

ECON 2200: Problem Set 5 — solutions

1. Take some player  $i$ . Since  $i$  is rational at every state, we know that for all  $\omega \in \Omega$ , her expected payoff at  $\omega$  from playing  $s_i(\omega)$  is at least as much as if she had chosen some other strategy,  $s'_i$ :

$$\sum_{\omega \in \mathcal{P}_i(\omega)} p(\omega | \mathcal{P}_i(\omega)) u_i(s_i(\omega), s_{-i}(\omega)) \geq \sum_{\omega \in \mathcal{P}_i(\omega)} p(\omega | \mathcal{P}_i(\omega)) u_i(s'_i, s_{-i}(\omega)), \text{ for all } s'_i \in S_i.$$

Since this holds for every  $\omega \in \Omega$ , it must hold for every information set  $\mathcal{P}_i(\omega)$  of player  $i$ . But this is precisely the definition of a correlated equilibrium (see F&T, p. 57, equn. 2.2). The theorem is straightforward to prove, but its interpretation is important: see Aumann, *Ecta*. 1987.

2.  $\delta = \frac{1}{2}$ : Let  $\bar{v}_1$  and  $\underline{v}_1$  be the highest and lowest continuation payoffs of player 1 in any SPE that begins with player 1 making an offer, and let  $\bar{w}_1$  and  $\underline{w}_1$  be the highest and lowest continuation payoffs for player 1 in any SPE that begins with player 2 making an offer. We define  $\bar{v}_2, \underline{v}_2, \bar{w}_2$ , and  $\underline{w}_2$  in the same way. Clearly, when 1 is making an offer, 2 will accept any offer that gives him at least  $\frac{1}{2}\bar{v}_2$ , since he cannot get more than  $\bar{v}_2$  in the continuation game following his refusal. It follows that  $\underline{v}_1 \geq 1 - \frac{1}{2}\bar{v}_2 - 0.01$  (note that player 1 may not be able to make an offer that gives player 2 *exactly*  $\frac{1}{2}\bar{v}_2$ ), and by a symmetric argument, we have  $\underline{v}_2 \geq 1 - \frac{1}{2}\bar{v}_1 - 0.01$ . Since  $\bar{w}_1 + \underline{v}_2 \leq 1$ , this gives us  $\bar{w}_1 \leq \frac{1}{2}\bar{v}_1 + 0.01$ . Now consider  $\bar{v}_1$ , the highest SPE payoff that player 1 can achieve when making an offer. We know that player 2 will reject any offer less than  $\frac{1}{2}\underline{v}_2$ , so the largest offer that might be accepted is for player 1 to keep  $1 - \frac{1}{2}\underline{v}_2$ . On the other hand, the highest payoff that player 1 can get if her offer is rejected is  $\frac{1}{2}\bar{w}_1$ . Thus

$$\bar{v}_1 \leq \max \left\{ 1 - \frac{1}{2}\underline{v}_2, \frac{1}{2}\bar{w}_1 \right\} \leq \max \left\{ 1 - \frac{1}{2}\underline{v}_2, \frac{1}{4}\bar{v}_1 \right\} = 1 - \frac{1}{2}\underline{v}_2.$$

By symmetry,  $\bar{v}_2 \leq 1 - \frac{1}{2}\underline{v}_1$ . Combining these inequalities, we obtain

$$\begin{aligned} \underline{v}_1 &\geq 1 - \frac{1}{2}\bar{v}_2 - 0.01 \geq 1 - \frac{1}{2} \left( 1 - \frac{1}{2}\underline{v}_1 \right) - 0.01 \\ \Rightarrow \underline{v}_1 &\geq \frac{98}{150}, \end{aligned}$$

and

$$\begin{aligned} \bar{v}_1 &\leq 1 - \frac{1}{2}\underline{v}_2 \leq 1 - \frac{1}{2} \left( 1 - \frac{1}{2}\bar{v}_1 - 0.01 \right) \\ \Rightarrow \bar{v}_1 &\leq \frac{101}{150}. \end{aligned}$$

Because  $\underline{v}_1$  and  $\bar{v}_1$  must be multiples of 0.01, we have  $\underline{v}_1 \geq 0.66$  and  $\bar{v}_1 \leq 0.67$ . Similarly,  $\underline{v}_2 \geq 0.66$  and  $\bar{v}_2 \leq 0.67$ .

