THE CAUCHY PROBLEM AND THE STABILITY OF SOLITARY WAVES OF A HYPERELASTIC DISPERSIVE EQUATION

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ABSTRACT. We prove that the Cauchy problem for a certain sixth order hyperelastic dispersive equation is globally well-posed in a natural space. We also show that there exist solitary wave solutions $u(x, y, t) = \phi_c(x - ct, y)$ that come from an associated variational problem. Such solitary waves are nonlinearly stable in the sense that if a solution is initially close to the set of such solitary waves, it remains close to the set for all time in the natural norm.

1. INTRODUCTION

In this article, we consider the following two-dimensional nonlinear dispersive elastic wave equation

$$[u_t - u_{xxt} + \delta u_{xxxxt} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx})]_x - \alpha u_{yy} + \beta u_{xxyy} = 0. \quad (1.1)$$

Equation (1.1) was derived by the author in [10] as a model for the deformations of a hyperelastic compressible plate relative to a uniformly pre-stressed state. In this model u represents vertical displacement of the plate relative to a uniformly pre-stressed state, while x and y are rescaled longitudinal and lateral coordinates in the horizontal plane. To reduce the full three-dimensional field equation to an approximate two-dimensional plate equation, an assumption has been made that the thickness of the plate is small in comparison to the other dimensions. It is also assumed that the small perturbations superimposed on the pre-stressed state only appear in the vertical direction (the z-direction) and in one horizontal direction (the x-direction). Hence the variation of waves in the transverse direction (the y-direction) is small. Equation (1.1) is obtained under the additional assumption that the wavelength in the x-direction is short. On the other hand, if the wavelength is large, we obtain the Kadomtsev-Petviashvili(KP) equation.

The parameters in equation (1.1) are all material constants. The scalar δ describes the stiffness of the plate which is nonnegative. The coefficients α and β are material constants that measure weak transverse effects. The material constant γ occurs as a consequence of the balance between the nonlinear and dispersive effects. Note that there is no dissipation in this model.

Equation (1.1) generalizes several well-known equations including the BBM equation [1] when $\delta = \alpha = \beta = \gamma = 0$, the regularized long-wave Kadomtsev-Petviashvili(KP) equation [3] (also referred as KP-BBM equation, see [34]) when $\delta = \beta = \gamma = 0$, and the Camassa-Holm (CH) equation [9] when $\delta = \alpha = \beta = 0, \gamma = 1$. In contrast to our derivation in [10] of nonlinear dispersive waves in a hyperelastic plate, these particular equations are usually derived as models of water waves. In equation (1.1), the two spatial dimensions make the analysis very different from the CH equation. The γ-terms include a nonlinear term of fourth order, which makes equation (1.1) very different from the KP-BBM equation. This is the reason why we need the stiffness δ-term and the higher-order dispersion β-term in y to overcome the technical difficulties in estimating the nonlinear terms.

One aim of the present paper is to establish the global well-posedness of (1.1) in the natural space suggested by the following conservation law. Multiplying (1.1) with u and integrating over the whole space leads to the formal conservation law

$$\frac{d}{dt}E(u) = \frac{d}{dt}\Big\{ \int_{\mathbb{R}^2} [u^2 + u_x^2 + \delta u_{xx}^2] dxdy \Big\} = 0, \tag{1.2}$$

hence we may consider the space $H_x^2 \subset L^2(\mathbb{R}^2)$ with the norm

$$\|u\|_{H_x^2}^2 = \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2,$$

as a good candidate for solving the global Cauchy problem for (1.1). Note the absence of the y derivatives.

For any $s \geq 0$, we also introduce the space $Y^s \subset H_x^2$ equipped with the norm

$$\|u\|_{Y^s} = \|\langle\xi\rangle^2 \langle|\xi| + |\eta|\rangle^s \hat{u}(\xi, \eta)\|_{L^2_{\xi\eta}}, \tag{1.3}$$

where for $x \in \mathbb{R}$, $\langle x \rangle = (1 + x^2)^{1/2}$ and $\hat{u}$ is the Fourier transform. Clearly, $Y^0 = H_x^2$. In addition, we define the "Bourgain space" $X^{b,s} \subset \mathcal{S}'(\mathbb{R}^3)$ equipped with the norm

$$\|u\|_{X^{b,s}} = \|\langle\tau - p(\xi,\eta)\rangle^b \langle\xi\rangle^2 \langle|\xi| + |\eta|\rangle^s \hat{u}(\tau, \xi, \eta)\|_{L^2_{\tau\xi\eta}}, \tag{1.4}$$

for any $b \in \mathbb{R}$, where

$$p(\xi, \eta) = \frac{\alpha\xi^{-1}\eta^2 + \beta\xi\eta^2}{1 + \xi^2 + \delta\xi^4}.$$

Theorem 1.1. *For any $\phi \in H_x^2$, there exists $u \in C(\mathbb{R}; H_x^2)$ which solves (1.1) with $u(0, x, y) = \phi(x, y)$ such that $E(u(t)) = E(\phi)$ for all $t \in \mathbb{R}$. Furthermore, there exists some $b > 1/2$ such that $u \in X^{b,0}$ and u is unique in the class $X^{b,0}$. Moreover, if $s > 0$, then the map $\phi \mapsto u(t)$ takes Y^s to $X^{b,s}$ continuously.*

Formally, equation (1.1) can be written in the Hamiltonian form

$$u_t + JF'(u) = 0, \tag{1.5}$$

where $J = \frac{1}{2}(1 - \partial_x^2 + \delta\partial_x^4)^{-1}\partial_x$ is a skew-symmetric operator and

$$F(u) = \int_{\mathbb{R}^2} [u^3 + \gamma u u_x^2 - \alpha(\partial_x^{-1} u_y)^2 - \beta u_y^2] dxdy \tag{1.6}$$

is the Hamiltonian. Here, $\partial_x^{-1} u_y$ is defined via the Fourier transform as

$$\widehat{\partial_x^{-1} u_y} = \frac{\eta}{\xi}\hat{u}(\xi, \eta).$$

Hence the functional $F(u)$ is formally conserved. Both $E(u)$ and $F(u)$ are crucial to the stability analysis.

Combining $E(u)$ and $F(u)$ gives us the space $W \subset H_x^2(\mathbb{R}^2)$ equipped with the norm

$$\|u\|_W^2 = \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|u_y\|_{L^2}^2 + \|\partial_x^{-1} u_y\|_{L^2}^2$$

Because of the last term, any $u \in W$ has (formally) a zero $x-$ average for each y. The space W will be the natural space for our stability theorem.

Theorem 1.2. *If $\phi \in W$ and $\alpha\beta > 0$, then the solution obtained in Theorem 1.1 satisfies $u \in C(\mathbb{R}; W)$. Moreover, for each $t \in \mathbb{R}$, $F(u(t)) = F(\phi)$.*

Beginning with Section 5 we study the solitary wave of equation (1.1). A solitary wave is a solution $u \in W$ to equation (1.1) of the form $u(x, y, t) = \phi_c(x - ct, y)$. Note that any translate of a solitary wave is another solitary wave. Solitary waves in solids are easy to detect because they do not change their shapes during propagation, and can be used for determination of material properties and flaw detection. Therefore it is of great interest to decide whether they are stable or not. The appropriate notion of stability here is orbital stability as follows.

Definition. Let $S \subset W$ be a set of solitary waves. It is called W**-stable** if for any $\varepsilon > 0$, there is a $\nu > 0$ such that for any $u_0 \in W$ with

$$\inf_{v \in S} \|u_0 - v\|_W < \nu,$$

the solution u of equation (1.1) with initial data $u(\cdot, 0) = u_0(\cdot)$ satisfies

$$\sup_{-\infty < t < \infty} \inf_{v \in S} \|u(t) - v\|_W < \varepsilon.$$

Otherwise, S is called W**-unstable**.

Substituting $\phi(x - ct, y)$ into equation (1.1) and integrating once in x we obtain the equation for any solitary wave to be

$$-c\phi_x + c\phi_{xxx} - c\delta\phi_{xxxxx} + 3\phi\phi_x - \gamma(2\phi_x\phi_{xx} + \phi\phi_{xxx}) - \alpha\partial_x^{-1}\phi_{yy} + \beta\phi_{xyy} = 0. \quad (1.7)$$

A solitary wave $\phi_c \in W$ is therefore a critical point of the functional $cE(u) - F(u)$. Let

$$d(c) = cE(\phi_c) - F(\phi_c). \quad (1.8)$$

This variational characterization allows us to apply the concentration-compactness principle for the existence of solitary waves. Our strategy is to minimize the sum of all the quadratic terms in the functional under the constraint that the sum of the cubic terms is a constant, that is, to minimize

$$G_c(u) = \int_{\mathbb{R}^2} [cu^2 + cu_x^2 + c\delta u_{xx}^2 + \alpha(\partial_x^{-1}u_y)^2 + \beta u_y^2] dxdy \quad (1.9)$$

subject to the constraint

$$K(u) = \int_{\mathbb{R}^2} [u^3 + \gamma u u_x^2] dxdy = \lambda. \quad (1.10)$$

Using this variational method, we have the following existence result

Theorem 1.3. *If $c, \alpha, \beta > 0$, then equation (1.1) admits nontrivial solitary waves $\phi_c(x - ct, y)$ which are multiples of the minimizers to this associated variational problem.*

Solitary waves thus obtained are called *ground states*. In Section 6 we investigate the stability of the solitary wave solutions to equation (1.1). The stability analysis makes use of the function $d(c)$. We show that for each $c > 0$, the function $d(c)$ defined in (1.8) does not depend on the choice of ground state ϕ_c and $d(c)$ is smooth. Moreover, the sign of $d''(c)$ determines the stability of the ground states. Our main result is the following

Theorem 1.4. *If $c, \alpha, \beta > 0$, then the set of ground states of (1.1) is W-stable.*

There have been several results regarding the well-posedness and stability for equation (1.1) in some special cases. When the space dimension is one ($\alpha = \beta = 0$) and the stiffness is zero ($\delta = 0$), the equation becomes the generalized Camassa-Holm (CH) equation. In [14] was proved the local well-posedness of CH equation in $H^s(\mathbb{R})$, $s \geq 3$. The result was improved to $s > 3/2$ in [23] and [32]. It is also discussed in [23] that a necessary and sufficient condition for a global solution u to exist in $H^s(\mathbb{R})$ is that the L^∞-norm of u_x remains bounded. A Besov space approach can be found in [18]. The phase plane analysis used in [17] shows that there are smooth solitary waves for $\gamma < 1$. In the case $\gamma = 1$, the solitary waves are peaked solitons ([9]). The stability of solitary waves was proven in ([15], [16]). The difference between those papers is that the first one is the exact CH equation ($\gamma = 1$) while the second one treats the case $\gamma < 1$. The methods of proof are necessarily very different however. The first paper [15] gave a quantitative estimate on the H^1 deviation from a translated peakon shape in terms of two conservation laws of the flow. In the second paper [16] the authors used a spectral analysis of the linearized Hamiltonian operator following the method of [21]. In contrast to the CH equation, this present paper considers an equation in two spatial dimensions.

In two dimensions the standard generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation. The first result regarding well-posedness for a KP type equation appeared in [37] which proved local well-posedness for KP-I and KP-II equations for initial data in $H^s(\mathbb{R}^2)$ for $s \geq 3$. A significant result on the global well-posedness was given by Bourgain by using an analysis of multiple Fourier series introduced in [6]-[8], in the context of Schrödinger, KdV or KP equations. He proved that the KP-II equation is globally well-posed for initial data in $L^2(\mathbb{R}^2)$. Bourgain's result was improved in an anisotropic Sobolev space in [35] and [36]. The gain in regularity for the KP-II equation was proved in [22]. Flows of the KP-I and KP-II equations, considered in the natural spaces, behave very differently in the sense that the KP-II equation can be solved by Picard iteration ([8]) while KP-I cannot, in any Sobolev class ([31]). In [19]) and [38], a global existence result for small initial data was obtained via inverse scattering techniques. The smallness assumption was later removed ([11]). It was improved in [33] that the solution is locally well-poseded initial data and their antiderivatives in H^s for $s \geq 3$. It was also shown later in [30] that one obtains global well-posedness provided more regular initial data. In [13] a well-posedness result for small data in a weighted Sobolev space with essentially H^2 regularity was obtained and the result was improved for data in the intersection of the energy space and a natural weighted L^2 space ([12]).

The situation is quite different for KP-BBM equations. It has been proved in [3] that the KP-BBM equation can be solved by iteration, yielding local and global well-posedness results. Later [34] showed the global well-posedness for less regular initial data without the extra constraint on the initial data used in [3]. In contrast to the KP and KP-BBM equations, the equation considered in the present paper involves much higher nonlinearities.

The equations for the solitary waves of the KP and KP-BBM equations are identical. The main result on the existence of solitary waves was obtained in [5] using the concentration-compactness method ([24]). For stability results for the KP equation, we recall the work of [27], [26] and [4]. For stability of KP-BBM equation, we recall the work of [3], [25] and [34].

This paper is organized as follows. We prove in the next three sections that the initial-value problem in $\mathbb{R}^2$ is globally well-posed using the Fourier transform restriction method. We first show that equation (1.1) is locally well-posed. This is accomplished by means of some bilinear estimates and the contraction-mapping principle in a suitably chosen space. The global existence is achieved by use of the two conservation laws. In later sections the prospect in view is the solitary wave problem. In Section 5 we give an existence result of solitary waves using a concentration-compactness argument. Next we prove that the solutions are regular. In Section 6 we show that all such solitary waves are orbitally stable when considered as solutions of the full evolution equation. In Section 7 we provide a condition for the nonexistence of solitary waves.

Notation. For $A, B \in \mathbb{R}$, the notation $A \sim B$ means $|A|/2 \leq |B| \leq 2|A|$. $A \vee B = \max\{A, B\}$ and $A \wedge B = \min\{A, B\}$. For a Lebesgue measurable set D, we denote by $|D|$ its measure. Constants are denoted by C and may change from line to line.

2. LOCAL WELL-POSEDNESS

In this section we consider the Cauchy problem for

$$\begin{cases} [u_t - u_{xxt} + \delta u_{xxxxt} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx})]_x - \alpha u_{yy} + \beta u_{xxyy} = 0 \\ u(x, y, 0) = \phi(x, y) \end{cases} \tag{2.1}$$

in $(x, y) \in \mathbb{R}^2$, $t \geq 0$. Integrating once in x, we see that the equation can also be written as

$$\begin{cases} (1 - \partial_x^2 + \delta\partial_x^4)u_t - \alpha\partial_x^{-1}u_{yy} + \beta u_{xyy} = -\frac{3}{2}(u^2)_x + \frac{\gamma}{2}[(u^2)_{xxx} - (u_x^2)_x] \\ u(x, y, 0) = \phi(x, y). \end{cases} \tag{2.2}$$

We formally solve for u_t and invert its linear part to see that equation (2.2) is equivalent to the following integral equation

$$u(t) = S(t)\phi - \frac{1}{2}\int_0^t S(t-t')\Big\{[3P_1(D_x) - \gamma P_2(D_x)](u^2(t')) + \gamma P_1(D_x)(u_x^2(t'))\Big\}dt', \tag{2.3}$$

where $S(t)$ is the Fourier multiplier operator with symbol $\exp\Big\{it(\alpha\xi^{-1}\eta^2 + \beta\xi\eta^2)(1 + \xi^2 + \delta\xi^4)^{-1}\Big\}$, $P_1(D_x)$ is the Fourier multiplier with symbol $\xi/(1+\xi^2+\delta\xi^4)$ and $P_2(D_x)$ is the Fourier multiplier with symbol $\xi^3/(1 + \xi^2 + \delta\xi^4)$. If u solves (2.3) locally, then it also solves (2.2) in sense of distributions.

Let $\psi(t)$ be a cut-off function such that $\psi \in C_0^\infty(\mathbb{R})$, supp$\psi \subset [-2, 2]$, $\psi = 1$ on $[-1, 1]$. For $T > 0$, let $\psi_T(t) = \psi(t/T)$. Let

$$f(u) = \Big[\frac{3}{2}P_1(D_x) - \frac{\gamma}{2}P_2(D_x)\Big](u^2) + \frac{\gamma}{2}P_1(D_x)(u_x^2).$$

We define the "temporally truncated" operator

$$Lu(t) = \psi(t)S(t)\phi - \psi_T(t)\int_0^t S(t-t')f(u(t'))dt' \tag{2.4}$$

for which we will estimate each term separately.

We use the idea in [20] to give the linear estimates.

