ECON 2200: Problem Set 3 — solutions

1. (a) If the game reaches the final round, Alice will offer $(100, 0)$; thus she will reject anything less than 100 in the second-last round, so Bob can do no better than offer $(100, 0)$; Alice will therefore offer $(100, 0)$ in the first round, which Bob will accept rather then have to pay 10 for making an offer himself.

   (b) If the game reaches the final round, Alice will offer $(100, 0)$; but she will accept any offer of $(20, 80)$ or more in the second round rather than have to pay 80 to make this offer; thus Bob will accept any offer of $(30, 70)$ or more in the first round. This is the offer Alice will make, and Bob will accept.

   (c) In round 3, both types of Alice will offer $(100, 0)$ in the final round, which will be accepted. In round 2, anything but $(100, 0)$ will be rejected by an Alice with $c = 0$ (Alice_0), while $(20, 80)$ or more will be accepted by Alice_{80}. Clearly $(20, 80)$ is the best offer for Bob to make as long as he thinks $\text{Prob.}(c = 80) > 0$. Can we have $\text{Prob.}(c = 80) = 0$ in equilibrium? The answer is no: this could only arise if we had a separating equilibrium, following a rejection of the offer made by Alice_0. But if a first-round offer identifies Alice as type $c = 0$, Bob should accept it (whatever it is) to avoid having to pay of cost of 10 to make his own offer. So in the second round Bob will offer $(20, 80)$, and Alice will accept if and only if $c = 80$.

   Now consider the first round. Clearly Alice_0 will offer $(100, 0)$, since she is happy to wait until round 3 when this is what she will get. Suppose Alice_{80} makes the same offer. Then Bob will reject it, since he is better off with a $\frac{1}{2}$ chance of having his second-round offer accepted, for a expected payoff of $\frac{1}{2}80 - 10 = 30$. This cannot be an equilibrium, since then Alice_{80} would prefer to offer $(30, 70)$ which Bob is sure to accept. What is Alice with $c = 0$ never offers $(100, 0)$? Then Bob will reject whatever offer(s) she does make, unless it gives him at least $(30, 70)$, but he will accept the offer of $(100, 0)$ since it must come from Alice_0; it follows that Alice_{80} would offer $(100, 0)$ after all. So Alice_{80} must mix between offering $(100, 0)$ (with probability p) and offering $(30, 70)$ with probability $(1 - p)$ (since offering anything other than $(100, 0)$ reveals her to be Alice_{80}, in which case Bob rejects any offer of less than $(30, 70)$). For Alice_{80} to be willing to mix between these two offers, she must be indifferent between the two; this requires that Bob mixes between accepting (with probability q) and rejecting the offer of $(100, 0)$. Then we have:

$$\begin{aligned} & \pi_{Alice_{80}}(x_A = 100) = \pi_{Alice_{80}}(x_A = 30) \\ \Rightarrow\quad & 100q + 20(1 - q) - 80 = 30 - 80 \\ \Rightarrow\quad & q = \frac{1}{8}. \end{aligned}$$

   Finally, faced with a first-round offer of $(100, 0)$, Bob believes it comes from Alice_0 with probability $\frac{1}{1+p}$, so he is willing to randomize between accepting and rejecting it if:

$$\begin{aligned} & \pi_{Bob}(\text{accept}) = \pi_{Bob}(\text{reject}) \\ \Rightarrow\quad & 0 = \frac{1}{1+p} \cdot 0 + \frac{p}{1+p} \cdot 80 - 10 \\ \Rightarrow\quad & p = \frac{1}{7}. \end{aligned}$$

   To summarize, the outcome is as follows:

if $c = 0$**:** Alice offers $(100, 0)$; Bob accepts with probability $\frac{1}{8}$ and rejects it with probability $\frac{7}{8}$. If he rejects, he offers $(20, 80)$ in the second round, which Alice rejects. In the third round, Alice offers $(100, 0)$, which Bob accepts.

if $c = 80$**:** Alice offers $(30, 70)$ with probability $\frac{6}{7}$ and $(100, 0)$ with probability $\frac{1}{7}$; if she offers $(30, 70)$, Bob accepts; if she offers $(100, 0)$, Bob accepts with probability $\frac{1}{8}$ and rejects it with probability $\frac{7}{8}$. In this case, Bob offers $(20, 80)$ in the second round, which Alice accepts.

