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Calculus of Variations

# Stability of the Camassa-Holm peakons in the dynamics of a shallow-water-type system

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**Abstract** The stability of the Camassa-Holm (periodic) peakons in the dynamics of an integrable shallow-water-type system is investigated. A variational approach with the use of the Lyapunov method is presented to prove the variational characterization and the orbital stability of these wave patterns. In addition, a sufficient condition for the global existence of strong solutions is given. Finally, a local-in-space wave-breaking criterion is illustrated in the periodic setting.

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## 1 Introduction

Many shallow water models have been proved to be appropriate approximations to the full Euler dynamics when the water depth is small compared to the horizontal wavelength scale

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[1,8]. Among these models, the two-component Camassa-Holm (2CH) system [7,21,25], which describes the evolution of both the horizontal velocity component and the free surface of the water waves, has brought up much attention recently, since it incorporates the interaction between the free surface and the horizontal velocity component and can present the phenomenon of wave breaking. Moreover, the 2CH system is integrable among the equivalent class of the model equations, allowing one to obtain a great amount of information about the underlying physical system from analyzing it by using the well-developed tools in the integrable theory.

More precisely, the 2CH system reads

$$\begin{cases} u_t - u_{txx} + \kappa u_x + 3uu_x - (2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.1}$$

for some constant $\kappa \in \mathbb{R}$. As mentioned above, system (1.1) is completely integrable [12, 21,25,26] since it can be written as a compatibility condition of two linear equations with a spectral parameter $\zeta$ (Lax pair), that is,

$$\Psi_{xx} = \left[-\zeta^2\rho^2 + \zeta\left(u - u_{xx} + \frac{\kappa}{2}\right) + \frac{1}{4}\right]\Psi, \quad \Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{1}{2}u_x\Psi,$$

and has a bi-Hamiltonian structure corresponding to the following Hamiltonian functionals

$$\mathcal{E}(u,\rho) = \int \left(u^2 + u_x^2 + \rho^2\right) dx, \quad \mathcal{F}_\kappa(u,\rho) = \int \left(u^3 + uu_x^2 + u\rho^2 + \kappa u^2\right) dx.$$

Through $\mathcal{F}_\kappa$, system (1.1) can be represented as the following abstract Hamiltonian form

$$\partial_t \begin{pmatrix} u \\ \rho \end{pmatrix} = \mathcal{J}\mathcal{F}_\kappa'(u,\rho), \tag{1.2}$$

where $\mathcal{J}$ is a closed skew symmetric operator given by

$$\mathcal{J} = \frac{1}{2}\begin{pmatrix} -\partial_x(1-\partial_x^2)^{-1} & 0 \\ 0 & -\partial_x \end{pmatrix}$$

and $\mathcal{F}_\kappa'(u,\rho) = (\delta\mathcal{F}_\kappa/\delta u,\ \delta\mathcal{F}_\kappa/\delta\rho)^T$ denotes the variational derivative of the functional $\mathcal{F}_\kappa$. Since system (1.1) admits $\int(u - u_{xx})\,dx$ as its Casimir, the following functional

$$\mathcal{H}(u,\rho) = \int u\,dx \tag{1.3}$$

is also conserved along the flow (1.1).

Similar to the classical Camassa-Holm (CH) equation, the 2CH system (1.1) exhibits a remarkable feature: when the linear dispersion is absent, corresponding to $\kappa = 0$, the 2CH system admits not only the peaked solitary wave solution $(\varphi_c(x-ct),\ 0) = \left(c\,e^{-|x-ct|},\ 0\right)$ with the speed $c > 0$ on the line, but also the periodic peaked solution $(\psi_c(x-ct),\ 0)$ with

$$\psi_c(x-ct) = c\,\frac{\cosh\left(\frac{1}{2} + [x-ct] - (x-ct)\right)}{\cosh\frac{1}{2}}, \quad c > 0.$$

Indeed when $\rho = 0$ and $\kappa = 0$, the 2CH system (1.1) reduces to the scalar CH equation [4,13]

$$u_t - u_{txx} + 3uu_x - (2u_x u_{xx} + uu_{xxx}) = 0, \tag{1.4}$$

and $\varphi_c$ (or $\psi_c$) is the (periodic) peakon solution to (1.4). Note that solutions $(\varphi_c, 0)$ (or $(\psi_c, 0)$) are not classical solutions of the 2CH system (1.1). They should be understood as weak solutions, since (1.1) can be written in the following conservation law

$$\begin{cases} u_t + uu_x + \partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.5}$$

where $p(x)$ is the corresponding kernel of the convolution operator $\left(1 - \partial_x^2\right)^{-1}$ for the whole line case or the periodic case. The associated Hamiltonian conserved functionals for (1.5) are

$$\mathcal{E}(u, \rho) = \int \left(u^2 + u_x^2 + \rho^2\right) dx \tag{1.6}$$

and

$$\mathcal{F}(u, \rho) = \int \left(u^3 + uu_x^2 + u\rho^2\right) dx. \tag{1.7}$$

The similarity between the 2CH system (1.5) and the CH equation (1.4) leads us to consider how much in common the corresponding peaked wave profiles share. In particular, we would like to address the issue on the stability of these wave profiles $(\varphi_c, 0)$ and $(\psi_c, 0)$. The stability of the CH (periodic) peakons $\varphi_c$ and $\psi_c$ [9,10,22,23] seems to suggest the analogous result to the wave profiles $(\varphi_c, 0)$ or $(\psi_c, 0)$ for the 2CH system (1.5). Yet the coupling between $u$ and $\rho$ in (1.5)–(1.7) makes it non-trivial to verify.

Due to the quasilinear nature of the system (1.5), the nonlinear part cannot be regarded as a higher-order perturbation of the linear terms. Hence, it is not clear how the linearization may hint on the nonlinear stability through the abstract Hamiltonian form (1.2) using the approach introduced in [14]. Furthermore, the non-differentiability at $(\varphi_c, 0)$ or $(\psi_c, 0)$ makes the spectral analysis hard to employ.

Another approach uses the variational structure and seeks to prove that the considered traveling waves are the unique (up to translation) minimizers of the constrained energy functional. Such an idea has been successfully applied to prove the stability of the CH (periodic) peakons, cf. [9,23]. For the line peaked wave profiles $(\varphi_c, 0)$, we consider here the following constrained minimization problem in the spirit of [9],

$$\mathcal{I} := \inf\left\{\mathcal{E}(u, \rho) \,|\, (u, \rho) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \quad \text{with } \mathcal{F}(u, \rho) = \mathcal{F}(\varphi_c, 0) = \frac{4c^3}{3}\right\}. \tag{1.8}$$

The usual method is Lions' concentration-compactness principle [24] (which is what was used in [9]). The difficulty comes from showing that the weak limit of the minimizing sequence satisfies the constraint $\mathcal{F}(u, \rho) = \mathcal{F}(\varphi_c, 0)$. However, from the definitions (1.6) of $\mathcal{E}$ and (1.7) of $\mathcal{F}$, we see that one can at best hope to obtain the weak $H^1 \times L^2$ convergence of the minimizing sequence, which is far from enough to ensure that the constraint can be fulfilled at the weak limit. To be more precise, the convergence of the integration of the two terms $uu_x^2$ and $u\rho^2$ in $\mathcal{F}$ is not easy to obtain. In [9], the authors made crucial use of the sign-preservation of the momentum density $u - u_{xx}$ initially, combined with a compensated compactness argument to saturate the constraint at the weak limit. Unfortunately, such a property does not hold for the system (1.5), leading to a serious gap in solving the constrained minimization problem.

The method we are using to prove the orbital stability of the 2CH (periodic) peaked wave profiles $(\varphi_c, 0)$ (or $(\psi_c, 0)$) is different from the above mentioned and has the flavor in the Lyapunov sense. Such an idea was first introduced by Constantin-Strauss [10] in the study of

CH line peakons $\varphi_c$, and was also adapted in treating the CH periodic peakons $\psi_c$ in [22]. We are still looking to prove the existence and uniqueness (up to translation) of the constrained minimization problem without restriction of the sign-preservation for $u - u_{xx}$. The main idea is to use the conservation laws of the system. The key observation is that the $H^1 \times L^2$ distance of the perturbed (periodic) solution $(u, \rho)$ to the (periodic) peaked wave profile $(\varphi_c, 0)$ (or $(\psi_c, 0)$) can be controlled by the difference between the corresponding energy functionals $\mathcal{E}$, with an error term expressed in terms of the pointwise difference between the peak of velocity component $u$ of the perturbed (periodic) solution and that of the line peakon $\varphi_c$ or the periodic peakon $\psi_c$, and is independent of the shape information of the surface component $\rho$, cf. (3.19) for the line case and (4.15) for the periodic case. The conservation law indicates that the difference between the energy functionals will remain small if it is small initially. Therefore the $H^1 \times L^2$ orbital stability relies on a delicate control of the height difference between $u$ and $\varphi_c$ or $\psi_c$.

