of text testify to this bold assertiveness.

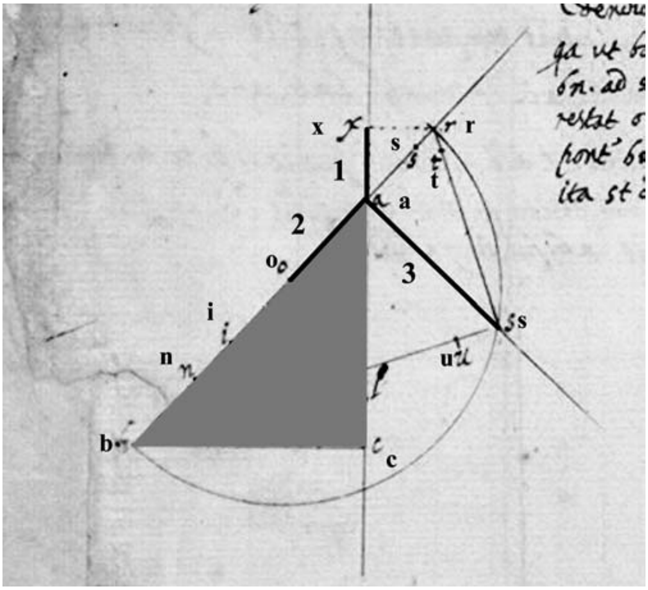



**Fig. 3.** A portion of folio 61 verso. Another diagram similar to those already seen. The diagram is drawn "correctly", i.e., the time segments satisfy the equality time **1** + time **2** = time **3**. To improve legibility I have emphasized some lettering in bold type (notice, however, that numbers 1, 2, 3 are not in the original)

In the first text, Galileo starts by assuming a unit measure of time such that segment **ac** be the time through vertical fall **ac**. Then he notices that the time of descent along incline **ab** after a fall from **r** must be equal to **ai**, . . . *ut verum est*, he exclaims concluding the text. Next he takes a different unit measure for time. He assumes that segment **xa** also is the time of descent along vertical **xa** itself. He then asserts the relation of times, that is times **1+2** must equal time **3**, and again exclaims to himself . . . *ut verum est*. Here are the two texts.

| | |
|---|---|
| Si *ac* sit tempus per *ac*, erit *bi* tempus per *xa*, et *ab* tempus per *ab*, et *bs* tempus per *rb*, et tempus per *ab* post *ra* erit excessus *bs* super *sa*, cui oportet ostendere aequari *ai*; *ut verum est*.<br><br>[If *ac* is the time through *ac*, *bi* will be the time through *xa*, and *ab* [will be] the time through *ab*, and *bs* the time through *rb*, and the time through *ab* after *ra* will be the excess of *bs* over *sa*, to which it is necessary to show that *ai* is equal; as is true.] | Si *xa* sit tempus per *xa*, erit *ar* tempus per *ra*, et *as* tempus per *ab* ex quiete in *a*, et *rs* tempus per totam *rb*, et excessus *rs* super *ra* (puta *ao*) erit tempus per *ab* post *ra*. Ostendendum ergo est, *xa* cum *ao* aequari *as*, *ut verum est*: . . .[13]<br>[If *xa* is the time through *xa*, *ar* will be the time through *ra*, and *as* [will be] the time through *ab* from rest at *a*, and *rs* the time through the whole *rb*, and the excess *rs* over *ra* (*ao*, say) will be the time through *ab* after *ra*. Thus, it must be shown that *xa* with *ao* is equal to *as*, as is true.] |

Now, how could Galileo possibly assert the "ut verum est" twice? How did he know that *it is true*? I believe that the answer is precisely the reading of the time segments off the diagrams themselves. Indeed, as already noted, the diagram on the folio is "correct" (within the precision of my reading in millimeters). In other words, Galileo must in some way have read the *ut verum est* off the diagrams, and quickly set out to find a way to confirm his finding.

He accomplished this on folio 94 recto (Fig. 4). On it we see a diagram and a short text. The diagram is "correct", in the same sense explained above. However, here there is no series of wrong diagrams converging to the right one. The diagram is spot on, correct from the very get-go. A "correct" diagram is indeed achievable very simply with ruler and pair of compasses empirically, i.e., by trial and error, as follows (again, "correct" within the precision of the empirical procedure, which involves physical instruments).

Start by drawing vertical segment **ab** (**1**, according to our nomenclature above), then add the incline of fall, or do vice versa (Fig. 5). Draw the perpendicular to the incline at the intersection of the latter with vertical segment **ab**. Draw the horizontal through the topmost point of the vertical segment (point **a**). Open the compasses and center it on point **e**, varying aperture until the circle cuts points **n** and **l**in such a way that the equality **ab+bn=bl** is satisfied. You can "see" the condition of equality with some training, and/or measure it with a ruler, or another pair of compasses. Then the circle through points **e**, **l**

[13] The texts are on folio 61 verso, *Manuscript 72*, published in Galilei 1890–1909, VIII, pp. 411. Italics are mine.

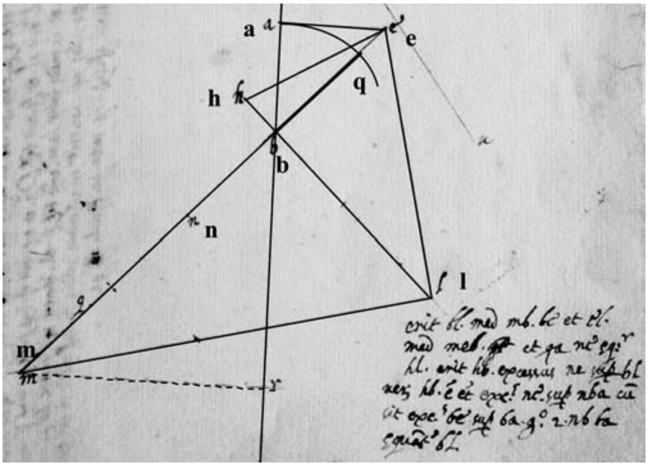



**Fig. 4.** Folio 94 recto. Galileo confirms the *ut verum est*. I have emphasized the lettering, and added letter **q** and the arc to show graphically that **ab** = **bq**

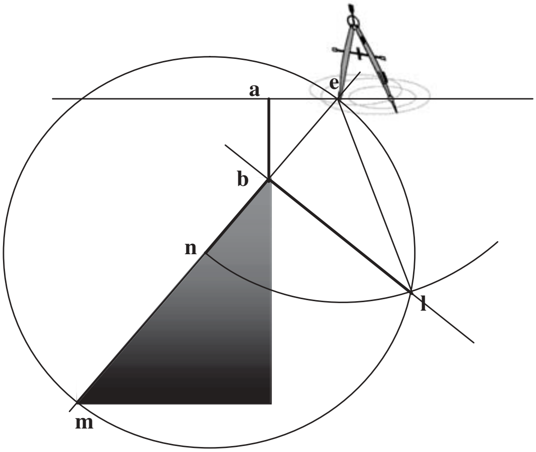



**Fig. 5.** A simple way to get the diagram right from the very beginning with ruler and pair of compasses

and center somewhere on the incline (which, once again, can be found by trial and error, centering the compasses on the incline and varying position of center and aperture) will intersect the incline at point **m**, thus defining the length of the incline. We need to pause here.

There are many "correct" diagrams in the manuscript (for example, folio 54 recto, folio 97 verso, folio 99 verso, etc.). All of these diagrams are spot on, no tentative preliminaries of the kind seen above. They suggest that the physical action of manipulating the compasses to find the right circle develops a special feeling in the mathematician. We might say that the "existence" of a solution (i.e., the answer to the question whether *the time of descent of a body starting from rest at the topmost point of the inclined plane* *might equal the time of descent of a body starting from rest at the topmost point of the vertical segment and deviating along the incline*) is "sensed" through the physical act of manipulating the compasses. One has to re-experience the actions involved in the manipulation of the compasses and ruler if one is to understand this processes of discovery. I said "sensed", which may seem rather vague, precisely because by repeating the actions myself I was rewarded by the discovery that this empirical procedure is a tantalizing process. It only gradually gives one a feeling. In fact the approximations in the drawing of the diagram, the tentative opening of the compasses, and the ruler-based checking of the equality, impose some limitations on how strong a feeling of a solution may be had. This depends on the fact that the uncertainties introduced by the physical instruments compresses, so to speak, the variability of the approximation around the solution-point.

