TIME-FREQUENCY ANALYSIS EE3528 REPORT

N.Krishnamurthy

Department of ECE
University of Pittsburgh
Pittsburgh, PA 15261

ABSTRACT

Time-Frequency analysis, is an important ingredient in signal analysis. It has a plethora of applications ranging from early detection of failure of machine parts to characterizing heart and muscle sounds to possibly detect anomalies in medicine . This project ties in the concepts learnt in theory to practice, by implementing routines that perform Short-time Fourier analysis and Wigner-Ville representation of test signals. Instantaneous properties of signals like instantaneous frequency, group-delay and bandwidth are also studied.

1. KEYWORDS

Time-Frequency Analysis, Wigner-Ville distribution, Spectrogram , Analytical signal, Instantaneous Frequency(IF), Group Delay(GD), Cohen-Lee Bandwidth

2. INTRODUCTION

2.1. What is Signal Analysis

Engineering problems are analyzed and solved by asking relevant questions. It is quite common to find multiple plausible hypotheses to a problem, when different questions are posed or when solutions are sought under different trains of thought.

Signal analysis is no different, many methods, transformations, generalizations have evolved to explain, and model the wide variety of signals that are encountered in radar, sonar, biomedical imaging and speech applications. These models are used to retrieve a signal that undergoes dispersion, attenuation, corruption with extraneous signals when it propagates through a medium.

2.2. Signal Decomposition: Time, Frequency and Scale Content

Decomposition of complex signals into simpler constituent units, and reconstruction of the original signal from the decomposed units has been the fundamental theme underlying signal analysis. Projection of the original signal onto different subsets of decompositions enhances different properties in the signal.

Fourier was the first to represent functions as a sum of sinusoids and Fourier coefficients represent the frequency content of the signal. The time localization of a band-limited signal is obtained by representing the original signal as sum of Dirac(Sinc) functions. Many more of these signal transformations including Scale transform and Wavelets have been proposed to study different characteristics of the original signal. We will now look into the properties of Short-Time Fourier Analysis[7] and Wigner-Ville representations[5][4].

3. SECOND MOMENT CHARACTERIZATION OF SIGNALS

Quadratic or Bilinear representations of signals are very important in signal analysis. They completely characterize a very common class of stochastic process in Engineering - the Normal or Gaussian process, the sum of a large number of random variables tend to be Normal by the central limit theorem. Speaking non-stochastically the quadratic representation is a Least Square approximation that optimizes the $(error)^2$ in trying to model the given signal.

The second moment characterization of a signal thus forms the cornerstone of characterizing non-stationary signals, like speech or FM with a time varying spectrum. The characterization takes two flavors, an energetic representation and the correlative representation, given by the distribution and its ambiguity function, we shall see the properties of both in Wigner-Ville Distribution 3.3. Cohen's generic formulation of Bilinear TFD is taken up next.

3.1. Cohen's General Bilinear Formulation

In 1966, Cohen[3] gave a unified formulation from which all TFD's can be obtained:

$$\mathscr{P}(t,\omega) = \iiint e^{j2\pi v(u-t)}\phi(v,\tau)s^*(u-\frac{\tau}{2})s(u+\frac{\tau}{2})e^{j2\pi f\tau}dvdud\tau \quad (1)$$

where $\phi(v,\tau)$ is called the kernel. v and τ are the frequency and time lag variables. s is the correlated signal, and the ambiguity function of the signal in 1:

$$\mathscr{A}_s(v,\tau) = \iint s^*(u-\frac{\tau}{2})s(u+\frac{\tau}{2})e^{j2\pi v(u-t)}dvd\tau \quad (2)$$

Different distributions can be obtained by taking different kernels. The kernel method has a number of advantages[3] such as :

- It is easy to generate the distributions by just choosing the kernel function. For example, the WVD is obtained by choosing $\phi(v,\tau) = 1$, while a Spectrogram can be obtained when $\phi(v,\tau) = \mathscr{A}_h(v,\tau)$, the *Ambiguity function of the window*
- The distributions with certain characteristics can be extracted by constraining the kernel see section 3.3.3
- The properties of an unknown distribution can be easily determined by examining the kernel. The properties of TFD's can be represented in terms of the kernel function see section 3.3.3 on WVD.