It is now easy to show that there are two SPE, both involving stationary strategies for each player:

- { player 1: always offer a split of (0.66, 0.34); accept only offers which give her at least 0.33
- { player 2: always offer a split of (0.33, 0.67); accept only offers which give him at least 0.34

and

- $$\left\{ \begin{array}{l} \text{player 1: always offer a split of } (0.67, 0.33); \text{ accept only offers which give her at least } 0.34 \\ \text{player 2: always offer a split of } (0.34, 0.66); \text{ accept only offers which give him at least } 0.33 \end{array} \right.$$

To show that these are the unique SPE strategies, observe that the continuation payoffs for player  $i$  must be the same in every odd round of the game, and also in every even round of the game; these continuation payoffs generate unique offer and acceptance strategies.

$\delta$  **close to 1**: As long as  $\delta \geq \frac{99}{100}$ , every feasible outcome is sustainable by some SPE strategy profile. For instance, the following (stationary) strategies result in the agreement  $(x, 1 - x)$  in round  $t = 0$ :

- $$\left\{ \begin{array}{l} \text{player 1: always offer a split of } (x, 1 - x); \text{ accept only offers which give her at least } x \\ \text{player 2: always offer a split of } (x, 1 - x); \text{ accept only offers which give him at least } 1 - x \end{array} \right.$$

It is easy to see that the players cannot gain a higher payoff by proposing more: their offer will be rejected and they will have to wait a round to get the same payoff. And furthermore the acceptance strategies are credible: e.g. accepting  $x - 0.01$  (for player 1) would give her  $x - 0.01$  in the current time period rather than  $x$  in the next time period, but  $x - 0.01 \leq \delta x$  since  $\delta \geq \frac{99}{100}$  and  $x \leq 1$ .

To see how to construct SPE strategies which result in delayed agreements and perpetual disagreement, see Proposition 2 of van Damme, Selten and Winter (*Games and Economic Behavior* 1990).

3. (a) Let  $U = \{(x_1, x_2) \mid x_1 + x_2 \leq 1 \text{ and } x_2 \leq \frac{1}{2}\}$ , and  $d = (0, 0)$ . Then the Nash bargaining solution  $f^N(U, d) = (\frac{1}{2}, \frac{1}{2})$ , while the Kalai-Smorodinsky solution  $f^{KS}(U, d) = (\frac{2}{3}, \frac{1}{3})$  (to see this, observe that  $b_1 = 1$  and  $b_2 = \frac{1}{2}$ ).
- (b) **INV** Suppose the bargaining problem  $(U', d')$  is obtained from the bargaining problem  $(U, d)$  as follows:

$$\begin{aligned} U' &= \{x' \mid (x'_1, x'_2) = (\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2) \text{ for some } x \in U\}, \\ d' &= ((\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)). \end{aligned}$$

Then if  $x$  is Pareto efficient in  $U$ ,  $x'$  must be Pareto efficient in  $U'$ . Furthermore,  $b'_i = \alpha_i b_i + \beta_i$ . Therefore:

$$\begin{aligned} \frac{x_1 - d_1}{x_2 - d_2} &= \frac{b_1 - d_1}{b_2 - d_2} \\ \Rightarrow \frac{(\alpha_1 x_1 + \beta_1) - (\alpha_1 d_1 + \beta_1)}{(\alpha_2 x_2 + \beta_2) - (\alpha_2 d_2 + \beta_2)} &= \frac{(\alpha_1 b_1 + \beta_1) - (\alpha_2 d_1 + \beta_1)}{(\alpha_2 b_2 + \beta_2) - (\alpha_2 d_2 + \beta_2)} \\ \Rightarrow \frac{x'_1 - d'_1}{x'_2 - d'_2} &= \frac{b'_1 - d'_1}{b'_2 - d'_2} \end{aligned}$$

and so if  $f^{KS}(U, d) = x$ , it must be the case that  $f^{KS}(U', d') = x'$ , as required.

**PAR**  $f^{KS}$  satisfies the Pareto axiom by construction.

**SYM** If  $(U, d)$  is symmetric, then  $b_1 = b_2$  and  $d_1 = d_2$ . Thus  $\frac{b_1 - d_1}{b_2 - d_2} = 1$ ; so we need  $\frac{x_1 - d_1}{x_2 - d_1} = 1$ , which implies  $x_1 = x_2$  as required. (Note that if  $d$  itself is Pareto efficient, then we have  $f^{KS}(U, d) = b = d$  and again symmetry is satisfied.)

**IIA** This axiom is not satisfied. Let  $U' = \{(x_1, x_2) \mid x_1 + x_2 \leq 1\}$  and  $d = (0, 0)$ . Then  $f^{KS}(U', d) = (\frac{1}{2}, \frac{1}{2})$ . Now consider the bargaining problem in the solution to (a) above.  $U \subseteq U'$  and  $f^{KS}(U', d) \in U$ . But  $f^{KS}(U', d) \neq f^{KS}(U, d)$ .