However, on folio 96 recto (Table 3) we find almost a smoking gun. Here we can see how the sense of getting closer to the right construction may be developed. Galileo draws a diagram which keeps one segment in the equality of times constant and progressively increases the angle of the incline. We find four inclines, which I believe were tried in the sequence 1,2,3,4. The correct diagram, number four, came after Galileo "overshot" the solution by just a little bit.

In the text accompanying the diagram on folio 94 recto (bottom right corner of Fig. 4), Galileo finds a way to visualize the error in the empirical procedures. He has no exact geometrical construction so far, beyond the empirical procedures I described, but he has a construction for the error. Let us see how (Fig. 6).

> Erit *bl* media *mb*, *be*, et *el* media *meb*; et quia *ne* aequatur *hl*, erit *hb* excessus *ne* super *bl*: verum *hb* est etiam excessus *ne* super *nba*, cum sit excessus *be* super *ba*: ergo 2 *nb*, *ba* aequantur *bl*.[14]
>
> [*bl* will be the mean [between] *mb* and *be*, and *el* the mean [between] *me* and *eb*; and since *ne* is equal to *hl*, *hb* will be the excess [of] *ne* over *bl*: but *hb* is also the excess [of] *ne* over *nba*, because it is the excess [of] *be* over *ba*: therefore the two *nb* and *ba* are equal to *bl*.]

[*Notational help* (":" ratio; "::" proportion)

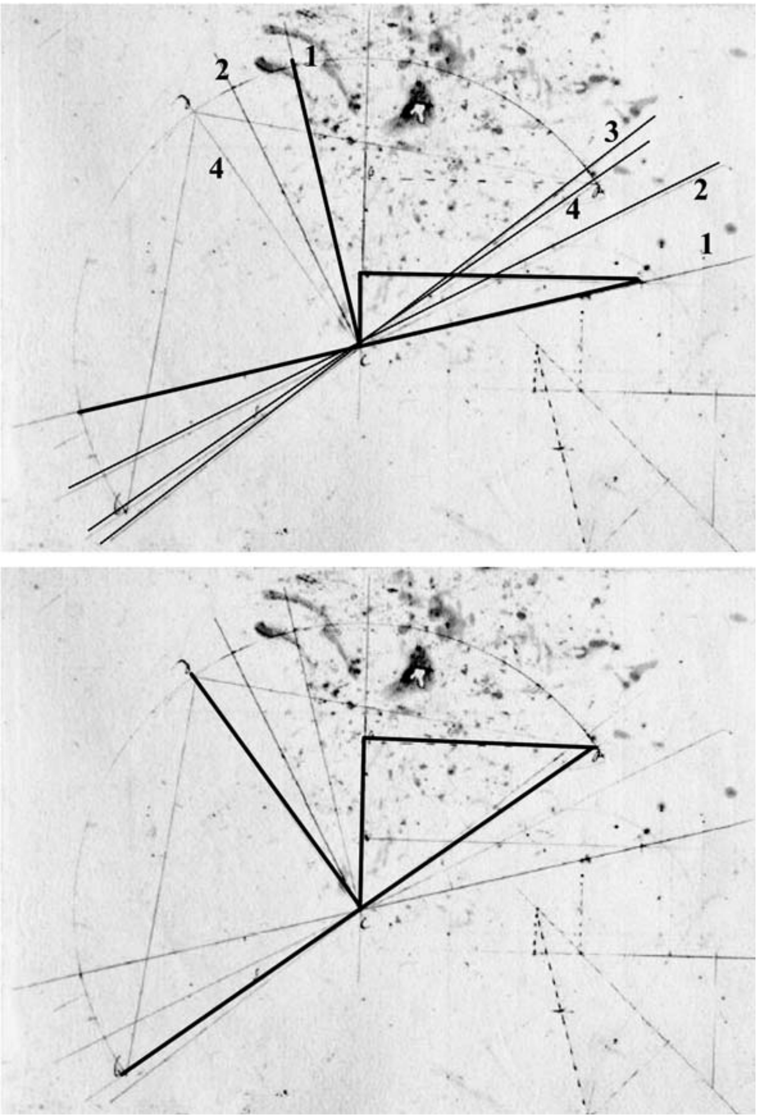
mb:bl :: bl:be
me:el :: el:eb
ne=hl
hb=ne – bl
*since* hb=be – ba ➔ hb=ne – (nb+ba)
*therefore* nb+ba=bl

]

This text is striking. The argument amounts to recognizing that the excess of one line (**ne**) over two lines (**bl**, **nq**, where I have added the letter **q** to Galileo's diagram to improve readability) is the same, and thus the two lines must be equal to each other. The two lines are times of descent, along the incline starting from **b**, and along the vertical starting from **a** plus the deviation along the incline, as in the cases already considered above. If you read the text out of the context discussed so far, as I erroneously tried to

[14] The texts are on folio 94 recto, *Manuscript 72*, published in Galilei 1890–1909, VIII, p. 403.

**Table 3.** Folio 96 recto. Searching for the right diagram: how the sense of getting closer to the right construction may be developed. Above: I have emphasized the first attempt. Below: I have emphasized the fourth and correct attempt. (The perpendicular to line 3, the third attempt, was not drawn by Galileo)




do for a long time, you end up concluding that Galileo is begging the question. Further, the fragment is disconnected from other relevant passages in the manuscript, and from what Galileo will publish in *Two new sciences*. How do we square all this?

The conclusion is that **nb+ba=bl**, i.e., the equality of times of descent. Thus it looks as if Galileo has found some kind of proof for the equality. Not so. The argument hinges

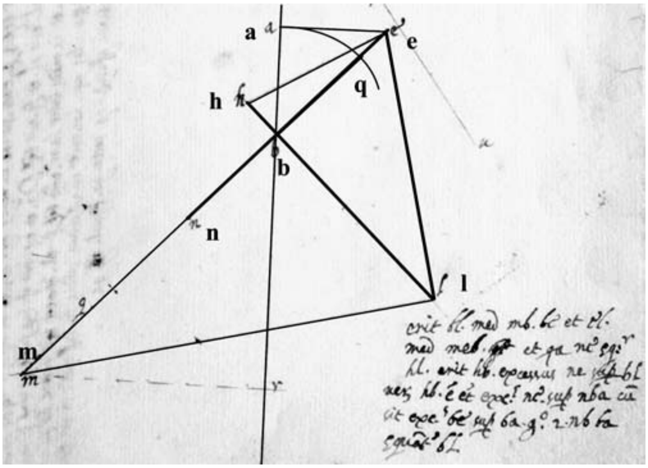



**Fig. 6.** Folio 94 recto. Galileo has found a simple procedure for quickly checking the "correctness" of the diagram on the diagram itself. Lines **le**, **en**, **lh**, are equal to each other by construction. As you can see, **hb** looks pretty close to **qe**. Thus the diagram is "correct"

on assuming that a certain quantity, the difference **be-ba**, is equal to another quantity, the difference **hl-bl**. So by construction the three lines **hl**, **le**, **en** must be equal to one another. Assuming that the difference **be-ba** is equal to the difference **hl-bl** seems thus tantamount to *ad hoc* assuming the equality of times of descent. Is it a circular form of reasoning? No, because this reasoning is not intended to prove the equality from independent premisses, as one might think at first glance looking at the conclusion (misleading, if read out of context), but simply to show that *if* the difference **be-ba** is equal to the difference **hl-bl** *then* the equality follows. In other words, Galileo is interested in what follows if the difference **be-ba** is equal to the difference **hl-bl**. What follows is the equality of times of descent. And when the difference **be-ba** is equal to the difference **hl-bl** *you have a simple way of quickly checking on the diagram the "correctness" of the diagram itself*, i.e., you have a simple way of drawing a right diagram, much simpler than checking the equality **nb+ba=bl** with a ruler or compasses,which involves an addition introducing another source of uncertainty.

