A NONLINEAR PARABOLIC EQUATION WITH DISCONTINUITY IN THE HIGHEST ORDER AND APPLICATIONS

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ABSTRACT. In this paper we establish a viscosity solution theory for a class of nonlinear parabolic equations with discontinuities of the sign function type in the second derivatives of the unknown function. We modify the definition of classical viscosity solutions and show uniqueness and existence of the solutions. These results are related to the limit behavior for the motion of a curve by a very small power of its curvature, which has applications in image processing. We also discuss the relation between our equation and the total variation flow in one space dimension.

1. INTRODUCTION

We study a class of fully nonlinear parabolic equations with a jump discontinuity in the second derivatives of the unknown. The general equation form of equations we discuss is

$$u_t + F(\nabla u, \operatorname{sgn}(f(\nabla^2 u))) = 0 \quad \text{in } \mathbb{T}^n \times (0, \infty) \tag{1.1}$$

with initial condition

$$u(x, 0) = u_0(x) \text{ for all } x \in \mathbb{T}^n, \tag{1.2}$$

where $\mathbb{T}^n$ denotes the n-dimensional torus, F and f are assumed to be continuous functions and satisfy the ellipticity and u_0 is a continuous function on $\mathbb{T}^n$. More detailed assumptions will be given later. The function sgn is formally understood as the usual sign function:

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a < 0. \end{cases}$$

We are particularly interested in the motion of a one dimensional graph with its normal velocity equal to the sign of its curvature:

$$u_t - \sqrt{1 + u_x^2}\, \operatorname{sgn}(u_{xx}) = 0. \tag{1.3}$$

It is not clear how one should handle such a discontinuity caused by the sign function of the second derivatives to obtain a unique solution of such equations. The classical theory of viscosity solutions (e.g., [11]) does not apply directly. In this work, we give a definition of viscosity solutions for (1.1) and show the uniqueness and existence of continuous solutions.

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1.1. **Motivations.** Our problem is closely related to the mathematical models for image processing. It is well-known that the following nonlinear equation in two space dimensions has important applications in image denoising [1, 8]:

$$u_t - |\nabla u| \left|\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right|^{\alpha-1} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0. \tag{1.4}$$

This equation in two dimensions gives a level-set formulation of the motion of a curve Γ governed by the law:

$$V = \kappa^\alpha,$$

where V denotes the normal velocity and κ denotes the curvature of Γ; see [28, 29, 30] for results related to this geometric motion.

The choice of the exponent $\alpha > 0$ reflects a particular purpose for practical use in image processing. For the purpose of shape analysis, we need to pick a small α to cancel the pixel effect as fast as possible; on the other hand, if our aim is image denoising, we may want to remove small details while keeping main features unchanged, for which a large α seems more suitable. We are particularly interested in the behavior of the operator when $\alpha \to 0$, which formally comes to the equation:

$$u_t - |\nabla u| \operatorname{sgn}\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right) = 0. \tag{1.5}$$

One may expect that the unique viscosity solution u^α of (1.4) converges to the solutions of (1.5). The proof of convergence is however not straightforward as in the classical case, since the nonlinear operator in (1.4) does not converge locally uniformly to the one in (1.5). Moreover, it is not clear whether continuous solutions of (1.5) uniquely exist for any given continuous initial data.

A more general class of such equations can be written as

$$u_t + F(\nabla u, \operatorname{sgn} f(\nabla u, \nabla^2 u)) = 0,$$

where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbf{S}^n \to \mathbb{R}$ are functions satisfying proper assumptions. We denote by $\mathbf{S}^n$ the set of all $n \times n$ symmetric matrices. As shown in the example (1.5), the difficulty of studying such equations lies in the discontinuities of the sign function and of $f(p, X)$ in p.

In this paper we focus our attention on the special case (1.1), where f only depends continuously on the second space derivatives $\nabla^2 u$, as exemplified by (1.3). The example (1.3) can also be viewed as the formal limit, as $\alpha \to 0$, of the graph version of (1.4):

$$u_t - (1 + u_x^2)^{\frac{1-3\alpha}{2}} |u_{xx}|^{\alpha-1} u_{xx} = 0.$$

We adapt the viscosity solution theory to show existence and uniqueness of continuous solutions in the new circumstances. Concerning discontinuities appearing in nonlinear PDEs, there are many related works using viscosity solution theory in the literature. Some of them discuss Hamilton-Jacobi equations with discontinuous Hamiltonians ([7, 6, 12, 9, 22, 23], etc.) and some others study second order parabolic equations with discontinuity in the first derivatives including level-set mean curvature flow equations and p-Laplace equations with $1 \le p \le \infty$ (e.g. [10, 13, 15, 24, 19]). But there are few results on well-posedness for second order equations with discontinuities in the highest order like (1.1).

1.2. **Assumptions and main results.** Here and in the sequel, F and f are assumed to satisfy the following conditions:

(H1) F is degenerate elliptic, i.e.,
$$F(p, y_1) \leq F(p, y_2)$$
for all $y_1 \geq y_2$ and $p \in \mathbb{R}^n$.

(H2) $F(p, y)$ is Lipschitz continuous, i.e., there exists an $L > 0$ such that
$$|F(p_1, y_1) - F(p_2, y_2)| \leq L(|p_1 - p_2| + |y_1 - y_2|)$$
for all $p_1, p_2 \in \mathbb{R}^n$ and $y_1, y_2 \in \mathbb{R}$.

(H3) f is uniformly elliptic; namely, there exists a $\mu > 0$ such that
$$f(X_1) - f(X_2) \geq \mu \operatorname{tr}(X_1 - X_2)$$
for all $X_1 \geq X_2$ in $\mathbf{S}^n$.

(H4) f is locally Lipschitz continuous in $\mathbf{S}^n$, i.e., for any $R > 0$, there exists a K_R such that
$$|f(X_1) - f(X_2)| \leq K_R \|X_1 - X_2\|$$
for all $X_1, X_2 \in \mathbf{S}^n$ with $\|X_1\|, \|X_2\| \leq R$.

(H5) $f(O) = 0$; in other words, any constant is a solution of $f(\nabla^2 u) = 0$ in $\mathbb{T}^n$.