2. (a) No. For these beliefs to be part of a consistent assessment, we would need to find a sequence of strictly mixed strategy profiles for players 1 and 2 such that the beliefs generated by these strategy profiles tend to α_1, β_1, and γ_1. Let $p_n(L)$, $p_n(M)$, $p_n(R)$, $p_n(l)$, $p_n(m)$, and $p_n(r)$ denote such a sequence, for $n = 1, 2, \ldots$. Then we need:

$$\frac{p_n(L)p_n(m)}{p_n(M)p_n(l)} \rightarrow \frac{\alpha_1}{\alpha_2} = 2 \quad (1)$$

$$\frac{p_n(M)p_n(r)}{p_n(R)p_n(m)} \rightarrow \frac{\beta_1}{\beta_2} = 2 \quad (2)$$

$$\frac{p_n(L)p_n(r)}{p_n(R)p_n(l)} \rightarrow \frac{\gamma_1}{\gamma_2} = 2 \quad (3)$$

But (1) and (2) give us:

$$\frac{p_n(L)p_n(m)}{p_n(M)p_n(l)} \cdot \frac{p_n(M)p_n(r)}{p_n(R)p_n(m)} = \frac{p_n(L)p_n(r)}{p_n(R)p_n(l)} \rightarrow 4$$

(since the product of two limits is the limit of the product), contradicting (3).

(b) Yes. For each information set, it easy to find strategies for player 3's opponents which generate these beliefs: e.g. $p(L) = p(M) = \frac{1}{2}$, $p(m) = \frac{2}{3}$, $p(l) = \frac{1}{3}$, $p(R) = p(l)$ for information set α and so on.

3. (a) In the subgame after entry, profits for the two firms are given by:

$$\pi_I = (10 - x_I - x_E - C)\, x_I - Cy$$
$$\pi_E = (10 - x_I - x_E - 4)\, x_E - 2$$

Best response functions are given by:

$$\frac{\partial \pi_I}{\partial x_I} = 10 - C - 2x_I - x_E = 0 \Rightarrow x_I = \frac{10 - C - x_E}{2}$$
$$\frac{\partial \pi_E}{\partial x_E} = 6 - x_I - 2x_E = 0 \Rightarrow x_E = \frac{6 - x_I}{2}$$

Solving,we find

$C = H = 4$**:** If there is entry, we have $x_I^* = x_E^* = 2$, so $p = 6$, $\pi_I = 4 - 4y$, and $\pi_E = 2$;
If there is no entry, we have $x_I^* = 3$ and $x_E^* = 0$, so $p = 7$, $\pi_I = 9 - 4y$, and $\pi_E = 0$.

$C = L = 1$**:** If there is entry, we have $x_I^* = 4$ and $x_E^* = 1$, so $p = 5$, $\pi_I = 16 - y$, and $\pi_E = -1$;
If there is no entry, we have $x_I^* = 4\frac{1}{2}$ and $x_E^* = 0$, so $p = 5\frac{1}{2}$, $\pi_I = 20\frac{1}{4} - y$, and $\pi_E = 0$.

(b) Suppose both types of incumbent choose $y = 0$, and the entrant believes $prob\,(H \mid y) = \frac{1}{2}$ for all values of y. Clearly these beliefs satisfy Bayes' rule. In the subgames following entry and no entry, outputs are chosen as in part (b) above. Clearly it is a best response for the entrant to enter (whatever value of y is chosen), since expected profits from so doing are $\frac{1}{2} > 0$. Therefore we have constructed a PBE in which there is always entry.

(c) The PBE described in part (b) does not pass the test of dominated messages. Any give-away $y > 1\frac{1}{4}$ is dominated for the high-cost type of incumbent, since even if such a give-away deters entry, the maximum profit thereby obtained is $9 - 4y < 4$. This is smaller than the profit that can be obtained from choosing $y = 0$ and accepting entry. Thus the test of dominated messages requires $prob\,(H \mid y) = 0$ for $1\frac{1}{4} < y \leqslant 4\frac{1}{4}$ (note that $y > 4\frac{1}{4}$ is dominated for the low-cost type as well).

(d) No it does not: by choosing any $y > 1\frac{1}{4}$ the low-cost incumbent should be able to distinguish itself from the high-cost incumbent and thereby deter entry; thus choosing $y = 2$ yields less profit than choosing and value of y such that $1\frac{1}{4} < y < 2$.