In the case of the line peaked wave profiles $(\varphi_c, 0)$, we are able to construct a Lyapunov function $P(M_u; u, \rho)$ where $M_u = \max_x u$. Notice that $M_u$ is well-defined and is positive when $\mathcal{F}(u, \rho)$ is near $\mathcal{F}(\varphi_c, 0)$. We can further prove that $P$ satisfies the following estimate

$$|M_u - c|^2 \lesssim P(M_u; \varphi_c, 0) \lesssim \mathcal{E}(u, \rho) - \mathcal{E}(\varphi_c, 0) + |\mathcal{F}(u, \rho) - \mathcal{F}(\varphi_c, 0)| . \tag{1.9}$$

Therefore, for the purpose of minimization issue, we see from (1.9) that any $(u, \rho)$ under the constraint $\mathcal{F}(u, \rho) = \mathcal{F}(\varphi_c, 0)$ satisfies

$$\mathcal{E}(u, \rho) \geq \mathcal{E}(\varphi_c, 0),$$

and hence $(\varphi_c, 0)$ is a minimizer. On the other hand, when concerning the stability, we know from the continuity of the two functionals $\mathcal{E}$ and $\mathcal{F}$ in $H^1 \times L^2$ that the right-hand side of (1.9) can be made small when the initial data is a small perturbation of the peaked wave profile $(\varphi_c, 0)$. Therefore $|M_u - c|$ is small, proving the stability. The case for the periodic peaked wave profile $(\psi_c, 0)$ can be treated similarly.

We would like to comment that in [10], what the authors proved can be translated into

$$|M_u - c|^2 \lesssim |\mathcal{E}(u) - \mathcal{E}(\varphi_c)| + |\mathcal{F}(u) - \mathcal{F}(\varphi_c)| .$$

Thus the stability follows but it is not clear from the above estimate whether peakons are minimizers of $\mathcal{E}$ subject to the constraint on $\mathcal{F}$. On the other hand, our improved estimate (1.9) indicates the existence of minimizers as the peaked wave profiles $(\varphi_c, 0)$ and their stability simultaneously.

Since the peaked wave solutions $(\varphi_c, 0)$ can be regarded as global weak solutions to (1.5), when considering their stability, we would like to consider perturbations that would generate solutions that can persist for all time as well. Hence we also investigate the global dynamics of the 2CH system (1.5). On the local well-posedness of the strong solutions of system (1.5), Escher-Lechtenfeld-Yin [11] provided the result in $H^s \times H^{s-1}$ for $s \geq 2$. Such a result was later improved by Gui-Liu [18,19] to Sobolev spaces $H^s \times H^{s-1}$ for $s > 3/2$. So far, most of the results on the global existence of strong solutions concern the case when $\rho \to 1$ as $|x| \to \infty$ and a sufficient condition is $\inf_x \rho_0(x) > 0$ (see, for example [27]). However, in our setup, $\rho$ decays to zero at infinity. Thus the previous results cannot be applied. From examining the dynamics along the characteristics, we are able to show that the previous condition $\inf_x \rho_0(x) > 0$ can be replaced by $\rho_0(x) > 0$ on $\mathbb{R}$, cf. Theorem 2.1 below. We want to point out that a similar result was also obtained by Grunert [15] recently from the Lagrangian point of view, which is quite different from the approach we take here. A more detailed comparison is given in Remark 2.1. We also note that the global weak and conservative solutions of (1.5) have been constructed in [16] and [17] by Grunert-Holden-Raynaud.

Finally, for a complete picture of the dynamics of the solutions, we also provide some blow-up analysis. In particular, we focus on the local-in-space type blow-up criterion introduced by Brandolese [2] and Brandolese-Cortez [3]. In the case $x \in \mathbb{R}$, such a problem was considered in [20]. For the periodic situation $x \in \mathbb{S}^1$, we derived in this paper that the solution blows up in finite time if the initial data satisfies that $u_{0,x}(x_0) < -\beta^*|u_0(x_0)|$ and $\rho_0(x_0) = 0$ for some $x_0 \in \mathbb{S}^1$, where $\beta^*$ is defined in (5.6). Technically, the difference compared with the whole line case lies in the different convolution estimates due to the change of the convolution kernel.

The rest of the paper is organized as follows. In Sect. 2, we provide a sufficient condition for the global existence of strong solutions to the system (1.5) on the whole line. In Sect. 3, we prove that the line peaked wave profiles $(\varphi_c,\ 0)$ are the unique (up to translation) minimizers of a constrained minimization problem and they are orbitally stable. The orbital stability for periodic peaked solutions $(\psi_c,\ 0)$ is established in Sect. 4 by a more delicate Lyapunov type argument. Finally in Sect. 5, we construct initial data in the periodic setting which leads to the local-in-space blow-up.

## 2 Global existence of solutions

This section is devoted to providing a sufficient condition for the global existence of strong solutions to the 2CH system (1.5). First of all, we focus our attention on the case of the whole line and consider the following Cauchy problem for system (1.5) written as

$$\begin{cases} u_t + uu_x + \partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) = 0, \\ \rho_t + (\rho u)_x = 0, \qquad t \in (0, T), \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), \qquad x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where the function $p(x)$ in (2.1) is the kernel of the convolution operator $\left(1 - \partial_x^2\right)^{-1}$ on the line defined by $p(x) = e^{-|x|}/2$.

The following lemma states that the infimum points of $u_x(t, x)$ form an absolutely continuous curve.

**Lemma 2.1** ([6]) *Let $T > 0$ and $v \in C^1\left([0, T), H^2(\mathbb{R})\right)$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ satisfying*

$$m(t) := \inf_{x\in\mathbb{R}}\{v_x(t, x)\} = v_x\left(t, \xi(t)\right).$$

*Furthermore, the function $m(t)$ is absolutely continuous on $(0, T)$ with*

$$\frac{dm(t)}{dt} = v_{tx}\left(t, \xi(t)\right) \quad \textit{a.e. on } (0, T).$$

In addition, the result on the precise blow-up scenario, usually expressed by wave-breaking mechanism, is also needed.

**Proposition 2.1** ([18]) *Let $(u_0\ \rho_0)$ be in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > 3/2$ and $(u,\ \rho)$ be the corresponding strong solution to (2.1) with $T$ being the maximal existence time. Then $(u,\ \rho)$ blows up in finite time if and only if*

$$\lim_{t\to T^-}\inf_{x\in\mathbb{R}} u_x(t,x) = -\infty.$$

Let's now give the following sufficient condition for the global existence of (2.1).

**Theorem 2.1** *Let $(u,\rho)$ be a strong solution of (2.1) with initial data $(u_0,\rho_0)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ for $s>3/2$, and $T>0$ be the maximal time of existence. If $\rho_0(x)>0$ for all $x\in\mathbb{R}$, then $T=+\infty$ and the solution $(u,\rho)$ is global.*

*Proof* From Proposition 2.1, we know that to obtain the global existence, we only need to control the lower bound of $u_x(t,x)$. A density argument indicates that it suffices to prove the desired results for $s\geq 3$. Hence, differentiating the first equation of (2.1) with respect to $x$ and using the identity $-\partial_x^2 p * f = f - p * f$, we have

$$\begin{cases} u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 = \frac{1}{2}\rho^2 + u^2 - p*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right), \\ \rho_t + u\rho_x = -\rho u_x. \end{cases} \tag{2.2}$$

For any fixed $x\in\mathbb{R}$, take a trajectory $q(t;x)$ as defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}q(t;x) = u(t,q(t;x)), \qquad 0<t<T, \\ \\ q(0;x) = x, \end{cases} \tag{2.3}$$

and denote

$$n(t) = u_x(t,q(t;x)), \quad \zeta(t) = \rho(t,q(t;x)).$$

Then for $t\in[0,T)$, along such a trajectory $q(t;x)$, the system (2.2) reads

$$n'(t) = -\frac{1}{2}n^2(t) + \frac{1}{2}\zeta^2(t) + f(t,q(t;x)) \tag{2.4}$$

and

$$\zeta'(t) = -n(t)\zeta(t), \tag{2.5}$$

where $'$ denotes the derivative with respect to $t$ along the trajectory and $f(t,q(t;x))$ is

$$f := u^2 - p*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)$$

evaluated along $(t,q(t;x))$.

Similarly as in [19], we can estimate $f$ in the following way

$$\begin{aligned} |f(t,q(t;x))| &\leq u^2(t,q(t;x)) + p*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)(t,q(t;x)) \\ &\leq p*\left(3u^2+\frac{3}{2}u_x^2+\frac{1}{2}\rho^2\right)(t,q(t;x)) \\ &\leq \frac{1}{2}\left\|3u^2+\frac{3}{2}u_x^2+\frac{1}{2}\rho^2\right\|_{L^1(\mathbb{R})} \leq \frac{3}{2}\|(u_0,\rho_0)\|^2_{H^1(\mathbb{R})\times L^2(\mathbb{R})} =: C_1, \end{aligned}$$

where we have used that $u^2/2\leq p*\left(u^2+u_x^2/2\right)$ (cf. [5]) and the conservation of $\mathcal{E}(u,\rho)$ (1.6).