Under the assumptions, we give a definition (Definition 2.1) of viscosity solutions by adapting the usual solutions involving upper and lower semicontinuous envelopes (cf. [11, 3, 21]) for discontinuous equations. We remark that those usual definitions cannot directly guarantee the uniqueness of continuous solutions in our current situation. For example, in the simplest case of (1.1) such as
$$u_t - \operatorname{sgn}(u_{xx}) = 0 \text{ in } \mathbb{T} \times (0, \infty), \tag{1.6}$$
one is tempted to define an envelope subsolution u (resp., supersolution) in the following way: whenever $u - \phi$ attains a maximum (resp., minimum) at $(x_0, t_0) \in \mathbb{T} \times (0, \infty)$ for some smooth function $\phi : \mathbb{T} \times [0, \infty) \to \mathbb{R}$, we have
$$\phi_t(x_0, t_0) - \operatorname{sgn}^*(\phi_{xx}(x_0, t_0)) \leq 0 \text{ (resp., } \phi_t(x_0, t_0) - \operatorname{sgn}_*(\phi_{xx}(x_0, t_0)) \geq 0).$$
Here
$$\operatorname{sgn}^*(a) = \begin{cases} 1 & \text{if } a \geq 0 \\ -1 & \text{if } a < 0 \end{cases} \text{ and } \operatorname{sgn}_*(a) = \begin{cases} 1 & \text{if } a > 0 \\ -1 & \text{if } a \leq 0. \end{cases} \tag{1.7}$$
We call u is an envelope solution if it is both an envelope subsolution and an envelope supersolution.

However, such envelope solutions are not unique when $u(\cdot, 0)$ is identically equal to any constant $C \in \mathbb{R}$. In fact, it is easy to see that
$$u(x, t) = C + bt$$
satisfy the definitions of both sub- and supersolutions above for any $b \in [-1, 1]$. It is natural to expect that the only correct solution is $u \equiv C$ in $\mathbb{T} \times [0, \infty)$.

Hence, we need to modify the definitions to fix the uniqueness issue. It turns out that the diffusion is so fast that solutions turn to be constant in space in finite time; we discuss this flattening behavior in more details later. Given any non-constant initial value for (1.1), one can still obtain the uniqueness of solutions up to the moment when they become constant in space. In order to get the uniqueness global in time, we only need to force the correct solution to keep its velocity equal to $F(0,0)$ after being completely flattened; see Definition 2.1.

Our main theorem of this work is the existence and uniqueness of our adapted viscosity solutions described above.

Theorem 1.1. *Assume that F and f satisfy (H1)–(H5). Let u_0 be Lipschitz continuous. Then there exists a unique Lipschitz continuous solution of (1.1) with $u(x,0)=u_0(x)$ for all $x\in\mathbb{T}^n$.*

We also discuss the wellposedness problem in a less regular function space, i.e. the space of continuous functions. We can show that with an additional boundedness assumption (cf. Remark 4.4), one is able to construct a unique continuous solution from any continuous initial data.

In Section 3, we prove the uniqueness by showing a comparison principle for continuous sub- and supersolutions. As mentioned above, we first establish comparison between a subsolution u and a supersolution v up to their first flattening moment.

The key issue in the proof is clearly how to avoid situations when the second derivatives of both u and v equal to zero, since comparison will be completely lost due to the semicontinuity envelopes taken in the definitions. It turns out that one may apply a perturbation technique for test functions with proper small quadratic terms so that a strict ordering of second derivatives is maintained.

A formal argument in one space dimension is given in what follows. Suppose u and v are smooth and $u-v$ is positive somewhere in $\mathbb{T}\times(0,T_0)$. Then

$$\Phi(x,t)=u(x,t)-v(x,t)-\frac{\sigma}{T\wedge T_0-t}$$

attains a maximum at $(x_0,t_0)\in\mathbb{T}\times(0,T_0)$, where $T>0$, $\sigma>0$ are fixed and T_0 denotes the first moment that both u and v become constant in space. If $\Phi(\cdot,t_0)$ is not identically equal to the maximum in space, then for any $h>0$, one may take $a>0$ small such that there exists a local maximizer $(\hat{x},\hat{t})$ of

$$u(x,t)-v(x,t)-\frac{\sigma}{T\wedge T_0-t}-h(t-t_0)^2+a|x-x_0|^2,$$

which yields,

$$u_{xx}(\hat{x},\hat{t})\leq v_{xx}(\hat{x},\hat{t})-2a.$$

This inequality implies $\operatorname{sgn}^* f(u_{xx}(\hat{x},\hat{t}))\leq \operatorname{sgn}_* f(v_{xx}(\hat{x},\hat{t}))$, which allows us to proceed to the usual comparison arguments of deriving a contradiction.

If, on the other hand, $\Phi(x,t_0)$ is constant in space, then it implies formally that $u_{xx}(x,t_0)\equiv v_{xx}(x,t_0)$ for all $x\in\mathbb{T}$. We will completely avoid the difficulty caused by the discontinuity of equations provided that $f(u_{xx}(x,t_0))=f(v_{xx}(x,t_0))\neq 0$ for some $x\in\mathbb{T}$. The remaining case is that $f(u_{xx})=f(v_{xx})\equiv 0$ and therefore $u_{xx}=v_{xx}$ in $\mathbb{T}\times\{t_0\}$, which cannot occur, since either u or v has not become constant in space yet.

Our formal proof above can be made rigorous in the framework of viscosity solution theory under the assumptions that u and v are continuous in time. The global uniqueness of solutions with time continuity follows from the extra restriction in our modified definition mentioned above.

In Section 4, we obtain the existence of continuous solutions by considering the limit for a sequence of approximating equations:

$$u_t + F(\nabla u, |f(\nabla^2 u)|^{\alpha-1} f(\nabla^2 u)) = 0 \tag{1.8}$$

with $\alpha > 0$. Noticing that there are no singularities in (1.8), we may apply the classical viscosity solution theory to show the existence of a unique solution u^α. We get a solution u of (1.1) by passing to the limit of u^α as $\alpha \to 0+$.

