

Lecture 14: Chapter 6, Section 2 Finding Probabilities: More General Rules

- General “And” Rule
- More about Conditional Probabilities
- Two Types of Error
- Independence

Looking Back: Review

- 4 Stages of Statistics
 - Data Production (discussed in Lectures 1-4)
 - Displaying and Summarizing (Lectures 5-12)
 - Probability
 - Finding Probabilities
 - Random Variables
 - Sampling Distributions
 - Statistical Inference

Probability Rules (Review)

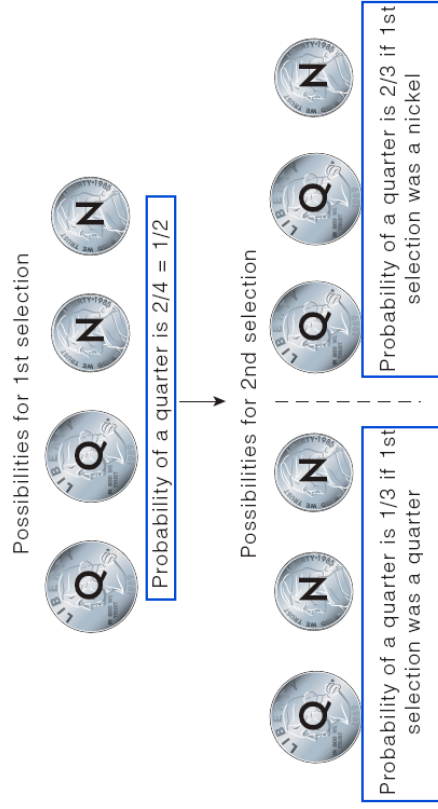
Non-Overlapping “Or” Rule: For any two *non-overlapping* events A and B,
 $P(A \text{ or } B) = P(A) + P(B)$.

Independent “And” Rule: For any two *independent* events A and B,
 $P(A \text{ and } B) = P(A) \times P(B)$.

General “Or” Rule: For any two events A and B,
 $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Need “And” Rule that applies even if events are *dependent*.

Example: When Probabilities Can't Simply be Multiplied



Definition and Notation

Conditional Probability of a second event, given a first event, is the probability of the second event occurring, assuming that the first event has occurred.

P(B given A) denotes the conditional probability of event B occurring, given that event A has occurred.

Looking Ahead: Conditional probabilities help us handle dependent events.

Example: Intuiting the General “And” Rule

- Background:** In a child’s pocket are 2 quarters and 2 nickels. He randomly picks a coin, does *not* replace it, and picks another.
- Question:** What is the probability that the first *and* the second coin are quarters?
- Response:** probability of first a quarter (), times (conditional) probability that second is a quarter, **given** first was a quarter ():

Example: Intuiting General “And” Rule with Two-Way Table

- Background:** Surveyed students classified by sex and ears pierced or not.

	Ears Pierced	Ears Not Pierced	Total
Female	270	30	300
Male	20	180	200
Total	290	210	500

- Question:** What are the following probabilities?
 - Probability of being male
 - Probability of having ears pierced, given a student is male
 - Probability of being male and having ears pierced
- Response:**
 - $P(M) =$
 - $P(E \text{ given } M) =$
 - $P(M \text{ and } E) =$

General “And” Rule (General Multiplication Rule)

For *any* two events A and B,

$$P(A \text{ and } B) = P(A) \times P(B \text{ given } A)$$

$$= P(B) \text{ if } A \text{ and } B \text{ are independent}$$

A Closer Look: In general, the word “and” in probability entails multiplication.

Example: Applying General “And” Rule

- **Background:** Studies suggest lie detector tests are “well below perfection”, 80% of the time concluding someone is a spy when he actually is, 16% of the time concluding someone is a spy when he isn't. Assume 10 of 10,000 govt. employees are spies.
- **Question:** What are the following probabilities?
 - Probability of being a spy and being detected as one
 - Probability of *not* being a spy but “detected” as one
 - Overall probability of a positive lie detector test
- **Response:** First “translate” to probability notation:
 $P(D \text{ given } S) = \underline{\hspace{2cm}}$; $P(D \text{ given not } S) = \underline{\hspace{2cm}}$; $P(S) = \underline{\hspace{2cm}}$; $P(\text{not } S) = \underline{\hspace{2cm}}$
 - $P(S \text{ and } D) = \underline{\hspace{2cm}}$
 - $P(\text{not } S \text{ and } D) = \underline{\hspace{2cm}}$
 - $P(D) = P(S \text{ and } D \text{ or not } S \text{ and } D) = \underline{\hspace{2cm}}$

Example: “Or” Probability as Weighted Average of Conditional Probabilities

- **Background:** Studies suggest lie detector tests are “well below perfection”, 80% of the time concluding someone is a spy when he actually is, 16% of the time concluding someone is a spy when he isn't. Assume 10 of 10,000 govt. employees are spies.
- **Question:** Should we expect the overall probability of being “detected” as a spy, $P(D)$, to be closer to $P(D \text{ given } S) = 0.80$ or to $P(D \text{ given not } S) = 0.16$?
- **Response:** Expect $P(D)$ closer to _____ because _____
(In fact, $P(D) = 0.16064$.)