Lemma 2.1. *Let $s \in \mathbb{R}$. There exists $C > 0$ such that*

$$\|\psi(t)S(t)\phi\|_{X^{b,s}} \leq C\|\phi\|_{Y^s}. \tag{2.5}$$

Proof. By definition of $\|\cdot\|_{X^{b,s}}$,

$$\begin{aligned} \|\psi(t)S(t)\phi\|_{X^{b,s}} &= \|\langle\tau - p(\xi,\eta)\rangle^b\langle\xi\rangle^2\langle|\xi| + |\eta|\rangle^s\mathcal{F}_t\Big(\psi(t)e^{itp(\xi,\eta)}\hat{\phi}(\xi,\eta)\Big)\|_{L^2_{\tau\xi\eta}} \\ &= \|\langle\xi\rangle^2\langle|\xi| + |\eta|\rangle^s\hat{\phi}(\xi,\eta)\Big(\|\langle\tau\rangle^b\hat{\psi}(\tau)\|_{L^2(\tau)}\Big)\|_{L^2_{\xi\eta}} \\ &= \|\phi\|_{Y^s}\|\psi\|_{H^b} \leq C\|\phi\|_{Y^s}. \end{aligned}$$

□

Similarly we get

$$\|\psi_T(t)S(t)\phi\|_{X^{b,s}} = \|\phi\|_{Y^s}\|\psi_T\|_{H^b} \le CT^{1/2-b}\|\phi\|_{Y^s}. \tag{2.6}$$

Lemma 2.2. *Let* $0<\varepsilon<1/4$, $b=1/2+\varepsilon$, $b'=1/2-2\varepsilon$. *Let* $g\in H^{-b'}(\mathbb{R})$. *Then*

$$\|\psi_T\int_0^t g(t')dt'\|_{H^b} \le CT^{\varepsilon}\|g\|_{H^{-b'}}. \tag{2.7}$$

Proof. First we split the integral into three parts as follows

$$\begin{aligned}
&\psi_T\int_0^t g(t')dt' = \psi_T\int_0^t\int_{\mathbb{R}} e^{it'\tau}\hat{g}(\tau)d\tau dt' \\
&= \psi_T\int_{\mathbb{R}}\hat{g}(\tau)\int_0^t e^{it'\tau}dt'd\tau = \psi_T\int_{\mathbb{R}}\frac{e^{it\tau}-1}{i\tau}\hat{g}(\tau)d\tau \\
&= \psi_T\int_{T|\tau|\le 1}\frac{e^{it\tau}-1}{i\tau}\hat{g}(\tau)d\tau - \psi_T\int_{T|\tau|\ge 1}\frac{\hat{g}(\tau)}{i\tau}d\tau + \psi_T\int_{T|\tau|\ge 1}\frac{e^{it\tau}}{i\tau}\hat{g}(\tau)d\tau \quad := I_1+I_2+I_3.
\end{aligned}$$

Now we estimate the contribution of the above three terms separately. The first term can be written as $I_1 = \psi_T\int_{T|\tau|\le 1}\sum_{k\ge 1}\frac{t^k}{k!}(i\tau)^{k-1}\hat{g}(\tau)d\tau$. Therefore

$$\begin{aligned}
\|I_1\|_{H^b} &= \Big\|\langle\tau\rangle^b\mathcal{F}\Big(\sum_{k\ge 1}\frac{t^k\psi_T}{k!}\int_{T|\tau|\le 1}(i\tau)^{k-1}\hat{g}(\tau)d\tau\Big)\Big\|_{L^2} \\
&= \Big\|\langle\tau\rangle^b\int_{\mathbb{R}} e^{-it\tau}\Big(\sum_{k\ge 1}\frac{t^k\psi_T}{k!}\int_{T|\tau'|\le 1}(i\tau)^{k-1}\hat{g}(\tau')d\tau'\Big)dt\Big\|_{L^2} \\
&= \Big\|\langle\tau\rangle^b\int_{\mathbb{R}} e^{-it\tau}\sum_{k\ge 1}\frac{t^k\psi_T}{k!}dt\Big\|_{L^2}\cdot\Big|\int_{T|\tau'|\le 1}(i\tau)^{k-1}\hat{g}(\tau')d\tau'\Big| \\
&\le \sum_{k\ge 1}\|\frac{t^k\psi_T}{k!}\|_{H^b}\cdot\int_{T|\tau|\le 1}|i\tau|^{k-1}|\hat{g}(\tau)|d\tau \\
&\le \sum_{k\ge 1}\|\frac{t^k\psi_T}{k!}\|_{H^b}\cdot T^{1-k}\cdot\|g\|_{H^{-b'}}\cdot\Big(\int_{T|\tau|\le 1}\langle\tau\rangle^{2b'}d\tau\Big)^{1/2} \\
&\le \sum_{k\ge 1}\frac{C}{k!}T^{k+1/2-b}\cdot T^{1-k}\cdot T^{-b'-1/2}\cdot\|g\|_{H^{-b'}} \\
&\quad (\text{ since we know from (2.6) that } \|t^k\psi_T\|_{H^b}\le CT^{k+1/2-b}\ ) \\
&\le CT^{1-b-b'}\|g\|_{H^{-b'}} = CT^{\varepsilon}\|g\|_{H^{-b'}}.
\end{aligned}$$

The second term I_2 can be bounded as follows.

$$\begin{aligned}
\|I_2\|_{H^b} &\le \|\psi_T\|_{H^b}\cdot\int_{T|\tau|\ge 1}\frac{|\hat{g}(\tau)|}{|\tau|}d\tau \\
&\le \|\psi_T\|_{H^b}\cdot\|g\|_{H^{-b'}}\cdot\Big(\int_{T|\tau|\ge 1}|\tau|^2\langle\tau\rangle^{-2b'}d\tau\Big)^{1/2} \\
&\le CT^{1-b-b'}\|g\|_{H^{-b'}} = CT^{\varepsilon}\|g\|_{H^{-b'}}.
\end{aligned}$$

Contribution of I_3

$$\|I_3\|_{H^b} = \Big\|\langle\tau\rangle^b \mathcal{F}\Big(\psi_T \int_{T|\tau|\geq 1} \frac{e^{it\tau}}{i\tau}\hat{g}(\tau)d\tau\Big)\Big\|_{L^2}$$
$$= \Big\|\langle\tau\rangle^b \mathcal{F}\Big[\psi_T(t)\cdot \mathcal{F}^{-1}\Big(\frac{\hat{g}(\tau)}{i\tau}1_{T|\tau|\geq 1}\Big)(t)\Big]\Big\|_{L^2}$$
$$= \Big\|\langle\tau\rangle^b \cdot \hat{\psi}_T * \Big(\frac{\hat{g}(\tau)}{i\tau}1_{T|\tau|\geq 1}\Big)\Big\|_{L^2}.$$

Let $\hat{G}(\tau) = \frac{\hat{g}(\tau)}{i\tau}1_{T|\tau|\geq 1}$. Then

$$\|G\|_{H^b} = \|\langle\tau\rangle^b \hat{G}\|_{L^2} \leq \|g\|_{H^{-b'}} \cdot \sup_{T|\tau|\geq 1} \frac{\langle\tau\rangle^{b+b'}}{|\tau|} \leq CT^{1-b-b'}\|g\|_{H^{-b'}} = CT^{\varepsilon}\|g\|_{H^{-b'}}.$$

Hence

$$\|I_3\|_{H^b} = \Big\|\langle\tau\rangle^b\Big(\hat{\psi}_T * \hat{G}\Big)\Big\|_{L^2} \leq C\Big(\|\langle\tau\rangle^b\hat{\psi}_T\|_{L^1}\|G\|_{L^2} + \|\hat{\psi}_T\|_{L^1}\|G\|_{H^b}\Big)$$
$$\leq CT^{1-b-b'}\|g\|_{H^{-b'}} = CT^{\varepsilon}\|g\|_{H^{-b'}}.$$

Altogether, we obtain

$$\|\psi_T \int_0^t g(t')dt'\|_{H^b} \leq CT^{1-b-b'}\|g\|_{H^{-b'}} = CT^{\varepsilon}\|g\|_{H^{-b'}}.$$

□

Lemma 2.3. *Let* $0<\varepsilon<1/4$, $b=1/2+\varepsilon$, $b'=1/2-2\varepsilon$. *Then*

$$\|\psi_T \int_0^t S(t-t')f(u(t'))dt'\|_{X^{b,s}} \leq CT^{\varepsilon}\|f(u)\|_{X^{-b',s}}. \tag{2.8}$$

Proof. It is easy to see that

$$\|u\|_{X^{b,s}} = \|S(-t)u\|_{H^{b,s}},$$

where $H^{b,s}$ denotes the subspace of $\mathcal{S}'(\mathbb{R}^3)$ with the norm

$$\|u\|_{H^{b,s}} = \|\langle\tau\rangle^b\langle\xi\rangle^2\langle|\xi|+|\eta|\rangle^s\hat{u}(\tau,\xi,\eta)\|_{L^2_{\tau\xi\eta}}.$$

Therefore

$$\|\psi_T \int_0^t S(t-t')f(u(t'))dt'\|_{X^{b,s}} = \|S(-t)\psi_T \int_0^t S(t-t')f(u(t'))dt'\|_{H^{b,s}}$$
$$= \|\psi_T \int_0^t S(-t')f(u(t'))dt'\|_{H^{b,s}}$$
$$= \Big\|\langle\tau\rangle^b\langle\xi\rangle^2\langle|\xi|+|\eta|\rangle^s\mathcal{F}_{t,x,y}\Big(\psi_T\int_0^t S(-t')f(u(t'))dt'\Big)\Big\|_{L^2_{\tau\xi\eta}}$$
$$\leq \Big\|\langle\xi\rangle^2\langle|\xi|+|\eta|\rangle^s\|\mathcal{F}_{x,y}\Big(\psi_T\int_0^t S(-t')f(u(t'))dt'\Big)\|_{H^b_t}\Big\|_{L^2_{\tau\xi\eta}}$$
$$\leq \Big\|\langle\xi\rangle^2\langle|\xi|+|\eta|\rangle^s\cdot CT^{1-b-b'}\|\mathcal{F}_{x,y}\Big(S(-t')f(u(t'))\Big)\|_{H_t^{-b'}}\Big\|_{L^2_{\tau\xi\eta}}$$

(from Lemma 2.2)

$$= CT^{1-b-b'}\|f(u)\|_{X^{-b',s}} = CT^{\varepsilon}\|f(u)\|_{X^{-b',s}}.$$

□

As stated in the introduction, the two functional $E(u)$ and $F(u)$ are formally conserved. Hence it is possible to draw some preliminary conclusions. First the conservation of $E(u)$ implies that if the initial data $\phi \in H_x^2$ then the corresponding solution u of (1.1) lies in H_x^2 for all t for which it exists. To draw an inference based on the invariance of $F(u)$, the following lemma is helpful. This lemma is closely related to the embedding theorems for anisotropic Sobolev spaces studied in [2].

Lemma 2.4.

$$\text{(i) } \|f\|_{L^\infty(\mathbb{R}^2)} \le C(\|f\|_{L^2(\mathbb{R}^2)} + \|f_{xx}\|_{L^2(\mathbb{R}^2)} + \|f_y\|_{L^2(\mathbb{R}^2)}). \tag{2.9}$$

$$\text{(ii) } \|f\|^3_{L^3(\mathbb{R}^2)} \le C\|f\|^{3/2}_{L^2(\mathbb{R}^2)}\|f\|_{H_x^1(\mathbb{R}^2)}\|\partial_x^{-1} f_y\|^{1/2}_{L^2(\mathbb{R}^2)}. \tag{2.10}$$

Proof. (i) We can easily see this from the following estimate

$$\begin{aligned}\int_{\mathbb{R}^2} |\hat{f}|d\xi d\eta &\le \Big(\int_{\mathbb{R}^2} (1+\xi^4+\eta^2)|\hat{f}|^2 d\xi d\eta\Big)^{1/2}\Big(\int_{\mathbb{R}^2} \frac{1}{1+\xi^4+\eta^2} d\xi d\eta\Big)^{1/2} \\ &\le C(\|f\|_{L^2(\mathbb{R}^2)} + \|f_{xx}\|_{L^2(\mathbb{R}^2)} + \|f_y\|_{L^2(\mathbb{R}^2)}).\end{aligned}$$

(ii) See [3] Lemma 2.1. □

Remark. Now suppose the initial data $\phi \in W$. Consider the case $\alpha\beta > 0$. For simplicity, we assume $\alpha, \beta > 0$ and $\delta \le 1$. The invariance of E infers that $u \in H_x^2$ and more precisely,

$$\int_{\mathbb{R}^2} [u^2 + u_x^2 + \delta u_{xx}^2] \ dxdy \le \frac{1}{\delta}\int_{\mathbb{R}^2} [\phi^2 + \phi_x^2 + \phi_{xx}^2] \ dxdy \le \frac{1}{\delta}\|\phi\|_W^2.$$

Thus we would like to show that $u(t) \in W$ for all t for which the solution exists. So it suffices to show that $\partial_x^{-1} u_y \in H_x^1(\mathbb{R}^2)$, that is, $\partial_x^{-1} u_y, u_y \in L^2(\mathbb{R}^2)$. The invariance of F means

$$\int_{\mathbb{R}^2} [u^3 + \gamma u u_x^2 - \alpha(\partial_x^{-1} u_y)^2 - \beta u_y^2] \ dxdy = \int_{\mathbb{R}^2} [\phi^3 + \gamma\phi\phi_x^2 - \alpha(\partial_x^{-1}\phi_y)^2 - \beta\phi_y^2] \ dxdy. \tag{2.11}$$

Hence by Lemma 2.4 we know that

$$\begin{aligned}
&\min\{\alpha,\beta\}\int_{\mathbb{R}^2} [(\partial_x^{-1} u_y)^2 + u_y^2] dxdy \\
&\quad\le \int_{\mathbb{R}^2} [u^3 + \gamma u u_x^2] dxdy - \int_{\mathbb{R}^2} [\phi^3 + \gamma\phi\phi_x^2] dxdy + \max\{\alpha,\beta\}\int_{\mathbb{R}^2} [(\partial_x^{-1}\phi_y)^2 + \phi_y^2] dxdy \\
&\quad\le \|u\|^3_{L^3} + |\gamma|\|u\|_{L^\infty}\|u_x\|^2_{L^2} + \|\phi\|^3_{L^3} + |\gamma|\|\phi\|_{L^\infty}\|\phi_x\|^2_{L^2} \\
&\qquad + \max\{\alpha,\beta\}\int_{\mathbb{R}^2} [(\partial_x^{-1}\phi_y)^2 + \phi_y^2] dxdy \\
&\quad\le \|u\|^3_{L^3} + \|\phi\|^3_{L^3} + C|\gamma|(\|u\|_{L^2} + \|u_{xx}\|_{L^2} + \|u_y\|_{L^2})\|u_x\|^2_{L^2} \\
&\qquad + C|\gamma|(\|\phi\|_{L^2} + \|\phi_{xx}\|_{L^2} + \|\phi_y\|_{L^2})\|\phi_x\|^2_{L^2} + \max\{\alpha,\beta\}\|\phi\|_W^2 \\
&\quad\le \|u\|^3_{L^3} + \|\phi\|^3_{L^3} + C|\gamma|\|\phi\|^2_{W_1}\|u_x\|_{L^2} + \frac{C|\gamma|}{\delta}\|\phi\|^3_W + \max\{\alpha,\beta\}\|\phi\|^2_W \\
&\quad\le C\|\phi\|_W^{5/2}\|\partial_x^{-1} u_y\|^{1/2}_{L^2} + C\|\phi\|^3_W + C|\gamma|\|\phi\|^2_W\|u_y\|_{L^2} + \frac{C|\gamma|}{\delta}\|\phi\|^3_{W_1} + \max\{\alpha,\beta\}\|\phi\|^2_W \\
&\quad\le C(\delta,\alpha,\beta,\gamma,\|\phi\|_W) + C(\gamma,\|\phi\|_W)\Big(\|\partial_x^{-1} u_y\|^{1/2}_{L^2} + \|u_y\|_{L^2}\Big).
\end{aligned}$$

Therefore $\sup_t \Big(\|\partial_x^{-1} u_y\|_{L^2}^2 + \|u_y\|_{L^2}^2 \Big) \le C(\delta, \alpha, \beta, \gamma, \|\phi\|_W)$. That is,

$$\sup_t (\|u\|_W) \le C(\delta, \alpha, \beta, \gamma, \|\phi\|_W).$$

This discussion leads to the following formal statement:

If a solution u of equation (1.1) that starts in the space W, it will remain in this space throughout its period of existence.