3.2. Short-Time Fourier Analysis(STFT and Spectrogram)

STFT is a natural extrapolation of the Fourier analysis technique for non-stationary or time-varying signals. It involves segmenting the signal using a real-window and obtaining the frequency content of the signal in that window (assuming the signal is stationary within the window).

3.2.1. *Definitions and Properties*

The STFT is obtained by taking the Fourier transform of the windowed signal, $w(n-m)$ is the time dependent real-window function, used to segment the signal $x(n)$.

$$STFT: \ \mathscr{X}_n(e^{j\omega}) = \sum_{m=-\infty}^{\infty} w(n-m)x(m)e^{-j\omega m} \quad (3)$$

The Spectrogram is the square of the magnitude of the STFT obtained in 3

$$Spectrogram: \ \mathscr{P}_{SP}(t,\omega) = |\mathscr{X}_n(e^{j\omega})|^2 \quad (4)$$

The spectrogram does not satisfy both marginals simultaneously. The broadband spectrogram uses narrow windows and has good time localization while the narrow-band spectrogram that uses wider windows trades-off the time localization for finer frequency resolution, see Figure [1] of simulations for confirmation of these properties.

3.3. Wigner-Distribution

The Wigner distribution was proposed in 1932 by E.P Wigner, as a new tool for quantum mechanics. J.Ville proposed Wigner distribution for the study certain stochastic signals in harmonic analysis in 1948.[5][4][3].

3.3.1. *CT Wigner Distribution*

$$CTWD: \ \mathscr{W}(t,\omega) = \int x^*(t-\frac{\tau}{2})x(t+\frac{\tau}{2})e^{j2\pi\omega\tau}d\tau \quad (5)$$

3.3.2. *DT Wigner Distribution*

Implementation of WD in digital computers entails a discrete-time formulation of the WD. Classen and Mecklenbrauker[1980] formulated the discrete version:

$$DTWD: \ \mathscr{W}(n,\theta) = 2\sum_m x^*(n-m)x(n+m)e^{j4\pi\theta m} \quad (6)$$

where θ is the normalized-frequency, with period $\frac{1}{2}$. Sampling theorem imposes restrictions on the *half-band* of the signal x to be $|\theta-\theta_0| < \frac{1}{4}$ to prevent aliasing.

3.3.3. *Properties of WD*

- WD is always Real, the necessary condition for a distribution to be real is the *Kernel* described in section 3.1 satisfies the condition: $\phi(v,\tau) = \phi^*(-v,-\tau)$
- WD satisfies time and frequency marginals. When the kernel described in section 3.1 is constrained such that $\phi(0,\tau) = 1$ frequency marginal is satisfied, while $\phi(v,0) = 1$ ensures the time marginal is met, and total-normalized energy is obtained when $\phi(0,0) = 1$
- WD exhibits good Weak-support properties, i.e the distribution is zero when the signal is zero, but not Strong-support properties see Figure [1] of simulations for confirmation of these properties.

4. CONDITIONAL MOMENTS:

4.1. Instantaneous Frequency and Group Delay

The conditional moments define the instantaneous quantities of a signal.

$$IF: <\omega>_t = \frac{\int_{-\infty}^{\infty}\omega\mathscr{W}(t,\omega)d\omega}{\int_{-\infty}^{\infty}\mathscr{W}(t,\omega)d\omega} \quad (7)$$

$$GD: <t>_\omega = \frac{\int_{-\infty}^{\infty}t\mathscr{W}(t,\omega)dt}{\int_{-\infty}^{\infty}\mathscr{W}(t,\omega)dt} \quad (8)$$

The Instantaneous frequency and Group delay are the first conditional moments of a joint TF distribution that satisfies marginals(in this case it is the *Wigner Distribution*). They can also be obtained from the *Analytical Signal*, which is a complex signal with no negative spectral components. The analytical signal is obtained by taking the *Hilbert transform* of the signal refer Picinobono[6] and Boashash[2] for more on IF and GD.