From (2.5), we know that $\zeta(t)$ and $\zeta(0)=\rho_0(x)$ are of the same sign. Therefore, the assumption of the theorem that $\rho_0(x)$ is a positive function on the line implies that $\zeta(t)>0$ for all $t\in[0,T)$. Therefore, we can define the following Lyapunov function (cf. [7])

$$w(t) = \zeta(0)\zeta(t) + \frac{\zeta(0)}{\zeta(t)}\left(1+n^2(t)\right).$$

By the definition of $w(t)$, we deduce that $w(t) > 0$ for all $t \in [0, T)$ and

$$w(0) = \zeta^2(0) + 1 + n^2(0) = \rho_0^2(x) + 1 + u_{0,x}^2(x) \leq 1 + \|(u_{0,x}, \rho_0)\|^2_{L^\infty(\mathbb{R})} =: C_2.$$

Moreover, using (2.4) and (2.5) a short computation shows

$$\begin{aligned} w'(t) &= \zeta(0)\zeta'(t) + \frac{2\zeta(0)}{\zeta(t)} n(t)n'(t) - \frac{\zeta(0)\zeta'(t)}{\zeta^2(t)}\left(1 + n^2(t)\right) \\ &= \frac{2\zeta(0)n(t)}{\zeta(t)}\left[f(t, q(t;x)) + \frac{1}{2}\right] \leq \frac{\zeta(0)}{\zeta(t)}\left(1 + n^2(t)\right)\left(C_1 + \frac{1}{2}\right) \leq C\,w(t), \end{aligned}$$

for some constant $C$. Thus

$$w(t) \leq w(0)e^{Ct} \leq C_2 e^{Ct}, \quad t \in [0, T).$$

This in turn gives the estimate of $u_x(t, q(t; x))$ as

$$|u_x(t, q(t;x))| = |n(t)| \leq \frac{w(t)}{|\zeta(0)|} \leq \frac{C_2}{|\rho_0(x)|} e^{Ct}, \quad t \in [0, T). \tag{2.6}$$

Suppose now the maximal time of existence $T$ is finite, $T < \infty$. Applying Lemma 2.1 and the wave-breaking criterion given in Proposition 2.1, we have for each $t \in [0, T)$ a point $\xi(t) \in \mathbb{R}$ such that

$$m(t) := \inf_{x\in\mathbb{R}} [u_x(t, x)] = u_x(t, \xi(t))$$

and

$$\lim_{t\to T^-} m(t) = \lim_{t\to T^-} u_x(t, \xi(t)) \to -\infty.$$

Since $u(t, \cdot) \in H^s(\mathbb{R})$ for $s \geq 3$, it decays at infinity together with $u_x(t, \cdot)$. We know that

$$\limsup_{t\to T^-} |\xi(t)| = M < \infty.$$

On the other hand, because $q(t; \cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism of the line, for each $\xi(t)$, $t \in [0, T)$, there exists a point $x^t \in \mathbb{R}$ such that

$$q(t;\ x^t) = \xi(t).$$

So from (2.6) we have

$$\frac{C_2}{|\rho_0(x^t)|} e^{Ct} \geq |u_x(t, q(t; x^t))| = |u_x(t, \xi(t))| = |m(t)| \to \infty, \quad \text{as } t \to T^-,$$

and therefore necessarily,

$$\lim_{t\to T^-} \rho_0(x^t) = 0.$$

By assumption, $\rho_0(x) > 0$ for all $x \in \mathbb{R}$. So the above asserts that

$$\lim_{t\to T^-} |x^t| = \infty. \tag{2.7}$$

Now consider the characteristic Eq. (2.3) for $q(t;\ x^t)$,

$$\begin{cases} \frac{d}{ds} q(s;\ x^t) = u(s, q(s; x^t)), \quad 0 < s < t, \\ q(0;\ x^t) = x^t, \quad q(t;\ x^t) = \xi(t). \end{cases}$$

Thus

$$\xi(t) = x^t + \int_0^t u(\tau, q(\tau, x^t))\, d\tau,$$

from which we have for $t \in [0, T)$ that

$$|x^t| \le |\xi(t)| + \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R})}\, d\tau \le M + \int_0^t \|u(\tau)\|_{H^1(\mathbb{R})}\, d\tau$$
$$\le M + T\|(u_0, \rho_0)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} < \infty,$$

which is a contradiction to (2.7). Therefore, we must have $T = \infty$. □

*Remark 2.1* In fact, the condition in Theorem 2.1 can be replaced by $\rho_0(x) \ne 0$ and the same argument as the above works. Such a condition was also derived in [15] in the weak solution setting. The method is based on the description of weak conservative solutions in the Lagrangian coordinates. Here we provide a direct approach in the Eulerian coordinates for strong solutions.

## 3 Stability of the line peaked wave profile $(\varphi_c, 0)$

We start with the case when the problem is posed on the whole line. The stability we discuss here is the orbital stability, defined by the following.

**Definition 3.1** Let $(\phi_c, \chi_c)$ be a solitary wave of (1.5) with speed $c > 0$. Then $(\phi_c, \chi_c)$ is orbitally stable if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for any $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, with $\|(u_0, \rho_0) - (\phi_c, \chi_c)\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \le \delta$ and such that $\rho_0(x) > 0$ for all $x \in \mathbb{R}$, the corresponding global solution $(u(t), \rho(t))$ of (2.1) with initial data $(u_0, \rho_0)$ satisfies

$$\sup_{t\ge 0} \inf_{r\in\mathbb{R}} \|(u(t,\cdot), \rho(t,\cdot)) - (\phi_c(\cdot - r), \chi_c(\cdot - r))\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})} \le \epsilon.$$

For the peaked wave profile of the form $(\varphi_c(x), 0) = (c\, e^{-|x|}, 0)$ with a positive speed $c$, it is clear that $\varphi_c(x)$ exhibits the peak at $x = 0$ with

$$\max_{x\in\mathbb{R}}\{\varphi_c(x)\} = \varphi_c(0) = c, \tag{3.1}$$

and by a direct computation

$$\mathcal{E}(\varphi_c, 0) = 2\, c^2, \qquad \mathcal{F}(\varphi_c, 0) = \frac{4\, c^3}{3}. \tag{3.2}$$

Now, we consider the minimization problem

$$\mathcal{I} := \inf\left\{\mathcal{E}(u,\rho)\,\Big|\,(u,\rho) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \ \text{ with } \mathcal{F}(u,\rho) = \frac{4\, c^3}{3}\right\}. \tag{3.3}$$

In the following proposition, we prove that this constraint variational problem has, up to translation, a unique solution $(\varphi_c(x), 0) = (c\, e^{-|x|}, 0)$.

**Proposition 3.1** *The solutions of the constraint variational problem* (3.3) *are exactly all translations* $(\varphi_c(x-\xi), 0) = (c\, e^{-|x-\xi|}, 0)$, $\xi \in \mathbb{R}$, *of the peaked wave profile.*

*Proof* For $(u, \rho) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that $\mathcal{F}(u, \rho) = 4c^3/3 = \mathcal{F}(\varphi_c, 0)$, we deduce from the expression (1.7) of the functional $\mathcal{F}$ that the component $u$ should admit the positive maximal value $M = \max_{x\in\mathbb{R}}\{u(x)\} > 0$. Let $M$ be obtained at $\xi \in \mathbb{R}$ and define an $L^2(\mathbb{R})$-function $g(x)$ by

$$g(x) = \begin{cases} u(x) - u_x(x), & x < \xi, \\ u(x) + u_x(x), & x > \xi, \end{cases} \tag{3.4}$$

which satisfies

$$\int_{\mathbb{R}} g^2(x)\,dx = \int_{-\infty}^{\xi} (u-u_x)^2\,dx + \int_{\xi}^{\infty} (u+u_x)^2\,dx = \int_{\mathbb{R}} (u^2+u_x^2)\,dx - 2M^2. \quad (3.5)$$

A further calculation leads to

$$\int_{\mathbb{R}} u(x)g^2(x)\,dx = \int_{-\infty}^{\xi} u(u-u_x)^2\,dx + \int_{\xi}^{\infty} u(u+u_x)^2\,dx = \int_{\mathbb{R}} (u^3+uu_x^2)\,dx - \frac{4}{3}M^3. \quad (3.6)$$

Hence, by the relations (3.5) and (3.6), the functionals $\mathcal{E}(u,\rho)$ and $\mathcal{F}(u,\rho)$ given by (1.6) and (1.7) satisfy the following identities

$$\mathcal{E}(u,\rho) = \int_{\mathbb{R}} \left(g^2(x)+\rho^2(x)\right)\,dx + 2M^2 \quad (3.7)$$

and

$$\begin{aligned}\mathcal{F}(u,\rho) - \frac{4}{3}M^3 &= \int_{\mathbb{R}} \left(u^3(x)+u(x)u_x^2(x)+u(x)\rho^2(x)\right)\,dx - \frac{4}{3}M^3 \\ &= \int_{\mathbb{R}} u(x)\left(g^2(x)+\rho^2(x)\right)\,dx \quad (3.8)\end{aligned}$$

with $\rho(x) \in L^2(\mathbb{R})$. Since for all $x \in \mathbb{R}$, $u(x) \le M$ due to the embedding $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, we deduce from (3.7) and (3.8) that

$$\mathcal{F}(u,\rho) - \frac{4}{3}M^3 \le M\int_{\mathbb{R}} \left(g^2(x)+\rho^2(x)\right)\,dx = M\,\mathcal{E}(u,\rho) - 2M^3,$$

which implies

$$M^3 - \frac{3}{2}M\,\mathcal{E}(u,\rho) + \frac{3}{2}\mathcal{F}(u,\rho) \le 0. \quad (3.9)$$

Now, define a cubic polynomial $P$ with respect to $y \in \mathbb{R}$ for $(u,\rho) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $\mathcal{F}(u,\rho) = \mathcal{F}(\varphi_c, 0)$ by

$$P(y;u,\rho) = y^3 - \frac{3}{2}\,y\,\mathcal{E}(u,\rho) + \frac{3}{2}\mathcal{F}(u,\rho). \quad (3.10)$$

On the one hand, associated with the profile $(\varphi_c, 0)$, we obtain using (3.2) that

$$P(y;\varphi_c,0) = y^3 - \frac{3}{2}\,y\,\mathcal{E}(\varphi_c,0) + \frac{3}{2}\mathcal{F}(\varphi_c,0) = y^3 - 3c^2y + 2c^3 = (y-c)^2(y+2c). \quad (3.11)$$

On the other hand, we take $y = \max_{x\in\mathbb{R}}\{u(x)\} = M$ and derive from (3.9), (3.10) and (3.11) that

$$\begin{aligned}(M-c)^2(M+2c) &= P(M;\varphi_c,0) \\ &= P(M;u,\rho) + \frac{3}{2}M\left(\mathcal{E}(u,\rho) - \mathcal{E}(\varphi_c,0)\right) - \frac{3}{2}\left(\mathcal{F}(u,\rho) - \mathcal{F}(\varphi_c,0)\right) \\ &\le \frac{3}{2}M\left(\mathcal{E}(u,\rho) - \mathcal{E}(\varphi_c,0)\right) - \frac{3}{2}\left(\mathcal{F}(u,\rho) - \mathcal{F}(\varphi_c,0)\right). \quad (3.12)\end{aligned}$$

From the fact that $M > 0$, it then follows from (3.12) and the constraint $\mathcal{F}(u,\rho) = \mathcal{F}(\varphi_c,0)$ that

$$(M-c)^2 \le \frac{3}{4c}M\,\left(\mathcal{E}(u,\rho) - \mathcal{E}(\varphi_c,0)\right),$$

which implies $\mathcal{E}(u,\rho) \geq \mathcal{E}(\varphi_c, 0) = 2\,c^2$ and

$$\mathcal{E}(u,\rho) = 2\,c^2 \qquad \Rightarrow \qquad M = c. \tag{3.13}$$

Hence, we claim that $\mathcal{I} = 2\,c^2$ and $(\varphi_c, 0)$ is a minimizer of (3.3) in terms of the properties (3.1) and (3.2).