The rigorous proof for existence is more or less similar to the standard stability theory of viscosity solutions, since comparison principle is available. But it is worth pointing out that one needs to include an argument to show the continuity of (semi-) limits of u^α, as required in our comparison principle. While the continuity in space is shown in a standard way, the continuity in time relies much on the boundedness of the sign function which is inherent in the special structure of our equation (1.1). For instance, one can easily see that any solution u of (1.6) formally satisfies

$$|u_t| = |\operatorname{sgn}(u_{xx})| \leq 1,$$

from which the Lipschitz bound of u in time follows immediately.

By an argument similar to the existence proof, we show that the unique solution u of (1.1)–(1.2) is stable with respect to u_0 in the class of Lipschitz functions; see Section 5.

1.3. **Connections with total variation flow.** A different viewpoint of our problem is related to the total variation flow. Let us take (1.6) as an example. By formally differentiating it with respect to x and substituting u_x with a new unknown function v, we obtain the one-dimensional total variation flow.

$$v_t - (\operatorname{sgn}(v_x))_x = 0, \quad \text{in } \mathbb{T} \times (0, \infty). \tag{1.9}$$

(The second derivative $w = u_{xx}$ formally satisfies the so-called sign fast diffusion [5]

$$w_t - \operatorname{sgn}(w)_{xx} = 0.)$$

It is well known that (1.9) is nonlocal and facets (pieces where $v_x = 0$) appear during the evolution. Among many works on this equation, Giga et al. constructed a viscosity solution theory for (1.9) over the years; see [15, 16, 20]. See also [2, 26, 17] and related results in [18, 19] etc. The novelty of their definition of viscosity solutions consists in extra nonlocal tests for facets; namely, the graphs of smooth test functions themselves also contain facets. Such extra tests play an important role in the proof of comparison principles.

In our definition for viscosity solutions of (1.6), we however do not need to include any nonlocal tests. This is more or less natural, since, from the viewpoint of viscosity solution theory, differentiating unknowns in space variables makes equations more complicated. For example, the definition for viscosity solutions of Hamilton-Jacobi equations is much simpler than that of Burger's equation.

In Section 6, we discuss (1.6) in more details and show that the derivative of its solution in space does satisfy (1.9) under certain regularity assumptions. Note that evolutions of graphs with a non-smooth interface energy like (1.9) were first studied by Fukui and Giga [14]. Our example (Example 6.6) suggests that it is also possible to study discontinuous solutions of (1.9) by differentiating solutions of (1.6); see [27] for recent results on discontinuous solutions of (1.9).

2. Definition of Viscosity Solutions

Let us begin with the definition of subsolutions and supersolutions of (1.1) with F and f satisfying (H1)–(H5).

Let sgn^* and sgn_* be defined as in (1.7). Here the notation $*$ is essentially consistent with that of the semicontinuous envelopes. It is clear that $\mathrm{sgn}_*(x) \leq \mathrm{sgn}^*(x)$ and

$$\mathrm{sgn}^*(x) \leq \mathrm{sgn}_*(x+a)$$

for any $x \in \mathbb{R}$ and $a > 0$.

Definition 2.1 (Definition of viscosity solutions)**.** A locally bounded upper semicontinuous (resp., lower semicontinuous) function $u : \mathbb{T}^n \times [0, \infty) \to \mathbb{R}$ is called a subsolution (resp., supersolution) of (1.1) if the following two conditions hold:

(1) Whenever there exist $\phi \in C^2(\mathbb{T}^n \times [0, \infty))$ and $(x_0, t_0) \in \mathbb{T} \times (0, \infty)$ such that

$$\max_{\mathcal{O}}(u-\phi) = (u-\phi)(x_0, t_0), \ \ (\text{resp., } \min_{\mathcal{O}}(u-\phi) = (u-\phi)(x_0, t_0),)$$

where $\mathcal{O}$ is an open set of $\mathbb{T}^n \times (0, \infty)$ containing (x_0, t_0), the function ϕ satisfies

$$\begin{gathered}\phi_t + F(\nabla\phi, \mathrm{sgn}^*(f(\nabla^2\phi))) \leq 0 \text{ at } (x_0, t_0)\\ \left(\text{resp., } \phi_t + F(\nabla\phi, \mathrm{sgn}_*(f(\nabla^2\phi))) \geq 0 \text{ at } (x_0, t_0)\right).\end{gathered}$$

(2) When $u(\cdot, t_0)$ is constant in $\mathbb{T}^n$ for some $t_0 \geq 0$, one has

$$u(x,t) \leq u(x,t_0) - C_F(t-t_0) \quad (\text{resp., } u(x,t) \geq u(x,t_0) + C_F(t-t_0))$$

for all $x \in \mathbb{R}^n$ and $t \geq t_0$, where $C_F = F(0, 0)$.

A locally bounded function u is a solution if it is both a subsolution and a supersolution.

Our definition above implies that if a solution u is constant in space at $t = t_0$, then it is constant in space at any $t \geq t_0$; in particular, $u(\cdot, t) = u(\cdot, t_0) - C_F(t-t_0)$ for all $t \geq t_0$.

Remark 2.1. Although our definition of subsolutions and supersolutions requires only semicontinuity, we strengthen their regularity in our comparison principle (Theorem 3.1).

Remark 2.2. As usual, we may rewrite the condition (1) in the definition of sub- and supersolutions involving the semijets [11, 21]:

$$\begin{gathered}\tau + F(p, \mathrm{sgn}^*(f(X))) \leq 0\\ (\text{resp., } \tau + F(p, \mathrm{sgn}_*(f(X))) \geq 0)\end{gathered}$$

for all $(x,t) \in \mathbb{T}^n \times (0,\infty)$ and $(p,X) \in J^{2,+}u(x,t)$ (resp., $(p,X) \in J^{2,-}u(x,t)$). Here we recall the definition of $J^{2,\pm}$:

$$J^{2,+}u(x,t) = \Big\{(\tau,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbf{S}^n : u(y,s) - u(x,t)$$
$$\leq \tau(s-t) + \langle p,(y-x)\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2)\Big\};$$
$$J^{2,-}u(x,t) = \Big\{(\tau,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbf{S}^n : u(y,s) - u(x,t)$$
$$\geq \tau(s-t) + \langle p,(y-x)\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2)\Big\}.$$