*Embedding time in diagrams.* As we have seen, it is easy to follow the argument in the short text, but there are more general presuppositions on which it is based that are not made explicit in the text, and whose discussion I have postponed so far. It is important to recognize that at this stage Galileo is, so to speak, working for himself. Thus to understand the argument fully those presuppositions must be revealed. Galileo is playing with times of descent embedded in diagrams. The *embedding of times of descent in diagrams* is the process that fascinates him. He literally *sees* times of descent within the diagrams. How does this work?

Consider an incline and a vertical having the same "elevation", i.e., such that the vertical distance covered by the falling body is the same (Fig. 7). Take as a unit of time a segment such that the time needed to descend along the incline is the incline itself. Then the time needed to descend along the vertical is the vertical itself. You can do vice versa,

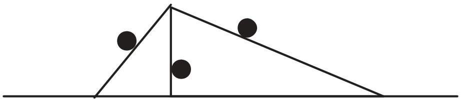


**Fig. 7.** The times of descent along a vertical are to those along inclines having the same elevation as the vertical is to the inclines

the converse is true as well. Next you can generalize. The time needed to fall along all inclines having the same "elevation", i.e., such that the vertical distance covered by the falling body is the same, is the incline itself. Briefly, Galileo knows that under certain circumstances time is *visible* within the diagram.

In *Two new sciences*, Galileo will publish a fundamental theorem concerning times of descent along inclines of the same elevation, proving that if

> the same moveable is carried from rest on an inclined plane, and also along a vertical of the same elevation, the times of the movements will be to one another as the lengths of the plane and of the vertical.[15]

It is a general statement. Take one time of descent along an incline to be the incline itself (this establishes a unit measure for time) and all inclines will automatically be their times of descent.

A glimpse of the tormented prehistory of the argument supporting this claim can be had from the tentative approaches surviving in *Manuscript 72*, folio 179 recto.[16] The argument is based on what we might call "a granular conception" of time and space continua. For example, Galileo talks of "innumera quaedam spaciola" and "gradus velocitatis". He relates degrees of speed at points on the vertical and on the incline. Then he performs a kind of summation of these degrees of speed in order to come up with Euclidean quantities. The intriguing complex of ideas beyond this approach need not concern us here, however.[17] The upshot of it all is the above theorem: *along inclines having the same elevation the times are to one another as the inclines*.

The question that looms large at this stage is as follows. Can a geometrical construction for the equality of times of descent be found? Could it really be possible that under certain conditions the time of descent of a body starting from rest at the topmost point of the incline equals the time of descent of a body starting from rest at the topmost point of the vertical and deviating along the incline?

## 3. Problem finding vs. problem solving

On folio 61 recto we find a series of texts, two of which wrap around a diagram (Fig. 8). The texts puzzled Winifred Wisan, the only author who studied the folio in

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[15] It is Theorem III, in the Third Day, see Galilei 1974, p. 175.

[16] Galilei 1890–1909, VIII, pp. 387–388.

[17] Cf. my *Galileo's construction of idealized fall in the void*, forthcoming in *History of Science*, for a discussion of the broader context in which Galileo's granular conception of continua developed.

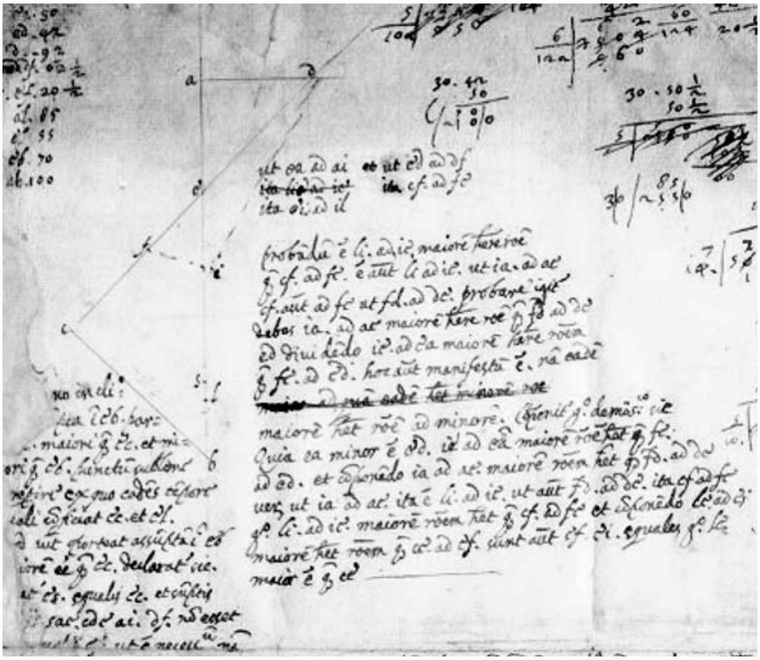

**Fig. 8.** A portion of folio 61 recto: two puzzling texts wrapping around a diagram

detail. She could not discover a plausible background for these texts.[18]. Indeed they do not seem to be correlated with any theorems and/or problems published in *Two new sciences*.

So this is a case of a wealth of material whose genesis cannot be framed according to the deductive sequence of a series of propositions. This is why, in my view, Wisan gave up searching for a plausible background. Her methodological approach prevented her from recognizing these texts' function in the economy of Galileo's processes of mathematical discovery. The texts seem to be floating in an empty space. This cannot be the case. Let see why.

In fact the interest that motivates the texts and diagram on folio 61 recto is once again the embedding of times within diagrams.

Consider Fig. 9, where I have emphasized the key elements of the diagram. Note that by construction **ec = es**, and angle **ecb** is a right angle (in the fig. I have added the dotted line). Galileo is interested in understanding what happens when bodies fall from point **a**. What happens if a body continues along the vertical, **eb**, and another body deviates along the incline, **ec**? Is it possible that two bodies starting from rest at point **a** and following these two different trajectories might reach their destinations, i.e., say,

[18] Wisan 1974, pp. 248–249. Anotnio Favaro, on the other hand, suggested that the texts might be related to Problem 13 (Galilei 1890–1909, VIII, p. 411). This is possible, but no direct evidence survives.

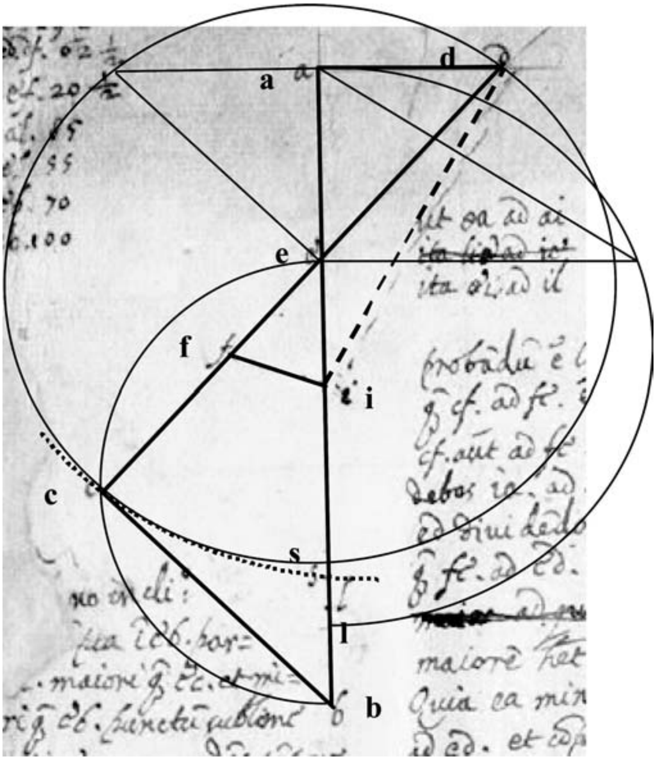


**Fig. 9.** The interests motivating folio 61 recto. Note that by construction **ec** = **es**, and angle **ecb** is a right angle. Also I have added the dotted arc, the circle, and the two semicircles. In addition, note that line **id** is drawn by Galileo apparently free-hand, and is not very precise (it was left out by the editor of the *National Edition*, Galilei 1890–1909, VIII, pp. 410, but is necessary for the comprehension of the accompanying text)

points **c** and **l**, at the same instant? The first part of the vertical trajectory, **ae**, is common to both descents.