Assume that $(u,\rho)$ is any other solution of (3.3). The relations (3.7) and (3.13) lead to

$$\int_{\mathbb{R}} \left(g^2(x) + \rho^2(x)\right)\,dx = \mathcal{E}(u,\rho) - 2\,c^2 = 0,$$

which together with the definition (3.4) of the function $g(x)$ implies $u(x) = ce^{-|x-\xi|}$, $\xi \in \mathbb{R}$, and $\rho(x) = 0$ a.e. in $\mathbb{R}$. The proof of this proposition is complete. □

It turns out from Proposition 3.1 that the configuration of the peaked wave profile $(\varphi_c, 0) = (c\,e^{-|x|}, 0)$ is a ground state. In other words, it is a state of the lowest energy with an appropriate constraint on the functional $\mathcal{F}$. Generally, the standard physical principle reveals the stability of the ground states. In the following theorem, we present a precise reformulation of the Definition 3.1 in terms of the peaked wave profile $(\varphi_c, 0)$ and give a direct proof of the orbital stability for these profiles in the similar spirit of the procedure introduced by Constantin-Strauss [10].

**Theorem 3.1** *Let $(\varphi_c, 0) = (c\,e^{-|x|}, 0)$ be the profile of the peaked solitary wave of system (1.5) with a positive speed c. Then $(\varphi_c, 0)$ is orbitally stable in the following sense. Suppose that $(u_0, \rho_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ for some $s > 3/2$, $\rho_0(x) > 0$ for all $x \in \mathbb{R}$, and*

$$\|u_0 - \varphi_c\|_{H^1(\mathbb{R})} + \|\rho_0\|_{L^2(\mathbb{R})} < \delta, \qquad 0 < \delta \ll 1. \tag{3.14}$$

*Then the corresponding global solution $(u(t,x), \rho(t,x))$ of the Cauchy problem for the 2CH system (2.1) on the line with initial data $(u_0, \rho_0)$ satisfies*

$$\sup_{t>0} \left(\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})}\right) < C\,\delta^{1/4},$$

*where $\xi(t) \in \mathbb{R}$ is the maximum point of the component $u(t,\cdot)$ and the constant $C > 0$ depends only on the speed c.*

*Proof* First of all, note that the conserved functionals $\mathcal{E}(u,\rho)$ and $\mathcal{F}(u,\rho)$ are continuous in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Hence, in terms of (3.14), we have for any $t \geq 0$

$$|\mathcal{F}(u(t), \rho(t)) - \mathcal{F}(\varphi_c, 0)| = |\mathcal{F}(u_0, \rho_0) - \mathcal{F}(\varphi_c, 0)| \leq C_1\delta, \tag{3.15}$$

where the constant $C_1$ depends on the speed $c > 0$. Taking $\delta$ small enough together with $\mathcal{F}(\varphi_c, 0) = 4c^3/3 > 0$, it follows that for the component $u(t,\cdot) \in C([0,\infty), H^s(\mathbb{R}))$ with $s > 3/2$, its spatial maximal value $\max_{x\in\mathbb{R}}\{u(t,x)\}$ denoted by $M(t)$ is positive and there exists a point $\xi(t)$ such that $u(t,\xi(t)) = M(t)$ for $t \geq 0$. We can thus deduce the following time-evolution version analogous to (3.12) for $M(t)$

$$\begin{aligned}
&(M(t) - c)^2\,(M(t) + 2c)\\
&\qquad \leq \frac{3}{2}M(t)\,(\mathcal{E}(u(t), \rho(t)) - \mathcal{E}(\varphi_c, 0)) - \frac{3}{2}\,(\mathcal{F}(u(t), \rho(t)) - \mathcal{F}(\varphi_c, 0))\\
&\qquad = \frac{3}{2}M(t)\,(\mathcal{E}(u_0, \rho_0) - \mathcal{E}(\varphi_c, 0)) - \frac{3}{2}\,(\mathcal{F}(u_0, \rho_0) - \mathcal{F}(\varphi_c, 0)), 
\end{aligned} \tag{3.16}$$

where we have used the fact that $\mathcal{E}(u(t), \rho(t))$ and $\mathcal{F}(u(t), \rho(t))$ are both conserved along the flow of system (1.5).

Furthermore, $M(t)$ satisfies the following uniform estimate

$$
\begin{aligned}
0 < M(t) &\leq \frac{\sqrt{2}}{2}\|u(t,\cdot)\|_{H^1(\mathbb{R})} \leq \frac{\sqrt{2}}{2}\sqrt{\mathcal{E}(u(t),\rho(t))} = \frac{\sqrt{2}}{2}\sqrt{\mathcal{E}(u_0,\rho_0)} \\
&\leq \frac{\sqrt{2}}{2}\sqrt{|\mathcal{E}(u_0,\rho_0)-\mathcal{E}(\varphi_c,0)|+2c^2} \leq \frac{\sqrt{2}}{2}\sqrt{C_2\delta+2c^2} \qquad (3.17)
\end{aligned}
$$

with a constant $C_2$ depending on $c$. Plugging (3.15) and (3.17) into (3.16) leads to

$$
\begin{aligned}
2c\,(M(t)-c)^2 &\leq \frac{3}{2}M(t)\,|\mathcal{E}(u_0,\rho_0)-\mathcal{E}(\varphi_c,0)| + \frac{3}{2}\,|\mathcal{F}(u_0,\rho_0)-\mathcal{F}(\varphi_c,0)| \\
&\leq \frac{3\sqrt{2}}{4}C_2\sqrt{C_2\delta+2c}\,\delta + \frac{3}{2}C_1\delta,
\end{aligned}
$$

which implies that for a constant $C_3 = C_3(c)$, there holds

$$
|M(t)-c| \leq C_3\delta^{1/2}. \qquad (3.18)
$$

On the other hand, a direct calculation shows that the corresponding global strong solution $(u(t),\rho(t))$ satisfies for $t \geq 0$,

$$
\begin{aligned}
&\|u(t,\cdot)-\varphi_c(\cdot-\xi(t))\|^2_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad = \int_{\mathbb{R}} \left(u^2(t,x)+u_x^2(t,x)+\rho^2(t,x)\right)\,dx - \int_{\mathbb{R}} \left(\varphi_c^2(x)+\varphi_{cx}^2(x)\right)\,dx \\
&\qquad - 4c\,\left(M(t)-c\right) \\
&\quad = \mathcal{E}(u(t),\rho(t)) - \mathcal{E}(\varphi_c,0) - 4c\,\left(M(t)-c\right) \\
&\quad \leq |\mathcal{E}(u_0,\rho_0)-\mathcal{E}(\varphi_c,0)| + 4c\,|M(t)-c|\,. \qquad (3.19)
\end{aligned}
$$

We conclude from (3.18) and (3.19) and the continuity of functional $\mathcal{E}(\cdot,\cdot)$ in $H^1(\mathbb{R})\times L^2(\mathbb{R})$ that there exists a positive constant $C$ depending only on the speed $c$ such that for all $t \in [0,\infty)$

$$
\|u(t,\cdot)-\varphi_c(\cdot-\xi(t))\|_{H^1(\mathbb{R})} + \|\rho(t,\cdot)\|_{L^2(\mathbb{R})} \leq C\,\delta^{1/4}
$$

and finish the proof of the theorem. □

## 4 Stability of the periodic peaked wave profile $(\psi_c,\ 0)$

In this section, we are concerned with the orbital stability for the periodic peaked wave profile of system (1.5). Such a periodic profile takes the form of $(\psi_c(x),0)$ with

$$
\psi_c(x) = c\,\psi(x) = c\,\frac{\cosh\left(\frac{1}{2}-x\right)}{\cosh\frac{1}{2}}, \quad c>0, \qquad (4.1)
$$

where $\psi_c(x)$ is defined for $x \in [0,1)$ and extends periodically to the whole real line. Henceforth, we denote $\mathbb{S}^1$ to be the interval $[0,1)$ and regard functions on $\mathbb{S}^1$ as periodic functions on the real line of period one.