To prove the local well-posedness of the Cauchy problem of (1.1), we need the following bilinear estimates.

Theorem 2.5. *For every $s \ge 0$, there exists $0 < \varepsilon < 1/4$ such that for $b = 1/2 + \varepsilon$, $b' = 1/2 - 2\varepsilon$, we have*

$$\|P_2(D_x)(uv)\|_{X^{-b',s}} \le C\Big(\|u\|_{X^{b,s}} \|v\|_{X^{b,0}} + \|u\|_{X^{b,0}} \|v\|_{X^{b,s}} \Big). \tag{2.12}$$

$$\|P_1(D_x)(uv)\|_{X^{-b',s}} \le C\Big(\|u\|_{X^{b,s}} \|v\|_{X^{b,0}} + \|u\|_{X^{b,0}} \|v\|_{X^{b,s}} \Big). \tag{2.13}$$

$$\|P_1(D_x)(u_x v_x)\|_{X^{-b',s}} \le C\Big(\|u\|_{X^{b,s}} \|v\|_{X^{b,0}} + \|u\|_{X^{b,0}} \|v\|_{X^{b,s}} \Big). \tag{2.14}$$

Proof. First we introduce some notations. Let

$$\zeta = (\tau, \xi, \eta), \quad \zeta_1 = (\tau_1, \xi_1, \eta_1), \quad \sigma(\zeta) = \tau - p(\xi, \eta).$$

Then estimate (2.12) is equivalent to

$$\begin{aligned} &\Big\| \frac{|\xi|^3}{1 + \xi^2 + \delta\xi^4} \frac{\langle \xi \rangle^2 \langle |\xi| + |\eta| \rangle^s}{\langle \sigma(\zeta) \rangle^{b'}} \int_{\mathbb{R}^3} \hat{u}(\zeta_1) \hat{v}(\zeta - \zeta_1) d\zeta_1 \Big\|_{L^2} \\ &\quad \le C\Big(\|u\|_{X^{b,s}} \|v\|_{X^{b,0}} + \|u\|_{X^{b,0}} \|v\|_{X^{b,s}} \Big). \end{aligned} \tag{2.15}$$

Now let

$$\begin{aligned} f_1(\zeta) &= \langle \sigma(\zeta) \rangle^b \langle \xi \rangle^2 \langle |\xi| + |\eta| \rangle^s \hat{u}(\zeta), \qquad g_1(\zeta) = \langle \sigma(\zeta) \rangle^b \langle \xi \rangle^2 \hat{v}(\zeta), \\ f_2(\zeta) &= \langle \sigma(\zeta) \rangle^b \langle \xi \rangle^2 \hat{u}(\zeta), \qquad g_2(\zeta) = \langle \sigma(\zeta) \rangle^b \langle \xi \rangle^2 \langle |\xi| + |\eta| \rangle^s \hat{v}(\zeta). \end{aligned}$$

Since for $s \ge 0$,

$$\frac{\langle |\xi| + |\eta| \rangle^s}{\langle |\xi_1| + |\eta_1| \rangle^s \langle |\xi - \xi_1| + |\eta - \eta_1| \rangle^s} \le C\Big(\frac{1}{\langle |\xi_1| + |\eta_1| \rangle^s} + \frac{1}{\langle |\xi - \xi_1| + |\eta - \eta_1| \rangle^s} \Big),$$

the left-hand side of (2.15) is no bigger than

$$C \Big\| \frac{|\xi|^3}{1 + \xi^2 + \delta\xi^4} \int_{\mathbb{R}^3} \frac{\langle \xi \rangle^2}{\langle \xi_1 \rangle^2 \langle \xi - \xi_1 \rangle^2} \frac{f_1(\zeta_1) g_1(\zeta - \zeta_1) + f_2(\zeta_1) g_2(\zeta - \zeta_1)}{\langle \sigma(\zeta) \rangle^{b'} \sigma(\zeta_1) \rangle^b \sigma(\zeta - \zeta_1) \rangle^b} d\zeta_1 \Big\|_{L^2}.$$

The right-hand side of (2.15) is equal to

$$C\Big(\|f_1\|_{L^2} \|g_1\|_{L^2} + \|f_2\|_{L^2} \|g_2\|_{L^2} \Big).$$

Therefore it suffices to show that for $i = 1, 2$,

$$\begin{aligned} &\Big\| \frac{|\xi|^3}{1 + \xi^2 + \delta\xi^4} \int_{\mathbb{R}^3} \frac{\langle \xi \rangle^2}{\langle \xi_1 \rangle^2 \langle \xi - \xi_1 \rangle^2} \frac{f_i(\zeta_1) g_i(\zeta - \zeta_1)}{\langle \sigma(\zeta) \rangle^{b'} \sigma(\zeta_1) \rangle^b \sigma(\zeta - \zeta_1) \rangle^b} d\zeta_1 \Big\|_{L^2} \\ &\quad \le C \|f_i\|_{L^2} \|g_i\|_{L^2}. \end{aligned} \tag{2.16}$$

Using L^2-duality, (2.16) is equivalent to

$$J = \Big| \int_{\mathbb{R}^6} \frac{|\xi|^3 \langle \xi \rangle^2}{(1+\xi^2+\delta\xi^4)\langle \xi_1 \rangle^2 \langle \xi - \xi_1 \rangle^2} \frac{f(\zeta_1) g(\zeta - \zeta_1) h(\zeta)}{\langle \sigma(\zeta) \rangle^{b'} \sigma(\zeta_1) \rangle^b \sigma(\zeta - \zeta_1) \rangle^b} d\zeta d\zeta_1 \Big|$$
$$\leq C \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}. \tag{2.17}$$

Without loss of generality we may assume $f, g, h \geq 0$ and hence can neglect the absolute value in the left-hand side of (2.17).

Define the dyadic levels

$$D^{KK_1K_2}_{MM_1M_2} = \{(\zeta, \zeta_1) : \langle \xi \rangle \sim M, \langle \xi_1 \rangle \sim M_1, \langle \xi - \xi_1 \rangle \sim M_2, \langle \sigma(\zeta) \rangle \sim K, \langle \sigma(\zeta_1) \rangle \sim K_1, \langle \sigma(\zeta - \zeta_1) \rangle \sim K_2\}, \tag{2.18}$$

where K, K_1, K_2, M, M_1, M_2 are all dyadic integers $2^n, n = 1, 2, 3, \ldots$. The set $D^{KK_1K_2}_{MM_1M_2}$ is not empty only if

$$M \leq C(M_1 + M_2), \quad M_1 \leq C(M + M_2), \quad M_2 \leq C(M + M_1). \tag{2.19}$$

Let $J^{KK_1K_2}_{MM_1M_2}$ be the contribution of $D^{KK_1K_2}_{MM_1M_2}$ to J. Then we have

$$J \leq C \sum_{K, K_1, K_2, M, M_1, M_2} J^{KK_1K_2}_{MM_1M_2}, \tag{2.20}$$

where the sum is taken over the dyadic integers such that (2.19) holds. Next we define the localizations on level sets of dispersion relation

$$f_{KM}(\zeta) = \begin{cases} f(\zeta), & \text{when } \langle \sigma(\zeta) \rangle \sim K, \langle \xi \rangle \sim M, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.21}$$

Now we write

$$J^{KK_1K_2}_{MM_1M_2} \leq C \frac{M^5}{(1+M^4) M_1^2 M_2^2 K^{b'} (K_1 K_2)^b} \langle f_{K_1M_1} * g_{K_2M_2}, h_{KM} \rangle_{L^2}, \tag{2.22}$$

where $\langle \cdot, \cdot \rangle_{L^2}$ is the scalar product in $L^2(\mathbb{R}^3)$. Before estimating the convolution in (2.22), we give the following two elementary lemmas. The proof is straightforward.

Lemma 2.6. *Let $C > 0$. Let the measurable set $\Lambda \subset I \times \mathbb{R}$, where $I \subset \mathbb{R}$ is measurable. Suppose*

$$\sup_{\xi \in I} \Big| \Lambda \bigcap \{(\xi, \eta) : \eta \in \mathbb{R}\} \Big| \leq C.$$

Then $|\Lambda| \leq C|I|$.

Lemma 2.7. *Let $a \neq 0, b, c \in \mathbb{R}$ and I an interval on $\mathbb{R}$, then*

$$|H| = \Big| \{x \in \mathbb{R} : ax^2 + bx + c \in I\} \Big| \leq 2\sqrt{\frac{|I|}{|a|}}. \tag{2.23}$$

The following lemma is crucial to the estimates.

Lemma 2.8. *Let $u_i, i = 1, 2$ be two functions on $\mathbb{R}^3$ such that for $\zeta \in \text{supp} u_i$, we have $\langle \sigma(\zeta) \rangle \sim K_i$ and $\langle \xi \rangle \sim M_i$. Then*

$$\|u_1 * u_2\|_{L^2} \leq C(K_1 \wedge K_2)^{1/2} (K_1 \vee K_2)^{1/4} (M_1 \wedge M_2)^{5/4} \|u_1\|_{L^2} \|u_2\|_{L^2}. \tag{2.24}$$

Proof. Take a map $q : L^2(\mathbb{R}^3) \mapsto L^2(\mathbb{R}^3)$ defined by $q(u)(\zeta) = u(-\zeta)$, hence q is isometric on $L^2(\mathbb{R}^3)$. Moreover, for real-valued u, v,

$$\|u * v\|_{L^2} = \|u * q(v)\|_{L^2}.$$

Hence we may assume $\xi \geq 0$ on $\text{supp} u_i$ in the proof of Lemma 2.8. Using the Cauchy-Schwartz inequality we get

$$\begin{aligned}\|u_1 * u_2\|_{L^2} &= \Big(\int_{\mathbb{R}^3} \Big| \int_{\mathbb{R}^3} u_1(\zeta_1) u_2(\zeta - \zeta_1) d\zeta_1 \Big|^2 d\zeta \Big)^{1/2} \\ &\leq \Big(\int_{\mathbb{R}^3} \Big(\int_{\mathbb{R}^3} |u_1(\zeta_1) u_2(\zeta - \zeta_1)| d\zeta_1 \Big)^2 d\zeta \Big)^{1/2} \\ &\leq \Big(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u_1(\zeta_1)|^2 d\zeta_1 \int_{\mathbb{R}^3} |u_2(\zeta - \zeta_1)|^2 d\zeta_1 \;\; d\zeta \Big)^{1/2} \\ &\leq C \Big(\sup_{\zeta \in \mathbb{R}^3} |A_\zeta| \Big)^{1/2} \|u_1\|_{L^2} \|u_2\|_{L^2}, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} A_\zeta = \Big\{ \zeta_1 \in \mathbb{R}^3 :\; & 1 \leq 1 + \xi_1 \sim M_1, 1 \leq 1 + \xi - \xi_1 \sim M_2, \\ & \langle \sigma(\zeta_1) \rangle \sim K_1, \langle \sigma(\zeta - \zeta_1) \rangle \sim K_2 \Big\}. \end{aligned} \tag{2.26}$$

Consider the set

$$\begin{aligned} B_\zeta = \Big\{ (\xi_1, \eta_1) \in \mathbb{R}^2 :\; & 1 \leq 1 + \xi_1 \sim M_1, 1 \leq 1 + \xi - \xi_1 \sim M_2, \\ & \tau - p(\xi_1, \eta_1) - p(\xi - \xi_1, \eta - \eta_1) \leq C(K_1 \vee K_2) \Big\}. \end{aligned} \tag{2.27}$$

In set A_ζ, we have

$$\tau - 2(K_1 \wedge K_2) \leq \tau_1 \leq \tau - \frac{1}{2}(K_1 \vee K_2),$$

therefore

$$|A_\zeta| \leq C(K_1 \wedge K_2)|B_\zeta|. \tag{2.28}$$

Now we estimate $|B_\zeta|$. First we have that $|\text{Proj}_{\xi_1}(B_\zeta)| \leq C(M_1 \wedge M_2)$. Now we fix ξ_1, then use Lemma 2.7 to get that the Lebesgue measure of the sections of B_ζ with lines parallel to the η_1-axis is bounded by

$$C(K_1 \vee K_2)^{1/2}(M_1 \wedge M_2)^{3/2},$$

since now the interval $|I| \sim K_1 \vee K_2$, $|a|^{-1} \sim |\xi_1|(1+\xi_1^2)$, and $|\xi - \xi_1|(1 + |\xi - \xi_1|^2) \leq C(M_1 \wedge M_2)^3$. Finally, we use Lemma 2.6 to obtain that

$$|B_\zeta| \leq C(K_1 \vee K_2)^{1/2}(M_1 \wedge M_2)^{5/2}, \tag{2.29}$$

which, combined with (2.25) and (2.28), completes the proof of Lemma 2.8. □

Proof of Theorem 2.5. Using (2.22) and Lemma 2.8, we can now write

$$\begin{aligned} & J^{KK_1K_2}_{MM_1M_2} \\ & \quad \leq C \frac{M^5 (K \wedge K_1 \wedge K_2)^{1/2} (K \vee K_1 \vee K_2)^{1/4} (M \wedge M_1 \wedge M_2)^{5/4}}{(1+M^4) M_1^2 M_2^2 K^{b'} (K_1 K_2)^b} \cdot \\ & \qquad \cdot \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}. \end{aligned} \tag{2.30}$$

with (2.19) satisfied. Hence now we pick $0 < \varepsilon < 1/4$ and choose $b = 1/2 + \varepsilon$, $b' = 1/2 - 2\varepsilon$. It's easy to see that we always have

$$\frac{(K \wedge K_1 \wedge K_2)^{1/2}(K \vee K_1 \vee K_2)^{1/4}}{K^{b'}(K_1K_2)^b} \leq \frac{1}{K^{1/4-2\varepsilon}K_1^{1/2+\varepsilon}K_2^{\varepsilon}}.$$

Also we obtain that

$$\frac{M^5(M \wedge M_1 \wedge M_2)^{5/4}}{(1+M^4)M_1^2M_2^2} \leq C\frac{1}{M^{1/2}M_1^{1/2}M_2^{3/4}}.$$

Thus choosing $\varepsilon < 1/8$ small enough, we may find some $\theta > 0$, for example, $\theta = 1/16$, so that

$$\begin{aligned} &\frac{M^5(K \wedge K_1 \wedge K_2)^{1/2}(K \vee K_1 \vee K_2)^{1/4}(M \wedge M_1 \wedge M_2)^{5/4}}{(1+M^4)M_1^2M_2^2K^{b'}(K_1K_2)^b} \\ &\quad \leq C(KK_1K_2MM_1M_2)^{-\theta}. \end{aligned} \tag{2.31}$$

Then the proof of (2.12) is completed by summing up with respect to the dyadic integers M, M_1, M_2, K, K_1, K_2 in (2.20).

Similarly we can obtain the bilinear estimate (2.13) for $P_1(D_x)(uv)$.

Now we are left with the estimate (2.14) for $P_1(D_x)(u_xv_x)$. First we rewrite (2.14) in the form

$$\begin{aligned} &\Big|\int_{\mathbb{R}^6}\frac{|\xi||\xi_1||\xi-\xi_1|\langle\xi\rangle^2}{(1+\xi^2+\delta\xi^4)\langle\xi_1\rangle^2\langle\xi-\xi_1\rangle^2}\frac{f(\zeta_1)g(\zeta-\zeta_1)h(\zeta)}{\langle\sigma(\zeta)\rangle^{b'}\sigma(\zeta_1)\rangle^b\sigma(\zeta-\zeta_1)\rangle^b}d\zeta d\zeta_1\Big| \\ &\quad \leq C\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \end{aligned} \tag{2.32}$$

Introducing $J^{KK_1K_2}_{MM_1M_2}$ as before, we have

$$J^{KK_1K_2}_{MM_1M_2} \leq C\frac{M^3}{(1+M^4)M_1M_2K^{b'}(K_1K_2)^b}\langle f_{K_1M_1} * g_{K_2M_2}, h_{KM}\rangle_{L^2}. \tag{2.33}$$

We now bound the term $\langle f_{K_1M_1} * g_{K_2M_2}, h_{KM}\rangle_{L^2}$ as in the proof of Theorem 2.5 and thus

$$\begin{aligned} &J^{KK_1K_2}_{MM_1M_2} \\ &\quad \leq C\frac{M^3(K \wedge (K_1 \wedge K_2))^{1/2}(K \vee (K_1 \vee K_2))^{1/4}(M \wedge (M_1 \wedge M_2))^{5/4}}{(1+M^4)M_1M_2K^{b'}(K_1K_2)^b}\cdot \\ &\quad \cdot \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}, \end{aligned} \tag{2.34}$$

Therefore we can choose ε and (b, b') satisfying the assumptions in the theorem so that for some $\theta > 0$ small enough as before, for example, $\theta = 1/16$,

$$\begin{aligned} &\frac{M^3(K \wedge (K_1 \wedge K_2))^{1/2}(K \vee (K_1 \vee K_2))^{1/4}(M \wedge (M_1 \wedge M_2))^{5/4}}{(1+M^4)M_1M_2K^{b'}(K_1K_2)^b} \\ &\quad \leq C(KK_1K_2MM_1M_2)^{-\theta}. \end{aligned} \tag{2.35}$$

Therefore the proof is complete. □

With the above estimates by hand, we can now finish the local version of Theorem 1.1. As mentioned in the beginning of this section, the Cauchy problem (2.1) is equivalent to the integral equation (2.3).

