ECON 2200: Problem Set 4 — Solutions

1. Consider the first round of elimination: it is easy to see that the strategy $s_i = K$ for player i is not a best response to any beliefs: at best it wins a split of the \$1 with all of the other players, in which case playing $s_i = K - 1$ would do strictly better; at worst, it wins nothing, in which case there is some other strategy which would at least win a share of the \$1. So $BR(S) = \{1, 2, \dots, K-1\}$ (note that $s_i = K - 1$ *is* a best response, to the belief that *all* of one's opponents will play $s_j = K$). We can repeat the argument until we obtain $BR(\dots(BR(S))) = \{1\}$. At this point, no further strategies can be eliminated. Therefore $R_i = \{1\}$ is the set of rationalizable strategies for each player.

2. (This example due to Tiberiu Dragu.) Consider the following game, where player 1 chooses a row and player 2 chooses a column:

	A	B	C	D
a	$0,0$	$2,0$	$0,2$	$1,1$
b	$0,2$	$0,0$	$2,0$	$1,1$
c	$2,0$	$0,2$	$0,0$	$1,1$
d	$1,1$	$1,1$	$1,1$	$3,3$

   This game has a unique Nash equilibrium, (d, D), yielding payoff of 3 for each player. There is a correlated equilibrium distribution in which each of the cells yielding $(0,2)$ or $(2,0)$ is played with probablity $\frac{1}{6}$. The expected payoff from this c.e.d. is 1 for each player. Furthermore, the most either player could receive (*ex post*) in this c.e.d. is 2.

3. (a) Their *ex ante* subjective probabilites for E are both $\frac{2}{9}$. Their posterior subjective probabilities (after receiving their private information), depend on what information they recieved, which depends on the true state. Letting p_i^ω denote i's posterior of E at state ω, we have:
   $p_A^\omega = \frac{1}{3}$ for $\omega = 1, \dots, 6$ and $p_A^\omega = 0$ for $\omega = 7, 8, 9$
   $p_B^\omega = \frac{1}{2}$ for $\omega = 1, \dots, 4$ and $p_A^\omega = 0$ for $\omega = 5, \dots, 9$

   (b) Ann: $\frac{1}{3}$; Bob: $\frac{1}{2}$; Ann: $\frac{1}{3}$; Bob: $\frac{1}{3}$; Ann: $\frac{1}{3}$; ...

   (c)

$\bullet^1$ $\bullet^2$ $\bullet^3$	$\bullet^4$	$\bullet^5$ $\bullet^6$	$\bullet^7$ $\bullet^8$ $\bullet^9$	$\mathcal{P}_A$

$\bullet^1$ $\bullet^2$ $\bullet^3$	$\bullet^4$	$\bullet^5$ $\bullet^6$	$\bullet^7$ $\bullet^8$	$\bullet^9$	$\mathcal{P}_B$

   (d) Yes: the element of the meet of $\mathcal{P}_A$ and $\mathcal{P}_B$ containing state 2 $\{1, 2, 3\}$ is a subset of F.

4. (a) Suppose that it is common knowledge at ω that $E_i[\phi] > E_j[\phi]$. Let P be the element of $\mathcal{P}_i \wedge \mathcal{P}_j$ that contains ω. Since it is common knowledge that $E_i[\phi] > E_j[\phi]$, it must be true at every ω in P. Thus $\sum_{\omega \in \mathcal{P}_i(\omega)} \phi(\omega) \cdot p(\omega) / p(\mathcal{P}_i(\omega)) > \sum_{\omega \in \mathcal{P}_j(\omega)} \phi(\omega) \cdot p(\omega) / p(\mathcal{P}_j(\omega))$ for all ω in P. Consider first the LHS. Multiplying by $p(\omega)$ and summing over all ω in Ω we obtain:

$$\sum_{\omega \in P} \left(p(\omega) \cdot \sum_{\omega \in \mathcal{P}_i(\omega)} \phi(\omega) \cdot p(\omega)/p(\mathcal{P}_i(\omega)) \right).$$

Write $P = \cup_k P_k$ where the P_k's are disjoint members of $\mathcal{P}_i$. Then the expression above is equal to:

$$\sum_{P_k} \sum_{\omega \in P_k} \left(p(\omega) \cdot \sum_{\omega \in \mathcal{P}_i(\omega)} \phi(\omega) \cdot p(\omega)/p(\mathcal{P}_i(\omega)) \right) \tag{1}$$

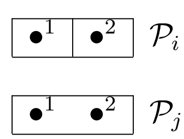
$$= \sum_{P_k} \sum_{\omega \in P_k} \left(\frac{p(\omega)}{p(P_k)} \cdot \sum_{\omega \in P_k} \phi(\omega) \cdot p(\omega) \right) \text{ (since if } \omega \in P_k, \text{ then } P_k = \mathcal{P}_i(\omega)\text{)} \tag{2}$$

$$= \sum_{P_k} \left(\sum_{\omega \in P_k} (\phi(\omega) \cdot p(\omega)) \cdot \sum_{\omega \in P_k} \frac{p(\omega)}{p(P_k)} \right) \text{ (since } \sum_{\omega \in P_k} \phi(\omega) \cdot p(\omega) \text{ is constant across } P_k\text{)} \tag{3}$$

$$= \sum_{P_k} \left(\sum_{\omega \in P_k} (\phi(\omega) \cdot p(\omega)) \right) = \sum_{\omega \in P} (\phi(\omega) \cdot p(\omega)). \tag{4}$$

Conducting the same operation for the RHS, we get the same expression, establishing a contradiction.

(b) Let $\Omega = \{1, 2\}$, $p(1) = p(2) = \frac{1}{2}$, and the agents' partitions be as follows:



Suppose $\phi(1) = 0$ and $\phi(2) = 2$. Then i's expectation of ϕ is 0 at state 1 and 2 at state 2; while j's expectation of ϕ is 1 at both states. Therefore i's expectation of ϕ differs from j's expectation at every state, and thus this fact is common knowledge at every state.

(c) Consider the same state space Ω and random variable ϕ as in part (b), but assume that the agents' information functions are $\mathcal{P}_i(1) = \mathcal{P}_i(2) = 1$, and $\mathcal{P}_j(1) = \mathcal{P}_j(2) = 2$. Then it is common knowledge at every state that i's expectation of ϕ is 0, and that j's expectation of ϕ is 2.