Observe that $\psi_c(x)$ is continuous on $\mathbb{S}^1$, exhibits its peak at $x=0$ and denote

$$
M_{\psi_c} = \max_{x\in\mathbb{S}^1}\{\psi_c(x)\} = \psi_c(0) = c, \quad L_{\psi_c} = \min_{x\in\mathbb{S}^1}\{\psi_c(x)\} = \psi_c\left(\frac{1}{2}\right) = \frac{c}{\cosh\frac{1}{2}}.
$$

In addition, the conservation law $\mathcal{H}$ given by (1.3) satisfies

$$\mathcal{H}(\psi_c(x), 0) = c \int_{\mathbb{S}^1} \psi(x)\, dx = 2c \tanh \frac{1}{2} > 0.$$

Furthermore, although $\psi(x)$ is smooth on $(0, 1)$, it satisfies the relation $\psi_{xx} = \psi - 2\tanh\,(1/2)\,\delta(x)$ on $\mathbb{S}^1$ with $\delta(x)$ being the usual Dirac delta distribution. Using this identity, we obtain after a direct computation the following Hamiltonian conservation laws for $(\psi_c, 0)$

$$\mathcal{E}(\psi_c, 0) = c^2 \int_{\mathbb{S}^1} \left(\psi^2 + \psi_x^2\right)\, dx = 2c^2 \tanh \frac{1}{2}$$

and

$$\mathcal{F}(\psi_c, 0) = c^3 \int_{\mathbb{S}^1} \left(\psi^3 + \psi\psi_x^2\right)\, dx = 2c^3 \tanh \frac{1}{2} \cdot \left(1 - \frac{1}{3} \tanh^2 \frac{1}{2}\right),$$

which are both positive since $c > 0$.

Here, we define the admissible set

$$\mathcal{A} = \left\{(u, \rho) \in H^1(\mathbb{S}^1) \times L^2(\mathbb{S}^1) \mid \mathcal{H}(u, \rho) = \mathcal{H}(\psi_c, 0) \text{ and } \mathcal{F}(u, \rho) = \mathcal{F}(\psi_c, 0)\right\} \quad (4.2)$$

and take into account the associated variational problem

$$\mathcal{J} := \inf\{\mathcal{E}(u, \rho) \mid (u, \rho) \in \mathcal{A}\}. \quad (4.3)$$

We aim to prove the following proposition which identifies the profile of the periodic peaked waves $(\psi_c, 0)$ as minima of the constrained energy.

**Proposition 4.1** *The solutions of the variational problem (4.3) are exactly all translation $(\psi_c(x - \xi), 0)$, $\xi \in \mathbb{R}$, of the profile of the periodic peaked wave.*

To prove this proposition, the following two lemmas are needed.

**Lemma 4.1** *For any $\rho \in L^2(\mathbb{S}^1)$ and each positive $u \in H^1(\mathbb{S}^1)$, define a function $G(M, L;\ u, \rho)$ on $\Gamma = \{(M, L) \in \mathbb{R}^2 \mid M \geq L > 0\}$ by*

$$\begin{aligned} G(M, L;\ u, \rho) = {} & L^2\mathcal{H}(u, \rho) + M\mathcal{E}(u, \rho) - \mathcal{F}(u, \rho) - \frac{2}{3}\left(M^2 + 2L^2\right)\sqrt{M^2 - L^2} \\ & - ML^2 + 2ML^2 \ln\left(\frac{M + \sqrt{M^2 - L^2}}{L}\right). \end{aligned} \quad (4.4)$$

*Then $M_u = \max_{x\in\mathbb{S}^1}\{u(x)\}$ and $L_u = \min_{x\in\mathbb{S}^1}\{u(x)\}$ satisfy*

$$G\left(M_u, L_u;\ u, \rho\right) \geq 0. \quad (4.5)$$

*Proof* For any positive function $u \in H^1(\mathbb{S}^1)$, it follows that $(M_u, L_u) \in \Gamma$. We take $\xi$ and $\eta$ in the same period such that $u(\xi) = M_u$ and $u(\eta) = L_u$. Making use of the periodic function $g(x)$ corresponding to such $u$, defined by

$$g(x) = \begin{cases} u_x(x) + \sqrt{u^2(x) - L_u^2}, & \xi < x \leq \eta, \\ u_x(x) - \sqrt{u^2(x) - L_u^2}, & \eta < x < \xi + 1 \end{cases}$$

and extended periodically to the real line, one obtains after the similar computation as in [22] the following identities

$$\int_{\mathbb{S}^1} g^2(x)\,dx = \int_{\mathbb{S}^1} \left(u^2(x)+u_x^2(x)\right)\,dx - L_u^2 - 2M_u\sqrt{M_u^2-L_u^2} + 2L_u^2\ln\left(\frac{M_u+\sqrt{M_u^2-L_u^2}}{L_u}\right) \tag{4.6}$$

and

$$\int_{\mathbb{S}^1} u(x)g^2(x)\,dx = \int_{\mathbb{S}^1} \left(u^3(x)+u(x)u_x^2(x)\right)\,dx - L_u^2\int_{\mathbb{S}^1} u(x)\,dx - \frac{4}{3}\left(M_u^2-L_u^2\right)^{3/2}. \tag{4.7}$$

Plugging the function $\rho(x)\in L^2(\mathbb{S}^1)$ into the above two relations (4.6) and (4.7), we have

$$\int_{\mathbb{S}^1} \left(g^2(x)+\rho^2(x)\right)\,dx = \mathcal{E}(u,\rho) - L_u^2 - 2M_u\sqrt{M_u^2-L_u^2} + 2L_u^2\ln\left(\frac{M_u+\sqrt{M_u^2-L_u^2}}{L_u}\right)$$

and

$$\int_{\mathbb{S}^1} u(x)\left(g^2(x)+\rho^2(x)\right)\,dx = \mathcal{F}(u,\rho) - L_u^2\mathcal{H}(u,\rho) - \frac{4}{3}\left(M_u^2-L_u^2\right)^{3/2}.$$

Therefore,

$$\begin{aligned}\mathcal{F}(u,\rho) &\le M_u\int_{\mathbb{S}^1}\left(g^2(x)+\rho^2(x)\right)\,dx + L_u^2\mathcal{H}(u,\rho) + \frac{4}{3}\left(M_u^2-L_u^2\right)^{3/2}\\ &= L_u^2\,\mathcal{H}(u,\rho) + M_u\,\mathcal{E}(u,\rho)\\ &\quad - \frac{2}{3}\left(M_u^2+2L_u^2\right)\sqrt{M_u^2-L_u^2} - M_uL_u^2 + 2M_uL_u^2\ln\left(\frac{M_u+\sqrt{M_u^2-L_u^2}}{L_u}\right),\end{aligned}$$

which proves (4.5) and finishes the proof of this lemma. □

Note that the function $g(x)$ vanishes when $u(x)$ is replaced by the periodic peaked function $\psi_c(x)$. This argument justifies the definition of the auxilliary function $g$. In the next lemma, some required properties of the function $G(M, L;\ \psi_c, 0)$ are presented. The proofs are similar as in [22,23] and hence we omit them.

**Lemma 4.2** *Associated with the profile of the periodic peaked waves* $(\psi_c, 0)$*, we define the set* $L$ *as in Lemma 4.1 and have*

(i)
$$G(M_{\psi_c}, L_{\psi_c};\ \psi_c, 0) = 0,$$
$$\frac{\partial}{\partial M}G(M,L;\ \psi_c,0)\Big|_{(M_{\psi_c},L_{\psi_c})} = \frac{\partial}{\partial L}G(M,L;\ \psi_c,0)\Big|_{(M_{\psi_c},L_{\psi_c})} = 0,$$
$$\frac{\partial^2}{\partial M^2}G(M,L;\ \psi_c,0)\Big|_{(M_{\psi_c},L_{\psi_c})} = \frac{\partial^2}{\partial L^2}G(M,L;\ \psi_c,0)\Big|_{(M_{\psi_c},L_{\psi_c})} = -4c\tanh\frac{1}{2},$$
$$\frac{\partial^2}{\partial M\partial L}G(M,L;\ \psi_c,0)\Big|_{(M_{\psi_c},L_{\psi_c})} = 0;$$

(ii) *The function* $G(M, L;\ \psi_c, 0)$ *admits only one critical point* $(M_{\psi_c}, L_{\psi_c})$ *in* $\Gamma$*;*

(iii) *For any* $(M, L) \in \Gamma$ *and* $(M, L) \neq (M_{\psi_c}, L_{\psi_c})$, $G(M, L;\ \psi_c, 0) < 0$*; Near the boundary of* $\Gamma$, $G(M, L;\ \psi_c, 0)$ *stays bounded away from zero; For any large* $(M, L) \in \Gamma$,

$$G(M, L;\ \psi_c, 0) < -a\|(M, L)\|^3_{\mathbb{R}^2}$$

*with a constant* $a > 0$ *depending on* $c$.

We are now in a position to prove Proposition 4.1.

