ECON 2200: Problem Set 2 solutions

1. There are two pure strategy Nash equilibria of the stage game: (T, L) and (M, C), and in a subgame perfect equilibrium, one of the equilibria must be played in each of the nine second-round subgames. Notice that if (B, R) is played in the first round, yielding the outcome $(4, 4)$, player 1 will be tempted to deviate and play T instead, but player 2 will not. In order to sustain this first round outcome, therefore, player 1 must be rewarded for playing B in the first round, and punished if she plays T. The following strategies will achieve this:

   - player 1: play B in round 1; and play T in round 2 if the first round outcome was (B, R); otherwise play M in round 2.
   - player 2: play R in round 1; and play L in round 2 if the first round outcome was (B, R); otherwise play C in round 2.

   If player 1 deviates in round 1 she gains 1 but loses 2 for a net loss of 1; if player 2 deviates in round 1, he loses 3 (at least) and gains 1 for a net loss of 2.

2. For this to be a subgame perfect equilibrium, we need to check that no one wants to deviate from any of the three possible future play paths. One of these paths is the punishment path $\{(C, C), (C, C), \ldots\}$. By deviating today only, a player can only lower his payoff today, and cannot affect his payoff in the future. Hence, staying on the path is optimal.

   The initial play path is $\{(A, A), (B, B), (A, A), (B, B), \ldots\}$, which yields the payoff stream $\{3, 2, 3, 2, \ldots\}$ for both players. The easiest way to compute the value of this payoff stream is to find the value x such that $\{3, 2, 3, 2, \ldots\}$ is worth the same as $\{x, x, x, x, \ldots\}$, which we know is worth x (in terms of average per period payoff). It's not hard to see that the x that yields indifference between the two payoff streams is the one which solves $3 + \delta(2) = x + \delta(x)$, which holds when $x = (3 + 2\delta)/(1 + \delta)$. (One can also compute this value by evaluating $\{3, 2, 3, 2, \ldots\}$ as the sum of $\{3, 0, 3, 0, \ldots\}$ and $\{0, 2, 0, 2, \ldots\}$.) If either player deviates in the initial period, he can obtain 5 in that period and obtains 1 thereafter. Hence, following the equilibrium is optimal when

$$\begin{aligned} & (3 + 2\delta)/(1 + \delta) \geq (1 - \delta)5 + \delta(1) \\ \Leftrightarrow \quad & 4\delta^2 + \delta - 2 \geq 0 \\ \Leftrightarrow \quad & \delta \geq (\sqrt{33} - 1)/8. \end{aligned}$$

   The other play path to consider is $\{(B, B), (A, A), (B, B), (A, A), \ldots\}$, which yields the payoff stream $\{2,\ 3,\ 2,\ 3,\ \ldots\}$ for both players. This payoff stream is worth $(2+3\delta)/(1+\delta)$. Deviating yields a payoff of 3 today and 1 thereafter. Hence, following the equilibrium is optimal when

$$\begin{aligned} & (2+3\delta)/(1+\delta) \geq (1-\delta)3 + \delta(1) \\ \Leftrightarrow\quad & 2\delta^2 + 2\delta - 1 \geq 0 \\ \Leftrightarrow\quad & \delta \geq (\sqrt{3}-1)/2. \end{aligned}$$

Observe that $(\sqrt{33}-1)/8 > (\sqrt{3}-1)/2$. Hence, the strategy profile is sustainable whenever $\delta \geq (\sqrt{33}-1)/8$.

3. The analysis is based on backward induction. We group decision nodes according to the values of n_i and n_j, the numbers of steps that remain for players i and j. In each statement of the backward induction argument to follow, we assume that it is currently player is turn to move.

   (i) If $n_i = 1$, then because of discounting, player i achieves his maximal payoff by finishing the race immediately; he therefore takes 1 step.

   (ii) Suppose $n_i \geq 2$ and that $n_j = 1$. If player i does nothing, then by (i), player j will finish in the next round. If $n_i \in \{2,3\}$, then it is profitable for player i to finish the race immediately, and so he does so by taking n_i steps. If $n_i \in \{4,5,6\}$, player i cannot finish the race this turn; since he knows that player j will finish next round, player i does nothing.

   (iii) Suppose $n_i = 2$ and $n_j \in \{2,3\}$. If player i advances one step, then by (ii), player j will finish next round. Given this, the best player i can do is to finish immediately, so he does so by taking 2 steps.

   (iv) Suppose $n_i = 3$ and $n_j = 2$. Then if player i advances 1 or 2 steps, (ii) and (iii) imply that j will finish in the next round. Given this, player i finishes immediately by taking 3 steps.

   (v) Suppose that $n_i = n_j = 3$. Then if player i advances 1 or 2 steps, (ii) and (iv) imply that j will finish in the next round. Given this, player i finishes immediately by taking 3 steps.

   (vi) Suppose $n_i = 2$ and $n_j \geq 4$. Then if player i takes 1 step, (ii) implies that he will win next round. Since taking 1 step twice costs less than taking 2 steps once, player i takes just 1 step.

   (vii) Suppose $n_i \geq 4$ and $n_j = 2$. Then regardless of what player i does, player j will eventually win, so i does nothing.

   (viii) If $n_i = 3$ and $n_j \geq 4$, then i takes 1 step (as in (vi)).

   (ix) If $n_i \geq 4$ and $n_j = 3$, then j does nothing (as in (vii)).

   (x) Suppose $n_i = 4$ and $n_j \geq 4$. Then in principle, the cheapest way for i to ultimately win is to take 1 step 4 times. In fact, if player i takes 1 step now, (ix) and other earlier statements imply that he will win in precisely this fashion. Therefore, player i takes 1 step.