IF and GD in terms of the *Analytical signal* $x_a(t)$ and its fourier transform $X_a(\omega)$ are :

$$IF : <\omega>_t = \frac{1}{2\pi}\frac{d\ arg\ x_a}{dt} \tag{9}$$

$$GD : <t>_\omega = \frac{1}{2\pi}\frac{d\ arg\ X_a(\omega)}{d\omega} \tag{10}$$

See Figure [3] for the IF of two tones of equal strength. The IF for $s(n) = A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n}$ was obtained by Cohen[3], where $\dot{\phi}(n)$ is the derivative of the instantaneous phase.

$$<\omega>_t\text{:}\ \dot{\phi}(n) = \frac{1}{2}(\omega_1+\omega_2)+\frac{1}{2}(\omega_1-\omega_2)\frac{A_2^2-A_1^2}{A^2(n)} \tag{11}$$

Note: When $A_1 = A_2$ i.e for equal strength sinusoids, the IF reduces to $\frac{1}{2}(\omega_1+\omega_2)$, see Figure [3] of simulations for confirmation of this property.

4.2. Instantaneous Bandwidth

The first conditional moments as described in the previous section gave IF and GD. Similarly the second conditional moment gives a complex instantaneous bandwidth expression:

$$\sigma^2_{\omega/t} = <\omega^2>_t - <\omega>_t^2 \tag{12}$$

where the first term is the second conditional moment and the second term in the $(IF)^2$. Cohen and Lee bandwidth for the signal $s(t) = A(t)e^{\phi(t)}$, is given by:

$$\sigma^2_{\omega/t} = \left(\frac{\dot{A}(t)}{A(t)}\right)^2 \tag{13}$$

The instantaneous bandwidth for Wigner-Ville representations is given by:

$$\sigma^2_{\omega/t} = \frac{1}{2}\left[\left(\frac{\dot{A}(t)}{A(t)}\right)^2 - \frac{\ddot{A}(t)}{A(t)}\right] \tag{14}$$

5. MATLAB SIMULATION

Matlab code in the appendix, was written using canned functions available in the time-frequency toolbox created by P.Flandrin, F.Auger e.t.al [1]. The annotations for generating the figures in the report are given below:

5.0.0.0.1. Spectrogram and Wigner-Ville representation of signals

Annotation for Figure[1]. The TFD Plots viz. WV and Spectrogram of test data are compared. A Gaussian window was used for the spectrogram, both narrow and broad windows are used for obtaining good resolution in time and frequency respectively. A minimum of 50 percent overlap is required for smooth plots here a 99 percent overlap is used to obtain the spectrogram plots, caxis is the equivalent of clims that thresholds gray-scale images to the specified thresholds. The plots obtained are inverse gray scale by using colormap(1-gray(256)) i.e dark regions have maximum amplitudes. Note: The axis command is used to restrict the plot to the desired regions, and fs the sampling frequency is used to scale the time and frequency axis

5.0.0.0.2. Marginals of WVD and Spectrogram

Annotation for Figure[2]. The marginals $|s(t)|^2$ and $|s(w)|^2$ are obtained from the original signal and the time and frequency marginals for the joint WVD and spectrogram is obtained by summing the TFD in frequency and time respectively. Note: WVD's marginals match the signals marginal while the spectrogram is only able to match either one, narrow-band gives the frequency marginal while broadband gives the time marginal.

5.0.0.0.3. Instantaneous Frequency and Bandwidth of Signals

Annotation for Figure[3]. The IF is obtained for two tones of equal strength by differentiating the phase of the analytical signal. Note:$IF = \frac{1}{2}(w_1 + w_2)$. The IF is also obtained as the first conditional moment of the WVD of the data, Note it matches the IF obtained as the first differential of the instantaneous phase obtained from the analytical signal. Confirms that WVD satisfies the IF and GD condition. The second conditional moment of the data signals is similarly calculated from the WVD of the data and the instantaneous bandwidth $\sigma_{\omega/t}$ is obtained. The standard deviation is obtained by taking expectation of the conditional moment or local average, $\sigma_\omega = \frac{1}{N}\sum \sigma_{\omega/t}$.