*Proof of Proposition 4.1* For any $(u, \rho) \in \mathcal{A}$ with $\mathcal{A}$ defined in (4.2), notice that

$$\mathcal{F}(u, \rho) = \mathcal{F}(\psi_c, 0) = 2c^3 \tanh \frac{1}{2} \cdot \left(1 - \frac{1}{3} \tanh^2 \frac{1}{2}\right) > 0,$$

then the maximal value $M_u = \max_{x\in\mathbb{S}^1}\{u(x)\}$ must be positive. In addition, the constraint

$$\mathcal{H}(u, \rho) = \int_{\mathbb{S}^1} u\, dx = \mathcal{H}(\psi_c, 0) = 2c \tanh \frac{1}{2}$$

implies

$$M_u \geq 2c \tanh \frac{1}{2}. \tag{4.8}$$

We first claim that if $\mathcal{E}(u, \rho) \leq \mathcal{E}(\psi_c, 0)$, then the function $u(x)$ is strictly positive in $\mathbb{S}^1$. In fact, using the relation $\psi_{xx} = \psi - 2\tanh\ (1/2)\ \delta(x)$, we deduce by a direct calculation that for all $\xi \in \mathbb{R}$,

$$\|u(\cdot) - \psi_c(\cdot - \xi)\|^2_{H^1(\mathbb{S}^1)} = \int_{\mathbb{S}^1} \left(u^2 + u_x^2\right)\, dx + c^2 \int_{\mathbb{S}^1} \left(\psi^2 + \psi_x^2\right)\, dx - 4c \tanh \frac{1}{2} \cdot u(\xi). \tag{4.9}$$

Taking $\xi \in \mathbb{R}$ with $u(\xi) = M_u$, we obtain together with $\rho \in L^2(\mathbb{S}^1)$ that

$$\|u(\cdot) - \psi_c(\cdot - \xi)\|^2_{H^1(\mathbb{S}^1)} + \|\rho\|^2_{L^2(\mathbb{S}^1)} + 4c \tanh \frac{1}{2} \cdot (M_u - c) = \mathcal{E}(u, \rho) - \mathcal{E}(\psi_c, 0). \tag{4.10}$$

Since $\mathcal{E}(u, \rho) \leq \mathcal{E}(\psi_c, 0)$ by assumption, it follows from (4.8) and (4.10) that

$$2c \tanh \frac{1}{2} \leq M_u \leq c. \tag{4.11}$$

On the other hand, using the estimate (cf. Lemma 2.6 in [22])

$$|u(x)| \leq \sqrt{\frac{\cosh\ (1/2)}{2 \sinh\ (1/2)}}\, \|u\|_{H^1(\mathbb{S}^1)}, \quad x \in \mathbb{S}^1,$$

we deduce that

$$\begin{aligned} (M_u - u(x))^2 &\leq \frac{1}{2 \tanh \frac{1}{2}} \|M_u - u\|^2_{H^1(\mathbb{S}^1)} \\ &= \frac{1}{2 \tanh \frac{1}{2}} \left( \int_{\mathbb{S}^1} \left(u^2 + u_x^2\right)\, dx - 2M_u \int_{\mathbb{S}^1} u\, dx + M_u^2 \right) \\ &< \frac{1}{2 \tanh \frac{1}{2}} \left(\mathcal{E}(u, \rho) - 2M_u \mathcal{H}(u, \rho) + M_u^2\right). \end{aligned} \tag{4.12}$$

Hence, combining (4.12) with (4.11), we know that

$$\begin{aligned}(M_u - L_u)^2 &\leq \frac{1}{2\tanh\frac{1}{2}}\left(\mathcal{E}(\psi_c, 0) - 2M_u\mathcal{H}(\psi_c, 0) + M_u^2\right) \\ &\leq \frac{1}{2\tanh\frac{1}{2}}\left(2c^2\tanh\frac{1}{2} - 8c^2\tanh^2\frac{1}{2} + c^2\right) \\ &= c^2\left(1 + \frac{1}{2\tanh\frac{1}{2}} - 4\tanh\frac{1}{2}\right)\end{aligned}$$

for $(u, \rho) \in \mathcal{A}$. It follows from the estimate (4.11) that

$$\begin{aligned}L_u &\geq M_u - c\sqrt{1 + \frac{1}{2\tanh\frac{1}{2}} - 4\tanh\frac{1}{2}} \\ &\geq c\left(2\tanh\frac{1}{2} - \sqrt{1 + \frac{1}{2\tanh\frac{1}{2}} - 4\tanh\frac{1}{2}}\right) > 0.\end{aligned}$$

So for $(u, \rho) \in \mathcal{A}$ with $\mathcal{E}(u, \rho) \leq \mathcal{E}(\psi_c, 0)$, the component $u(x)$ is a strictly positive periodic function on $\mathbb{S}^1$.

Subsequently, we show that $\mathcal{E}(u, \rho)$ cannot be strictly smaller than $\mathcal{E}(\psi_c, 0)$ for all $(u, \rho) \in \mathcal{A}$. Otherwise, assume $\mathcal{E}(u, \rho) < \mathcal{E}(\psi_c, 0)$ for some $(u, \rho) \in \mathcal{A}$. Since in this case the function $u$ is positive, we infer from the structure of $G$ given by (4.4) that the condition $\mathcal{E}(u, \rho) < \mathcal{E}(\psi_c, 0)$ implies that $G(M, L;\ u, \rho) < G(M, L;\ \psi_c, 0)$ for $(M, L) \in \Gamma$. Hence, we deduce from Lemma 4.2 that

$$G(M, L;\ u, \rho) < 0, \quad \text{for all } (M, L) \in \Gamma,$$

which contradicts the fact that $G(M_u, L_u;\ u, \rho) \geq 0$ for $(M_u, L_u) \in \Gamma$ stated in Lemma 4.1. Therefore, it holds that $\mathcal{E}(u, \rho) \geq \mathcal{E}(\psi_c, 0)$ for $(u, \rho) \in \mathcal{A}$, which implies that the periodic peaked wave profile $(\psi_c, 0)$ is in fact one solution of the variational problem (4.3).

Finally, we conclude that for any $(u, \rho) \in \mathcal{A}$ with $\mathcal{E}(u, \rho) = \mathcal{E}(\psi_c, 0)$, $(u, \rho)$ is equal to a translation of the profile of periodic peaked wave profile $(\psi_c, 0)$. Indeed, since $\mathcal{E}(u, \rho) = \mathcal{E}(\psi_c, 0)$, $\mathcal{H}(u, \rho) = \mathcal{H}(\psi_c, 0)$ and $\mathcal{F}(u, \rho) = \mathcal{F}(\psi_c, 0)$, we have that the component $u$ is positive in $\mathbb{S}^1$ and $G(M, L;\ u, \rho) = G(M, L;\ \psi_c, 0)$ for all $(M, L) \in \Gamma$. Furthermore, from the statement in Lemma 4.1 that $G(M_u, L_u;\ u, \rho) \geq 0$ and the properties given in Lemma 4.2 (ii.) that $G(M, L;\ \psi_c, 0)$ admits $\left(M_{\psi_c}, L_{\psi_c}\right)$ as its unique critical point, we find that $M_u$ must satisfy $M_u = M_{\psi_c} = c$. We conclude from (4.10) that for $\xi \in \mathbb{R}$ being such that $u(\xi) = M_u$,

$$\|u(\cdot) - \psi_c(\cdot - \xi)\|^2_{H^1(\mathbb{S}^1)} + \|\rho\|^2_{L^2(\mathbb{S}^1)} = 0.$$

Therefore $u(x) = \psi_c(x - \xi)$, $x \in \mathbb{S}^1$ and $\rho(x) = 0$, a.e. $x \in \mathbb{S}^1$. Consequently, we prove this proposition. □

Now, we present the precise description for the orbital stability of the periodic peaked wave profile $(\psi_c, 0)$ and prove the following theorem.

**Theorem 4.1** *Let $(\psi_c, 0)$ be the profile of the periodic peaked solution defined by (4.1) with a positive speed c. Then $(\psi_c, 0)$ is orbitally stable in the following sense. Suppose that $(u_0(x), \rho_0(x)) \in H^s(\mathbb{S}^1) \times H^{s-1}(\mathbb{S}^1)$ for some $s > 3/2$ and $\rho_0(x) > 0$ for all $x \in \mathbb{S}^1$. Then for every $\epsilon > 0$, there exists $\delta > 0$, such that the initial condition*

$$\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} + \|\rho_0\|_{L^2(\mathbb{S}^1)} < \delta$$

*implies that the corresponding global periodic solution* $(u(t,x), \rho(t,x))$ *of the Cauchy problem for the 2CH system* (1.5) *with periodic initial data* $(u_0, \rho_0)$ *satisfies*

$$\sup_{t>0} \left( \|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{S}^1)} + \|\rho(t,\cdot)\|_{L^2(\mathbb{S}^1)} \right) < \epsilon,$$

*where* $\xi(t) \in \mathbb{R}$ *is any spatial maximal point of the component* $u(t,x)$.

*Remark 4.1* Note that in the periodic case the two conditions $\rho_0(x) > 0$ for all $x \in \mathbb{S}^1$ and $\inf_{x\in\mathbb{S}^1} \rho_0(x) > 0$ are equivalent, and thus the global existence of the periodic strong solutions can be inferred from (2.6).