We define the localized space $X^{b,s}_T$, equipped with the norm $\|u\|_{X^{b,s}_T} = \inf \|v\|_{X^{b,s}}$ where the infimum is taken over all $v \in X^{b,s}$ such that $v = u$ on $[-T, T] \times \mathbb{R}^2$.

Theorem 2.9. *Let $s \geq 0$. For any $\phi \in Y^s$, there exist $b > 1/2$, $T = T(\|\phi\|_{H_x^2})$ independent of s and a unique solution $u \in X_T^{b,s}$ of (2.1). Moreover, for each $t \in [-T, T]$, the flow map $\phi \mapsto u(t)$ is Lipschitz continuous from bounded sets of Y^s to Y^s.*

Proof. Fix $\phi \in Y^s$ not identically zero. Let $\nu = \|\phi\|_{H_x^2}/\|\phi\|_{Y^s}$. We start with the truncated problem (2.4), from Lemma 2.1 and 2.3 and the bilinear estimates theorems, we have

$$\|Lu\|_{X^{b,0}} \leq C\|\phi\|_{H_2^x} + CT^{\varepsilon}\|u\|_{X^{b,0}}^2,$$
$$\|Lu\|_{X^{b,s}} \leq C\|\phi\|_{Y^s} + CT^{\varepsilon}\|u\|_{X^{b,s}}\|u\|_{X^{b,0}},$$

where L is defined in (2.4). Define the space

$$Z = \{u \in X^{b,s} : \ \|u\|_Z = \|u\|_{X^{b,0}} + \nu\|u\|_{X^{b,s}} < \infty\}.$$

Then we obtain that

$$\|Lu\|_Z \leq C(\|\phi\|_{H_x^2} + \nu\|\phi\|_{Y^s}) + CT^{\varepsilon}\|u\|_Z^2, \tag{2.36}$$

also we have the contraction

$$\|Lu - Lv\|_{X^{b,0}} \leq CT^{\varepsilon}\|u - v\|_{X^{b,0}}\|u + v\|_{X^{b,0}}, \tag{2.37}$$
$$\|Lu - Lv\|_{X^{b,s}} \leq CT^{\varepsilon}(\|u - v\|_{X^{b,0}}\|u + v\|_{X^{b,s}} + \|u - v\|_{X^{b,s}}\|u + v\|_{X^{b,0}}). \tag{2.38}$$

Combining the above two we deduce that

$$\|Lu - Lv\|_Z \leq CT^{\varepsilon}\|u + v\|_Z\|u - v\|_Z. \tag{2.39}$$

Setting

$$T = \frac{1}{4C^2(\|\phi\|_{H_x^2} + \nu\|\phi\|_{Y^s})^{\varepsilon}} = \frac{1}{(8C^2\|\phi\|_{H_x^2})^{\varepsilon}},$$

we deduce from (2.36) and (2.39) that the mapping L is strictly contractive on the ball of radius $4C\|\phi\|_{H_x^2}$ in Z. This gives the existence and uniqueness of solution to the truncated problem (2.4), hence also proves the existence of solution $u \in X^{b,s}$ to the full integral equation (2.3) on the time interval $[-T, T]$ with $T = T(\|\phi\|_{H_x^2})$. Choosing T small enough to make $\psi, \psi_T = 1$ on $[-T, T]$, we deduce the local existence and uniqueness of solution to equation (2.3).

To show that the flow map $\phi \mapsto u(t)$ is Lipschitz continuous from bounded sets of Y^s to Y^s, we consider u, v are two solutions on $[-T, T]$ with initial data ϕ and ψ respectively. Similarly to the derivation of (2.38), we have

$$\begin{aligned}\|u - v\|_{X^{b,s}} &\leq \|\phi - \psi\|_{Y^s} + \left\|\psi_T \int_0^t S(t - t')[f(u(t')) - f(v(t'))]dt'\right\|_{X_T^{b,s}} \\ &\leq \|\phi - \psi\|_{Y^s} + CT^{\varepsilon}\|f(u_1) - f(u_2)\|_{X^{-b',s}} \\ &\leq \|\phi - \psi\|_{Y^s} + CT^{\varepsilon}\|u + v\|_{X^{b,s}}\|u - v\|_{X^{b,s}}. \end{aligned} \tag{2.40}$$

Hence on any bounded set of Y^s, say a ball of radius R, we have

$$\sup_{-T \leq t \leq T}\|u(t) - v(t)\|_{Y^s} \leq \|\phi - \psi\|_{Y^s} + 2CRT^{\varepsilon}\sup_{-T \leq t \leq T}\|u(t) - v(t)\|_{Y^s}, \tag{2.41}$$

which immediately gives the Lipschitz continuity of the flow map. Thus we complete the proof of Theorem 2.9. ∎

3. GLOBAL WELL-POSEDNESS IN Y^s

In this section we prove Theorem 1.1.

In Theorem 2.9 we know that the existence time T depends only on the H_x^2-norm of the initial data. As long as $\|u(t)\|_{H_x^2}$ does not blow up, we can always reiterate the result of Theorem 2.9. Hence the global well-posedness will follow from the conservation of $\|u(t)\|_{H_x^2}$.

Lemma 3.1. *Let $\phi \in H_x^2$. Then the solution u of (2.1) obtained in Theorem 2.9 satisfies the conservation law $E(u(t)) = E(\phi)$ on $[-T, T]$.*

Proof. First consider the case when $\phi \in Y^2$. From Theorem 2.9, we get a solution $u(t) \in X_T^{b,2}$ for some $b > 1/2$. We use a regularization argument due to Molinet ([29]). For $\varepsilon > 0$, we define the function φ^ε as

$$\widehat{\varphi^\varepsilon} = \begin{cases} 1, & \text{if } \varepsilon < |\xi|, |\eta| < 1/\varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Denote $u^\varepsilon(t) = \varphi^\varepsilon * u(t)$. Then $u^\varepsilon(t)$ satisfies the equation

$$\begin{aligned} &u_t^\varepsilon - u_{xxt}^\varepsilon + \delta u_{xxxxt}^\varepsilon + \frac{3}{2}\varphi^\varepsilon * (u^2)_x - \frac{\gamma}{2}[\varphi^\varepsilon * (u^2)_{xxx} - \varphi^\varepsilon * (u_x^2)_x] \\ &\quad - \alpha\partial_x^{-1}u_{yy}^\varepsilon + \beta u_{xyy}^\varepsilon = 0, \end{aligned} \tag{3.2}$$

where ∂_x^{-1} is defined as the Fourier multiplier with symbol $(-i\xi)^{-1}$. Multiplying (3.2) by $u^\varepsilon(t)$ and integrating over $\mathbb{R}^2$, after several integrations by parts we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E(u^\varepsilon) = &\frac{3}{2}\int_{\mathbb{R}^2}[\varphi^\varepsilon * u^2 - (u^\varepsilon)^2]u_x^\varepsilon dxdy + \frac{\gamma}{2}\int_{\mathbb{R}^2}[\varphi^\varepsilon * u_x^2 - (u_x^\varepsilon)^2]u_x^\varepsilon dxdy \\ &- \frac{\gamma}{2}\int_{\mathbb{R}^2}[\varphi^\varepsilon * u^2 - (u^\varepsilon)^2]u_{xxx}^\varepsilon dxdy. \end{aligned}$$

Integrating on $[0, t]$ for $t \in [-T, T]$ yields

$$\begin{aligned} E(u^\varepsilon) - E(\phi^\varepsilon) = &3\int_0^t\int_{\mathbb{R}^2}[\varphi^\varepsilon * u^2 - (u^\varepsilon)^2]u_x^\varepsilon dxdydt' + \gamma\int_0^t\int_{\mathbb{R}^2}[\varphi^\varepsilon * u_x^2 - (u_x^\varepsilon)^2]u_x^\varepsilon dxdydt' \\ &- \gamma\int_0^t\int_{\mathbb{R}^2}[\varphi^\varepsilon * u^2 - (u^\varepsilon)^2]u_{xxx}^\varepsilon dxdydt'. \end{aligned} \tag{3.3}$$

From the Sobolev embedding theorem we know that $Y^2 \subset L^\infty$ and moreover, $u, u_x \in L^\infty$. Thus

$$\begin{aligned} \|\varphi^\varepsilon * u^2 - (u^\varepsilon)^2\|_{L^2} &\le \|\varphi^\varepsilon * u^2 - u^2\|_{L^2} + \|u^2 - (u^\varepsilon)^2\|_{L^2} \\ &\le \|\varphi^\varepsilon * u^2 - u^2\|_{L^2} + \|u - u^\varepsilon\|_{L^2}\|(u + u^\varepsilon)\|_{L^\infty} \\ &\longrightarrow 0, \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\lim_{\varepsilon\to 0}\|\varphi^\varepsilon * u_x^2 - (u_x^\varepsilon)^2\|_{L^2} = 0. \tag{3.5}$$

We also have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2}[\varphi^\varepsilon * u^2-(u^\varepsilon)^2]u^\varepsilon_{xxx}dxdy\Big|\\
&\qquad=2\Big|\int_{\mathbb{R}^2}[\varphi^\varepsilon *(uu_x)-u^\varepsilon u^\varepsilon_x]u^\varepsilon_{xx}dxdy\Big|\le 2\|\varphi^\varepsilon *(uu_x)-u^\varepsilon u^\varepsilon_x\|_{L^2}\|u^\varepsilon_{xx}\|_{L^2}\\
&\qquad\le 2\Big\{\|\varphi^\varepsilon *(uu_x)-uu_x\|_{L^2}+\|uu_x-u^\varepsilon u_x\|_{L^2}+\|u^\varepsilon u_x-u^\varepsilon u^\varepsilon_x\|_{L^2}\Big\}\|u^\varepsilon_{xx}\|_{L^2}\\
&\qquad\le 2\Big\{\|\varphi^\varepsilon *(uu_x)-uu_x\|_{L^2}+\|u-u^\varepsilon\|_{L^\infty}\|u_x\|_{L^2}+\|u^\varepsilon\|_{L^\infty}\|u_x-u^\varepsilon_x\|_{L^2}\Big\}\|u^\varepsilon_{xx}\|_{L^2}.
\end{aligned}$$

Since $Y^2\subset L^\infty$, $uu_x\in L^2$, therefore

$$\lim_{\varepsilon\to 0}\|\varphi^\varepsilon *(uu_x)-uu_x\|_{L^2}=0,\quad \lim_{\varepsilon\to 0}\|u-u^\varepsilon\|_{L^\infty}=0.$$

Fixing $t'\in[-T,T]$, from all the above estimates we know

$$\begin{aligned}
\lim_{\varepsilon\to 0}\quad & 3\int_{\mathbb{R}^2}[\varphi^\varepsilon * u(t')^2-(u^\varepsilon(t'))^2]u^\varepsilon_x(t')dxdy\\
&+\gamma\int_{\mathbb{R}^2}[\varphi^\varepsilon * u_x^2(t')-(u^\varepsilon_x(t'))^2]u^\varepsilon_x(t')dxdy\\
&-\gamma\int_{\mathbb{R}^2}[\varphi^\varepsilon * u^2(t')-(u^\varepsilon(t'))^2]u^\varepsilon_{xxx}(t')dxdy=0.
\end{aligned}\tag{3.6}$$

Moreover since $u(t)\in X_T^{b;2}$ for some $b>1/2$, from the Sobolev embedding we that $u\in L^\infty([-T,T];Y^2)$. Therefore the integrals in (3.6) are uniformly bounded on $[-T,T]$. Thus by the Lebesgue dominated convergence theorem, passing to the limit in (3.3) we obtain that

$$E(u(t))=E(\phi),$$

for all $\phi\in Y^2$.

Now we approximate any $\phi\in H^2_x$ by a sequence in Y^2 and use the local well-posedness theorem 2.9 to get the conservation of $E(u)$ for data in H^2_x. □

Now combining Theorem 2.9 and Lemma 3.1, we obtain the global well-posedness in Y^s for all $s\ge 0$, hence completing the proof of Theorem 1.1.

4. Global well-posedness in W

In the previous section we made use of the conservation law $E(u)$ to establish the global well-posedness result in the space Y^s for all $s\ge 0$. As we pointed out in the introduction, there is another formal conservation law $F(u)$, which, together with $E(u)$, suggest to us another function space W to work on. As before, we define the Bourgain space W^b associated to the space W by the norm

$$\|u\|_{W^b}=\|\langle\tau-p(\xi,\eta)\rangle^b\langle\xi\rangle\langle|\xi|+|\xi|^{-1}|\eta|\rangle\hat{u}(\tau,\xi,\eta)\|_{L^2_{\tau\xi\eta}}.\tag{4.1}$$

To get the well-posedness, we first establish the bilinear estimates as before.

Lemma 4.1. *There exists $0<\varepsilon<1/4$ such that for $b=1/2+\varepsilon$, $b'=1/2-2\varepsilon$, we have*

$$\|P_1(D_x)(uv)\|_{W^{-b'}}\le C\|u\|_{W^b}\|v\|_{W^b}.\tag{4.2}$$

$$\|P_2(D_x)(uv)\|_{W^{-b'}}\le C\|u\|_{W^b}\|v\|_{W^b}.\tag{4.3}$$

$$\|P_1(D_x)(u_xv_x)\|_{W^{-b'}}\le C\|u\|_{W^b}\|v\|_{W^b}.\tag{4.4}$$

Proof. The proof is similar to that of Theorems 2.5. We first prove estimate (4.3). Denote

$$k(\xi,\eta) = \langle \xi\rangle \langle |\xi| + |\xi|^{-1}|\eta|\rangle.$$

As with (2.17) we see that estimate (4.3) is equivalent to

$$\Big|\int_{\mathbb{R}^6} \bar{m}(\xi,\xi_1,\eta,\eta_1)\frac{f(\zeta_1)g(\zeta-\zeta_1)h(\zeta)}{\langle\sigma(\zeta)\rangle^{b'}\sigma(\zeta_1)\rangle^{b}\sigma(\zeta-\zeta_1)\rangle^{b}}d\zeta d\zeta_1\Big| \le C\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}, \quad (4.5)$$

where

$$\bar{m}(\xi,\xi_1,\eta,\eta_1) = \frac{|\xi|^3 k(\xi,\eta)}{(1+\xi^2+\delta\xi^4)k(\xi_1,\eta_1)k(\xi-\xi_1,\eta-\eta_1)},$$

or equivalently, we may replace $\bar{m}$ by m defined as

$$m(\xi,\xi_1,\eta,\eta_1) = \frac{|\xi|^3\langle\xi\rangle l(\xi,\eta)}{(1+\xi^2+\delta\xi^4)\langle\xi_1\rangle l(\xi_1,\eta_1)\langle\xi-\xi_1\rangle l(\xi-\xi_1,\eta-\eta_1)}, \quad (4.6)$$

where $l(\xi,\eta) = 1+|\xi|+|\xi|^{-1}|\eta|$. Since $\frac{1+|\xi|}{l(\xi_1)l(\xi-\xi_1)} \le C\frac{\langle\xi\rangle}{\langle\xi_1\rangle\langle\xi-\xi_1\rangle}$, we have

$$m(\xi,\xi_1,\eta,\eta_1) \le \frac{|\xi|^2|\eta|\langle\xi\rangle}{(1+\xi^2+\delta\xi^4)\langle\xi_1\rangle l(\xi_1)\langle\xi-\xi_1\rangle l(\xi-\xi_1)} + C\frac{|\xi|^3\langle\xi\rangle}{(1+\xi^2+\delta\xi^4)\langle\xi_1\rangle\langle\xi-\xi_1\rangle}.$$

