MATH 2301-ANALYSIS I, FALL 2013, HOMEWORK 2

DUE ON MONDAY SEPT. 23, 2013, AT THE BEGINNNING OF CLASS

If you use a theorem from the class, you should quote it and justify all of its assumptions.

1. Let $0 < \delta < 1$. Construct a Cantor-type subset of the interval $[0,1]$ with a similar procedure except that at stage k, each interval removed has length $3^{-k}\delta$. Show that the resulting set is perfect, Lebesgue measurable of measure $1-\delta$ and contains no intervals.

2. If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of sets with $\sum_{k=1}^{\infty} \mathcal{L}^*(E_k) < +\infty$, show that $\liminf_{k\to\infty} E_k$ and $\limsup_{k\to\infty} E_k$ have Lebesgue measure zero.

3. Assume E_1 and E_2 are Lebesgue measurable subsets of $\mathbb{R}$.

a) Show that $\mathcal{L}(E_1 \cup E_2) + \mathcal{L}(E_1 \cap E_2) = \mathcal{L}(E_1) + \mathcal{L}(E_2)$.

b) Show that $E_1 \times E_2$ is Lebesgue measurable in $\mathbb{R}^2$ and that $\mathcal{L}(E_1 \times E_2) = \mathcal{L}(E_1) \cdot \mathcal{L}(E_2)$, with the convention that $\infty \cdot 0 = 0$.

4. Prove that the Cantor-Lebesgue function is not Lipschitz.

5. Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz. Prove that if $E \subset \mathbb{R}^n$ is Lebesgue measurable, then $f(E)$ is also measurable. What happens if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$? Is the statement true for all continuous functions?

6. We define the inner Lebesgue measure of $E \subset \mathbb{R}^n$ by $\mathcal{L}_*(E) := \sup \mathcal{L}(F)$ over all closed sets $F \subset E$. Prove that for all E, $\mathcal{L}_*(E) \leq \mathcal{L}^*(E)$ and if $\mathcal{L}^*(E) < +\infty$, the equality happens iff E is Lebesgue measurable.