In other words, might there exist a point along the vertical, point **l**, say, such that the vertical portion **el** would be covered by a body (starting from rest at point **a**) in the same time as that necessary for the body to cover incline **ec**, if it deviated at **e** after falling from **a**?

To understand more of the broader scenario behind these interests let us recall that by the time he works on folio 61 recto Galileo has long known the so-called *isochronism* of the chords of circles. So much has been noted before and seems uncontroversial.[19] It is a very simple and elegant result. In our case, it means that bodies falling from rest at **e** (the summit of circle **ecb**) will reach points **c** and **b** at the same instant. Hence it is clear how Galileo let his interests play on folio 61 recto. By choosing **cb** perpendicular to **ce** he is setting the boundaries of the problem. Also note that **es** is equal to **ec**. This is

[19] Wisan 1974, pp. 162ff.

nowhere explicitly stated by Galileo, but can easily be gathered by measuring the two lengths in the diagram on the manuscript folio.

Galileo begins one of the texts on the folio with a question in the form of a problem.

> [In pla]no inclinato [assum]pta in *eb* par[te] maiori quam *ec* et minori quam *eb*, punctum sublime reperire, ex quo cadens tempore [aeq]uali conficiat *ec* et *el*.[20]

> [In an inclined plane let a part [*el*] be taken greater than *ac* and less than *eb*, to find a point above from which a falling [body] will go through *ec* and *el* in the same time.]

The question is very ambitious. Galileo will not find an answer. The text simply goes on to show that the point sought must fall in between points **s** and **b**, and that a certain condition must be satisfied for a solution to the problem to exist. What is this condition and how does Galileo conclude that the point must fall in between points **s** and **b**? If you consider how I have completed the diagram in Fig. 9, you will immediately recognize the characteristic elements that we have seen playing a specific role so far. Thus **ef** and **ei** are the times of descent along **ec**, **el**, after falling from **a** and **d**, respectively. The answer to the question then reduces to showing that **ef** is equal to **ei**. This is the condition that must be satisfied for a solution to exist. That the point being sought must fall in between points **s** and **b** is shown by Galileo as follows, with a double *reductio ad absurdum*. (I have used □ for *quadratum* and □□ for *rectangulum*, and their inflected forms, according to Galileo's own symbols in the manuscripts. Galileo also uses a v-like symbol for "triangulus" and its inflections, which I have not been able to reproduce).

> [Qu]od autem oporteat, assumptam in *eb* [ma]iorem esse quam *ec*, declaratur sic. [Duc]atur *es* aequalis *ec* et sumptis [med]iis *sae, cde, ai, df*, non esset... aequalis *ef*, ut est necessarium: nam si id esset, foret quoque *si* aequalis *cf*; et cum sit ut *cf* ad *fe*, ita *fd* ad *de* et *ia* ad *ae*, esset, dividendo, *fe* ad *ed* ut... *fc* ad *ea*, et esset *ea* aequalis *ed*, quod est falsum. Quod autem oporteat, assumptam minorem esse quam *be*, sic ostenditur. Nam si *fe* aequatur *ei*, anguli *efi*, *eif* erunt aequales, et angulus *fid* maior angulo *f*, et latus *fd* maius latere *di*, et □$^{um}$ *fd* maius □$^{is}$ *iad*, et □□ *cde* maius □□$^{o}$ *bae* cum □$^{o}$ *ad*, et □□ *cde* cum □$^{o}$ *ed* maius □□$^{o}$ *bea* cum □□ *ead*, et demptis tribus □$^{is}$ *ead*, □□ *ced* maius □□$^{o}$ *bea*, quod est falsum, cum angulus *c* sit rectus. Data igitur *el* maiori *es* et minori *eb*, quaeratur *ea*, ex qua cadens temporibus aequalibus conficiat *aec* et *ael*, sive *ec* et *el* post *ae*: quod erit dum *ef*, *ei* sint aequales, positis *ai*, *df* mediis *lae*, *cde*. [21]

> [On the other hand, that it is necessary that [the line, *el*] taken in *eb* be greater than *ec* will be shown as follows. Let *es* be drawn equal to *ec*, and assuming the means, *sae*, *cde*, *ai*, *df*, it would not be... equal to *ef*, as is necessary: for, if this were the case, *si* would also be equal to *cf*; and since as *cf* is to *fe* so *fd* is to *de* and *ia* to *ae*, it would be, by *dividendo*, *fe* to *ed* as... *fc* is to *ea*, and *ea* would be equal to *ed*, which is false. But that it is necessary that [the line] taken is less than *be* will be shown as follows. For, if *fe* is equal to *ei*, angles *efi*, *eif* will be equal, and angle *fid* greater than angle *f*, and side *fd* greater than side *di*, and the square of *fd* greater than the squares of *ia*, *ad*, and rectangle *cde* greater than rectangle *bae* with the square of *ad*, and rectangle *ced* with the square of *ed* greater than rectangle *bea* with the squares of *ea*, *ad*, and subtracted three squares of *ea*, *ad*, rectangle *ced* [will

[20] Galilei 1890–1909, VIII, p. 410. The text in the manuscript is lacunose.
[21] Galilei 1890–1909, VIII, pp. 410–411.

be] greater than rectangle *bea*, which is false, since angle *c* is right. Therefore given *el* greater than *es* and less than *eb*, *ea* must be searched, from which a falling [body] will go through *aec* and *ael*, i.e., *ec* and *el* after *ae*, in the same times, which will be the case when *ef* and *ei* are equal, if *ai*, *df* are the means [between] *la*, *ae*, and *cd*, *de*.]

[*Notational help starting from:* But that it is necessary...
*if* fe=ei ➔ angle efi= angle eif, *and*
angle fid> angle f *and*
fd>di *and*
$fd^2 > ia^2 + ad^2$ *and*
$cd \cdot de > ba \cdot ae + ad^2$ *and*
$ce \cdot ed + ed^2 > be \cdot ea + ea^2 + ad^2$
➔ $ce \cdot ed > be \cdot ea$ (since $ed^2 = ea^2 + ad^2$, right triangle aed), which is false since angle **c** is right (by construction)
]

Note that one basic assumption is that **fe** equals **ei**, i.e., that the times of descent are equal to each other (*Nam si fe aequatur ei*...). In other words, the assumption here is the very result which is being sought. In this case Galileo is moving *forward* deductively, i.e., from the result being sought to a conclusion following from it. Another assumption is that angle **fcb** is right, which is the case by construction.

In addition, it is worth noting that Galileo is using a language with a marked tendency to ostensive meanings (*declaro*, *ostendo*, *attendo*, etc.). Why? Because he is in fact "showing" to himself a possible path of inquiry. He has not found a problem yet. We might say that in the midst of the basic process of embedding times of descent in diagrams, Galileo gradually zeros in on possible emerging questions as paths of inquiry. No wonder then that the text concludes with what the problem *must* be, ...[d]ata igitur *el* maiori *es* et minori *eb*, *quaeratur ea*, ex qua cadens temporibus aequalibus conficiat *aec* et *ael*... Indeed "quaeratur", not "quaeritur", says Galileo. The subjunctive, though not uncommon in these verbal forms of mathematics, suggests what the problem must be, not what the problem is.

Folio 61 recto is an extraordinary document. In it we see a mathematician progressively illuminating a complex search space. We see the formation of promising paths of inquiry. One path leads to nowhere. Another leads to a problem and its solution. I will examine them in turn.

*A path to nowhere.* As we have seen, at some point Galileo was struck by one peculiar fact. It could be possible that time **3** equals time **1+2** (cf. Fig. 2). On folio 144 recto Galileo starts out with a wrong diagram, "wrong" in the sense that the time segments do not satisfy the usual equality, i.e., time **3** equals time **1+2** (Fig. 10). Here is the text accompanying the diagram.