In view of Theorem 2.5, it suffices to prove (4.5) with

$$m(\xi,\xi_1,\eta,\eta_1) = \frac{|\xi|^2|\eta|\langle\xi\rangle}{(1+\xi^2+\delta\xi^4)\langle\xi_1\rangle l(\xi_1)\langle\xi-\xi_1\rangle l(\xi-\xi_1)}. \quad (4.7)$$

Introduce $J_{MM_1M_2}^{KK_1K_2}$ as before. Using $|\eta| \le |\eta_1| + |\eta-\eta_1|$ and Lemma 2.8, we obtain that the substitute of (2.22) in the context of (4.3) is

$$\begin{aligned}
&J_{MM_1M_2}^{KK_1K_2} \\
&\le C\frac{M^3}{(1+M^4)M_1M_2}\Big(\frac{M_1}{M_2}+\frac{M_2}{M_1}\Big)\frac{1}{K^{b'}(K_1K_2)^b}\langle f_{K_1M_1} * g_{K_2M_2}, h_{KM}\rangle_{L^2} \\
&\le C\Big(\frac{M_1}{M_2}+\frac{M_2}{M_1}\Big)\frac{M^3(K\wedge K_1\wedge K_2)^{1/2}(K\vee K_1\vee K_2)^{1/4}(M\wedge M_1\wedge M_2)^{5/4}}{(1+M^4)M_1M_2K^{b'}(K_1K_2)^b} \\
&\quad \times \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2} \\
&\le C\frac{(M_1\vee M_2)M^3(K\wedge K_1\wedge K_2)^{1/2}(K\vee K_1\vee K_2)^{1/4}(M\wedge M_1\wedge M_2)^{5/4}}{(1+M^4)M_1M_2K^{b'}(K_1K_2)^b} \\
&\quad \times \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.
\end{aligned} \quad (4.8)$$

Hence we may choose proper $\varepsilon > 0$ and for $b = 1/2+\varepsilon, b' = 1/2-2\varepsilon$ so that

$$\frac{(M_1\vee M_2)M^3(K\wedge K_1\wedge K_2)^{1/2}(K\vee K_1\vee K_2)^{1/4}(M\wedge M_1\wedge M_2)^{5/4}}{(1+M^4)M_1M_2K^{b'}(K_1K_2)^b} \le C(KK_1K_2MM_1M_2)^{-\theta}.$$

for some $\theta > 0$ sufficiently small (for example, $\theta = 1/16$). Therefore the sum over the dyadic integers M, M_1, M_2, K, K_1, K_2 satisfying (2.19) is bounded by $C\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$, which gives (4.3). We can apply the same method to get (4.2). As for (4.4), the substitute of (4.8) is

$$\begin{aligned}
&J^{KK_1K_2}_{MM_1M_2}\\
&\quad\le C\Big(\frac{M_1}{M_2}+\frac{M_2}{M_1}\Big)\frac{M(K\wedge K_1\wedge K_2)^{1/2}(K\vee K_1\vee K_2)^{1/4}(M\wedge M_1\wedge M_2)^{5/4}}{(1+M^4)K^{b'}(K_1K_2)^b}\\
&\qquad\times\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}\\
&\quad\le C\frac{(M_1\vee M_2)M(K\wedge K_1\wedge K_2)^{1/2}(K\vee K_1\vee K_2)^{1/4}(M\wedge M_1\wedge M_2)^{5/4}}{(1+M^4)K^{b'}(K_1K_2)^b}\\
&\qquad\times\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \qquad (4.9)
\end{aligned}$$

Therefore we have completed the proof of (4.4). □

With Lemma 4.1 in hand, we deduce the local well-posedness lemma for data in W. The proof is similar to the proof of Theorem 2.9.

Lemma 4.2. *Let $\phi\in W$. There exist $b>1/2$, $T=T(\|\phi\|_W)$ and a unique solution $u\in C([-T,T];W)\bigcap X^{b,0}_T$ of (2.1). Moreover, for each $t\in[-T,T]$, the flow map $\phi\mapsto u(t)$ is Lipschitz continuous from bounded sets of W to W.*

Now we are ready to prove the conservation of $F(u)$.

Lemma 4.3. *Let $\phi\in W$ and $\alpha\beta>0$. Then the solution u of (2.1) obtained in Lemma 4.2 satisfies the conservation law $F(u(t))=F(\phi)$ on $[-T,T]$.*

Proof. First we prove the conservation for initial data $\phi\in V$ where

$$V=\{f\in L^2(\mathbb{R}^2): f\in H^4 \text{ and } \partial_x^{-1}f\in H^2\}.$$

In view of Theorem 2.9 we know that there exists a unique solution $u(t)\in C([-T,T];H^4)\bigcap X^{b,0}_T$. Using the Duhamel's integral we may write $u(t)$ as

$$u(t)=S(t)\phi-\frac{1}{2}\int_0^t S(t-t')(1-\partial_x^2+\delta\partial_x^4)^{-1}\partial_x[3u^2(t')-\gamma(u^2(t'))_{xx}+\gamma u_x^2(t')]dt'.$$

Hence

$$\partial_x^{-1}u(t)=S(t)\partial_x^{-1}\phi-\frac{1}{2}\int_0^t S(t-t')(1-\partial_x^2+\delta\partial_x^4)^{-1}[3u^2(t')-\gamma(u^2(t'))_{xx}+\gamma u_x^2(t')]dt',$$

which implies that $\partial_x^{-1}u\in H^2$. Let P denote the operator $1-\partial_x^2+\delta\partial_x^4$. From equation (2.2) we know that

$$u_t=-P^{-1}\Big\{\frac{3}{2}(u^2)_x-\frac{\gamma}{2}[(u^2)_{xxx}-(u_x^2)_x]-\alpha\partial_x^{-1}u_{yy}+\beta u_{xyy}\Big\}. \qquad (4.10)$$

Therefore $u\in V$ implies that $u_t\in H^4_x$.

Introducing φ^ε as in (3.1) and $u^\varepsilon=\varphi^\varepsilon*u$ the convolution, we know that u^ε satisfies equation (3.2). Define

$$w^\varepsilon=-P^{-1}\Big\{\frac{3}{2}\varphi^\varepsilon*u^2-\frac{\gamma}{2}[\varphi^\varepsilon*(u^2)_{xx}-\varphi^\varepsilon*(u_x^2)]-\alpha\partial_x^{-2}u^\varepsilon_{yy}+\beta u^\varepsilon_{yy}\Big\}. \qquad (4.11)$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}F(u^\varepsilon) &= \int_{\mathbb{R}^2} \Big\{\frac{3}{2}(u^\varepsilon)^2 - \frac{\gamma}{2}[(u^\varepsilon)^2_{xx} - (u^\varepsilon_x)^2] - \alpha\partial_x^{-2}u^\varepsilon_{yy} + \beta u^\varepsilon_{yy}\Big\}u^\varepsilon_t \ dxdy \\ &= \int_{\mathbb{R}^2} (-Pw^\varepsilon)u^\varepsilon_t \ dxdy + \frac{3}{2}\int_{\mathbb{R}^2} [(u^\varepsilon)^2 - \varphi^\varepsilon * u^2]u^\varepsilon_t \ dxdy \\ &\quad - \frac{\gamma}{2}\int_{\mathbb{R}^2} [(u^\varepsilon)^2 - \varphi^\varepsilon * u^2]u^\varepsilon_{xxt} \ dxdy + \frac{\gamma}{2}\int_{\mathbb{R}^2} [(u^\varepsilon_x)^2 - \varphi^\varepsilon * u^2_x]u^\varepsilon_t \ dxdy.\end{aligned} \tag{4.12}$$

Since $u_t \in H^4_x$, from (3.4) and (3.5) we know that the last three terms converge to zero as ε tends to zero for any fixed $t \in [-T, T]$. From the definition of w^ε we know that $w^\varepsilon_x = u^\varepsilon_t$. Therefore the first term in (4.12) is equal to

$$\int_{\mathbb{R}^2} -w^\varepsilon P u^\varepsilon_t \ dxdy = \int_{\mathbb{R}^2} -w^\varepsilon P w^\varepsilon_x \ dxdy = 0.$$

Therefore by the Lebesgue dominated convergence theorem as used in Lemma 3.1 together with $F(u^\varepsilon) \to F(u), \quad \text{as } \varepsilon \to 0$, we obtain that

$$F(u(t)) = F(\phi).$$

Now for general initial data $\phi \in W$, we use a sequence $\{\phi_n\} \subset V$ converging to ϕ in W with corresponding solutions $u_n \subset V$. From Lemma 4.2 we know that $u_n(t) \to u(t)$ in W for all $t \in [-T, T]$ where $u(t)$ is the solution to equation (2.1) associated with initial data ϕ and $T = T(\|\phi\|_W)$. From the embedding Lemma 2.4 we obtain that

$$F(\phi_n) \to F(\phi), \quad F(u_n(t)) \to F(u(t)).$$

Thus $F(u(t)) = F(\phi)$ for all $t \in [-T, T]$. $\square$

Combining Lemma 4.2, Lemma 4.3 and the remark after Lemma 2.4, we complete the proof of Theorem 1.2.

5. Existence of solitary waves

The focus of the development of the following sections is the solitary wave of (1.1), defined in (1.7). Localized, travelling-wave solutions of nonlinear wave equations are known in many circumstances to play a distinguished role in the long-time evolution of an initial disturbance.

In this section we prove the existence of solitary waves for positive α and β. The result is the following:

Theorem 5.1. *Let α and β be positive. For any $c > 0$, the equation (1.7) possesses a nontrivial solution $\phi_c \in W$.*

We will prove existence of solitary waves in the space W by considering the following variational problem. Define for any $u \in W$, $\rho(u) = cu^2 + cu^2_x + c\delta u^2_{xx} + \alpha(\partial_x^{-1}u_y)^2 + \beta u^2_y$. Define

$$G_c(u) = \int_{\mathbb{R}^2} \rho(u) dxdy, \tag{5.1}$$

$$K(u) = \int_{\mathbb{R}^2} [u^3 + \gamma u u^2_x] dxdy. \tag{5.2}$$

Then if the minimization problem:

$$I_\lambda = \inf\Big\{G_c(u) \ \Big| \ u \in W, K(u) = \lambda\Big\} \tag{5.3}$$

has a nontrivial solution $\psi_c \in W$ for some $\lambda > 0$, then there is a Lagrange multiplier $\mu \neq 0$ such that

$$-c\psi_c + c\partial_x^2\psi_c - c\delta\partial_x^4\psi_c - \alpha\partial_x^{-2}\partial_y^2\psi_c + \beta\partial_y^2\psi_c = \mu\Big[-\frac{3}{2}\psi_c^2 + \frac{\gamma}{2}((\partial_x\psi_c)^2 + 2\psi_c\partial_x^2\psi_c)\Big] \text{ in } W', \tag{5.4}$$

where $\partial_x^{-2}\partial_y^2\psi_c$ is the element of W'(the dual space of W in L^2-duality) such that for any $f \in W$,

$$\langle \partial_x^{-2}\partial_y^2\psi_c, f\rangle_{W',W} = (\partial_x^{-1}\partial_y\psi_c, \partial_x^{-1}\partial_y f)_{L^2}.$$

By taking the x-derivative of (5.4) in $\mathcal{D}'(\mathbb{R}^2)$, and performing the scaling $\phi_c = \mu\psi_c$, one can easily see that ϕ_c satisfies the equation(1.7) in $\mathcal{D}'(\mathbb{R}^2)$. We call such solutions *ground state* solutions and denote the set of all ground state solutions S_c. By homogeneity of G_c and K we know that ground states also achieve the minimum

$$I_1 = \inf\Big\{\frac{G_c(u)}{K(u)^{2/3}} \ \Big|\ u \in W, K(u) > 0\Big\}. \tag{5.5}$$

It then follows that

$$I_\lambda = \lambda^{2/3} I_1. \tag{5.6}$$

First we show that I_λ is bounded from below.

Lemma 5.2. *For any $\lambda > 0$, $I_\lambda > 0$.*

Proof. First it is obvious to see that for every positive c, α, β, there are positive c_1, c_2 such that

$$c_1\|u\|_W^2 \le \int_{\mathbb{R}^2} \rho(u)dxdy \le c_2\|u\|_W^2. \tag{5.7}$$

From the embedding Lemma 2.4, we get that $\|u\|_{L^q} \le C\|u\|_W$, for $2 \le q \le \infty$. Hence

$$\lambda = \int_{\mathbb{R}^2} [u^2 + \gamma u u_x^2]dxdy \le C\|u\|_W^3,$$

and then $I_\lambda \ge c_1\left(\frac{\lambda}{C}\right)^{2/3} > 0$ for any $\lambda > 0$. □

We say that a sequence $\{u_n\} \subset W$ is a *minimizing sequence* if for some $\lambda > 0$,

$$\lim_{n\to 0} K(u_n) = \lambda, \qquad \lim_{n\to 0} G_c(u_n) = I_\lambda. \tag{5.8}$$

Proof of Theorem 5.1. From (5.6) we see that the subadditivity condition holds

$$I_\lambda < I_{\lambda_1} + I_{\lambda-\lambda_1}, \quad \text{for } \lambda_1 \in (0, \lambda). \tag{5.9}$$

Let u_n be a minimizing sequence for (5.3). Then from the anisotropic Sobolev embedding (2.9), we can find a sequence $\varphi_n \in L^\infty_{loc}(\mathbb{R}^2)$ such that $u_n = \partial_x\varphi_n$ and $v_n = \partial_y\phi_n = \partial_x^{-1}\partial_y u_n$. We denote $\rho_n = \rho(u_n)$. Hence we know that $\int_{\mathbb{R}^2} \rho_n dxdy \to I_\lambda > 0$ as $n \to \infty$.

(i) Assume first that "vanishing" happens, i.e. for any $R > 0$,

$$\lim_{n\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_R} \rho_n dxdy = 0, \tag{5.10}$$

where the B_R is the ball of radius R centered at the origin. In [2] the authors give a complete proof for the local version of anisotropic embedding (see [2], p.187). Here for $u = \varphi_x$, we denote $\Omega = (x, y) + B_1$. Then for any $q \ge 2$ this local version becomes

$$\|\varphi_x\|_{L^q(\Omega)} \le C\Big[\|\varphi_x\|_{L^2(\Omega)} + \|\varphi_y\|_{L^2(\Omega)} + \|\varphi_{xxx}\|_{L^2(\Omega)}\Big], \tag{5.11}$$

i.e. for $u \in W$,

$$\|u\|_{L^q(\Omega)} \le C\Big[\|u\|_{L^2(\Omega)} + \|\partial_x^{-1}u_y\|_{L^2(\Omega)} + \|u_{xx}\|_{L^2(\Omega)}\Big],$$

where in (5.11) the positive constant C is independent of $(x,y) \in \mathbb{R}^2$. Therefore we know that for $u \in W$,

$$\int_{(x,y)+B_1} |u|^q dxdy \le C\Big(\int_{(x,y)+B_1} \rho \ dxdy\Big)^{q/2},$$

$$\int_{(x,y)+B_1} |uu_x^2| dxdy \le C\Big(\int_{(x,y)+B_1} |u|^3 dxdy\Big)^{1/3} \cdot \Big(\int_{(x,y)+B_1} |u_x|^3 dxdy\Big)^{2/3}.$$

Applying Lemma 2.4 we get

$$\|u_x\|_{L^3} \le C\|u\|_W. \tag{5.12}$$

Covering $\mathbb{R}^2$ by balls of radius 1 such that each point of $\mathbb{R}^2$ is contained in at most 3 balls, we obtain that for any $u \in W$

$$\int_{\mathbb{R}^2} |u|^q dxdy \le 3C\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} \rho \ dxdy\Big)^{q/2-1} \cdot \|u\|_W^2,$$

$$\int_{\mathbb{R}^2} |uu_x^2| dxdy \le C\|u\|_W^2 \Big[3C\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} \rho \ dxdy\Big)^{1/2} \|u\|_W^2\Big]^{1/3}.$$

Hence from (5.10), we get

$$u_n \to 0 \text{ in } L^3, \ \int_{\mathbb{R}^2} |u_n(\partial_x u_n)^2| dxdy \to 0,$$

which contradicts the constraint in I_λ.

(ii) Assume now that “dichotomy" occurs. We define the usual concentration function

$$Q(t) = \lim_{n\to\infty} \sup_{(x_0,y_0)\in\mathbb{R}^2} \int_{(x_0,y_0)+B_t} \rho_n dxdy \quad \text{where for } t \ge 0.$$

“Dichotomy" means that there is a $\theta \in (0, I_\lambda)$ such that $\lim_{t\to\infty} Q(t) = \theta$. We will show that (**??**) will give a contradiction provided that it leads to the splitting of u_n into two sequences u_n^1 and u_n^2 in W with disjoint supports. We will construct u_n^1 and u_n^2 by localizing φ_n.