4. (a) Let  $\lambda_i$  be the belief of the seller that the buyer is of type  $H$ , in round  $i = 1, 2$ . Clearly  $\lambda_1 = \lambda$ . Consider the seller's second round offer. The only offers we need to consider are  $p_2 = v_H$  (which yields  $\lambda_2 \delta v_H$ ) and  $p_2 = v_L$  (which yields  $\delta v_L$ ). The first is better if  $\lambda_2 > \frac{v_L}{v_H}$ , the second if  $\lambda_2 < \frac{v_L}{v_H}$ , and otherwise and mix between the two. It follows that type  $L$  never obtains any surplus in the second period, and therefore behaves myopically in the first period; type  $v_H$  obtains surplus only if  $\lambda_2 \leq \frac{v_L}{v_H}$ . Now consider the first round. An offer of  $p_1 = v_L$  will be accepted by both types, but any offer  $p_1 > v_L$  will be rejected by type  $L$  as it would give him negative surplus. The behavior of type  $H$  in equilibrium depends on the value of  $\lambda$ ; we consider two cases (and ignore the case where  $\lambda = \frac{v_L}{v_H}$ ):

$\lambda < \frac{v_L}{v_H}$  Since type  $L$  rejects and offer  $p_1 > v_L$ , the seller's second-period belief as a function of  $p_1$  must be  $\lambda_2(p_1) \leq \lambda$  for  $p_1 > v_L$ . It follows that the seller will charge  $p_2 = v_L$ . Therefore type  $H$  will accept any  $p_1$  such that  $v_H - p_1 \geq \delta(v_H - v_L) \Leftrightarrow p_1 \leq (1 - \delta)v_H + \delta v_L$ . So the seller will either offer  $p_1 = v_L$ , obtaining a payoff of  $\pi_S = v_L$ , or he will offer  $p_1 = (1 - \delta)v_H + \delta v_L$ , obtaining an (expected) payoff of  $\pi_S = \lambda[(1 - \delta)v_H + \delta v_L] + (1 - \lambda)\delta v_L$ . But:  $\lambda[(1 - \delta)v_H + \delta v_L] + (1 - \lambda)\delta v_L = \lambda v_H + \delta(v_L - \lambda v_H) < v_L$  because  $\lambda < \frac{v_L}{v_H}$ . To summarize: the seller will offer  $p_1 = v_L$  and  $p_2 = v_L$ , and his offer will be accepted by both types of buyer in the first round.

$\lambda > \frac{v_L}{v_H}$  Consider any first-round offer  $p_1 \in ((1 - \delta)v_H + \delta v_L, v_H]$ . Type  $H$  cannot reject this offer with probability 1 in equilibrium, because then we would have  $\lambda_2 = \lambda > \frac{v_L}{v_H}$ , implying a second-period offer of  $p_2 = v_H$ , in which type  $v_H$  would have done at least as well to accept the first-period offer. Nor can  $v_H$  accept this offer with probability 1: if he did, we would have  $\lambda_2 = 0$ , implying a second-period offer of  $p_2 = v_L$ , and so type  $H$  should have rejected the original offer. So type  $H$  must mix between accepting (prob.  $q$ ) and rejecting (prob.  $1 - q$ ) so that  $\lambda_2 = \frac{v_L}{v_H}$ . Since type  $L$  always rejects the first-round offer, we have  $\lambda_2 = \frac{(1 - q)\lambda}{(1 - q)\lambda + (1 - \lambda)} = \frac{v_H}{v_L} \Rightarrow q = \frac{v_H - \lambda v_L}{\lambda(v_H - v_L)}$ . And to make type  $H$  indifferent between accepting and rejecting  $p_1$ , we need to have the seller mixing between  $p_2 = v_L$  (prob.  $r$ ) and  $p_2 = v_H$  (prob.  $1 - r$ ), where  $v_H - p_1 = \delta r(v_H - v_L) \Rightarrow r = \frac{v_H - p_1}{\delta(v_H - v_L)}$ . So in equilibrium, the first-period price must take one of the following three values:

$$\begin{aligned} p_1 &= v_L, \text{ with } \pi_S = v_L \\ p_1 &= (1 - \delta)v_H + \delta v_L, \text{ with } \pi_S = \lambda[(1 - \delta)v_H + \delta v_L] + \delta(1 - \lambda)v_L \\ p_1 &= v_H, \text{ with } \pi_S = \lambda(q + \delta(1 - q))v_H \end{aligned}$$

(to get the third payoff, notice that  $r = 0$  when  $p_1 = v_H$ ). Depending on the parameter values, either of these expressions could be largest. Notice that if the third is largest, the seller never sells to type  $L$ .