(xi) Suppose $n_i \geq 5$ and $n_j = 4$. Then (x) (and (viii) and (vi)) tell us that i will win if and only if he advances to at least $n_i = 3$ during this turn. If $n_i = 5$, then advancing 2 steps immediately and then winning the race is the only way i can earn a positive profit, so he takes 2 steps. If $n_i = 6$, then he cannot earn a positive profit by taking 3 steps now (since $15 + 2 + 2 + 2 = 21 > 20$), so he does nothing.

(xii) Suppose $n_i = n_j = 5$. Then i advances 2 steps (as in (xi)).

(xiii) If $n_i = 6$ and $n_j = 5$, then i does nothing (as in (xi)).

(xiv) If $n_i = 5$ and $n_j = 6$, then i advances 1 step (as in (x)).

(xv) If $n_i = n_j = 6$, then i advances 1 step (as in (x)).

4. Consider the following game of incomplete information, where nature determines the type of player 1: type a or type b with equal probability. This type is private information to player 1.

	L	M	R
U	$0,0$	$0,2$	$1,3$
D	$2,4$	$2,6$	$0,7$

type a

	L	M	R
U	$1,3$	$0,2$	$0,0$
D	$0,7$	$2,6$	$2,4$

type b

Player 2's payoffs against player 1's four pure strategies are as follows:

$$\begin{array}{rcl}
\text{against } U(a), U(b) & : & EU_2(L) = 1.5;\ EU_2(M) = 2;\ EU_2(R) = 1.5 \\
\text{against } U(a), D(b) & : & EU_2(L) = 3.5;\ EU_2(M) = 4;\ EU_2(R) = 3.5 \\
\text{against } D(a), U(b) & : & EU_2(L) = 3.5;\ EU_2(M) = 4;\ EU_2(R) = 3.5 \\
\text{against } D(a), D(b) & : & EU_2(L) = 5.5;\ EU_2(M) = 6;\ EU_2(R) = 5.5
\end{array}$$

Thus player 2 has a dominant strategy to play M, and player 1's best response is to play $D(a)$, $D(d)$. This is the unique Bayesian Nash equilibrium, yileding a payoff of 6 for player 2. Both of the related games, however, have a unique Nash equilibrium in which player 2 obtains a payoff of 3.

5. (a) Strategies: Nature chooses types v_1 and v_2 independently from a uniform distribution on $[0,1]$; player i of type v_i chooses a bid $b_i(v_i) \in [0, \infty)$. Payoffs are given by $u_i(v_i) = \int_0^1 I(v_j) \cdot (v_i - b(v_i))\, dv_j$, where $I(v_j)$ is a function which takes the value 1 if $b_j(v_j) \leqslant b_i(v_i)$ and the value 0 otherwise.

(b) Suppose bidder 2 bids according $b_2(v_2) = \alpha + \beta v_2$. Bidder 1 of type v_1 obtains an expected payoff from bidding b_1 of:

$$\begin{array}{rl}
 & (v_1 - b_1) \cdot \text{Prob}\,(\text{bidder 1 wins}) \\
 & (v_1 - b_1) \cdot \text{Prob}\,(b_1 > \alpha + \beta v_2) \\
= & (v_1 - b_1) \cdot \dfrac{b_1 - \alpha}{\beta}
\end{array}$$

Maximizing this expression with respect to b_1, we obtain:

$$b_1 = \frac{v_1 + \alpha}{2}.$$

Given that the bidding functions are symmetric ($b_1(v) = b_2(v)$), we have $\alpha = 0$, so $b_i(v_i) = \frac{v_i}{2}$ (i.e. $\beta = \frac{1}{2}$).

6. (a) Suppose bidder 2 bids according $b_2(v_2) = \gamma + \delta v_2 + \phi v_2^2$. Bidder 1 of type v_1 obtains an expected payoff from bidding b_1 of:

$$\begin{aligned} & b_1 + v_1 \cdot \text{Prob}\,(\text{bidder 1 wins}) \\ & b_1 + v_1 \cdot \text{Prob}\left(b_1 > \gamma + \delta v_2 + \phi v_2^2\right) \\ = \quad & b_1 + v_1 \cdot \frac{-\delta + \sqrt{\delta^2 + 4b_1\phi - 4\phi\gamma}}{2\phi} \end{aligned}$$

Maximizing this expression with respect to b_1, we obtain:

$$b_1 = \frac{-\delta^2 + 4\phi\gamma + v_1^2}{4\phi}.$$

Since the bidding functions are symmetric, it must be the case that $\gamma = \delta = 0$, and so $b_i(v) = \phi v^2 = \frac{v^2}{4\phi} \Rightarrow \phi = \frac{1}{2}$. Thus $b_i(v_i) = \frac{v^2}{2}$.

(b) In the all-pay auction, the bidders bid less than in the first-price auction. Against a given bidding function of your opponent, the benefit of raising your bid in the all- pay auction is the same as before (increased probability of winning), but the costs are higher, since you have to pay your bid even if you lose.

(c) Let $\bar{v}$ denote the higher of the two valuations. Then in the first-price auction, expected revenue is given by:

$$E[b(\bar{v})] = \frac{E[\bar{v}]}{2} = \frac{1}{3}.$$

In the all-pay auction, expected revenue is given by:

$$E[b(v_1) + b(v_2)] = 2E\left[\frac{v_1^2}{2}\right] = 2\int_0^1 \frac{v_1^2}{2} dv_1 = 2\left[\frac{v_1^3}{6}\right]_0^{\bar{v}} = \frac{1}{3}.$$