For example, if $u - \phi$ attains a maximum at (x_0,t_0) for some function $\phi(x,t)$ twice differentiable in x and differentiable in t at (x_0,t_0), then

$$(\phi_t(x_0,t_0), \nabla\phi(x_0,t_0), \nabla^2\phi(x_0,t_0)) \in J^{2,+}u(x_0,t_0).$$

For our later use, we also recall the definitions of the "closures" of $J^{2,+}u$ and $J^{2,-}u$. We set for any $(x,t) \in \mathbb{T}^n \times (0,\infty)$

$$\overline{J}^{2,\pm}u(x,t) = \{(\tau,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbf{S}^n : \text{there exist } (x_n,t_n) \in \mathbb{T}^n \times (0,\infty) \text{ and}$$
$$(\tau_n,p_n,X_n) \in J^{2,\pm}u(x_n,t_n) \text{ with } (\tau_n,p_n,X_n) \to (\tau,p,X) \text{ and } u(x_n,t_n) \to u(x,t)\}.$$

It is well-known that one may use the semicontinuous envelope to define viscosity solutions for elliptic or parabolic equations with discontinuity. Our definition here adapts that idea. Note that

$$G^*(p,X) = F(p, \operatorname{sgn}_*(f(X))),$$
$$G_*(p,X) = F(p, \operatorname{sgn}^*(f(X))),$$

for $G(p,X) = F(p, \operatorname{sgn}(f(X)))$, due to (H1),(H2) and (H4). However, as pointed out in Section 1, uniqueness does not hold under such a definition of solutions. We add the condition (2) for this purpose.

In fact, even if the initial condition is not a constant, it is still possible that solutions become spatially constant in finite time. For any function $u : \mathbb{T} \times [0,\infty) \to \mathbb{R}$, we denote by $T(u)$ the first time t for $u(x,t)$ becoming constant in x, i.e,

$$T(u) = \inf\{t \in [0,\infty), u(x,t) \equiv C \text{ for some } C \in \mathbb{R}\}.$$

Note that in general $T(u)$ might be infinity. The condition (2) in the definition of subsolutions amounts to saying

$$u(x,t) \leq u(x,T(u)) - C_F(t - T(u))$$

for any $x \in \mathbb{T}^n$ and $t \geq T(u)$.

3. Uniqueness of Solutions

Let us present a comparison theorem for (1.1) up to the first flattening time.

Theorem 3.1 (Comparison before flattening). *Assume (H1)–(H5). Let u and v be respectively a subsolution and a supersolution of (1.1). Assume in addition that u*

and v are uniformly continuous in time; that is, there exists modulus of continuity ω such that

$$|u(x,t)-u(x,s)|\le \omega(|t-s|), \quad |v(x,t)-v(x,s)|\le \omega(|t-s|) \tag{3.1}$$

for all $x\in\mathbb{R}^n$ and $t,s\in[0,\infty)$. If $u(x,0)\le v(x,0)$ for all $x\in\mathbb{T}^n$, then $u\le v$ in $\mathbb{T}^n\times[0,T)$ for any $T\in[0,T_0]$, where $T_0=\max\{T(u),T(v)\}$.

In the proof of Theorem 3.1, we need the following property about sup- and inf-convolutions, which is a parabolic version of [11, Lemma A.5].

Proposition 3.2 (Properties of sup- and inf-convolutions)**.** *Let $\lambda>0$ and u be an upper (resp. lower) semicontinuous function in $\mathbb{R}^n\times[0,\infty)$. Let*

$$u^\lambda(x,t)=\sup_{\xi\in\mathbb{R}^n}\left\{u(\xi,t)-\frac{\lambda}{2}|x-\xi|^2\right\}$$

$$\left(\textit{resp.}, \ u_\lambda(x,t)=\inf_{\xi\in\mathbb{R}^n}\left\{u(\xi,t)+\frac{\lambda}{2}|x-\xi|^2\right\}\right).$$

If there exists $(\tau,p,X)\in J^{2,+}u^\lambda(x_0,t_0)$ (resp., $(\tau,p,X)\in J^{2,-}u_\lambda(x_0,t_0)$) for some $(x_0,t_0)\in\mathbb{R}^n\times(0,\infty)$, then

$$(\tau,p,X)\in J^{2,+}u(x_0+p/\lambda,t_0) \quad \left(\textit{resp.}, \ (\tau,p,X)\in J^{2,-}u(x_0-p/\lambda,t_0)\right).$$

Proof. We only prove the result on u^λ; the proof for u_λ is symmetric. If $(\tau,p,X)\in J^{2,+}u^\lambda(x_0,t_0)$, then

$$\begin{aligned} u(\xi,t)-\frac{\lambda}{2}|x-\xi|^2 &\le u^\lambda(x,t)\\ &\le u^\lambda(x_0,t_0)+\tau(t-t_0)+\langle p,x-x_0\rangle+\frac{1}{2}\langle X(x-x_0),x-x_0\rangle\\ &\quad +o(|x-x_0|^2)+o(|t-t_0|) \end{aligned} \tag{3.2}$$

for any $x,\xi\in\mathbb{R}^n$. Let $\xi_0\in\mathbb{R}^n$ such that

$$u^\lambda(x_0,t_0)=u(\xi_0,t_0)-\frac{\lambda}{2}|\xi_0-x_0|^2.$$

It follows from (3.2) that

$$\begin{aligned} u(\xi,t)-\frac{\lambda}{2}|x-\xi|^2 \le & u(\xi_0,t_0)-\frac{\lambda}{2}|\xi_0-x_0|^2+\tau(t-t_0)+\langle p,x-x_0\rangle\\ &+\frac{1}{2}\langle X(x-x_0),x-x_0\rangle+o(|x-x_0|^2)+o(|t-t_0|). \end{aligned} \tag{3.3}$$

Setting $x=\xi-\xi_0+x_0$, we get

$$\begin{aligned} u(\xi,t)\le u(\xi_0,t_0)+\tau(t-t_0)+\langle p,\xi-\xi_0\rangle+\frac{1}{2}\langle X(\xi-\xi_0),\xi-\xi_0\rangle\\ +o(|\xi-\xi_0|^2)+o(|t-t_0|). \end{aligned}$$

It remains to show that $\xi_0=x_0+p/\lambda$. We take $\xi=\xi_0$, $t=t_0$ and $x=x_0+\beta(\lambda(x_0-\xi_0)+p)$ in (3.3) for an arbitrary $\beta\in\mathbb{R}$ small and get

$$0\le\beta|\lambda(x_0-\xi_0)+p|^2+O(\beta^2).$$

This yields $\lambda(x_0-\xi_0)+p=0$. $\square$

The following result is elementary.