General “And” Rule Leads to Rule of Conditional Probability

Recall: For any two events A and B,

$$P(A \text{ and } B) = P(A) \times P(B \text{ given } A)$$

Rearrange to form **Rule of Conditional Probability:**

$$P(B \text{ given } A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example: Applying Rule of Conditional Probability

- **Background:** For the lie detector problem, we have
 - Probability of being a spy: $P(S) = 0.001$
 - Probability of spies being detected: $P(D \text{ given } S) = 0.80$
 - Probability of non-spies detected: $P(D \text{ given not } S) = 0.16$
 - Probability of being a spy and detected: $P(D \text{ and } S) = 0.0008$
 - Overall probability of positive lie detector: $P(D) = 0.16064$
- **Question:** If the lie-detector indicates an employee is a spy, what is the probability that he actually is one?
- **Response:** $P(S \text{ given } D) = \underline{\hspace{2cm}}$

Note: $P(S \text{ given } D)$ is very different from $P(D \text{ given } S)$.

A Closer Look: Bayes Theorem uses conditional probabilities to find probability of earlier event, given later event is known to occur.

Two Types of Error in Lie Detector Test

- 1st Type of Error:** Conclude employee is a spy when he/she actually is not.
- 2nd Type of Error:** Conclude employee is not a spy when he/she actually is.

Example: Two Types of Error in Lie Detector Test

- Background:** For the lie detector problem, we have
 - Probability of spies being detected: $P(D \text{ given } S) = 0.80$
 - Probability of non-spies detected: $P(D \text{ given not } S) = 0.16$
- Questions:**
 - What is probability of 1st type of error (conclude employee is spy when he/she actually is not)?
 - What is probability of 2nd type of error (conclude employee is not a spy when he/she actually is)?
- Responses:**
 - 1st type:
 - 2nd type:

Testing for Independence

The concept of independence is tied in with conditional probabilities.

Looking Ahead: Much of statistics concerns itself with whether or not two events, or two variables, are dependent (related).

Example: Intuiting Conditional Probabilities When Events Are Dependent

- Background:** Students are classified according to gender, M or F, and ears pierced or not, E or not E.

	Ears Pierced	Ears Not Pierced	Total
Female	270	30	300
Male	20	180	200
Total	290	210	500

- Questions:**
 - Should gender and ears pierced be dependent or independent? If dependent, which should be less, $P(E)$ or $P(E \text{ given } M)$?
 - What are the above probabilities, and which is less?
- Responses:**
 - Expect $P(E \text{ given } M)$ _____ $P(E)$ because fewer _____ have pierced ears.
 - $P(E \text{ given } M) =$ _____ $P(E) =$ _____

Example: Intuiting Conditional Probabilities When Events Are Independent

- **Background:** Students are classified according to gender, M or F, and whether they get an A in stats.

	A	Not A	Total
Female	0.15	0.45	0.60
Male	0.10	0.30	0.40
Total	0.25	0.75	1.00
- **Questions:**
 - Should gender and getting an A or not be dependent or independent? How should $P(A)$ and $P(A \text{ given } F)$ compare?
 - What are the above probabilities, and how do they compare?
- **Responses:**
 - _____ . Expect $P(A \text{ given } F)$ $\underline{\hspace{1cm}}$ $P(A)$ because knowing a student's gender doesn't impact probability of getting an A.
 - $P(A) = \underline{\hspace{1cm}}$; $P(A \text{ given } F) = \underline{\hspace{1cm}}$

Independence and Conditional Probability

Rule:

A and B independent $\rightarrow P(B) = P(B \text{ given } A)$

Test:

$P(B) = P(B \text{ given } A) \rightarrow$ A and B are independent

$P(B) \neq P(B \text{ given } A) \rightarrow$ A and B are dependent

Independent \leftrightarrow regular and conditional probabilities are equal (occurrence of A doesn't affect probability of B)

Independence and Product of Probabilities

Rule:

Independent $\rightarrow P(A \text{ and } B) = P(A) \times P(B)$

Test:

$P(A \text{ and } B) = P(A) \times P(B) \rightarrow$ independent

$P(A \text{ and } B) \neq P(A) \times P(B) \rightarrow$ dependent

Independent \leftrightarrow probability of both equals product of individual probabilities

Table of Counts Expected if Independent

- For A, B independent, $P(A \text{ and } B) = P(A) \times P(B)$.
- This Rule dictates what counts would appear in two-way table if the variable A or not A is independent of the variable B or not B:
- If independent, count in category-combination A and B must equal total in A times total in B, divided by overall total in table.

Example: Counts Expected if Independent

- Background: Students are classified according to gender and ears pierced or not. A table of expected counts ($174 = \frac{290 \times 300}{500}$, etc.) has been produced.

		Counts expected if gender and pierced ears were independent			Counts actually observed		
		E	not E	Total	E	not E	Total
not M		174	126	300	270	30	300
M		116	84	200	20	180	200
Total		290	210	500	290	210	500

- Question: How different are the observed and expected counts?
- Response: Observed and expected counts are very different (270 vs. 174, 20 vs. 116, etc.) because

Example: Counts Expected if Independent

- Background: Students are classified according to gender and grade (A or not). A table of expected counts ($15 = \frac{25 \times 60}{100}$, etc.) has been produced.

Exp		A	not A	Total	Obs		A	not A	Total
F		15	45	60	F		15	45	60
M		10	30	40	M		10	30	40
Total		25	75	100	Total		25	75	100

- Question: How different are the observed and expected counts?
- Response: Counts are identical because

Lecture Summary

(Finding Probabilities; More General Rules)

- General “And” Rule
- More about Conditional Probabilities
- Two Types of Error
- Independence
 - Testing for independence
 - Rule for independent events
 - Counts expected if independent