*Proof of Theorem.4.1* Since the functionals $\mathcal{H}(u(t), \rho(t))$, $\mathcal{E}(u(t), \rho(t))$ and $\mathcal{F}(u(t), \rho(t))$ are all conserved along the flow of (1.5) and continuous in $(u, \rho) \in H^1(\mathbb{S}^1) \times L^2(\mathbb{S}^1)$, we deduce that there exists a positive constant $A$ depending only on the positive speed $c$ such that

$$\begin{aligned} &\max\{|\mathcal{H}(u(t), \rho(t)) - \mathcal{H}(\psi_c, 0)|, \ |\mathcal{E}(u(t), \rho(t)) - \mathcal{E}(\psi_c, 0)|, \\ &|\mathcal{F}(u(t), \rho(t)) - \mathcal{F}(\psi_c, 0)|\} < A\,\delta, \qquad \forall t \geq 0, \end{aligned} \tag{4.13}$$

whenever

$$\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} + \|\rho_0\|_{L^2(\mathbb{S}^1)} < \delta.$$

In this case, we claim that the component $u(t,x)$ is a strictly positive function in $\mathbb{S}^1$ for all $t \geq 0$, provided $\delta$ is small enough. To prove this claim, we use the similar approach as in the proof of Proposition 4.1 and denote $M_u(t) = \max_{x\in\mathbb{S}^1}\{u(t,x)\}$ and $L_u(t) = \min_{x\in\mathbb{S}^1}\{u(t,x)\}$. On the one hand, since

$$|\mathcal{F}(u(t), \rho(t)) - \mathcal{F}(\psi_c, 0)| < A\,\delta,$$

if $\delta$ is small enough, then $\mathcal{F}(u(t), \rho(t)) > 0$ for all $t \geq 0$. We thus deduce from the structure of the functional $\mathcal{F}$ that $M_u(t) > 0$, $t \geq 0$. In addition, based on

$$\mathcal{H}(u(t), \rho(t)) = \int_{\mathbb{S}^1} u(t,x)\,dx > \mathcal{H}(\psi_c, 0) - A\,\delta = 2c \tanh\frac{1}{2} - A\,\delta,$$

we obtain for small $\delta$ that

$$M_u(t) > 2c \tanh\frac{1}{2} - A\,\delta > 0. \tag{4.14}$$

On the other hand, we exploit the time evolution version of relation (4.9), that is

$$\begin{aligned} &\|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|^2_{H^1(\mathbb{S}^1)} \\ &\quad = \int_{\mathbb{S}^1} \left(u^2(t,x) + u_x^2(t,x)\right) dx + c^2 \int_{\mathbb{S}^1} \left(\psi^2 + \psi_x^2\right) dx - 4c \tanh\frac{1}{2} \cdot u(t, \xi(t)), \end{aligned}$$

to deduce for $\xi(t)$ with $u(t, \xi(t)) = M_u(t)$ that

$$\begin{aligned} &\|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|^2_{H^1(\mathbb{S}^1)} + \|\rho(t,\cdot)\|^2_{L^2(\mathbb{S}^1)} + 4c \tanh\frac{1}{2} \cdot (M_u(t) - c) \\ &\quad = \mathcal{E}(u(t), \rho(t)) - \mathcal{E}(\psi_c, 0). \end{aligned} \tag{4.15}$$

Since

$$|\mathcal{E}(u(t), \rho(t)) - \mathcal{E}(\psi_c, 0)| < A\,\delta,$$

it follows from (4.15) that

$$M_u(t) \le c + \frac{A}{4c\tanh\frac{1}{2}}\,\delta, \qquad t \ge 0. \tag{4.16}$$

Furthermore, using the time evolution version of inequality (4.12), which takes the form

$$(M_u(t) - u(t,x))^2 \le \frac{1}{2\tanh\frac{1}{2}}\left(\mathcal{E}(u(t),\rho(t)) - 2M_u(t)\mathcal{H}(u(t),\rho(t)) + M_u^2(t)\right),$$

we derive from the estimates (4.14) and (4.16) the following inequality with $x$ chosen such that $u(t,x) = L_u(t)$

$$\begin{aligned}
&(M_u(t) - L_u(t))^2 \\
&\quad \le \frac{1}{2\tanh\frac{1}{2}}\left(\mathcal{E}(\psi_c,0) + A\,\delta + 2\left(-2c\tanh\frac{1}{2} - A\,\delta\right)(\mathcal{H}(\psi_c,0) + A\,\delta)\right. \\
&\qquad \left. + \left(c + \frac{A}{4c\tanh\frac{1}{2}}\,\delta\right)^2\right) \\
&\quad = \frac{1}{2\tanh\frac{1}{2}}\left(c^2 + 2c^2\tanh\frac{1}{2} - 8c^2\tanh^2\frac{1}{2} + \left(1 + 8c\tanh\frac{1}{2} + \frac{1}{2\tanh\frac{1}{2}}\right)A\delta\right. \\
&\qquad \left. + \left(-2 + \frac{1}{16\,c^2\tanh^2\frac{1}{2}}\right)A^2\delta^2\right) \\
&\quad = c^2\left(1 + \frac{1}{2\tanh\frac{1}{2}} - 4\tanh\frac{1}{2}\right) + \mathcal{O}(\delta).
\end{aligned} \tag{4.17}$$

Hence, combining (4.14) with (4.17), we deduce that for any $t \ge 0$

$$\begin{aligned}
L_u(t) &\ge M_u(t) - c\sqrt{1 + \frac{1}{2\tanh\frac{1}{2}} - 4\tanh\frac{1}{2}} + \mathcal{O}(\delta^{1/2}) \\
&\ge c\left(2\tanh\frac{1}{2} - \sqrt{1 + \frac{1}{2\tanh\frac{1}{2}} - 4\tanh\frac{1}{2}}\right) + \mathcal{O}(\delta^{1/2}) > \bar{A} > 0,
\end{aligned}$$

provided $\delta$ is small enough, where the positive constant $\bar{A}$ depends on $\delta$ and the speed $c$. Therefore, we prove that for the global solution $(u(t),\rho(t))$, if the initial data $(u,\rho)$ satisfies

$$\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} + \|\rho_0\|_{L^2(\mathbb{S}^1)} < \delta$$

with $\delta$ small enough, then the component $u(t,x)$ is bounded below uniformly by a positive constant.

Subsequently, in terms of the solution $(u(t),\rho(t))$, we obtain for the corresponding function $G(M,L;\,u(t),\rho(t))$, after a direct computation together with (4.13), that

$$\begin{aligned}
G(M,L;\,u(t),\rho(t)) &= G(M,L;\,\psi_c,0) + M\,(\mathcal{E}(u(t),\rho(t)) - \mathcal{E}(\psi_c,0)) \\
&\quad + L^2\,(\mathcal{H}(u(t),\rho(t)) - \mathcal{H}(\psi_c,0)) - (\mathcal{F}(u(t),\rho(t)) - \mathcal{F}(\psi_c,0)) \\
&= G(M,L;\,\psi_c,0) + (M + L^2 - 1)\mathcal{O}(\delta).
\end{aligned} \tag{4.18}$$

Hence, $G(M,L;\,u(t),\rho(t))$ is a small perturbation of $G(M,L;\,\psi_c,0)$ in $\Gamma$. We argue that such perturbation could be arbitrarily small on any bounded subset of $\Gamma$ provided $\delta$ is small enough. While for large $(M, L) \in \Gamma$, since in this case, $G(M, L;\ \psi_c, 0) < -a\|(M, L)\|^3_{\mathbb{R}^2}$ as claimed in Lemma 4.2 and the perturbation in (4.18) is $\mathcal{O}(\|(M, L)\|^2_{\mathbb{R}^2})$, the function $G(M, L;\ u(t), \rho(t))$ is definitely negative. Therefore, the point $(M_u(t), L_u(t)) \in \Gamma$ with which $G(M_u(t), L_u(t);\ u(t), \rho(t)) \geq 0$ should lie in an arbitrarily small neighborhood of $\left(M_{\psi_c}, L_{\psi_c}\right) = (c,\ c/\cosh(1/2))$ and satisfies the following estimate

$$|M_u(t) - c| < \delta, \qquad \forall\, t \geq 0.$$

Now, given $\epsilon > 0$, the preceding $\delta$ could be chosen further smaller such that

$$|M_u(t) - c| < \frac{\epsilon^2}{32c \tanh \frac{1}{2}} \quad \text{and} \quad |\mathcal{E}(u_0, \rho_0) - \mathcal{E}(\psi_c, 0)| < \frac{\epsilon^2}{8}.$$

We then conclude from (4.15) that

$$\begin{aligned}
\|u(t, \cdot) - &\psi_c(\cdot - \xi(t))\|^2_{H^1(\mathbb{S}^1)} + \|\rho(t, \cdot)\|^2_{L^2(\mathbb{S}^1)} \\
&= \mathcal{E}(u(t), \rho(t)) - \mathcal{E}(\psi_c, 0) - 4c \tanh \frac{1}{2} \cdot (M_u(t) - c) \\
&\leq |\mathcal{E}(u_0, \rho_0) - \mathcal{E}(\psi_c, 0)| + 4c \tanh \frac{1}{2} \cdot |M_u(t) - c| \\
&< \frac{\epsilon^2}{8} + \frac{\epsilon^2}{8} = \frac{\epsilon^2}{4},
\end{aligned}$$

where $\xi(t)$ is chosen such that $u(t, \xi(t)) = M_u(t)$. Therefore, the theorem is proved. □

## 5 Wave breaking

The goal of this section is to complete the analysis of the well-posedness of the 2CH system (1.5). To this end, we investigate the precise blow-up scenario. As is pointed out in Sect. 2, the profile of $\rho_0$ strongly affects the existence time. In fact the solutions can develop finite-time singularities when $\rho_0$ touches zero (see [7, 15, 18]). Such a formation of singularities can be understood as being induced by certain local structure of the data, which is referred to be the local-in-space type blow-up. See [20] for the case when the system is posed on the whole real line. Here we would like to further confirm that this phenomenon also happens in the periodic setting.

We consider the Cauchy problem for the periodic 2CH system written as follows

$$\begin{cases} u_t + uu_x + \partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) = 0, \\ \rho_t + (\rho u)_x = 0, \qquad t \in (0, T), \quad x \in \mathbb{S}^1, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), \qquad x \in \mathbb{S}^1, \end{cases} \tag{5.1}$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} = [0, 1)$ is the unit circle as before. The function $p(x)$ in (5.1) is the periodic kernel of the convolution operator $\left(1 - \partial_x^2\right)^{-1}$, which as a continuous 1-periodic function is given by

$$p(x) = \frac{\cosh\left(x - [x] - \frac{1}{2}\right)}{2 \sinh \frac{1}{2}} \tag{5.2}$$

with $[\cdot]$ denoting the integer part.