- (b) A strategy for the seller describes the offer  $p_1$  to be made in the first period, and the offer  $p_2(p_1)$  to be made in the second period if  $p_1$  is rejected in the first period. We must also describe the firms beliefs: let  $\mu_1(\cdot)$  denote first-round beliefs, and  $\mu_2(\cdot \mid p_1)$  denote second-round beliefs, given that an offer of  $p_1$  has been rejected in the first round. Similarly, let  $A_1(p_1 \mid v)$  denote the action of a buyer of type  $v$  facing an offer of  $p_1$  in

the first round, where  $A_1$  is the probability of acceptance, and let  $A_2(p_2 | v, p_1)$  be the action of a buyer of type  $v$  who has rejected an offer of  $p_1$  and now faces an offer of  $p_2$  in the second round.

It is easy to see that  $A_2(p_2 | v, p_1) = 1$  if and only if  $v \geq p_2$ ; otherwise  $A_2(p_2 | v, p_1) = 0$ . Now consider the seller's choice of  $p_2(p_1)$ ; suppose he believes that  $v$  is uniformly distributed on  $[v_L, \bar{v}]$  (we shall see soon that this is always the case). Then the optimal offer is  $\frac{\bar{v}}{2}$  or  $v_L$ , whichever is higher.

Now return to the first period. If the seller offers  $p_1$  in the first period and is expected to offer  $p_2$  in the first period, the first offer will be accepted if and only if  $v \geq p_1$  and  $v - p_1 \geq \delta(v - p_2) \Rightarrow v \geq \frac{p_1 - \delta p_2}{1 - \delta}$ , i.e.  $p_1$  will be accepted if and only if  $v \geq \max\left\{\frac{p_1 - \delta p_2}{1 - \delta}, p_1\right\}$ . This tells us that the seller's second-period beliefs must be that  $v$  is uniform on  $[v_L, \bar{v}]$ , where  $\bar{v} = \max\left\{\frac{p_1 - \delta p_2}{1 - \delta}, p_1\right\}$  and therefore his second-period offer  $p_2(p_1)$  will be  $\max\{v_L, \bar{v}/2\}$ . So  $\bar{v}$  must satisfy the following equation:

$$\bar{v} = \max\left\{\frac{p_1 - \delta \max\{v_L, \bar{v}/2\}}{1 - \delta}, p_1\right\}.$$

To solve this implicit equation, suppose that  $p_1 \geq \frac{p_1 - \delta \max\{v_L, \bar{v}/2\}}{1 - \delta}$ . Then  $\bar{v} = p_1$ , giving us  $p_1 = \frac{p_1 - \delta \max\{v_L, p_1/2\}}{1 - \delta}$ , so  $p_1 = \frac{p_1 - \delta p_1/2}{1 - \delta}$  or  $p_1 = \frac{p_1 - \delta v_L}{1 - \delta}$ ; clearly the first is not possible if  $\delta > 0$ , nor is the second if  $p_1 > v_L$  as well, which must be the case in equilibrium. So  $\bar{v} = \frac{p_1 - \delta \max\{v_L, \bar{v}/2\}}{1 - \delta}$ : if  $\frac{\bar{v}}{2} \geq v_L$ , then  $\bar{v} = \frac{2p_1}{2 - \delta}$  and  $p_2(p_1) = \frac{p_1}{2 - \delta}$ ; if  $v_L \geq \frac{\bar{v}}{2}$ , then  $\bar{v} = \frac{p_1 - \delta v_L}{1 - \delta}$  and  $p_2(p_1) = v_L$ ; thus  $p_2(p_1) = \max\left\{v_L, \frac{p_1}{2 - \delta}\right\}$ . We have now fixed  $p_2$  in terms of  $p_1$ ; it remains to find the optimal value of  $p_1$  for the seller:

$$\max_{p_1} p_1 \cdot \Pr\{\text{buyer accepts } p_1\} + \delta \max\left\{v_L, \frac{p_1}{2 - \delta}\right\} \cdot \Pr\{\text{firm rejects } p_1 \text{ but accepts } p_2\}$$

Depending on parameter values, the solution to this problem is  $p_1 = v_L$ ,  $p_1 = \frac{(1 - \delta)v_L + 2\delta v_L}{2}$ , or  $p_1 = \frac{(2 - \delta)^2}{2(4 - 3\delta)}v_H$  (corresponding to the three kinds of solution to part (a) above). The remainder of the specification of the equilibrium has already been described.