Lemma 3.3. *Assume $f : \mathbf{S}^n \to \mathbb{R}$ is continuous and satisfies (H3)–(H5). If $w \in C^{1,1}(\mathbb{T}^n)$ satisfies $-f(\nabla^2 w) = 0$ a.e. in $\mathbb{T}^n$, then w is constant in $\mathbb{T}^n$.*

Proof. We claim that w is actually a viscosity solution of $-f(\nabla^2 w) = 0$ in $\mathbb{T}^n$. We only show the subsolution verification. The supersolution part is handled by a symmetric argument.

Suppose there exists $\phi \in C^2(\mathbb{T}^n)$ and $x_0 \in \mathbb{T}^n$ such that $w - \phi$ attains a maximum at x_0. Then by Jensen's lemma (cf., [11, Lemma A.3]), for $r, \delta > 0$, the set

$$\{x \in B_r(x_0) : \text{there exists } p \in B_\delta(0) \text{ for which}$$
$$w(x) - \phi(x) - \langle p, x\rangle \text{ has a local maximum at } x\}$$

has a positive measure. We therefore can take $x_k \to x_0$, $p_k \to 0$ as $k \to \infty$ such that $w(x) - \phi(x) - \langle p, x\rangle$ attains a local maximum at x_k and $f(\nabla^2 w(x_k)) = 0$. This implies that

$$-f(\nabla^2 \phi(x_k)) \leq 0.$$

Letting $k \to \infty$, we get

$$-f(\nabla^2 \phi(x_0)) \leq 0,$$

which indicates that f is a subsolution.

We conclude the proof of this lemma by noting that Lipschitz viscosity solutions of $-f(\nabla^2 w) = 0$ in $\mathbb{T}^n$ must be constant, since the strong maximum principle [25, 4] holds thanks to the assumptions (H3)–(H5). □

We next present a proof of Theorem 3.1.

Proof of Theorem 3.1. Suppose by contradiction that $u - v$ has a positive value in $\mathbb{T}^n \times (0, T)$. Then we may assume that

$$\Phi(x,t) = u(x,t) - v(x,t) - \frac{\sigma}{T-t}$$

attains a maximum $\delta > 0$ for some small $\sigma > 0$. Take the sup-convolution u^λ of u and the inf-convolution v_λ for $\lambda > 0$ in space; namely,

$$u^\lambda(x,t) = \sup_{\xi \in \mathbb{T}^n} \left\{ u(\xi, t) - \frac{\lambda}{2}|x-\xi|^2 \right\},$$
$$v_\lambda(y,s) = \inf_{\eta \in \mathbb{T}^n} \left\{ v(\eta, s) + \frac{\lambda}{2}|y-\eta|^2 \right\}.$$

We then have

$$\delta_\lambda := \max_{\mathbb{T}^n \times [0,T)} \left(u^\lambda(x,t) - v_\lambda(x,t) - \frac{\sigma}{T-t} \right) > 0$$

.

It is not difficult to see that the auxiliary function

$$\Phi_\varepsilon(x,y,t,s) = u^\lambda(x,t) - v_\lambda(y,s) - \frac{|x-y|^2}{2\varepsilon} - \frac{|t-s|^2}{2\varepsilon} - \frac{\sigma}{T-t}$$

also attains a maximum at some $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \in \mathbb{T}^{2n} \times [0,T)^2$ with

$$\max_{\mathbb{T}^{2n} \times [0,T)^2} \Phi_\varepsilon = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \geq \delta$$

for all $\lambda > 0$. In addition, due to the maximum at $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$, we have

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} + \frac{|t_\varepsilon - s_\varepsilon|^2}{2\varepsilon} \leq u^\lambda(x_\varepsilon, t_\varepsilon) - v_\lambda(y_\varepsilon, s_\varepsilon) - \frac{\sigma}{T - t_\varepsilon} - \delta_\lambda. \tag{3.4}$$

Note that the right hand side is bounded uniformly in ε. By taking a subsequence, still indexed by ε, we get $x_\varepsilon, y_\varepsilon \to x_0$ and $t_\varepsilon, s_\varepsilon \to t_0$ for $(x_0, t_0) \in \mathbb{T}^n \times [0, T)$. It follows from (3.4) that

$$\limsup_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} + \frac{|t_\varepsilon - s_\varepsilon|^2}{2\varepsilon} \leq (u^\lambda - v_\lambda)(x_0, t_0) - \frac{\sigma}{T - t_0} - \delta_\lambda \leq 0,$$

which implies that

$$\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \to 0, \qquad \frac{|t_\varepsilon - s_\varepsilon|^2}{2\varepsilon} \to 0 \tag{3.5}$$

and

$$\max_{\mathbb{T}^{2n} \times [0,T)^2} \Phi_\varepsilon = \Phi_\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon) \to \max_{\mathbb{T}^n \times [0,T)} \Phi(x, t) > 0, \tag{3.6}$$

as $\varepsilon \to 0$.

For $\lambda > 0$ large, the comparison assumption on the initial values implies that $t_\varepsilon \neq 0$ and $s_\varepsilon \neq 0$ when $\varepsilon > 0$ is sufficiently small. Indeed, let (x_0, x_0, t_0, t_0) be a limit of $(x_\varepsilon, y_\varepsilon, t_\varepsilon, s_\varepsilon)$ along a subsequence ε_k. Then we get

$$\limsup_{k \to \infty} \max \Phi_{\varepsilon_k} \leq u^\lambda(x_0, 0) - v_\lambda(x_0, 0),$$

where the left hand side is greater than δ but the right hand side is small when $\lambda > 0$ is taken large. Also it is easily seen that $t_0 = s_0 < T$.