Tempus per *ab*, *ab*; tempus per *be* ex *a*, *bf*, posita *df* media inter *ed*, *db*; ergo tempora per *abe* erunt *abf*. [P]onatur media inter *eb*, *bd* ipsa *bh*; erit *bh* tempus *be* ex *b*: oportet igitur facere ut *bh* sit aequalis duabus *abf*, hoc est ut *ab* sit aequalis ipsi *fh*. [F]actum sit ut tempus per 2 *abe* sit aequale tempori per solam *be*. Divisa *de* bifariam, semicirculus

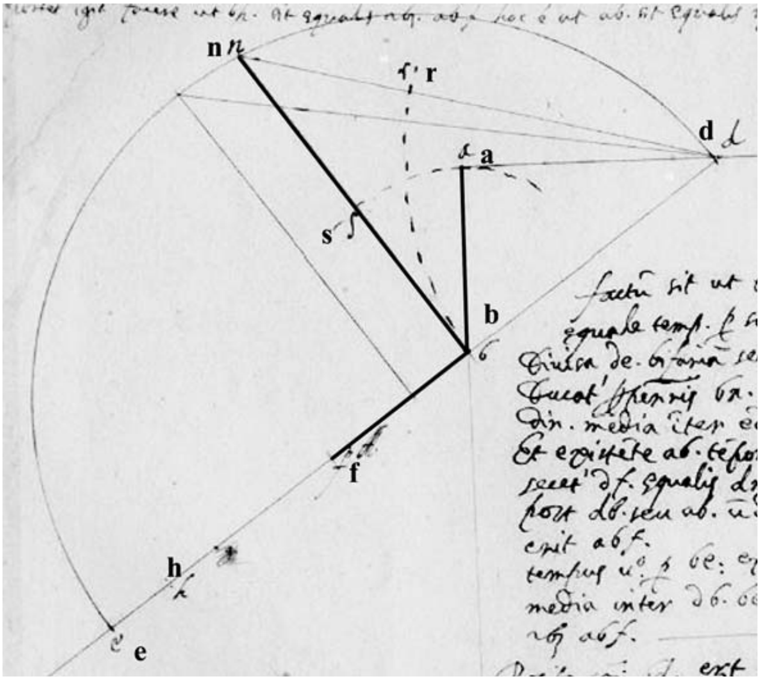


**Fig. 10.** Folio 144 recto. I have emphasized the usual key segments in Galileo's original diagram. The lettering is legible

describatur, et ducatur perpendicularis *bn*, et iungatur *dn*: erit *dn* media inter *ed*, *db*, et *bn* inter *ebd*. Et existente *ab* tempore per *ab*, et *db* per *db*, secetur *df* aequalis *dn*; erit *bf* tempus per *be* post *db*, seu *ab*: unde tempus per 2 *abe* erit *abf*, tempus vero per *be* ex *b* erit *bn*, cum sit media inter *db*, *be*: ergo *bn* aequatur duabus *abf*. Posita communi *bd*, erunt 3 *db, bf, ba*, hoc est 2 *dn*, *ba*, aequales duabus *nb*, *bd*; et ablatis aequalibus *db*, *dr* et *bs*, *ba*, reliqua *rn* reliquae *ns* erit aequalis.[22]

[[Let] time through *ab* [be] *ab*; [let] time through *be* from *a*, [be] *bf*; *df* is the mean between *ed*, *db*; thus the times through *ab*, *be* will be *ab*, *bf*. Let the mean between *eb*, *bd* be *bh* itself; *bh* will be the time through *be* from *b*: thus it is necessary that *bh* is equal to the two *abf*, i.e., that *ab* is equal to *fh* itself. Let it be that the time through the two *abe* is equal to the time through *be* only. Divide *de* into two equal lines, describe the semicircle, and draw perpendicular *bn*; join *dn*: *dn* will be the mean between *ed*, *db*, and *bn* between *eb*, *bd*. And since *ab* is the time through *ab*, and *db* through *db*, let *df* be cut equal to *dn*; *bf* will be the time through *be* after *db*, or *ab*: so that the time through the two *ab*, *be*, will be *abf*, while the time through *be* from *b* will be *bn*, since it is the mean between *db*, *de*: therefore, *bn* is equal to the two *ab*, *bf*. Let the common *bd* be assumed, the three *db*, *bf*,

[22] Galilei 1890–1909, VIII, p. 414.

*ba*, i.e., the two *dn*, *ba*, will be equal to the two *nb*, *bd*; and, subtracted the equals, *db*, *dr*, and *bs*, *ba*, the remaining *rn* will be equal to the remaining *ns*.]

According to the diagram lettering, the time equality will be **ab+bf=bn**. In the familiar way Galileo chooses a unit time such that time through **ab** is **ab** itself, so that time through **db** is **db** itself. Then **df** (cut equal to **dn**, the geometric mean between **de** and **eb**) will be the time through **de** from **d**. And **bf** will be the time through the mixed path **abe** from rest at **a**. From equality **ab+bf=bn** it must also follow that **ab=fh**.

In most of the text Galileo reiterates the conditions that must hold in order that equality **ab+bf=bn** is satisfied. It is only in the last sentence that he takes a deductive step. With simple additions and subtractions of quantities he concludes that **rn** must be equal to **ns** (you can see that since the diagram is "wrong" the two segments are not equal, cf. Fig. 10).

*A path to a problem and its solution.* On folio 54 recto we find another text in which further deductive steps are taken (Fig. 11).

This time Galileo starts out again with a correct diagram (as can easily be seen by measuring). Afterwards he adds a circle to his basic diagram, with center in **f** and radius **fb**. He also draws two other circles symmetrically with respect to **fb** (which, remember, is perpendicular to **be**), centers in **e** and **m**, and radii **ie** and **m*** (the one on the left is incomplete, drawn with a dotted line, and only center **m** is marked with a letter). The whole diagram, though only partially drawn, has an appealing *symmetrical* appearance ("symmetry" in the sense of left and right correspondence, or correspondence with respect to an axis). It must have been this appearance that captured Galileo's imagination. But, first of all, why these added circles? How do they pop up? In order to understand them we first need to "see" another equality of lines that is embedded in the diagram, and which must have been immediately visible to Galileo's trained eye. I have emphasized the segments involved. The sum of the two dotted segments is equal to the sum of the two solid segments. This follows immediately from the assumption of the main equality of times, **ab+bk=fb**, where **ek** equals **ef** (it is indeed another form of expressing the same fact). Thus it equally easily follows that **ab = bg** when **ei=eg**, i.e., when a circle is drawn with radius **eg** and center **g**, where point **g** is such that **gb=ab**.

A glimpse can at last be had of a possible construction, *and of a possible problem.*

*A possible problem.* If in the diagram you have the incline, that is the inclined plane (grey triangle, cf. Fig. 11), and fix **ab**, i.e., if you fix the length of the vertical, then you can construct circle **igh** (since its radius is **ig**, equal to **ab**). Then perhaps you can find a construction path toward fixing **bm**. The latter would be the length of an inclined plane such that the time of descent of a body starting from rest at the topmost point of the inclined plane might equal the time of descent of a body starting from rest at the topmost point of the vertical segment and deviating along the incline. At last you may have a problem.

We can now read the relevant text. In the text, Galileo launches in a sequence of operations, which are tantamount to algebraic transformations carried out without the help of any special notation (except for the pictographic abbreviations already mentioned), wholly in natural language.