6. CONCLUSIONS

The exercise of analyzing signals using TF analysis techniques viz. Spectrogram and Wigner-Ville, confirms several properties of these techniques. Listed below are certain observations from this exercise:

1. WVD satisfies the time and frequency marginals, while the Spectrogram can only meet one of them

2. The analytical signal is helpful in obtaining the Instantaneous attributes of a signal viz. amplitude and phase. The derivative of the instantaneous phase is the Instantaneous frequency(IF). The IF of two equal strength tones was verified. IF from the analytical signal and from the conditional mean of a WVD matched for both data1 and data2, which confirms another property of WVD that it meets the IF and GD condition.

3. When you look at the plots for the IF and IB, you would notice several strange things, IF need not be in the spectrum of the signal, for example in data1 signal IF exists even when the signal doesn't, and IF can go negative. These gray areas in interpreting the analytical signal and the negativity of the WV distribution were also observed.

4. Signals are multicomponent if they exhibit multiple trajectories in the TF plane, there is no clear definition of what a multicomponent signal as any signal can be represented as a sum of arbitrary constituent signals. Data2 seems to have 2 hyperbolic chirps that merge eventually is it multicomponent? from the above definition I guess it is not.

7. REFERENCES

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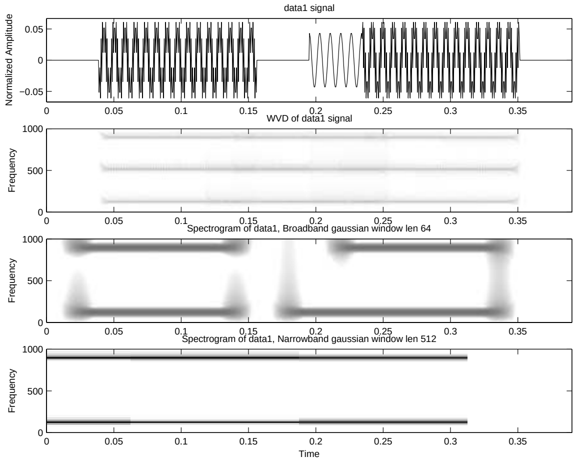
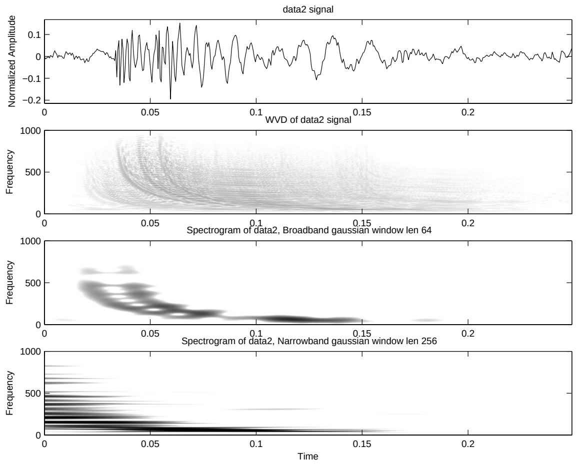
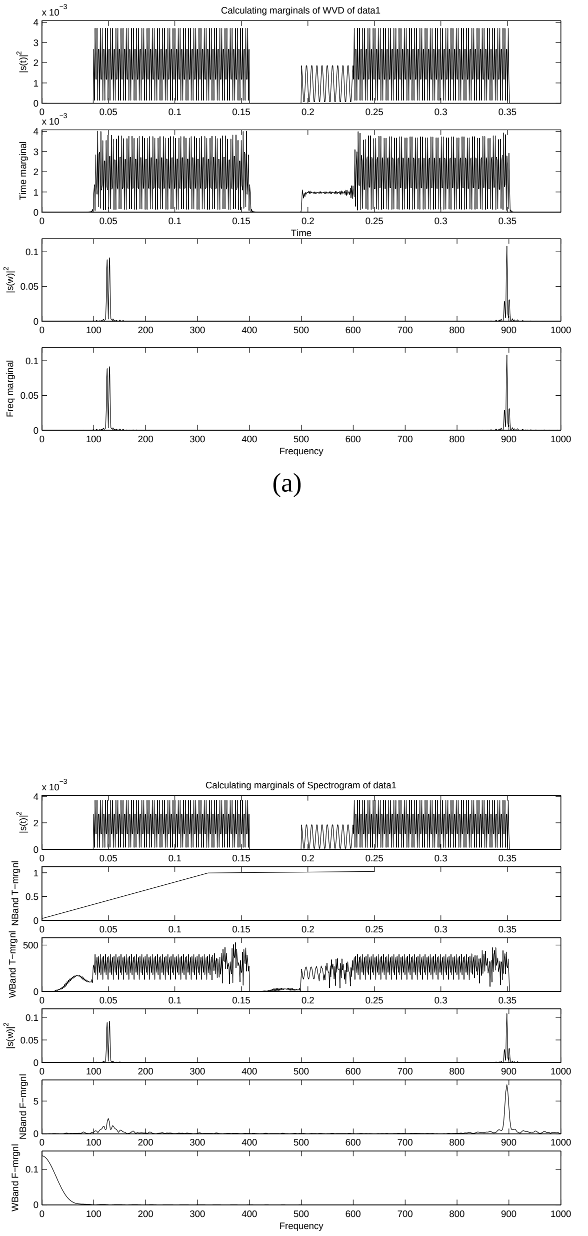