Set

$$\phi_1(x) = \max_{(y,t,s) \in \mathbb{T}^n \times [0,T)^2} \Phi_\varepsilon(x, y, t, s);$$
$$\phi_2(y) = \max_{(x,t,s) \in \mathbb{T}^n \times [0,T)^2} \Phi_\varepsilon(x, y, t, s).$$

It is clear that

$$\max_{\mathbb{T}} \phi_1 = \max_{\mathbb{T}^{2n} \times [0,T)^2} \Phi_\varepsilon.$$

We next discuss the following cases.

Case 1. Suppose that for any $\lambda > 0$ large, there exists a subsequence indexed by ε_k with $\varepsilon_k \to 0$ as $k \to \infty$, such that for all k, either of the following holds:

$$\mathcal{M}^1_{\varepsilon_k} = \{x \in \mathbb{T}^n : \max \Phi_{\varepsilon_k}(x, y, t, s) = \phi_1(x)\} \neq \mathbb{T}^n; \tag{3.7}$$
$$\mathcal{M}^2_{\varepsilon_k} = \{y \in \mathbb{T}^n : \max \Phi_{\varepsilon_k}(x, y, t, s) = \phi_2(y)\} \neq \mathbb{T}^n. \tag{3.8}$$

For simplicity of notation, we use the index ε instead of ε_k. In what follows, we derive a contradiction when (3.7) holds. The proof for the case (3.8) is similar.

We choose $h > 0$ to be determined later such that

$$\max_{y \in \mathbb{T}^n} \Phi(x_\varepsilon, y, t_\varepsilon, s_\varepsilon) > \max_{y \in \mathbb{T}^n} \Phi_\varepsilon(x, y, t, s) - h(t - t_\varepsilon)^2 - h(s - s_\varepsilon)^2$$

for any $t, s \in (0, T)$ and any $x \in M^1_\varepsilon$, which implies the existence of an $a > 0$ sufficiently small such that

$$\max_{y \in \mathbb{T}^n} \Phi_\varepsilon(x, y, t, s) - h(t - t_\varepsilon)^2 - h(s - s_\varepsilon)^2 + a|x - x_\varepsilon|^2$$

attains a local maximum at $(\hat{x}, \hat{t}, \hat{s}) \in \mathbb{T}^n \times (0,T)^2$ near the set $M^1_\varepsilon \times \{t_\varepsilon\} \times \{t_\varepsilon\}$; see [21, Lemma 2.2.5]. It amounts to saying that

$$\Phi_\varepsilon(x,y,t,s) - h(t-t_\varepsilon)^2 - h(s-s_\varepsilon)^2 + a|x-x_\varepsilon|^2$$

attains a local maximum at $(\hat{x}, \hat{y}, \hat{t}, \hat{s})$ for some $\hat{y} \in \mathbb{T}^n$.

We then apply the Crandall-Ishii lemma (cf. [11]) to get

$$(\tau^\lambda, p^\lambda, X^\lambda) \in \overline{J}^{2,+} u^\lambda(\hat{x}, \hat{t}) \text{ and } (\rho^\lambda, q^\lambda, Y^\lambda) \in \overline{J}^{2,-} v_\lambda(\hat{y}, \hat{s})$$

satisfying

$$\begin{aligned}
&\tau^\lambda = \frac{\hat{t}-\hat{s}}{\varepsilon} + \frac{\sigma}{(T-\hat{t})^2} + 2h(\hat{t}-t_\varepsilon), \quad \rho^\lambda = \frac{\hat{t}-\hat{s}}{\varepsilon} - 2h(\hat{s}-s_\varepsilon),\\
&p^\lambda = \frac{\hat{x}-\hat{y}}{\varepsilon} - a(\hat{x}-x_\varepsilon), \quad q^\lambda = \frac{\hat{x}-\hat{y}}{\varepsilon},\\
&X^\lambda \le Y^\lambda - 2aI.
\end{aligned}$$

The last inequality above implies that

$$f(X^\lambda) < f(Y^\lambda). \tag{3.9}$$

Take a sequence $(x_j, y_j, t_j, s_j) \in \mathbb{T}^{2n} \times [0,T)^2$ satisfying

$$\begin{aligned}
&(\tau_j, p_j, X_j) \in J^{2,+} u^\lambda(x_j, t_j), \quad (\rho_j, q_j, Y_j) \in J^{2,-} v^\lambda(y_j, s_j)\\
&(x_j, y_j, t_j, s_j) \to (\hat{x}, \hat{y}, \hat{t}, \hat{s})\\
&(\tau_j, p_j, X_j) \to (\tau^\lambda, p^\lambda, X^\lambda) \text{ and } (\rho_j, q_j, Y_j) \to (\rho^\lambda, q^\lambda, Y^\lambda) \text{ as } j \to \infty.
\end{aligned}$$

By Proposition 3.2, we get $(\tau_j, p_j, X_j) \in J^{2,+} u(x_j + p_j/\lambda, t_j)$ and $(\rho_j, q_j, Y_j) \in J^{2,-} u(y_j - q_j/\lambda, s_j)$.

We apply the definition of viscosity subsolutions and supersolutions to get

$$\begin{aligned}
\tau_j + F(p_j, \mathrm{sgn}^*(f(X_j))) &\le 0;\\
\rho_j + F(q_j, \mathrm{sgn}_*(f(Y_j))) &\ge 0.
\end{aligned}$$

Letting $j \to \infty$ and taking the difference of both inequalities, we have

$$\tau^\lambda - \rho^\lambda \le F(q^\lambda, \mathrm{sgn}_*(f(Y^\lambda))) - F(p^\lambda, \mathrm{sgn}^*(f(X^\lambda))),$$

which yields by (3.9), (H1) and (H2),

$$\frac{\sigma}{(T-\hat{t})^2} + 2h(\hat{t}-t_\varepsilon) + 2h(\hat{s}-s_\varepsilon) \le L|p^\lambda - q^\lambda| = La|\hat{x}-x_\varepsilon|.$$

Sending $a \to 0$ and choosing $h \le \frac{\sigma}{8T(T-\hat{t})^2}$, we reach a contradiction.