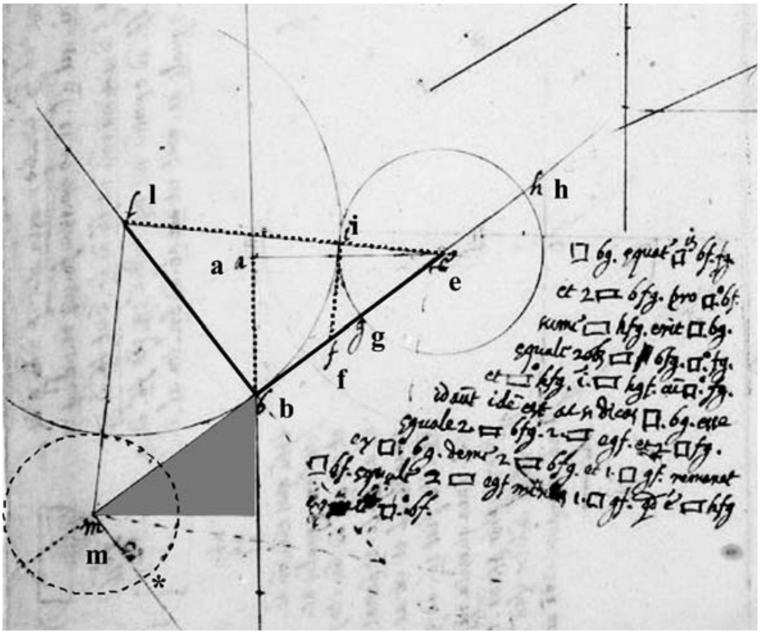


**Fig. 11.** Folio 54 verso (I have emphasized a few elements and added the gray triangle representing the inclined plane). Notice the text wrapping around the diagram, a sign that the diagram was drawn first. Notice also the small squares and rectangles used by Galileo in the text as abbreviations of "quadratum" and "rectangulum". The diagram from this folio published in Galilei 1890–1909, VIII, p. 415, is incomplete

□ *bg* aequatur □$^{is}$ *bf*, *fg* et 2 □□ *bfg*: pro □$^{o}$ *bf* sume □□ *hfg*; erit □ *bg* aequale duobus □□ *bfg*, □$^{o}$ *fg*, et □□$^{o}$ *hfg*, idest □□ *hfg* cum □$^{o}$ *fg*: id autem id est ac si dicas, □ *bg* esse aequale 2 □□ *bfg*, 2 □□ *egf*, et 2 □ *fg*. Ex □$^{o}$ *bg* deme 2 □□ *bfg* et 1 □ *gf*; remanet □ *bf* aequale 2 □□ *egf* minus 1 □ *gf*, quod est □□ *hfg* aequale □$^{o}$ *bf*.[23]

[The square of *bg* is equal to the squares of *bf*, *fg* and 2 rectangles *bfg*: for the square of *bf* assume rectangle *hfg*; the square of *bg* will be equal to two rectangles *bfg*, the square of *fg*, and rectangle *hfg*, i.e., rectangle *hfg* with the square of *fg*: this is the same as if I said, that the square of *bg* is equal to 2 rectangles *bfg*, rectangle *egf*, and 2 squares of *fg*. From the square of *bg* subtract 2 rectangles *bfg* and 1 square of *gf*; the square of *bf* will then be equal to 2 rectangles *egf* minus 1 square of *gf*, which is to say that rectangle *hfg* is equal to the square of *bf*.]

[*Notational help*
$bg^2 = bf^2 + fg^2 + 2\,(bf\cdot fg)$
$bf^2 = hf\cdot fg$ (since by construction bf=fi, and by Euclid's *Elements*, III, 36)
$bg^2 = 2(bf\cdot fg) + fg^2 + ((hf\cdot fg) = (hf\cdot fg) + fg^2)$
*if* $bg^2 - 2(bf\cdot fg) - gf^2 \Rightarrow bf^2 = 2(eg\cdot gf) - gf^2 \Rightarrow hf\cdot fg = bf^2$
]

[23] Galilei 1890–1909, VIII, p. 415.

The conclusion of this adventure in spoken algebra is astonishing. The path of inquiry takes a decisive step forward. A *commitment* to a final form begins to emerge. The algebraic acrobatics has paid off. The conclusion, i.e., "...quod est □□hfg aequale □$^{o}$ bf", indicates that rectangle **hfg** (that is, in notational language, the product, **hf**, **fg**) is equal to the square of **bf**. This means that a condition has almost been found that can fix point **f** on line **be**. It is as follows.

Segments **hg**, **gb** are given, so all that is needed is divide **gb** in such a way that the rectangle, that is the product, **hf**, **fg** is equal to the square of **bf**. For this to occur a proportion has to hold such that **hf** is to **fb** as **bf** is to **fg**. By simply applying the rules of *componendo* and *permutando* to this proportion one finds that as **hb** is to **bg** so **bf** is to **fg**.[24] So segment **bg** being given equal to **ba** (which may be given as datum in the problem), one has simply to divide it into two segments such that they are to each other in a certain given ratio, in the following way.

*The problem solved.* In the diagram (Fig. 11), there is an important give-away. It signals what Galileo is now seeing in the diagram. Line **fi** (dotted) is drawn but never spoken of in the text. Thus by drawing **fi** tangent to circle **igh** from **f** one can prolong line **ei** to point **l**, at the intersection with line **lb**, and by drawing the perpendicular to line **le** from **l** one finally determines point **m** on the incline. A geometrical construction has thus been found for the length of an inclined plane such that the time of descent of a body starting from rest at the topmost point of the inclined plane might equal the time of descent of a body starting from rest at the topmost point of the vertical segment and deviating along the incline. *Problem solved.*

Eventually, in *Two new sciences*, Galileo will publish this result as Problem 14.

> Given a line at an angle to a given vertical, to find a part therein through which alone, from rest, motion is made in the same time as through it and the vertical together.[25]

*A bizarre puzzle*. As we have seen, Wisan excludes Problem 14 from her considerations, because in *Manuscript 72* a partial *analysis* (in Wisan's terminology) of Problem 14 appears to be written by Galileo, on folio 97 verso, in a handwriting style which seems to be later than the style of an autograph of the corresponding *synthetic* proof (also eventually published in *Two new sciences*). Indeed I believe that Wisan is right. The handwriting style really seems to me to be later than that of the synthesis (the latter is on folio 142 recto).[26]

Could then the analysis have been thought up and/or written later than the synthesis? This question became a bizarre puzzle for Wisan, since on the assumption that the analysis itself *is the discovery process* she could not see how the analysis could have

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[24] The rules of *componendo* and *permutando* are simple manipulation rules for proportions expounded by Euclid in the fifth book of the *Elements*. See Euclid 1956, II, pp.114–115. Galileo uses the sequence *componendo- permutando* quite often.

[25] Galilei 1974, p. 209.

[26] Of course it is difficult to prove this fact. We are left with an indeterminacy. Cf. Drake 1979, pp. 107 and 108, for a printed facsimile of the two folios. Drake does not discuss the question of the handwriting style raised by Wisan.

**Fig. 12.** Folio 97 verso. Galileo's exposition of an "analysis" for Problem 14. The handwriting looks fluid and homogeneous and the text polished, as if jotted down in one session only

been written later. This assumption must be rejected. The analysis written by Galileo on folio 97 verso does *not* relate to the creative process of the discovery (Fig. 12). Why?

First of all, we note that the text appears to be one block of writing, fluently progressing from top to bottom. No apparent discontinuities can be detected, or changes in handwriting style. The block of text is well justified, right and left. The text does not wrap around the diagram, which is located below. Briefly, this text must have been written down by Galileo *ex post facto*. The question of "writing" a text has never been considered in the Greek analysis literature. So what is the function of Galileo's writing the text? If the modeling of the mathematical object has already occurred, and if the discovery has already been accomplished, why write the text of an "analysis"?

Around 1618 Galileo started to collect materials in view of a possible publication. We have a number of texts copied by disciples still mixed up in *Manuscript 72*. It is quite possible that the students who were commissioned the editing of the materials, all of whom were interested in Galileo's new science, asked the master to clarify the process of discovery that had led him to the solution to Problem 14. This request Galileo might have satisfied by writing down an account in the analysis style of the text on folio 97 verso. That the account is written in the analysis style does not come as a surprise. The account moves from the geometrical situation presenting a "rational", a-posteriori reconstruction of the creative process. We know that Galileo was familiar with Pappus's *Collections*, published in 1588.[27] From Book VII of Pappus's *Collections* he must have learned the rhetoric of the value of the analytic style in Greek geometry.[28] The analytic

[27] I have shown this fact in Palmieri 2003.

[28] Indeed, in the celebrated *Dialogo* of 1632, Galileo echoes a passage from Pappus' Book VII concerning analysis and its value in what Galileo calls "demonstrative sciences". It is perfectly suited to the rhetoric with which Galileo's mouthpiece, Salviati, needs to convince Simplicio that

style starts from the geometric situation as a *given*, thus presenting the pupil with a starting point for exploring the mathematical search space. The synthetic style, on the other hand, presents the pupil with a de facto solution, which has no obvious connection with the *given*, i.e., the initial geometric situation.