For any fixed $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$, R_0,$R_n > 0$, with $R_n \nearrow +\infty$, and points $\vec{A}_n \in \mathbb{R}^2$, such that

$$\theta \ge \int_{\vec{A}_n + B_{R_0}} \rho_n dxdy \ge \theta - \varepsilon, \ \ Q_n(2R_n) \le \theta + \varepsilon, \quad \text{for all } n \ge n_0,$$

where

$$Q_n(t) = \sup_{(x_0,y_0)\in\mathbb{R}^2} \int_{(x_0,y_0)+B_t} \rho_n dxdy.$$

Hence $\int_{R_0 \le |\vec{x}-\vec{A}_n| \le 2R_n} \rho_n dxdy \le 2\varepsilon$.

Now let $\xi, \eta \in C_0^\infty(\mathbb{R}^2)$ be as follows: $0 \le \xi, \eta \le 1$, $\xi \equiv 1$ on B_1, supp$\xi \subset B_2$; $\eta \equiv 1$ on $\mathbb{R}^2 \backslash B_2$, supp$\xi \subset \mathbb{R}^2 \backslash B_1$. We set $\xi_n = \xi\Big(\frac{\cdot - \vec{A}_n}{R_1}\Big)$, $\quad \eta_n = \eta\Big(\frac{\cdot - \vec{A}_n}{R_n}\Big)$. and we construct

$$u_n^1 = \partial_x[\xi_n(\varphi_n - a_n)], \quad u_n^2 = \partial_x[\eta_n(\varphi_n - b_n)],$$

where the $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers to be chosen later. Also we can set

$$v_n^1 = \partial^{-1}(u_n^1)_y = \partial_y[\xi_n(\varphi_n - a_n)], \quad v_n^2 = \partial^{-1}(u_n^2)_y = \partial_y[\eta_n(\varphi_n - b_n)],$$

and then we have

$$\|u_n^1 + u_n^2 - u_n\|_{L^2} \leq \\ \|(\partial_x \xi_n) \cdot (\varphi_n - a_n)\|_{L^2} + \|(\partial_x \eta_n) \cdot (\varphi_n - b_n)\|_{L^2} + \sqrt{2\varepsilon}$$

and

$$\|(\partial_x \xi_n) \cdot (\varphi_n - a_n)\|_{L^2} = \Big(\int_{R_1 \leq |\vec{x} - \vec{A}_n| \leq 2R_1} |\partial_x \xi_n|^2 |\varphi_n - a_n|^2 dxdy \Big)^{1/2} \\ \leq \|\partial_x \xi_n\|_{L^p} \Big(\int_{R_1 \leq |\vec{x} - \vec{A}_n| \leq 2R_1} |\varphi_n - a_n|^q dxdy \Big)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. In order to determine a_n and b_n, we need the following lemma:

Lemma 5.3. *Let $q \geq 2$, then there exists a positive constant C such that for all $f \in L^1_{loc}(\mathbb{R}^2)$ with $\nabla f \in L^2_{loc}(\mathbb{R}^2)$, for all $R > 0$ and for all $\vec{x}_0 \in \mathbb{R}^2$*

$$\Big(\int_{R \leq |\vec{x} - \vec{x}_0| \leq 2R} |f(\vec{x}) - m_R(f)|^q d\vec{x} \Big)^{1/q} \leq CR^{2/q} \Big(\int_{R \leq |\vec{x} - \vec{x}_0| \leq 2R} |\nabla f|^2 d\vec{x} \Big)^{1/2},$$

where

$$m_R(f) = \frac{1}{\text{vol}(\Omega_{\vec{x}_0,R})} \int_{R \leq |\vec{x} - \vec{x}_0| \leq 2R} f(\vec{x}) d\vec{x}, \ \ \vec{x} = (x, y) \in \mathbb{R}^2$$

and

$$\Omega_{\vec{x}_0,R} = \{\vec{x} \in \mathbb{R}^2 : \ \ R < |\vec{x} - \vec{x}_0| < 2R\}.$$

Proof of Lemma 5.3. The lemma is proved by applying the Poincar*é* inequality for zero mean-value H^1 functions on the bounded open set $\Omega_{\vec{x}_0,R}$. Then using the Sobolev embedding theorem we obtain the existence of a positive constant $C(\vec{x}_0, R)$ such that

$$\Big(\int_{R \leq |\vec{x} - \vec{x}_0| \leq 2R} |f(\vec{x}) - m_R(f)|^q d\vec{x} \Big)^{1/q} \leq C(\vec{x}_0, R) \Big(\int_{R \leq |\vec{x} - \vec{x}_0| \leq 2R} |\nabla f|^2 d\vec{x} \Big)^{1/2}.$$

Then the translation invariance of Lebesgue measure and the scale change $f \mapsto f(\frac{\cdot}{R})$ show that $C(\vec{x}_0, R) = CR^{2/q}$ where C is independent of $\vec{x}_0$ and R. □

Now we continue the proof of Theorem 5.1. We pick

$$a_n = m_{R_1}(\varphi_n) = \frac{1}{\text{vol}(\Omega_{\vec{A}_n,R_1})} \int_{R_1 \leq |\vec{x} - \vec{A}_n| \leq 2R_1} \varphi_n(\vec{x}) d\vec{x}.$$

Applying Lemma 5.3 we get

$$\|(\partial_x \xi_n) \cdot (\varphi_n - a_n)\|_{L^2} \\ \leq CR_1^{\frac{2}{p} + \frac{2}{q} - 1} \Big(\int_{R_1 \leq |\vec{x} - \vec{A}_n| \leq 2R_1} [|u_n|^2 + |v_n|^2] d\vec{x} \Big)^{1/2} \leq C' \sqrt{\varepsilon}.$$

In the same way, we can choose $b_n = m_{R_n}(\varphi_n)$ to get the bound:

$$\|(\partial_x \eta_n) \cdot (\varphi_n - b_n)\|_{L^2} \\ \leq C \Big(\int_{R_n \leq |\vec{x} - \vec{A}_n| \leq 2R_n} [|u_n|^2 + |v_n|^2] d\vec{x} \Big)^{1/2} \leq C\sqrt{\varepsilon}.$$

The above two bounds imply that $\|u_n^1 + u_n^2 - u_n\|_{L^2} \leq C\sqrt{\varepsilon}$. Similarly we obtain $\|v_n^1 + v_n^2 - v_n\|_{L^2} \leq C\sqrt{\varepsilon}$. Now we consider

$$\begin{aligned}
&\|\partial_x u_n^1 + \partial_x u_n^2 - \partial_x u_n\|_{L^2} \\
&= \|\partial_x^2\Big(\xi_n(\varphi_n - a_n)\Big) + \partial_x^2\Big(\eta_n(\varphi_n - b_n)\Big) - \partial_x^2\varphi_n\|_{L^2} \\
&\leq \|(\partial_x^2\xi_n)(\varphi_n - a_n)\|_{L^2} + \|(\partial_x^2\eta_n)(\varphi_n - b_n)\|_{L^2} + \|(1-\xi_n-\eta_n)\partial_x u_n\|_{L^2} \\
&\quad + 2\|(\partial_x\xi_n)u_n\|_{L^2} + 2\|(\partial_x\eta_n)u_n\|_{L^2}.
\end{aligned}$$

The first three terms in the right hand side of the above inequality are bounded as the preceding ones. For the last two terms, one may consider for example

$$\|(\partial_x\xi_n)u_n\|_{L^2} \leq \|\partial_x\xi_n\|_{L^\infty}\Big(\int_{R_1\leq|\vec{x}-\vec{A}_n|\leq 2R_1}|u_n|^2 d\vec{x}\Big)^{1/2} \leq C\sqrt{\varepsilon}.$$

Hence we have $\|\partial_x u_n^1 + \partial_x u_n^2 - \partial_x u_n\|_{L^2} \leq C\sqrt{\varepsilon}$, $\|\partial_y u_n^1 + \partial_y u_n^2 - \partial_y u_n\|_{L^2} \leq C\sqrt{\varepsilon}$. Finally, we estimate

$$\begin{aligned}
&\|\partial_x^2 u_n^1 + \partial_x^2 u_n^2 - \partial_x^2 u_n\|_{L^2} \\
&\leq \|(\partial_x^3\xi_n)(\varphi_n - a_n)\|_{L^2} + \|(\partial_x^3\eta_n)(\varphi_n - b_n)\|_{L^2} + \|(1-\xi_n-\eta_n)\partial_x^2 u_n\|_{L^2} \\
&\quad + 3\|(\partial_x\xi_n)\partial_x u_n\|_{L^2} + 3\|(\partial_x\eta_n)\partial_x u_n\|_{L^2} + 3\|(\partial_x^2\xi_n)u_n\|_{L^2} + 3\|(\partial_x^2\eta_n)u_n\|_{L^2} \\
&\leq C\sqrt{\varepsilon}.
\end{aligned}$$

Therefore we have proved that for any $\varepsilon > 0$, there is a $\sigma(\epsilon) > 0$(with $\sigma(\epsilon) \to 0$ as $\epsilon \to 0$) such that we can find u_n^1 and u_n^2 in W satisfying that for all $n \geq n_0$:

$$\int_{\mathbb{R}^2} \rho(u_n^1 + u_n^2 - u_n)dxdy \leq \sigma(\epsilon) \tag{5.13}$$

Similarly we can get

$$\Big|\int_{\mathbb{R}^2} \rho(u_n^1)dxdy - \theta\Big| \leq \sigma(\epsilon), \quad \Big|\int_{\mathbb{R}^2} \rho(u_n^2)dxdy - (I_\lambda - \theta)\Big| \leq \sigma(\epsilon). \tag{5.14}$$

and

$$\text{dist}(\text{supp } u_n^1, \text{supp } u_n^2) \to \infty \text{ as } n \to \infty, \ \ \text{supp } u_n^1 \bigcap \text{supp } u_n^2 = \emptyset$$

So

$$\Big|\int_{\mathbb{R}^2} [(u_n^1)^3 + (u_n^2)^3 - (u_n)^3]dxdy\Big| \leq \sigma(\varepsilon), \tag{5.15}$$

$$\Big|\int_{\mathbb{R}^2} [u_n^1(\partial_x u_n^1)^2 + u_n^2(\partial_x u_n^2)^2 - u_n(\partial_x u_n)^2]dxdy\Big| \leq \sigma(\varepsilon). \tag{5.16}$$

Now by taking subsequences if necessary, we may assume that as $n \to \infty$,

$$\int_{\mathbb{R}^2} [(u_n^1)^3 + \gamma u_n^1(\partial_x u_n^1)^2]dxdy \to \lambda_1(\varepsilon), \ \int_{\mathbb{R}^2} [(u_n^2)^3 + \gamma u_n^2(\partial_x u_n^2)^2]dxdy \to \lambda_2(\varepsilon),$$

with $|\lambda_1(\varepsilon) + \lambda_2(\varepsilon) - \lambda| \leq \sigma(\varepsilon)$.

(a) Assume first that $\lim_{\varepsilon\to 0}\lambda_1(\varepsilon) = 0$, then choosing ε small enough, for n sufficiently large, we get $k_n = \int_{\mathbb{R}^2} (u_n^2)^3 + \gamma u_n^2(\partial_x u_n^2)^2 dxdy > 0$. So by considering $w_n = \Big(\lambda_2(\varepsilon)/k_n\Big)^{1/3} u_n^2$, we get

$$I_{\lambda_2(\varepsilon)} \leq \liminf_{n\to\infty}\int_{\mathbb{R}^2} \rho(w_n)dxdy = \liminf_{n\to\infty}\int_{\mathbb{R}^2} \rho(u_n^2)dxdy \leq I_\lambda - \theta + \sigma(\varepsilon),$$

which contradicts the condition $\lim_{\varepsilon\to 0}\lambda_2(\varepsilon)=\lambda$.

(b) Therefore we may assume that $\lim_{\varepsilon\to 0}|\lambda_1(\varepsilon)|>0$, $\lim_{\varepsilon\to 0}|\lambda_2(\varepsilon)|>0$. In the same way, we get

$$I_{\lambda_1(\varepsilon)}+I_{\lambda_2(\varepsilon)}\le \liminf_{n\to\infty}\int_{\mathbb{R}^2}\rho(u_n^1)dxdy+\liminf_{n\to\infty}\int_{\mathbb{R}^2}\rho(u_n^2)dxdy\le I_\lambda+\sigma(\varepsilon).$$

Hence by letting ε go to zero, and by using the fact that for $\lambda>0$, $I_\lambda=\lambda^{2/3}I_1$, we reach a contradiction. This ends to rule out the “dichotomy" case.

(iii) So by [24] we can only have “compactness". Thus there exists a sequence $\{\vec{x}_n\}\subset\mathbb{R}^2$ such that for any $\varepsilon>0$, there is a finite $R>0$ and $n_0>0$ such that

$$\int_{\vec{x}_n+B_R}\rho_n dxdy\ge I_\lambda-\varepsilon \quad \text{for } n\ge n_0.$$

Let $H^1_{x,loc}(\mathbb{R}^2)=\{u\in L^2_{loc}(\mathbb{R}^2): u_x\in L^2_{loc}(\mathbb{R}^2)\}$. Since u_n is bounded in W, we may assume that $u_n(\cdot-\vec{x}_n)$ converges weakly in W to some $\psi_c\in W$. And for large n, we have

$$\int_{\vec{x}_n+B_R}|u_n|^2dxdy\ge\int_{\mathbb{R}^2}|u_n|^2dxdy-2\varepsilon,\ \int_{\vec{x}_n+B_R}|\partial_x u_n|^2dxdy\ge\int_{\mathbb{R}^2}|\partial_x u_n|^2dxdy-2\varepsilon.$$

So this implies that

$$\|\psi_c\|^2_{H^1_x}\le\liminf_{n\to\infty}\|u_n\|^2_{H^1_x}\le\liminf_{n\to\infty}\int_{\vec{x}_n+B_R}[|u_n|^2+|\partial_x u_n|^2]dxdy+2\varepsilon. \tag{5.17}$$

Now we need the following lemma to show that the injection $W\subset H^1_{x,loc}(\mathbb{R}^2)$ is compact.

