ECON 1905: Problem Set 1

Due 1/29/08 in class

Throughout this problem set, let N denote the set of individuals, with $n \geq 2$ elements, and let X denote the set of social states, with $k \geq 3$ elements. We use R_i,P_i and I_i to denote individual i's weak preference relation, strict preference relation, and indifference relation respectively, and assume that R_i is complete and transitive. R, P and I denote the social preference relations.

1. This question is asking you to investigate the likelihood that there is a Condorcet winner, i.e. a social state x which beats every other state in pairwise comparisons: $|\{i \mid xP_iy\}| > \frac{n}{2}$ for all $y \neq x$. Assuming that all preferences are strict (i.e. there is no indifference):

   (a) How many possible preference orderings are there for each individual?

   (b) How many combinations of preference orderings are there for the whole society of n individuals?

   (c) Let $\Pr(n, k)$ denote the probability that there is no Condorcet winner, if each combination of preference orderings is equally likely. It is difficult to compute $\Pr(n, k)$ for the general case, but see if you can find an expression for $\Pr(3, k)$. Treat the case where k is odd and the case where k is even separately.

2. (a) Find an example where there is a Condorcet winner, but a different state would be chosen by the first-past-the-post method (which selects the state ranked first by the most individuals).

   (b) Find an example where the Condorcet winner does not seem to be the most reasonable choice.

3. Suppose social preferences R are determined by the *unanimity rule*: xRy if there is some agent i for whom xR_iy, where $x, y \in X$. Explain why this rule does not provide a counter-example to Arrow's theorem.

4. Call a binary relation P on a set X *cyclic* if and only if there is a sequence of distinct elements $x_1, x_2, \ldots, x_k \in X$ such that $k > 1$ and x_jPx_{j+1} for all $j \in \{1, 2, \ldots, k-1\}$ and x_kPx_1. Suppose that strict social preferences are determined by the rule: xPy if and only if there are at least m agents for whom xP_iy.

   (a) Show that if there are four agents and four social states, and $m = 3$, then P may be cyclic.

   (b) Show that if there are four agents and three social states, and $m = 3$, then P cannot be cyclic.

5. Ancient Rome was (for a while) ruled by two *consuls*, each with veto power over the other's actions. We can represent this situation by the social choice function such that xRy if and only if xR_1y or xR_2y (x and y are social states, and R_1 and R_2 are the weak preference orderings of the two consuls; the preference of the other citizens, $3, \ldots, N$ do not matter). There are at least three social states.

   (a) Show that this social choice function satisfies the conditions WP, IIA, and ND of Arrow's theorem.

   (b) Show that the social choice ordering R generated by this rule need not be transitive.

   (c) Show that the social choice ordering R is *quasi-transitive*, i.e. strict preference P is transitive but indifference I might not be.

6. Are either of the following sets of preference orderings single-peaked?

   (a)

1:	x	y	z
2:	x	y	z
3:	y	z	x
4:	z	x	y

   (b)

1:	x	y	z	w
2:	y	z	w	x
3:	z	w	x	y