These circumstances would explain why the handwriting of the analytic piece is later than the style of the synthesis by Galileo himself. In brief the "analysis" on folio 97 verso is for the historian misleading. It had a didactic value for Galileo's pupils. It has nothing to do with the creative process of problem-finding and problem-solving. The formation of the complex *problem-solution* is a gradual phenomenon, in which commitment to the final form of the problem, solution to the problem, and deductive sequences gradually come together in a wholly integrated process.

One must conclude that the analysis text relates a narrative of processes which do not reflect the actual finding of the solution. The analysis text lures you into believing that the creative process unfolds within the text as the text itself unfolds before your eyes as you read. It is an illusion.

There is indeed a clue in the text, suggesting that the analytic text itself is misleading. In fact the analysis text alone leaves us with zero explanation as to why, at one point in the creative process, circle **psf** should magically pop up (I have emphasized the circle in the diagram, cf. Fig. 13). Let's explore the magic. Here is the analysis text (Fig. 13). It is worth reading it entirely.

> [P]osito *ab* esse tempus per *ab*, erit *eb* tempus per *eb*, et tempus per futuram *bx* ex quiete in *b* erit media inter *eb*, *bx*; et ideo erigitur perpendicularis *bo*, ut in ea notetur media. Tempus vero per totam *ebx* futurum est media inter *xe*, *eb*, quae erit *eo*, cuius excessus super *eb* erit tempus per *bx* post *eb*, qui excessus cum *ba* (tempora scilicet per *abx*) debent aequari mediae *bo*. Cum autem hoc fuerit (nempe excessum mediae *oe*, seu *en*, super *eb* una cum *ab*, dico *nba*, esse aequales ipsi *bo*), posita communi *be*, erunt 2 *ne*, idest *oe*, *ab*, aequales duabus *ob*, *be*: auferantur *os*, *ba*, aequales duabus *ob*, *bf*; reliqua *fe* (quae

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it is one thing to prove a truth and quite another to find it. To see the closeness to Pappus the passage must be read in the original Italian. It's as follows. "Cotesto, che voi [Simplicio] dite, èil metodo col quale egli [Aristotle] ha scritta la sua dottrina, ma non credo già che e' sia quello col quale egli la investigò, perchè io tengo per fermo ch' e' proccurasse prima, per via de' sensi, dell'esperienze e delle osservazioni, di assicurarsi quanto fusse possibile della conclusione, e che doppo andasse ricercando i mezi da poterla dimostrare, perchè così si fa per lo piú nelle *scienze dimostrative*: e questo avviene perchè, quando la conclusione è vera, servendosi del *metodo resolutivo*, agevolmente si incontra qualche proposizione già dimostrata, o si arriva a qualche principio per sè noto; ma se la conclusione sia falsa, si può procedere in infinito senza incontrar mai verità alcuna conosciuta, se già altri non incontrasse alcun impossibile o assurdo manifesto" (emphasis mine). What in today's literature is referred to as "Pappusian analysis" here Galileo calls "metodo resolutivo". See Galilei 1890–1909, VII, p. 75. Compare Galileo's passage with the Latin text of Pappus he would have used. "Resolutio igitur est via a quaesito concesso per ea, quae deinceps consequuntur ad aliquod concessum in compositione: in resolutione enim id quod quaeritur tanquam factus ponentes, quid ex hoc contingat, consideramus: et rursum illius antecedens, quousque ita progredientes incidamus in aliquod iam cognitum, vel quod sit e numero principiorum. Et huiusmodi processum resolutionem appellamus, veluti ex contrario factam solutionem" (Pappus 1660, pp. 240–241). (For this passage I have consulted Pappus 1660, a second edition of Pappus 1588).

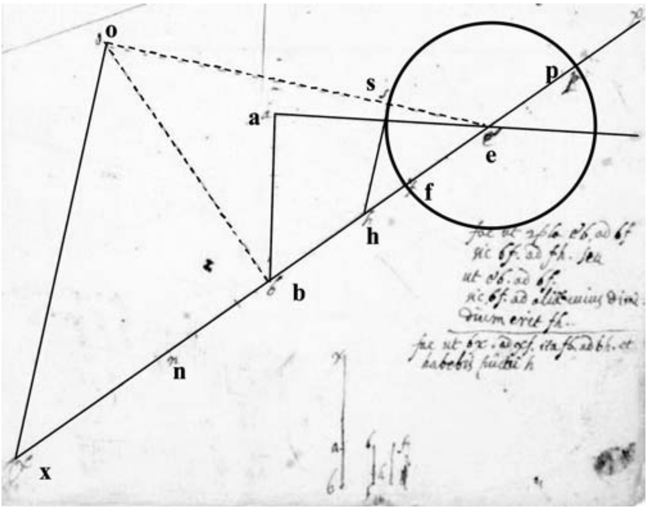


**Fig. 13.** The diagram accompanying the analysis text on folio 97 verso. The key circle is emphasized

datur) erit aequalis reliquae *es*. Redactum ergo est opus, ut, centro *e*, intervallo *ef*, circulo descripto, a centro *e* educatur linea occurens *bo*, ita ut pars extra circulum, qualis est *so*, sit aequalis *ob*. Dum autem hoc fuerit, perpendicularis ex *s* ad *so* occurret *bf*, ut in *h*, et erit aequalis *hb*, et tanget circulum, eritque eius □ aequale □□ *phf*: cumque datae sint *pf*, *fb*, oportet ita dividere *fb*, ut □□ *phf* sit aequale □ *hb*; quod erit dum *ph* ad *hb* sit ut *bh* ad *hf*, et, componendo, ut *pb* ad *bh*, ita *bf* ad *fh*, et, permutando, ut *pb* ad *bf*, ita *bh* ad *hf*. Dantur autem *pb*, *bf*: ergo dabitur *bh*. Duc igitur a puncto *h* tangentem *hs*, et per *s*, *eso*, etc.[29]

[Let *ab* be the time through *ab*, *eb* will be the time through *eb*, and the future time through *bx* from rest at *b* will be the mean between *eb*, *bx*; thus let perpendicular *bo* be drawn so that the mean can be marked in it. The time through the whole *ebx* is the mean between *xe*, *eb*, which is *eo*, and the excess of which over *eb* will be the time through *bx* after *eb*, which excess with *ba* (that is the times through *abx*) must be equal to the mean *bo*. When this is the case (namely, when the excess of mean *oe*, that is *en*, over *eb* together with *ab*, I say *nba*, are equal to *bo* itself), by assuming the common *be*, 2 *ne*, i.e., *oe*, *ab*, will be equal to the two *ob*, *be*: let *os*, *ba* be taken away, which are equal to the two *ob*, *bf*; the remaining *fe* (which is given) will be equal the remaining *es*. The work is thus reduced to the following, i.e., that, center *e*, radius *ef*, a circle be described such that from it a line emerges from center *e* cutting *bo*, such that the part [of it] outside the circle, which is *so*, be equal to *ob*. When this obtains, the perpendicular from *s* to *so* will intersect *bf*, at *h*, say, while being equal to *hb*, and tangent to the circle, and such that its square will be equal to rectangle *phf*. Since *pf*, *fb* are given it is necessary to divide *fb* in such a way that rectangle *phf* be equal to the square of *hb*, which will be the case when *ph* is to *hb* as *bh* is *hf*, so that, *componendo*, as *pb* is to *bh*, so *bf* is to *fh*, and, *permutando*, as *pb* is to *bf*, so

[29] Galilei 1890–1909, VIII, pp. 415–416.

*bh* is to *hf*. But *pb*, *bf* are given: therefore *bh* will be given. Thus, draw tangent *hs* from point *h*, and through *s*, *eso*, etc.]