Case 2. Suppose that there exists $\lambda_k \to \infty$ as $k \to \infty$ such that $\mathcal{M}^1_\varepsilon = \mathbb{T}^n$ and $\mathcal{M}^2_\varepsilon = \mathbb{T}^n$ for all λ_k and $\varepsilon > 0$. This means that

$$u^{\lambda_k}(x,t) - v_{\lambda_k,\varepsilon}(x,s) - \frac{|t-s|^2}{2\varepsilon} - \frac{\sigma}{T-t}$$

attains a maximum at any point in $\mathbb{T}^n \times \{t_\varepsilon\} \times \{s_\varepsilon\}$, where

$$v_{\lambda_k,\varepsilon}(x,s) = \inf_{y \in \mathbb{T}^n} \left\{ v_{\lambda_k}(y,s) + \frac{|x-y|^2}{2\varepsilon} \right\}.$$

In particular, we find that $u^{\lambda_k}(x,t_\varepsilon) - v_{\lambda_k,\varepsilon}(x,s_\varepsilon) = C$ in $\mathbb{T}^n$ for some constant C, which yields $u^{\lambda_k}(\cdot,t_\varepsilon), v_{\lambda_k,\varepsilon}(\cdot,s_\varepsilon) \in C^{1,1}(\mathbb{T}^n)$.

Case 2a. If $u^{\lambda_k}(\cdot, t_\varepsilon) = v_{\lambda_k,\varepsilon}(\cdot, s_\varepsilon) + C$ is not constant in $\mathbb{T}^n$ for some λ_k and ε, then by Lemma 3.3, there exists $x_\varepsilon \in \mathbb{T}^n$ such that $u^{\lambda_k}(\cdot, t_\varepsilon) = v_{\lambda_k,\varepsilon}(\cdot, s_\varepsilon)$ is twice differentiable at x_ε, and

$$p_\varepsilon = \nabla u^{\lambda_k}(x_\varepsilon, t_\varepsilon) = \nabla v_{\lambda_k,\varepsilon}(x_\varepsilon, s_\varepsilon);$$
$$X_\varepsilon = \nabla^2 u^{\lambda_k}(x_\varepsilon, t_\varepsilon) = \nabla^2 v_{\lambda_k,\varepsilon}(x_\varepsilon, s_\varepsilon)$$

with $f(X_\varepsilon) \neq 0$. This implies that

$$(\tau_\varepsilon, p_\varepsilon, X_\varepsilon) \in J^{2,+} u^{\lambda_k}(x_\varepsilon, t_\varepsilon)$$
$$(\rho_\varepsilon, p_\varepsilon, X_\varepsilon) \in J^{2,-} v_{\lambda_k,\varepsilon}(x_\varepsilon, s_\varepsilon),$$

where

$$\tau_\varepsilon = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \frac{\sigma}{(T - t_\varepsilon)^2} \text{ and } \rho_\varepsilon = \frac{t_\varepsilon - s_\varepsilon}{\varepsilon}.$$

We again use Proposition 3.2 to obtain that

$$(\tau_\varepsilon, p_\varepsilon, X_\varepsilon) \in J^{2,+} u(x_\varepsilon + \frac{p_\varepsilon}{\lambda_k}, t_\varepsilon)$$

and

$$(\rho_\varepsilon, p_\varepsilon, X_\varepsilon) \in J^{2,-} v(x_\varepsilon - \varepsilon p_\varepsilon - \frac{p_\varepsilon}{\lambda_k}, s_\varepsilon).$$

We apply the definitions of sub- and supersolutions and get

$$\tau_\varepsilon + F(p_\varepsilon, \operatorname{sgn}^*(f(X_\varepsilon))) \leq 0,$$
$$\rho_\varepsilon + F(p_\varepsilon, \operatorname{sgn}_*(f(X_\varepsilon))) \geq 0.$$

Since $\operatorname{sgn}^*(f(X_\varepsilon)) = \operatorname{sgn}_*(f(X_\varepsilon)) = \operatorname{sgn}(f(X_\varepsilon))$, the difference of the inequalities yields a contradiction.

Case 2b. If $u^{\lambda_k}(\cdot, t_\varepsilon)$ and $v_{\lambda_k,\varepsilon}(\cdot, s_\varepsilon)$ are both constants for all λ_k and ε, then passing to a limit, we deduce, by the continuity assumption (3.1), that $u(\cdot, t_0)$ and $v(\cdot, s_0)$ are both constants for $t_0 = s_0 < T$, which contradicts the definition of T. $\square$

Remark 3.1. Here we used the time-continuity assumption (3.1) on the solutions. In the next section we provide sufficient conditions (cf. Proposition 4.1 and Remark 4.2) for the time-continuity of solutions to hold true.

An immediate consequence of Theorem 3.1 is the following uniqueness result.

Corollary 3.4 (Uniqueness before flattening)**.** *Assume (H1)–(H5). If u_1 and u_2 are both solutions of* (1.1) *with $u_1(\cdot, 0) = u_2(\cdot, 0) = u_0$, and u_1 and u_2 are uniformly continuous in time, then $u_1 = u_2$ in $\mathbb{T}^n \times [0, T_0)$ with $T_0 = \max\{T(u), T(v)\}$.*

The uniqueness after solutions becoming flat requires the extra condition (2) in the definition and their continuity near the first flattening time.