Lemma 5.4. *Let u_n be a bounded sequence in W, and let $R>0$. Then there is a subsequence u_{n_k} which converges strongly to u in $H^1_x(B_R)$.*

Proof. Let u_n be a bounded sequence in W, with $u_n=\partial_x\varphi_n$, $\varphi_n\in L^2_{loc}(\mathbb{R}^2)$, and let $v_n=\partial_y\varphi_n\in L^2_{loc}(\mathbb{R}^2)$. Multiplying φ_n by a cutoff function $\psi\in C_0^\infty(\mathbb{R}^2)$ with $0\le\psi\le 1$, $\psi\equiv 1$ on B_R and supp $\psi\subset B_{2R}$, we may assume that supp $\varphi_n\subset B_{2R}$. Thus supp $u_n\subset B_{2R}$. Now since u_n is bounded in W, we may assume that $u_n\rightharpoonup u=\partial_x\varphi$ weakly in W, and replacing φ_n by $\varphi_n-\varphi$, we can also assume that $\varphi=0$. Then we have

$$\begin{aligned}\|u_n\|^2_{H^1_x(B_{2R})}&=\int_{B_{2R}}[|u_n|^2+|\partial_x u_n|^2]dxdy=\int_{\mathbb{R}^2}(1+|\xi|^2)|\hat{u}_n|^2d\xi d\eta\\&=\int_{\{|\xi|\le R_1,|\eta|\le R_1\}}(1+|\xi|^2)|\hat{u}_n|^2d\xi d\eta+\int_{\{|\xi|\ge R_1\}}(1+|\xi|^2)|\hat{u}_n|^2d\xi d\eta\\&\quad+\int_{\{|\xi|\le R_1,|\eta|\ge R_1\}}(1+|\xi|^2)|\hat{u}_n|^2d\xi d\eta,\end{aligned} \tag{5.18}$$

where $\hat{u}_n(\xi,\eta)$ is the Fourier transform of $u_n(x,y)$. The third term in (5.18) satisfies

$$\begin{aligned}\int_{\{|\xi|\le R_1,|\eta|\ge R_1\}}(1+|\xi|^2)|\hat{u}_n|^2d\xi d\eta&=\int_{\{|\xi|\le R_1,|\eta|\ge R_1\}}\frac{1}{|\eta|^2}(|\hat{v}_n|^2+|\eta|^2|\hat{u}_n|^2)d\xi d\eta\\&\le\frac{1}{R_1^2}(\|v_n\|^2_{L^2(\mathbb{R}^2)}+\|\partial_y u_n\|^2_{L^2(\mathbb{R}^2)})\end{aligned}$$

The second term is bounded in the following way

$$\int_{\{|\xi|\geq R_1\}}(1+|\xi|^2)|\hat{u}_n|^2 d\xi d\eta \leq \frac{1}{R_1^2}(\|\partial_x u_n\|^2_{L^2(\mathbb{R}^2)}+\|\partial_x^2 u_n\|^2_{L^2(\mathbb{R}^2)})$$

So for a fixed $\varepsilon > 0$, we can choose R_1 sufficiently large to get

$$\int_{\{|\xi|\geq R_1\}}(1+|\xi|^2)|\hat{u}_n|^2 d\xi d\eta + \int_{\{|\xi|\leq R_1, |\eta|\geq R_1^2\}}(1+|\xi|^2)|\hat{u}_n|^2 d\xi d\eta \leq \frac{\varepsilon}{2}.$$

We then use the Lebesgue dominated convergence theorem for the first term, having noticed that since u_n tends to 0 weakly in $H_x^1(\mathbb{R}^2)$, $\hat{u}_n(\xi,\eta) = \int_{B_{2R}} e^{-ix\xi - iy\eta} u_n(x,y)dxdy$ tends to zero as $n \to \infty$, for a.e. $(\xi,\eta) \in \mathbb{R}^2$, and that $|\hat{u}_n| \leq \|u_n\|_{L^1(B_{2R})}$. Therefore the first term in (5.18) also approaches zero as $n \to \infty$. Hence $u_n \to 0$ strongly in $H_x^1(B_{2R})$, thus proving the Lemma. □

By Lemma 5.4, we can assume that $u_n(\cdot - \vec{x}_n) \to \psi_c$ strongly in $H^1_{x,loc}(\mathbb{R}^2)$. Then (5.17) shows that in fact

$$\|\psi_c\|^2_{H^1_x} \leq \liminf_{n\to\infty} \|u_n\|^2_{H^1_x} \leq \|\psi_c\|^2_{H^1_x(B_R)} + 2\varepsilon.$$

Therefore by taking a subsequence we obtain that $u_n(\cdot - \vec{x}_n) \to \psi_c$ strongly in $H^1_x(\mathbb{R}^2)$. Using the embedding $W \subset L^q(\mathbb{R}^2)$ for $q \geq 2$, we get $u_n(\cdot - \vec{x}_n) \to \psi_c$ strongly in $L^q(\mathbb{R}^2)$ for $q \geq 2$, hence $\int_{\mathbb{R}^2} u_n^3 dxdy \to \int_{\mathbb{R}^2} \psi_c^3 dxdy$. In the same way, we obtain

$$\partial_x u_n(\cdot - \vec{x}_n) \to \partial_x \psi_c \text{ strongly in } L^2(\mathbb{R}^2).$$

So

$$\begin{aligned}
&\Big|\int_{\vec{x}_n+B_R}[u_n(\partial_x u_n)^2 - \psi_c(\partial_x\psi_c)^2]dxdy\Big| \\
&= \Big|\int_{\vec{x}_n+B_R}\{(u_n-\psi_c)(\partial_x u_n)^2 + \psi_c[(\partial_x u_n)^2 - (\partial_x\psi_c)^2]\}dxdy\Big| \\
&\leq \|u_n-\psi_c\|_{L^3(\vec{x}_n+B_R)}\|\partial_x u_n\|^2_{L^3} + \|\psi_c\|_{L^\infty}\|\partial_x u_n - \partial_x\psi_c\|_{L^2(\vec{x}_n+B_R)} \\
&\leq \|u_n-\psi_c\|_{L^3(\vec{x}_n+B_R)}\|u_n\|^2_{W} + \|\psi_c\|_{L^\infty}\|\partial_x u_n - \partial_x\psi_c\|_{L^2(\vec{x}_n+B_R)} \quad \text{(by (5.12))} \\
&\quad \longrightarrow 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Therefore $\int_{\mathbb{R}^2} u_n(\partial_x u_n)^2 dxdy \to \int_{\mathbb{R}^2} \psi_c(\partial_x\psi_c)^2 dxdy$. So

$$\int_{\mathbb{R}^2}[\psi_c^3 + \gamma\psi_c(\partial_x\psi_c)^2]dxdy = \lambda.$$

Since

$$\int_{\mathbb{R}^2}\rho(\psi_c)dxdy \leq \liminf_{n\to\infty}\int_{\mathbb{R}^2}\rho(u_n)dxdy = I_\lambda,$$

we can conclude that ψ_c is a solution of the minimization problem (5.3). Thus (1.7) admits a nontrivial solution. □

Next we prove that any solitary wave of (1.1) is smooth. More precisely we have

Theorem 5.5. *Any solitary wave solution of (1.1) is in H^∞ provided $\alpha, \beta > 0$.*

Proof. The solitary wave equation of (1.1) can be written as the following "elliptic" equation:

$$\Big[-c\delta\partial_x^6 + c\partial_x^4 + \beta\partial_x^2\partial_y^2 - c\partial_x^2 - \alpha\partial_y^2\Big]u = -3\partial_x(uu_x) + \gamma\partial_x^3(uu_x) - \frac{\gamma}{2}\partial_x^2(u_x^2). \quad (5.19)$$

The difficulty arises from the nonisotropy of the symbol of the linear "elliptic" operator $L = -c\delta\partial_x^6 + c\partial_x^4 + \beta\partial_x^2\partial_y^2 - c\partial_x^2 - \alpha\partial_y^2$. We will proceed by "bootstrapping", using the embedding theorems for anisotropic Sobolev spaces ([2]), and the following variant due to Lizorkin ([28]) of the Mikhlin-Hörmander multiplier theorem.

Proposition 5.6. -[28] *Let* Φ*:* $\mathbb{R}^n \to \mathbb{R}$ *be* C^n *for* $|\xi_j| > 0$, $j = 1, 2, \ldots$*. Assume that there exists*$M > 0$ *such that*

$$\left|\xi_1^{k_1}\cdots\xi_n^{k_n}\frac{\partial^k\Phi}{\partial\xi_1^{k_1}\cdots\partial\xi_n^{k_n}}(\xi)\right| \le M, \tag{5.20}$$

with $k_i = 0$ *or* 1, $k = k_1 + \cdots + k_n = 0, 1, \ldots, n$*. Then* $\Phi \in M_q(\mathbb{R}^n)$, $1 < q < \infty$*, i.e.* Φ *is a Fourier multiplier on* $L^q(\mathbb{R}^n)$*.*

Set $g = -uu_x$, $h = -u_x^2$. (5.19) yields that

$$\hat{u} = -\frac{i(3\xi + \gamma\xi^3)\hat{g} + \frac{\gamma}{2}\xi^2\hat{h}}{c\delta\xi^6 + c\xi^4 + \beta\xi^2\eta^2 + c\xi^2 + \alpha\eta^2}, \tag{5.21}$$

Thus

$$\widehat{u_{xx}} = \frac{i(3\xi^3 + \gamma\xi^5)\hat{g} + \frac{\gamma}{2}\xi^4\hat{h}}{c\delta\xi^6 + c\xi^4 + \beta\xi^2\eta^2 + c\xi^2 + \alpha\eta^2}. \tag{5.22}$$

Lemma 5.7. *Let* $u \in W$ *be a solitary wave solution of (1.1). Then*

$$u \in \{f \in L^q(\mathbb{R}^2)\ \ \forall q \in [2,\infty], f_x \in L^6(\mathbb{R}^2)\bigcap L^3(\mathbb{R}^2), f_{xx}, f_{xxx}, f_y, f_{xy} \in L^3(\mathbb{R}^2)\}.$$

Proof of Lemma 5.7. A stronger embedding result than Lemma 2.4 can be found in [2] (Theorem 15.7, p.323) which states that in fact

$$u \in L^q, \text{ for } \ q \ge 2, \quad u_x \in L^{q_1}, \text{ for } \ q_1 \in [2, 6].$$

Therefore $g = -uu_x \in L^{q_1}(\mathbb{R}^2)$, and $h = -u_x^2 \in L^3(\mathbb{R}^2)$. It is easily checked that

$$\Phi_1 = \frac{3\xi + \gamma\xi^3}{c\delta\xi^6 + c\xi^4 + \beta\xi^2\eta^2 + c\xi^2 + \alpha\eta^2}, \Phi_2 = \frac{\frac{\gamma}{2}\xi^2}{c\delta\xi^6 + c\xi^4 + \beta\xi^2\eta^2 + c\xi^2 + \alpha\eta^2}$$

satisfy the assumption of Proposition 5.6, yielding that $u \in L^3(\mathbb{R}^2)$. Similarly we obtain that $u_{xx}, u_{xxx}, u_y, u_{xy} \in L^3(\mathbb{R}^2)$. □

Let $w = u_x$. Lemma 5.7 implies that $u, u_{xx}, u_y \in L^3(\mathbb{R}^2)$ and $w, w_{xx}, w_y \in L^3(\mathbb{R}^2)$. By [2], Theorem 10.2, one has $u, w \in L^\infty(\mathbb{R}^2)$. And by interpolation between $w \in L^\infty(\mathbb{R}^2)$ and $w_{xx} \in L^3(\mathbb{R}^2)$, one has $w_x \in L^6(\mathbb{R}^2)$. So

$$g_{xx} = -(uu_{xxx} + 3u_xu_{xx}) \in L^3(\mathbb{R}^2), \ \ g_y = -(u_yu_x + uu_{xy}) \in L^3(\mathbb{R}^2).$$
$$h_{xx} = -2(u_{xx}^2 + u_xu_{xxx}) \in L^3(\mathbb{R}^2), \quad h_y = -2u_xu_{xy} \in L^3(\mathbb{R}^2).$$

Another application of Proposition 5.6 leads to

$$u_{xx}, u_{xxxx}, u_{xxy}, u_{yy}, w_{xx}, w_{xxxx}, w_{xxy}, w_{yy} \in L^3(\mathbb{R}^2).$$

Now let $u_1 = u_{xx}, w_1 = w_{xx}$, then $u_1, \partial_x^2 u_1, \partial_y u_1, w_1, \partial_x^2 w_1, \partial_y w_1 \in L^3(\mathbb{R}^2)$, which implies that $u_1, w_1 \in L^\infty(\mathbb{R}^2)$. Thus we obtain $g_{xx}, h_{xx} \in L^\infty(\mathbb{R}^2)$. Similarly, one can get $g_y, h_y \in L^\infty(\mathbb{R}^2)$ by considering $u_2 = u_y, w_2 = w_y$. Hence Lizorkin's Theorem implies that $u_{xxxx}, u_{xxy}, u_{yy} \in L^\infty(\mathbb{R}^2)$, $w_{xxxx}, w_{xxy}, w_{yy} \in L^\infty(\mathbb{R}^2)$, which implies that $g_{xxx}, g_{xxy}, h_{xxx}, h_{xxy} \in L^\infty(\mathbb{R}^2)$. Iteration of the process leads to the conclusion of Theorem 5.5. □

6. STABILITY OF SOLITARY WAVES

In the previous sections we proved that there exists a nontrivial smooth solitary wave solution ϕ_c to equation (1.7) with positive speed c, i.e., $\phi_c \in W$ solves the equation

$$-cu + cu_{xx} - c\delta u_{xxxx} + \frac{3}{2}u^2 - \gamma(\frac{1}{2}u_x^2 + uu_{xx}) - \alpha\partial_x^{-2}u_{yy} + \beta u_{yy} = 0. \qquad (6.1)$$

We also know that equation (1.1) can be written in Hamiltonian form and has the invariants

$$E(u) = \int_{\mathbb{R}^2} [u^2 + u_x^2 + \delta u_{xx}^2] \; dxdy, \qquad (6.2)$$

$$F(u) = \int_{\mathbb{R}^2} [u^3 + \gamma uu_x^2 - \alpha(\partial_x^{-1}u_y)^2 - \beta u_y^2] \; dxdy. \qquad (6.3)$$

A central role will be played by the functionals $G_c(u)$ and $K(u)$, where

$$G_c(u) = \int_{\mathbb{R}^2} [cu^2 + cu_x^2 + c\delta u_{xx}^2 + \alpha(\partial_x^{-1}u_y)^2 + \beta u_y^2] dxdy, \;\; K(u) = \int_{\mathbb{R}^2} [u^3 + \gamma uu_x^2] dxdy$$

are defined for $u \in W$. Note that the functional $K(u)$ is well-defined on W by Sobolev embedding theorem. Equation (6.1) is the Euler-Lagrange equation of the functional

$$L_c(u) = \frac{1}{2}[G_c(u) - K(u)] = \frac{1}{2}[cE(u) - F(u)].$$

Recall that from Section 5 we have introduced the ground states to be the solutions to the solitary wave equation that come from the associated variational problem. Multiplying equation (5.4) by ψ_c and integrating yields that

$$I_\lambda = G_c(\psi_c) = \frac{3}{2}K(\psi_c).$$

Using homogeneity, we have

$$I_\lambda = \lambda^{2/3}I_1, \quad G_c(\psi_c) = \frac{1}{\mu^2}G_c(\phi_c), \quad K(\phi_c) = \mu^3 K(\psi_c).$$

This implies that

$$G_c(\phi_c) = \frac{4}{9}I_1^3 = \frac{3}{2}K(\phi_c). \qquad (6.4)$$

Thus we may characterize the set of ground states S_c as

$$S_c = \{\phi_c \in W : G_c(\phi_c) = \frac{4}{9}I_1^3 = \frac{3}{2}K(\phi_c)\}. \qquad (6.5)$$

Define the scalar function d of the wavespeed c as introduced in [21] to be

$$d(c) = cE(\phi_c) - F(\phi_c)$$

for $\phi_c \in S_c$. It's easy to see that $d(c)$ depends only on c and not on $\phi_c \in S_c$. In fact, from the definition of d and the characterization of S_c, it follows that

$$d(c) = G_c(\phi_c) - K(\phi_c) = \frac{1}{2}K(\phi_c) = \frac{4}{27}I_1^3. \qquad (6.6)$$

Consider the equation

$$-w + w_{xx} - \delta w_{xxxx} + \frac{3}{2}w^2 - \gamma(\frac{1}{2}w_x^2 + ww_{xx}) - \alpha\partial_x^{-2}w_{yy} + \beta w_{yy} = 0.$$

By the theorem of existence of solitary waves we know that there is a ground state solution w to this equation and obviously w does not depend on c. Using the transformation $\phi_c(x,y) = cw(x, \sqrt{c}y)$, we get that $\phi_c(x,y)$ is a ground state of (6.1). Hence we consider

$$G_1(w) = \int_{\mathbb{R}^2} [w^2 + w_x^2 + \delta w_{xx}^2 + \alpha(\partial_x^{-1} w_y) + \beta w_y^2] dxdy,$$

then $G_1(w)$ is positive and independent of c and $G_c(\phi_c) = c^{5/2} G_1(w)$. Hence

$$d(c) = \frac{1}{6} G_c(\phi_c) = \frac{1}{6} c^{5/2} G_1(w), \;\; d''(c) = \frac{5}{8}\sqrt{c} G_1(w) > 0.$$

We will prove Theorem 1.4 in the following argument. The following two lemmas are helpful in order to prove the stability.

Lemma 6.1. *Let $c > 0$ and suppose $d''(c) > 0$, then there exists $\delta_0 > 0$ such that if $|c - c_1| < \delta_0$ then*

$$d(c_1) > d(c) + d'(c)(c_1 - c) + \frac{1}{4} d''(c)(c_1 - c)^2.$$

Proof. The functionals E and F are C^∞-mappings from W to $\mathbb{R}$. and the mapping $c \mapsto G_c$ is also C^∞ from $\mathbb{R}^+$ to $C^\infty(W;\mathbb{R})$. Hence the value I_1 varies smoothly with $c \in (0, \infty)$, and consequently d is a smooth function of $c > 0$. So the lemma is just a direct application of the Taylor's Theorem. □

Now for $\varepsilon > 0$ and $c > 0$, we define the "ε-tube" of the set of solitary waves of speed c to be $S_{c,\varepsilon} = \Big\{u \in W : \inf_{\phi_c \in S_c} \|u - \phi_c\|_W < \varepsilon\Big\}$. Since $d'(c) > 0$, by the Implicit Function Theorem, for each $c > 0$, there corresponds a tube S_{c,ε_0} and C^1-mapping $f : S_{\varepsilon_0,c} \longrightarrow \mathbb{R}^+$ such that

$$\begin{cases} d(f(u)) = \frac{1}{2} K(u) \\ f(\phi_c) = c. \end{cases} \tag{6.7}$$

In fact, from the previous calculation we may write out f explicitly in the form that $f(u) = \left[\frac{3K(u)}{G_1(w)}\right]^{2/5}$.