[*Notational help* (":" ratio; "::" proportion)
let ab=time through ab
➔ eb=time through eb, ➔ time through bx will be
bx= *mean*(eb, bx)
time through (eb, bx) together will be = *mean* (xe, eb)[30] =
eo
eo-eb= time through bx after eb
eo-eb + ba *must be* = bo
2ne = oe+ab = ob+be ➔ fe (*given*) = es
Construct the circle as *exlplained above*
sh=hb ➔ $sh^2 = ph \cdot hf$
pf, fb *are given*
divide fb so that $ph \cdot hf = hb^2$
*this holds true if*
ph:hb::bh:hf
pb:bh::bf:fh (*componendo*)
pb:bf::bh:hf (*permutando*)
➔ bh will be given
]

"Redactum ergo est opus, ut, centro e, intervallo ef, *circulo descripto*...", exclaims Galileo. The whole problem boils down to drawing a circle such that. . .Where does this circle come from? Intuition? Educated guesswork? *Nothing in the previous development of the analytic text hints at this possibility*. If you think of analysis itself as a heuristics you cannot explain the popping up of this circle. Look instead again at the diagram in Fig. 11. There are three circles. The one on the left is apparently purposeless. Galileo draws circles with profligacy and a sense of symmetry. The one on the right turned out to be the missing key to the solution.

*Problem 13.* We now need to look briefly at how the other problem, Problem 13, was eventually shaped during the process of problem finding and problem solving.

Galileo, as we have seen, drew the diagrams of folio 96 verso according to the sequence 1, 2, 3, 4 (see Fig. 1 and Table 1). In Fig. 14, we have diagram 1 enlarged. It became a transitional diagram. The relevant text wraps around the diagram. The diagram has significant give-aways revealing its transitional nature. It's as if the diagrammatic object were a subtle palimpsest of diagrams serving slightly different purposes. One purpose we have seen when discussing the process of modeling developing around the

[30] "Mean" between segments is calculated geometrically by Galileo according to Euclid in a geometrical way. See Euclid's *Elements*, VI, Proposition 13. Galileo expounds and justifies this technique of calculating times geometrically in *Two new sciences*, Theorem II, Corollary 2 (Galilei 1974, pp. 170–171).

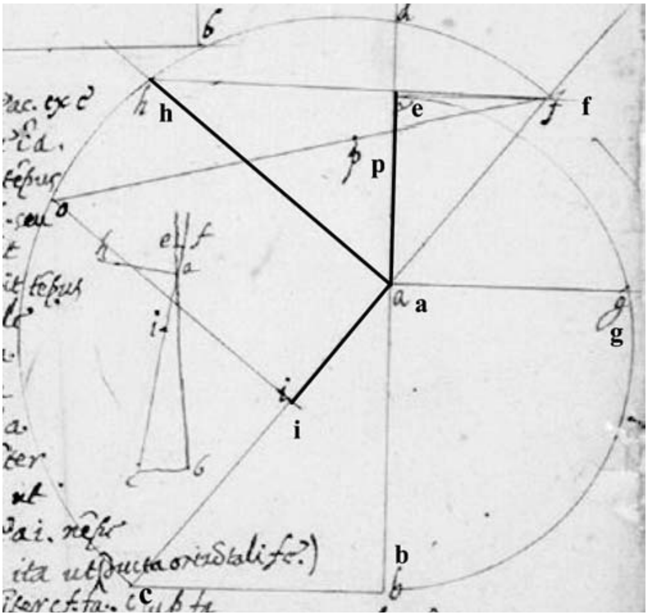


**Fig. 14.** Folio 96 verso. A transitional diagram. The version of this diagram published in Galilei 1890–1909, VIII, p. 398, is incomplete. I have emphasized the three line segments, **ea**, **ai**, **ah**, of the familiar time relation of Problem 13

diagrams in sequence. Later on Galileo must have realized that other possibilities were nested in the diagram and could be brought to light.

In the text accompanying the diagram, first Galileo sets down the conditions for the equality of times that we have discussed so far. But in the second part he moves on to a different interpretation of the diagram. Instead of assuming that the unit time is line **ea**, so that time of descent along line **ea** is line **ea** itself, he assumes that the time unit is line **ca**, so that the time of descent along line **ca** is line **ca** itself. Thus the time of descent along the vertical **ab** will be the vertical **ab** itself (for the same reasons already considered above). Then line **ag** will be the time of descent along vertical **ea**. Hence the diagram incorporates in another form the same time relation whose geometric elements I have emphasized in Fig. 15. With simple considerations which we need not dwell on here, Galileo realizes that in order that the time relation holds true another equality must hold true, namely, $\mathbf{pf} + \mathbf{ag} = \mathbf{ac}$.

The text and the reasoning make no further progress. However, the diagram's incorporating the same time relation in another form shifts Galileo's attention to a portion of the diagram which was not considered in the processes leading up to Problem 13.
This is all the evidence that survives relating to the form that Problem 13 and its proof finally took in *Two new sciences*.

> Given an inclined plane and a vertical, both with the same high point; to find in the vertical extended upward a higher point from which a movable that descends and is then deflected

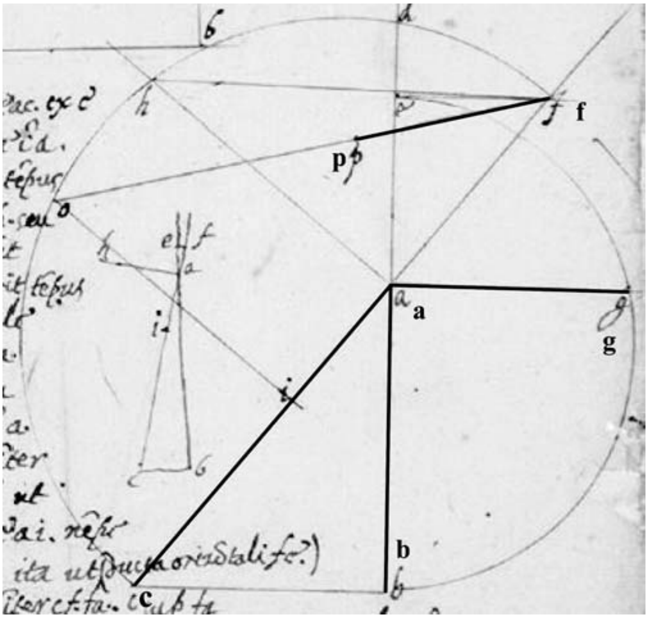


**Fig. 15.** Folio 96 verso. A transitional diagram. Here I have emphasized the three line segments, **ac**, **ab**, **ag**, of the second form of the time relation

> into the inclined plane traverses both in the same time as the inclined plane along [is traversed] from rest at its high point.[31].

In brief, Galileo moved the focus of attention to another portion of the diagram that had been the prime mover of the explorations leading to Problem 14. By searching for times of descent within diagrams he was rewarded by discovering two possible forms of presenting the same basic time relation. In the process he found two formidable problems and two elegant geometric constructions for their solutions.

## 4. Conclusion

To sum up, we have seen that mathematical creativity, at least in Galileo's case, is a phenomenon in which problem-*finding* and problem *solving* are strictly interconnected. Traditional views on the heuristic power of the Greek method of analysis (to be found in contemporary literature on the history and philosophy of mathematics), which, as suggested by W. Wisan, Galileo would have followed, must be called into question. Analysis, if such a thing ever existed, did not provide Galileo with a heuristics, or with a rough guide to finding problems. Galileo did not address Problems 13 and 14 because he had found them in a mathematical tradition. He had to excogitate them in the first

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31 Galilei 1974, p. 207

place. Strikingly, the manuscript analysis of Problem 14 might have been no more than a didactic device for presenting impervious material to Galileo's students.

Thus, the manuscript folios reveal that texts written in an analytic style may serve different functions. They tend to be *ex post facto* summaries, personal memos, and/or teaching material. The processes they relate are by no means a mirror of the creative processes behind Galileo's mathematical practice.

In the examples considered in this paper we have seen the power of a simple, fundamental activity based on diagrams, i.e., the embedding of times of descent into the representational structure of the diagrams themselves. Other paths of inquiry are disseminated throughout *Manuscript 72* which we cannot examine here. They are for future research. One thing is clear, then. We need to emphasize the relevance of the study of the the processes behind the deductive style of presentation of mathematical proofs. This approach might hold in store new discoveries, which, I am convinced, will further our understanding of how *exactness* is achieved in mathematics.

## Note

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Department of History and Philosophy of Science
University of Pittsburgh
Pittsburgh, PA 15260, USA
pap7@pitt.edu

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