(a)



(b)

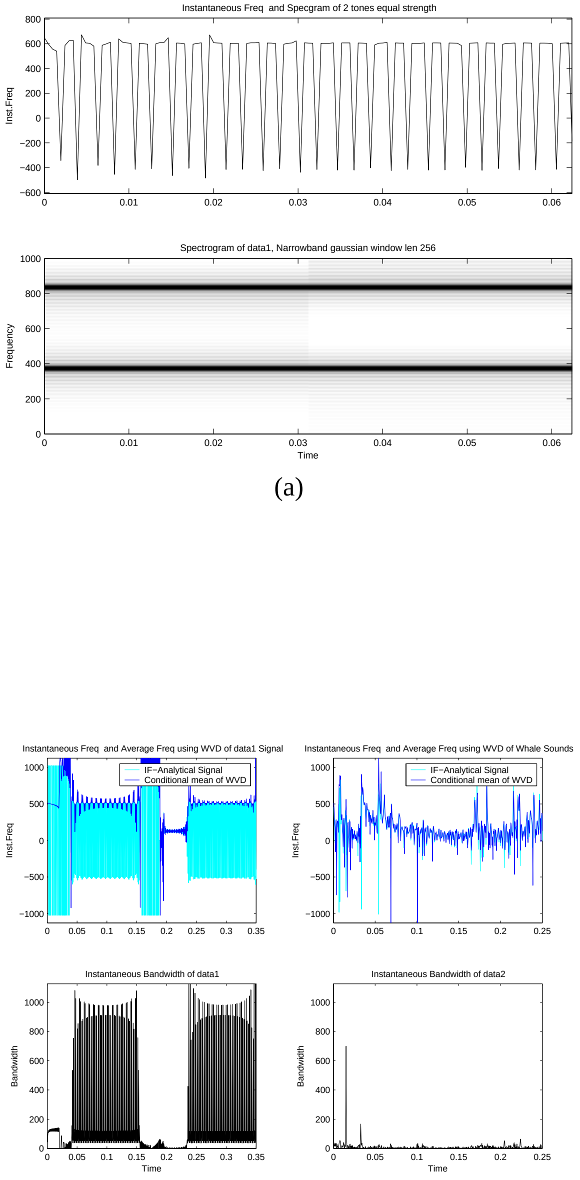
Fig. 1. Plot of Normalized Signal, WVD and Spectrogram(using Gaussian window) of test signals, (a) data1 - Two Tones, (b) data2 - Whale Sounds; A comparison of Time-Frequency representation



(a)

(b)

Fig. 2. Marginals of Spectrogram and WVD, (a) Time and Frequency Marginal of WVD matches $|s(t)|^2$ and $|s(\omega)|^2$ obtained from signal,(b) Wide-band Spectrogram has good time marginal while Narrow-band Spectrogram has good frequency marginal



(a)

(b)

Fig. 3. Instantaneous Frequency and Bandwidth using WVD and the Analytical signal. (a) IF of equal strength tones, Note:$IF = \frac{1}{2}(\omega_1 + \omega_2)$, (b) IF and IB of data1 and data2, IF found using $\dot{\phi}(t)$-derivative of phase of analytical signal and using the expression[7] for the first conditional moment of a joint TFD. The IB is obtained using expression[12] for second conditional moment of a joint TFD