Lemma 6.2. *Suppose $d''(c) > 0$ for some $c > 0$. Then there exists $\varepsilon_0 > 0$ such that for any $u \in S_{c,\varepsilon_0}$ and $\phi_c \in S_c$*

$$f(u)\Big[E(u) - E(\phi_c)\Big] - \Big[F(u) - F(\phi_c)\Big] \geq \frac{1}{4} d''(c)|f(u) - c|^2, \tag{6.8}$$

where $f(u)$ is defined in (6.7) above.

Proof. We know that

$$K(u) = 2d(f(u)) = 2\Big[fE(\phi_{f(u)}) - F(\phi_{f(u)})\Big] = 2[G_f(\phi_f) - K(\phi_f)] = K(\phi_f)$$

since $G_c(\phi_c) = 3K(\phi_c)$ for all c.

By definition, $\phi_{f(u)}$ minimizes $G_{f(u)}$ subject to the constraint $K(u) = K(\phi_{f(u)})$. This implies that $G_{f(u)}(\phi_{f(u)}) \leq G_{f(u)}(u)$. Applying Lemma 6.1 we obtain

$$\begin{aligned} f(u)E(u) - F(u) &= G_{f(u)}(u) - K(u) \\ &\geq G_{f(u)}(\phi_{f(u)}) - K(\phi_{f(u)}) = d(f(u)) \\ &\geq d(c) + d'(c)\Big(f(u) - c\Big) + \frac{1}{4}d''(c)\Big(f(u) - c\Big)^2 \\ &= f(u)E(\phi_c) - F(\phi_c) + \frac{1}{4}d''(c)\Big(f(u) - c\Big)^2, \end{aligned}$$

this is

$$f(u)\Big[E(u) - E(\phi_c)\Big] - \Big[F(u) - F(\phi_c)\Big] \geq \frac{1}{4}d''(c)|f(u) - c|^2.$$

□

Proof of Theorem 1.4. Suppose S_c is W-unstable. This means that there exists $\sigma > 0$ and initial data $u_n(0) \in S_{c,1/n}$ and times $t_n > 0$, $n = 1, 2, \ldots$ such that

$$\inf_{\phi \in S_c} \|u_n(\cdot, t_n) - \phi\|_W = \sigma. \tag{6.9}$$

Because the functional E and F are continuous on W and are conserved, there are elements $\{\phi_n\} \subset S_c$ such that as $n \to \infty$

$$|E(u_n(\cdot, t_n)) - E(\phi_n)| = |E(u_n(0)) - E(\phi_n)| \to 0, \tag{6.10}$$

$$|F(u_n(\cdot, t_n)) - F(\phi_n)| = |F(u_n(0)) - F(\phi_n)| \to 0. \tag{6.11}$$

Pick $\sigma < \varepsilon_0$ small enough so that Lemma 6.2 applies, which is to say that for all $n = 1, 2, \ldots$

$$f(u_n(t_n))\Big[E(u_n(t_n)) - E(\phi_n)\Big] - \Big[F(u_n(t_n)) - F(\phi_n)\Big] \geq \frac{1}{4}d''(c)|f(u_n(t_n)) - c|^2. \tag{6.12}$$

Observe that for any $n \geq 1$

$$\begin{aligned} \|u_n(t_n)\|_W &\leq \|\phi_n\|_W + 2\sigma \\ &\leq CG_c^{1/2}(\phi_n) + 2\sigma \quad \text{(from (5.7))} \\ &= \frac{2C}{3}I_1^{3/2} + 2\sigma \quad \text{(from (6.5))} \; < +\infty. \end{aligned}$$

Therefore the sequence $\{u_n(\cdot, t_n)\}$ is uniformly bounded in W. It follows immediately from (6.7) that $\{K(u_n(\cdot, t_n))\}$ is bounded and hence so is $\{f(u_n(t_n))\}$ since $f(u) = \Big[3K(u)/G_1(w)\Big]^{2/5}$. Combining this with (6.10)-(6.12) yields

$$f(u_n(t_n)) \to c \text{ as } n \to \infty. \tag{6.13}$$

This relation implies in turn that

$$K(u_n(t_n)) = 2d(f(u_n(t_n))) \to 2d(c) = \frac{8}{27}I_1^3, \text{ as } n \to \infty, \qquad \text{from (6.6).}$$

Hence

$$\begin{aligned} G_c(u_n(t_n)) &= cE(u_n(t_n)) - F(u_n(t_n)) + K(u_n(t_n)) \\ &\to d(c) + 2d(c) = 3d(c) = \frac{4}{9}I_1^3. \end{aligned}$$

Now define w_n by $w_n = K(u_n(t_n))^{-1/3}u_n(t_n)$, Then $K(w_n) = 1$ and $G_c(w_n) = K(u_n(t_n))^{-2/3}G_c(u_n(t_n)) \to I_1$. Therefore the sequence $\{w_n\}$ minimizes G_c subject to the constraint $K = 1$.

In the proof of Theorem 5.1 we showed that for a minimizing sequence $\{\psi_n\} \subset W$ with the constraint $K(\psi_n) = 1$, there exist a subsequence of $\{\psi_n\}$ (still denoted $\{\psi_n\}$) and a sequence of translation vectors $\{\vec{x}_n\}$ such that $\psi_n(\cdot - \vec{x}_n)$ converges to a minimizer. Hence there exists a subsequence of $\{w_n\}$, still denoted $\{w_n\}$, a sequence of translation vectors $\{\vec{x}_n\}$, and a $w \in W$ with $K(w) = 1$ such that $\lim_{n\to\infty} \|w_n(\cdot - \vec{x}_n) - w\|_W = 0$, that is $\lim_{n\to\infty} \|w_n - w(\cdot + \vec{x}_n)\|_W = 0$. So let $\phi_n = \frac{1}{3}I_1 w(\cdot + \vec{x}_n) \in S_c$. This in turn implies that $\lim_{n\to\infty} \|u_n(t_n) - \phi_n\|_W = 0$, which contradicts (6.9). Hence we obtain the stability. □

7. Nonexistence of solitary waves

In contrast to the existence theorem of the solitary waves, we also provide a nonexistence result of the solitary waves. The main result of this section is the following theorem:

Theorem 7.1. *The equation (1.7) with $\delta \geq 0$ does not admit any nontrivial solitary wave satisfying $u \in W$, $u \in H^2(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$, u_{xxx}, $u_{yy} \in L^2_{loc}(\mathbb{R}^2)$ if $\alpha \leq 0$ and $\beta \leq 0$.*

Proof. The proof is based on the Pohojaev type identities. The regularity assumptions in Theorem 7.1 are need to justify the identities by the following standard truncation argument. Let $f_0 \in C_0^\infty(\mathbb{R})$, $0 \leq f_0 \leq 1$, $f_0(t) = 1$ if $0 \leq |t| \leq 1$, $f_0(t) = 0$ if $|t| \geq 2$. Set $f_j = f_0(|\cdot|/j^2)$, $j = 1, 2, \cdots$

Multiplying (1.7) by xf_ju and then integrating over $\mathbb{R}^2$ we get

$$
\begin{aligned}
&-c\int_{\mathbb{R}^2} xf_j(\frac{u^2}{2})_x dxdy + c\int_{\mathbb{R}^2} xf_j uu_{xxx} dxdy - c\delta\int_{\mathbb{R}^2} xf_j uu_{xxxxx} dxdy + \int_{\mathbb{R}^2} xf_j(u^3)_x dxdy \\
&-\gamma\int_{\mathbb{R}^2} xf_j(u^2u_{xx})_x dxdy - \alpha\int_{\mathbb{R}^2} xf_j u\partial_x^{-1}u_{yy} dxdy + \beta\int_{\mathbb{R}^2} xf_j uu_{xyy} dxdy = 0,
\end{aligned}
$$

By using several integration by parts and Lebesgue dominated convergence theorem, we can get that as $j \to \infty$

$$
\begin{aligned}
&-c\int_{\mathbb{R}^2} xf_j(\frac{u^2}{2})_x dxdy \to \int_{\mathbb{R}^2} \frac{c}{2}u^2 dxdy, && \int_{\mathbb{R}^2} xf_j(u^3)_x dxdy \to -\int_{\mathbb{R}^2} u^3 dxdy, \\
&-\alpha\int_{\mathbb{R}^2} xf_j u\partial_x^{-1}u_{yy} dxdy \to \int_{\mathbb{R}^2} -\frac{\alpha}{2}(\partial_x^{-1}u_y)^2 dxdy, && \beta\int_{\mathbb{R}^2} xf_j uu_{xyy} dxdy \to \int_{\mathbb{R}^2} \frac{\beta}{2}u_y^2 dxdy,
\end{aligned}
$$

$$
\begin{aligned}
c\int_{\mathbb{R}^2} xf_j uu_{xxx} dxdy &= -c\Big[\int_{\mathbb{R}^2} f_j uu_{xx} dxdy + \int_{\mathbb{R}^2} x(f_j)_x uu_{xx} dxdy + \int_{\mathbb{R}^2} xf_j(\frac{u_x^2}{2})_x dxdy\Big] \\
&= -c\Big[-\int_{\mathbb{R}^2} (f_ju)_x u_x dxdy - \int_{\mathbb{R}^2} (x(f_j)_x u)_x u_x dxdy - \int_{\mathbb{R}^2} (xf_j)_x \frac{u_x^2}{2} dxdy\Big] \\
&= c\Big[\int_{\mathbb{R}^2} (f_j)_x(\frac{u^2}{2})_x dxdy + \int_{\mathbb{R}^2} f_j u_x^2 dxdy + \int_{\mathbb{R}^2} x(f_j)_{xx}(\frac{u^2}{2})_x dxdy \\
&\qquad \int_{\mathbb{R}^2} x(f_j)_x u_x^2 dxdy + \int_{\mathbb{R}^2} f_j \frac{u_x^2}{2} dxdy + \int_{\mathbb{R}^2} x(f_j)_x \frac{u_x^2}{2} dxdy\Big] \\
&\longrightarrow \int_{\mathbb{R}^2} \frac{3c}{2} u_x^2 dxdy.
\end{aligned}
$$

Similarly we get

$$c\delta\int_{\mathbb{R}^2} xf_j u u_{xxxxxx}dxdy \longrightarrow c\delta\int_{\mathbb{R}^2} -\frac{u_{xx}^2}{2}dxdy.$$

Now

$$\begin{aligned}
-\gamma\int_{\mathbb{R}^2} xf_j(u^2u_{xx})_x dxdy &= \gamma\Big[\int_{\mathbb{R}^2} f_j u^2 u_{xx}dxdy + \int_{\mathbb{R}^2} x(f_j)_x u^2 u_{xx}dxdy\Big]\\
&= -\gamma\int_{\mathbb{R}^2}(f_j)_x u^2 u_x dxdy - \gamma\int_{\mathbb{R}^2} f_j\cdot 2uu_x^2 dxdy + \gamma\int_{\mathbb{R}^2} x(f_j)_x u^2 u_{xx}dxdy\\
&\longrightarrow \int_{\mathbb{R}^2} -2\gamma u u_x^2 dxdy.
\end{aligned}$$

Putting all the above together we obtain

$$\int_{\mathbb{R}^2} \frac{c}{2}u^2 + \frac{3c}{2}u_x^2 + \delta\frac{cu_{xx}^2}{2} - u^3 - 2\gamma u u_x^2 - \frac{\alpha}{2}(\partial_x^{-1}u_y)^2 + \frac{\beta}{2}u_y^2 \;\; dxdy = 0. \qquad (7.1)$$

Multiplying (1.7) by $y\partial_x^{-1}u_y$ and integrating over $\mathbb{R}^2$

$$\begin{aligned}
-c\int_{\mathbb{R}^2} u_x y\partial_x^{-1}u_y dxdy + c\int_{\mathbb{R}^2} u_{xxx}y\partial_x^{-1}u_y dxdy + \int_{\mathbb{R}^2} 3uu_x y\partial_x^{-1}u_y dxdy&\\
-\gamma\int_{\mathbb{R}^2}(2u_xu_{xx}+uu_{xxx})y\partial_x^{-1}u_y dxdy - \alpha\int_{\mathbb{R}^2}\partial_x^{-1}u_{yy}y\partial_x^{-1}u_y dxdy&\\
+\beta\int_{\mathbb{R}^2} u_{xyy}y\partial_x^{-1}u_y dxdy = 0,&
\end{aligned}$$

Again, integrating by parts we get

$$\begin{aligned}
&-c\int_{\mathbb{R}^2} u_x y\partial_x^{-1}u_y dxdy = \int_{\mathbb{R}^2} -\frac{c}{2}u^2 dxdy, \qquad c\int_{\mathbb{R}^2} u_{xxx}y\partial_x^{-1}u_y dxdy = \int_{\mathbb{R}^2} -\frac{c}{2}u_x^2 dxdy,\\
&-c\delta\int_{\mathbb{R}^2} u_{xxxxx}y\partial_x^{-1}u_y dxdy = \int_{\mathbb{R}^2} -\frac{c\delta}{2}u_{xx}^2 dxdy, \qquad \int_{\mathbb{R}^2} 3uu_x y\partial_x^{-1}u_y dxdy = \int_{\mathbb{R}^2}\frac{1}{2}u^3 dxdy,\\
&-\alpha\int_{\mathbb{R}^2}\partial_x^{-1}u_{yy}y\partial_x^{-1}u_y dxdy = \int_{\mathbb{R}^2}\frac{\alpha}{2}(\partial_x^{-1}u_y)^2 dxdy, \qquad \beta\int_{\mathbb{R}^2} u_{xyy}y\partial_x^{-1}u_y dxdy = \int_{\mathbb{R}^2}\frac{\beta}{2}u_y^2 dxdy,
\end{aligned}$$

$$\begin{aligned}
-\gamma\int_{\mathbb{R}^2}(2u_xu_{xx}+uu_{xxx})y\partial_x^{-1}u_y dxdy &= -\frac{\gamma}{2}\int_{\mathbb{R}^2}[(u^2)_{xxx} - (u_x^2)_x]y\partial_x^{-1}u_y dxdy\\
&= -\frac{\gamma}{2}\int_{\mathbb{R}^2} u_x^2 y u_y - (u^2)_{xx}yu_y dxdy\\
&= -\frac{\gamma}{2}[\int_{\mathbb{R}^2}(uu_x^2)_y y dxdy - \int_{\mathbb{R}^2} 2uu_xu_{xy}y + (u^2)_{xx}yu_y dxdy\\
&= \int_{\mathbb{R}^2}\frac{\gamma}{2}uu_x^2 dxdy.
\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^2}\frac{c}{2}u^2 + \frac{c}{2}u_x^2 + \delta\frac{c}{2}u_{xx}^2 - \frac{1}{2}u^3 - \frac{\gamma}{2}uu_x^2 - \frac{\alpha}{2}(\partial_x^{-1}u_y)^2 - \frac{\beta}{2}u_y^2 \;\; dxdy = 0. \qquad (7.2)$$

To get the third identity, we first notice that if $u \in W$ satisfies (1.7) in $\mathscr{D}'(\mathbb{R}^2)$, then u satisfies

$$-cu + cu_{xx} - c\delta u_{xxxx} + \frac{3}{2}u^2 - \frac{\gamma}{2}(u_x^2 + 2uu_{xx}) - \alpha\partial_x^{-1}v_y + \beta u_{yy} = 0 \;\text{ in } W',$$

where $v = \partial_x^{-1} u_y \in L^2$ and $\partial_x^{-1} v_y \in W'$. Now taking the $W - W'$ duality product of the last equation with $u \in W$ we have

$$\int_{\mathbb{R}^2} cu^2 + cu_x^2 + \delta c u_{xx}^2 - \frac{3}{2}u^3 - \frac{3\gamma}{2} u u_x^2 + \alpha(\partial_x^{-1} u_y)^2 + \beta u_y^2 \ dxdy = 0. \tag{7.3}$$

(7.2)·3−(7.3) we obtain

$$\int_{\mathbb{R}^2} \frac{c}{2}(u^2 + u_x^2 + \delta u_{xx}^2) - \frac{5}{2}[\alpha(\partial_x^{-1} u_y)^2 + \beta u_y^2] \ dxdy = 0, \tag{7.4}$$

which rules out the case that $\alpha \le 0$, $\beta \le 0$. □

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