

## Gibbard-Satterthwaite theorem

Due date: 2/19/08

Complete the proof of the Gibbard-Satterthwaite theorem (note: this is a weaker version than we proved in class, and hence a little easier to prove).

Notation:

$N$  set of individuals, with  $|N| = n \geq 2$

$X$  set of social states, with  $|X| = k \geq 3$

$\mathcal{P}$  set of all strict orderings on  $X$

$P_i$  individual  $i$ 's preferences (complete, transitive and asymmetric)

$\rho = (P_1, \dots, P_n)$  generic element of  $\mathcal{P}^n$

A *voting rule* picks a single alternative on the basis of a reported preference profile:  $\phi : \mathcal{P}^n \rightarrow X$ .

A voting rule is *unanimous* if it always selects an alternative that is top-ranked by everyone (if such an alternative exists).

A voting rule is *strategy-proof* if it is a (weakly) dominant strategy for an individual truthfully to report her preferences.

**Theorem 1** *The only unanimous, strategy-proof voting rule is a dictatorship.*

A *do-it-yourself* proof

**Fact 1:** Suppose that some alternative  $a \in X$  is selected given some preference profile  $\rho$ . Modify the profile by raising some alternative  $x \in X$  in individual  $i$ 's ranking (holding everything else fixed). Then either  $a$  or  $x$  is now selected. Try to prove this.

**Step 1:** Start with an arbitrary profile in which everyone ranks  $b$  last. Explain why  $b$  cannot be selected by the voting rule (you have to use strategy-proofness and unanimity). Now starting with individual 1 and continuing in order with the other voters, one at a time, have  $b$  jump from the bottom of each ranking to the top. Let  $r \in N$  be the *pivotal* individual for whom the jump causes  $b$  to be selected, i.e.  $b$  is not selected under profile 1 below, but it is selected under profile 2:

<b>1</b>	<b>2</b>	...	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	...	<b>n</b>
$b$	$b$	$b \dots$	$b$	$k$	$a$	...	$c$
$d$	$e$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t$	$f$		$s$	$b$	$b$	$b \dots$	$b$

profile 1 —  $b$  is chosen

<b>1</b>	<b>2</b>	...	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	...	<b>n</b>
$b$	$b$	$b \dots$	$b$	$b$	$a$	...	$c$
$d$	$e$		$\vdots$	$k$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$t$	$f$		$s$		$b$	$b \dots$	$b$

profile 2 —  $b$  is not chosen

Prove the following statements:

- (\*)  $b$  is chosen whenever the first  $r$  individuals rank  $b$  first;
- (\*\*)  $b$  is not chosen whenever individuals  $r$  through  $n$  rank  $b$  last.

We now show that individual  $r$  is, in fact, a dictator.

**Step 2:** Consider any profile of the form

<b>1</b>	<b>2</b>	...	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	...	<b>n</b>
?	?	?...	?	$k$	?	...	?
?	?		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$b$	$b$	$b$	$b$	$b$	$b$	$b \dots$	$b$

profile 3

First, raise  $k$  to the top position for all voters. By unanimity,  $k$  is chosen. Next, raise  $b$  to the top for voters 1 through  $r - 1$  one at a time, leaving  $b$  on the bottom for everyone else:

<b>1</b>	<b>2</b>	$\dots$	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	$\dots$	<b>n</b>
$b$	$b$	$b \dots$	$b$	$k$	$k$	$k \dots$	$k$
$k$	$k$	$k \dots$	$k$	?	?	$? \dots$	?
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
?	?		?	$b$	$b$	$b \dots$	$b$

profile 4

By (\*\*) alternative  $b$  is not chosen, so by Fact 1 alternative  $k$  is still chosen. Finally, raise  $b$  to the second position for voter  $r$ . Explain why  $k$  is still chosen., i.e. why  $k$  is still chosen under profile 5

<b>1</b>	<b>2</b>	$\dots$	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	$\dots$	<b>n</b>
$b$	$b$	$b \dots$	$b$	$k$	$k$	$k \dots$	$k$
$k$	$k$	$k \dots$	$k$	$b$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
?	?		?		$b$	$b \dots$	$b$

profile 5

Now, reconsider profile 3 and suppose  $g \neq k$  is chosen. Raise  $b$  to the top position for the first  $r - 1$  voters, one at a time. By (\*\*),  $b$  is not chosen; explain why  $g$  must still be chosen. Now raise  $b$  in voter  $r$ 's profile to the second position:

<b>1</b>	<b>2</b>	$\dots$	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	$\dots$	<b>n</b>
$b$	$b$	$b \dots$	$b$	$k$	?	$\dots$	?
?	?		$\vdots$	$b$	$\vdots$	$\dots$	$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
?	?		?		$b$	$b \dots$	$b$

profile 6

Suppose  $b$  is not selected. Explain why  $g$  must still be chosen. However,  $b$  will be selected when it is raised to the top of  $r$ 's profile (why?). Hence, voter  $r$  should falsely report this in profile 6, since she prefers  $b$  to  $g \neq k$ , contradicting strategy-proofness. So  $b$  must be selected in profile 6. now raise  $k$  to the second position for voters 1 through  $r - 1$ , and to the first position for voters  $r + 1$  through  $n$ . Explain why  $b$  must still be chosen. But this modified profile is the same as profile 5, where  $k$  is chosen! This yields a contradiction, so it cannot be the case that  $g \neq k$  is chosen in profile 3 (i.e.  $k$  must in fact be chosen).

**Step 3:** Now consider an arbitrary profile in which  $r$  ranks some alternative  $k \neq b$  on top. First, modify the profile by dropping  $b$  to the bottom for all voters. From step 2, alternative  $k$  is chosen. Now restore  $b$  to its original position for all voters (one at a time). By Fact 1, either  $k$  or  $b$  is chosen. Now consider:

<b>1</b>	<b>2</b>	$\dots$	<b>r - 1</b>	<b>r</b>	<b>r + 1</b>	$\dots$	<b>n</b>
<i>b</i>	<i>b</i>	<i>b</i> ...	<i>b</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>a</i>
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i>	<i>c</i> ...	<i>c</i>

profile 7

where  $c \neq b$  and  $c \neq k$ . Just as in step 1, have  $c$  jump in the rankings of the voters one at a time until we discover the pivotal  $m$  for alternative  $c$ . Symmetrically to step 2, this pivotal voter has his top choice selected with profile 7. On the other hand, from (\*) we also know that with profile 7, alternative  $b$  is chosen. Hence, the pivotal voter must be such that  $m \leq r$ . But a symmetric argument (beginning with  $m$  and then finding  $r$ ) shows that  $r \leq m$ , so  $m = r$  and voter  $r$  is pivotal with respect to  $c$  as well as  $b$ . Thus, in addition to knowing that  $k$  or  $b$  is chosen for our initial arbitrary profile, we know that either  $k$  or  $c$  is chosen. Since  $c \neq b$ , alternative  $k$  must be chosen.

Finally, if  $k = b$  a similar argument shows that voter  $r$  is pivotal for  $a$  (as well as  $c$ ) and that  $b$  is selected. Hence,  $r$  is a dictator. ■