4. (a) Clearly, the normal monopolist (M_N) will acquiesce and the tough monopolist (M_T) will fight if entry occurs. It follows that the entrant will enter if $b\,(1-p) + (b-1)\,p > 0 \Leftrightarrow p < b$; otherwise the entrant will stay out.

(b) We consider two cases in turn (ignoring the case where $p = b$):

$p > b$ First observe that there is an equilibrium where both types of M acquiesce in the first round: clearly the first entrant, E_1 will enter in this case; if the second entrant, E_2, believes (out-of-equilibrium) that an M who fights is tough with probability $q < b$, he will enter following fighting in the first round and stay out following acquiescence in the first round (in this case there has been no updating). Since $0 + a > -1 + 0$ and $-1 + a > 0 + 0$, both types of M will acquiesce in the first round, as required. But this equilibrium does not pass the intuitive criterion: in equilibrium, M_N is gets a payoff of $0+a$, more than he could possibly get if he fights in the first round; on the other hand, M_T could improve his payoff by fighting, if doing so actually deterred entry. So we ignore this equilibrium.

So, suppose that at least one of the types fights with strictly positive probability in the first round. If it is M_N, then it must be because fighting increases the chances of deterring entry in the second round, in which case M_T will *always* fight in the first round; if on the other hand M_N never fights in the first round but M_T sometimes does, then fighting will always deter second-round entry, and again M_T will always fight in the first round. So either way, M_T will always fight in the first round. So whatever M_N does, E_2's posterior belief that M is tough is $q > p$, and since $p > b$, E_2 will stay out following fighting in the first round. Can it be an equilibrium for M_N to acquiesce in the first round with strictly positive probability? If he does, he gets a (continuation) payoff of $0 + 0$, since E_2 will enter; but by fighting, he gets the higher payoff of $-1 + a$. So M_N fights in the first round as well, and entry is deterred in both rounds. It remains to specify out-of-equilibrium beliefs for E_2 after acquiescence in the first round. There must be such that he would enter, so any probability $q < b$ that M is tough will work.

$p < b$ In this case, the first (weird) equilibrium does not exist, since if both types of M acquiesce in the first round, there is no updating and entry takes place in the second round; clearly M_T would have done better to fight.

Having eliminated this case, by the same reasoning as before M_T must always fight in the first round. If M_N always fights as well, E_2 will enter after fighting and M_N would have preferred to acquiesce; if M_N never fights, E_2 will stay out after fighting and M_N would have preferred to fight. So M_N must mix between fighting (with probability r) and acquiescing (prob. $1-r$), so that E_2 is indifferent between entering and not, i.e. we need E_2's posterior belief that M is tough following fighting ($q = \frac{p}{p+r(1-p)}$) to equal b, i.e. $r = \frac{p(1-b)}{b(1-p)}$. We also need to make sure that M_N is indifferent between fighting and acquiescing (in the first round). Suppose that E_2 enters after first-period fighting with probability s. Then by fighting M_N gets a (continuation) payoff of $-1 + s \cdot 0 + (1-s)\, a$; by acquiescing, he gets $0 + 0$, so for M_N to be indifferent we need $s = 1 - \frac{1}{a}$. It remains to calculate whether E_1 will enter or not. The overall probability of being fought is $p + (1-p)\,\frac{p(1-b)}{b(1-p)} = p\left(1 + \frac{(1-b)}{b}\right)$; if this is less than b (i.e. $p < b^2$) he will enter; if it is larger than b (i.e. $p > b^2$) he will stay out.

(c) If the second entrant cannot observe the first game, then the equilibrium from (a) is repeated twice. There are three cases to consider:

$p > b$ In this case, it does not make any difference whether the second entrant can observe the outcome of the first game, since entry is always deterred in either case.

$b^2 < p < b$ In this case, entry takes place in both rounds if the second entrant does not observe the first game. Both types of M get a payoff of 0. But if the second entrant does observe the outcome of the first game, then the first entrant will stay out (since M_N sometimes fights, for strategic reasons). Thus both types of M are better off.

$p < b^2$ As above, entry takes place in both rounds if the second entrant does not observe the first game. Both types of M get a payoff of 0. If the second entrant does observe the outcome of the first game, entry still always takes place in the first round. M_T always fights it, sometimes deterring entry in the second round and improving his payoff to a value greater than 0. M_N fights first-round entry only some of the time, and is indifferent between doing so and not, so his expected payoff is 0 as before.