# Arrow's impossibility theorem

Due date: 2/19/08

Complete the proof of Arrow's impossibility theorem

Notation:

$N$ set of individuals, with $|N| = n \geq 2$

$X$ set of social states, with $|X| = k \geq 3$

$\mathcal{R}$ set of all weak orderings on $X$

$R_i$ individual $i$'s preferences (complete and transitive)

$R$ social preferences

$\rho = (R_1, \ldots, R_n)$ generic element of $\mathcal{R}^n$

An *Arrowian social welfare function* (henceforth swf) is a function $f : \mathcal{R}^n \to \mathcal{R}$.

Consider the following conditions on $f$:

**UD** *Unrestricted domain*: For all $\rho \in \mathcal{R}^n$, $f(\rho) = R$ is complete and transitive

**WP** *Weak Pareto principle*: for all $x, y \in X$, if $xP_iy$ for all $i \in N$, then $xPy$

**IIA** *Independence of irrelevant alternatives*: suppose $\rho$ and $\rho'$ agree on $x, y \in X$ (i.e. $xR_iy$ if and only if $xR'_iy$, for all $i \in N$); then $xRy$ if and only if $xR'y$

**ND** *Non-dictatorship*: there is no individual $i \in N$ such that for all $x, y \in X$ and all $\rho \in \mathcal{R}^n$, $xP_iy$ implies $xPy$

**Theorem 1** *There is no swf satisfying* ***UD****,* ***WP****,* ***IIA****, and* ***ND***

*A do-it-yourself proof*

Assume that **UD**, **WP**, and **IIA** are satisfied. We will show that **ND** is not.

**Step 1**: Consider some social state $c \in X$, and profile of orderings $\rho$ such that $xP_ic$ for all $i \in N$ and for all $x \neq c$ (i.e. $c$ is at the bottom of everyone's rankings). Explain why $c$ must be at the bottom of the social ranking, $R = f(\rho)$ (so $xPc$ for all $x \neq c$).

**Step 2**: Now consider a profile of orderings $\rho'$ which is the same as $\rho$, except that everyone now places $c$ at the *top* (i.e. for all $i \in N$ and all $x, y \neq c$, $cP'_ix$, $cP'_iy$, and $xR'_iy$ if and only if $xR_iy$). For $j = 0, 1, \ldots, n$, define:

$$\rho^j = \left(R'_1, \ldots, R'_j, R_{j+1}, \ldots, R_n\right).$$

Note that $\rho^0 = \rho$, so $c$ is at the bottom of $f(\rho^0)$, while $\rho^n = \rho'$, so $c$ is at the top of $f(\rho^n)$ (why?). Explain why, for every $j = 0, \ldots, n$, $c$ is either at the bottom or at the top of $f(\rho^j)$. Let $j^*$ be the smallest $j$ such that $c$ is at the top of $f(\rho^j)$.

**Step 3**: Take any distinct states $a, b \neq c$, and consider the ranking

$$\rho'' = \left(R'_1, \ldots, R'_{j^*-1}, R''_{j^*}, R_{j^*+1}, \ldots, R_n\right),$$

where $R''_{j^*}$ is a ranking for $j^*$ such that $aP''_{j^*}c$ and $cP''_{j^*}b$ (so $aP''_{j^*}b$). Explain why we must have $aP''c$ and $cP''b$ (where $P'' = f(\rho'')$). Infer that $aP''b$.

**Step 4**: Now observe that, in our orignal choice of $\rho$, the individual rankings of $a$ and $b$ were arbitrary. Explain why, if we had started with a different profile of orderings also satsifying the condition on $c$ specified in step 1, the procedure described above would have picked out the *same* agent $j^*$. Infer that whenever $aP_{j^*}b$, we have $aPb$. Note that this is true for *all* pairs of states $a, b \neq c$.

**Step 5**: Repeat the entire procedure starting with some alternative state $d \neq c$ at the bottom of everyone's rankings. Explain why $j^*$ must be a dictator.