For a strong solution $(u, \rho)$ to (5.1) with initial data $(u_0, \rho_0) \in H^s(\mathbb{S}^1) \times H^{s-1}(\mathbb{S}^1)$, $s > 3/2$, and $T > 0$ being the maximal time of existence, the associated characteristic of (5.1) is given by $q(t; x)$ satisfying

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} q(t;x) = u(t,\ q(t;x)), & 0 < t < T, \\ q(0;x) = x \in \mathbb{S}^1. \end{cases} \tag{5.3}$$

For any $x \in \mathbb{S}^1$, the flow map $q(t;x)$ is well defined and continuously differentiable in the entire time interval $(0, T)$.

Moreover, the precise blow-up scenario of strong solution $(u,\ \rho)$ is presented in terms of the wave-breaking [18]. In other words, the solution $(u,\ \rho)$ blows up in finite time $T > 0$ if and only if

$$\lim_{t \to T^-} \inf_{x \in \mathbb{S}^1} u_x(t,x) = -\infty. \tag{5.4}$$

In particular, a local-in-space type of blow-up mechanism introduced by Brandolese *et al* in [2,3] considerably simplifies the classical results and characterizes how the local structure of the solution both in periodic and non-periodic cases can affect the formation of singularities. In this section, we study in the two-component system case the issue that under what kind of local-in-space structure condition the particular periodic initial data could generate the wave-breaking solution. Before the main theorem is given, we first present some notation and useful results obtained in [3].

For any real constant $\beta$, define $I(\beta) \geq -\infty$ by

$$I(\beta) = \inf \left\{ \int_0^1 (p + \beta\, p_x)\,(2u^2 + u_x^2)\,dx \mid u \in H^1(\mathbb{S}^1),\ u(0) = u(1) = 1 \right\} \tag{5.5}$$

and the quantity $\beta^* \in [0, +\infty]$ by

$$\beta^* = \inf \left\{ \beta \in (0, +\infty) \mid \beta^2 + I(\beta) - 2 \geq 0 \right\} \tag{5.6}$$

with the usual convention that $\beta^* = +\infty$ if the infimum is taken on the empty set.

In [3], the authors proved that $I(\beta)$ is even with respect to the variable $\beta \in \mathbb{R}$ and $I(\beta) > -\infty$ if and only if

$$-\frac{e+1}{e-1} \leq \beta \leq \frac{e+1}{e-1}. \tag{5.7}$$

Especially, if $|\beta| < (e+1)/(e-1)$, then $I(\beta)$ is in fact a minimum with only one minimizer $u \in H^1(\mathbb{S}^1)$ with $u(0) = u(1) = 1$. In addition, $\beta^*$ in (5.6) was computed numerically as the zero point of the function $\beta \mapsto \beta^2 + I(\beta) - 2$ by

$$\beta^* = 0.513\ldots. \tag{5.8}$$

More importantly, they established the following convolution estimates, which is the key technical issue for the blow-up analysis.

**Lemma 5.1** ([3]) *For any $\beta \in \mathbb{R}$ and all $u \in H^1(\mathbb{S}^1)$, the following convolution estimate holds*

$$(p \pm \beta\, p_x) * (2u^2 + u_x^2)(x) \geq I(\beta)\, u^2(x), \qquad \forall\, x \in \mathbb{S}^1, \tag{5.9}$$

*and $I(\beta)$ is the best possible constant.*

We are now in a position to give the following local-in-space criterion for finite time wave-breaking mechanism to (5.1).

**Theorem 5.1** *Let $(u_0,\ \rho_0) \in H^s(\mathbb{S}^1) \times H^{s-1}(\mathbb{S}^1)$ with $s > 3/2$ and assume that there exists $x_0 \in \mathbb{S}^1$, such that*

$$\rho_0(x_0) = 0 \quad and \quad u_{0,x}(x_0) < -\beta^* |u_0(x_0)|, \tag{5.10}$$

*where $\beta^*$ is defined by (5.6) and (5.8). Then the corresponding solution $(u,\ \rho)$ of system (5.1) arising from $(u_0,\ \rho_0)$ blows up in finite time.*

*Proof* As usual, due to the well-posedness results, we reduce to the case $s \geq 3$ and carry out the analysis along the characteristic (5.3) emanating from $x_0 \in \mathbb{S}^1$. First, differentiating the equation

$$u_t + uu_x = -\partial_x p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right), \tag{5.11}$$

with respect to the $x$ variable and applying the identity $p * f - \partial_x^2 p * f = f$, we have

$$u_{xt} + uu_{xx} = u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right). \tag{5.12}$$

Denote

$$\omega(t) = u(t, q(t; x_0)), \quad n(t) = u_x(t, q(t; x_0)) \quad \text{and} \quad \zeta(t) = \rho(t, q(t; x_0)).$$

Then, we use the Eqs. (5.11), (5.12) and $\rho_t + (u\,\rho)_x = 0$ to get the following time derivatives along the flow $q(t; x_0)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega(t) = -p_x * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)(t, q(t; x_0)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}n(t) = \omega^2(t) - \frac{1}{2}n^2(t) + \frac{1}{2}\zeta^2(t) - p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)(t, q(t; x_0)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t) = -n(t)\,\zeta(t).$$

From the last equation above and the initial condition on $\rho_0$, we see that

$$\zeta(t) = \zeta(0)\, e^{-\int_0^t n(\tau)\,d\tau} = \rho_0(x_0)\, e^{-\int_0^t n(\tau)\,d\tau} = 0. \tag{5.13}$$

Moreover, with an arbitrary constant $\beta$ and using (5.13), we get from the time evolution of $\omega(t)$ and $n(t)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta\,\omega(t) = -\frac{1}{2}\beta p_x * \left(2u^2 + u_x^2 + \rho^2\right)(t, q(t; x_0))$$

and

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}n(t) = \tfrac{1}{2}\Big[ &(\beta\,\omega(t) - n(t))\,(\beta\,\omega(t) + n(t)) \\ &-\beta^2\omega^2(t) + 2\,\omega^2(t) - p * \left(2u^2 + u_x^2 + \rho^2\right)(t, q(t; x_0))\Big].\end{aligned}$$

Hence, for the following two functions of time variable with parameter $\beta \in \mathbb{R}$,

$$f(t) = \beta\,\omega(t) - n(t) \quad \text{and} \quad g(t) = -\beta\,\omega(t) - n(t),$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(t) = \frac{1}{2}\Big[&f(t)g(t) + \beta^2\omega^2(t) + (p - \beta p_x) * \left(2u^2 + u_x^2\right)(t, q(t; x_0)) - 2\,\omega^2(t) \\ &+ (p - \beta p_x) * \rho^2(t, q(t; x_0))\Big]\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}g(t) = \frac{1}{2}\Big[&f(t)g(t) + \beta^2\omega^2(t) + (p + \beta p_x) * \left(2u^2 + u_x^2\right)(t, q(t; x_0)) - 2\,\omega^2(t) \\ &+ (p + \beta p_x) * \rho^2(t, q(t; x_0))\Big].\end{aligned} \tag{5.15}$$

Assume that the parameter $\beta$ satisfies (5.7). On the one hand, applying the symmetric property of $I(\beta)$ and the convolution estimate (5.9), we deduce that

$$\beta^2\omega^2(t) + (p \pm \beta p_x) * \left(2u^2 + u_x^2\right)(t,\ q(t;x_0)) - 2\,\omega^2(t) \geq \left(\beta^2 + I(\beta) - 2\right)\omega^2(t).$$

On the other hand, using the definition (5.2) of $p(x)$, we derive under the assumption (5.7) of $\beta$ the following inequality for all $x \in \mathbb{S}^1$,

$$(p \pm \beta p_x)(x) = \frac{1}{2(e-1)}\left[(1 \pm \beta)e^x + (1 \mp \beta)e^{1-x}\right] \geq \frac{e+1}{2(e-1)} \pm \frac{\beta}{2} \geq 0,$$

which implies

$$(p \pm \beta p_x) * \rho^2(t,\ q(t;x_0)) \geq 0.$$

Therefore, if we take $\beta = \beta^*$ defined by (5.6) and (5.8) such that

$$(\beta^*)^2 + I(\beta^*) - 2 = 0,$$

then for $f(t) = \beta^*\omega(t) - n(t)$ and $g(t) = -\beta^*\omega(t) - n(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \geq \frac{1}{2}f(t)g(t) + \frac{1}{2}\left((\beta^*)^2 + I(\beta^*) - 2\right)\omega^2(t) = \frac{1}{2}f(t)g(t). \tag{5.16}$$

In the same manner,

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) \geq \frac{1}{2}f(t)g(t). \tag{5.17}$$

Furthermore, the assumption (5.10) on the initial data $u_0$ that

$$u_{0,x}(x_0) < -\beta^*|u_0(x_0)|$$

guarantees that

$$f(0) = \beta^*\omega(0) - n(0) > 0 \quad \text{and} \quad g(0) = -\beta^*\omega(0) - n(0) > 0. \tag{5.18}$$

Now, define $h(t) = \sqrt{f(t)g(t)}$. Using the inequality $f(t) + g(t) \geq 2\,h(t)$ and the relation (5.16) and (5.17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = \frac{1}{2\,h(t)}\left(g(t)\frac{\mathrm{d}}{\mathrm{d}t}f(t) + f(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t)\right) \geq \frac{f(t)g(t)}{4\,h(t)}(f(t) + g(t)) \geq \frac{1}{2}h^2(t).$$

Since $h(0) = \sqrt{u_{0,x}(x_0)^2 - (\beta^* u_0(x_0))^2} > 0$, it is clear to see that the solution $(u, \rho)$ blows up in finite time.

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