Theorem 3.5 (Global uniqueness)**.** *Assume (H1)–(H5). Then the solutions of* (1.1)*–*(1.2) *satisfying* (3.1) *are unique.*

Proof. Suppose that u_1 and u_2 are both solutions of (1.1). Assume that u_1 and u_2 satisfy the continuity assumption (3.1). By Corollary 3.4, we have $u_1 \leq u_2$ in $\mathbb{T}^n \times [0, T_0)$, where $T_0 = \max\{T(u), T(v)\}$. Our proof is complete if $T_0 = \infty$. If $T_0 < \infty$, it follows from the time-continuity assumption that $T(u_1) = T(u_2) = T_0$

and therefore $u_1(\cdot, T_0)$ and $u_2(\cdot, T_0)$ are both constant in $\mathbb{T}^n$. By Definition 2.1, we have, for all $x \in \mathbb{T}^n$ and $t \geq T_0$,

$$u_1(x,t) = u_2(x,t) = u_1(x, T_0) - C_F(t - T_0).$$

□

4. Existence of Solutions

In order to show the existence of solutions of (1.1) with $u(x, 0) = u_0(x)$ for all $x \in \mathbb{T}^n$, we consider an approximation with $\alpha > 0$:

$$u_t + F\left(\nabla u, |g(\nabla u, \nabla^2 u)|^{\alpha-1} g(\nabla u, \nabla^2 u)\right) = 0 \text{ in } \mathbb{T}^n \times (0, \infty) \tag{4.1}$$

with

$$u(x, 0) = u_0(x) \text{ for } x \in \mathbb{T}^n,$$

where $g : \mathbb{R}^n \times \mathbf{S}^n \to \mathbb{R}$ is a continuous function such that

$$\operatorname{sgn} g(p, X) = \operatorname{sgn} f(X) \text{ for any } p \in \mathbb{R}^n \text{ and } X \in \mathbf{S}^n. \tag{4.2}$$

The simplest choice of g is certainly

$$g(p, X) = f(X)$$

for all $p \in \mathbb{R}^n$ and $X \in \mathbf{S}^n$. We however allow g to depend also on p for our general applications later.

It is clear that (4.2), together with (H3), implies that $g(p, O) = 0$ for any $p \in \mathbb{R}^n$. Note that (4.1) is not singular when $\alpha > 0$. Therefore the classical theory of viscosity solutions applies under the assumptions below:

(H3') $g(p, X)$ is elliptic locally in p, i.e., for any $R > 0$, there exists a $\mu > 0$ such that

$$g(p, X_1) - g(p, X_2) \geq \mu \operatorname{tr}(X_1 - X_2)$$

for all $X_1 \geq X_2$ in $\mathbf{S}^n$ and $p \in B_R$.

(H4') $g : \mathbb{R}^n \times \mathbf{S}^n \to \mathbb{R}$ is continuous and $g(p, X)$ are locally Lipschitz continuous; that is, for any $R > 0$, there exists an $L_R > 0$ such that

$$|g(p_1, X) - g(p_2, X)| \leq K_R(|p_1 - p_2| + \|X_1 - X_2\|)$$

for all $p_1, p_2 \in B_R$ and $\|X_1\|, \|X_2\| \leq R$.

In particular, the usual comparison principle holds in this case and there exists a unique continuous viscosity solution u^α of (4.1) if u_0 is continuous in $\mathbb{T}^n$; consult [11] for more details.

We intend to send the limit, as $\alpha \to 0+$, to get a solution of (1.1) with the same initial condition. For later use, let us recall the relaxed half limits of u^α [11] for any given $(x, t) \in \mathbb{T}^n \times [0, \infty)$:

$$\begin{aligned}
\overline{u}(x,t) &= \limsup_{\alpha \to 0}{}^* u^\alpha(x,t) \\
&= \lim_{\delta \to 0} \sup \left\{ u^\alpha(y,s) : (y,s) \in \mathbb{T}^n \times [0,\infty),\ |x-y| + |t-s| \leq \delta,\ \alpha \leq \delta \right\}; \\
\underline{u}(x,t) &= \liminf_{\alpha \to 0}{}_* u^\alpha(x,t) \\
&= \lim_{\delta \to 0} \inf \left\{ u^\alpha(y,s) : (y,s) \in \mathbb{T}^n \times [0,\infty),\ |x-y| + |t-s| \leq \delta,\ \alpha \leq \delta \right\}.
\end{aligned}$$

Proposition 4.1 (Lipschitz continuity). *Assume (H1), (H2), (H3') and (H4'). Let u^α be the solution of (4.1) with Lipschitz initial value u_0, i.e., there exists $L > 0$ such that*

$$u_0(x) - u_0(y) \leq L|x-y| \text{ for all } x, y \in \mathbb{T}^n.$$

Then

$$|\overline{u}(x,t) - \overline{u}(y,s)| \leq L|x-y| + M_F|t-s|,$$
$$|\underline{u}(x,t) - \underline{u}(y,s)| \leq L|x-y| + M_F|t-s|,$$

for all $x, y \in \mathbb{T}^n$ and $t, s \geq 0$, where

$$M_F = \max\{|F(p,\theta)| : |p| \leq L, |\theta| \leq 1\}. \tag{4.3}$$

Proof. 1. We first show the Lipschitz continuity in space. For any $h \in \mathbb{T}^n$, let $v^\alpha(x,t) := u^\alpha(x+h,t) + L|h|$. It is clear that v is a viscosity solution of (1.1). Since u_0 is Lipschitz continuous and

$$v(x,0) = u_0(x+h) + Lh \geq u_0(x),$$

by comparison principle, we have $v^\alpha(x,t) \geq u^\alpha(x,t)$, which implies

$$u^\alpha(x+h,t) - u^\alpha(x,t) \geq -L|h|$$

for any $x, h \in \mathbb{T}^n$ and $\alpha > 0$. A symmetric argument yields

$$u^\alpha(x+h,t) - u^\alpha(x,t) \leq L|h|.$$

Taking the relaxed limits, we get the desired Lipschitz continuity for $\overline{u}$ and $\underline{u}$ in space.

2. We next prove the Lipschitz continuity in time. We approximate u_0 by a smooth function w_0 such that there exists a $\delta > 0$ satisfying

$$w_0 - \delta \leq u_0 \leq w_0 + \delta \text{ and } \max_{\mathbb{T}^n} |\nabla w_0| \leq L + \delta.$$

Since g is continuous, we may assume $\max_{\mathbb{T}^n} |g(\nabla w_0, \nabla^2 w_0)| \leq C$ for some $C > 0$. Set

$$M_\delta^\alpha := \max\{|F(p,\theta^\alpha)| : |p| \leq L + \delta, |\theta| \leq C\}.$$

It is not difficult to see that

$$w_\delta(x,t) = w_0(x) - \delta - M_\delta^\alpha t$$

is a subsolution of (4.1). Indeed we have

$$|\nabla w_\delta| \leq L + \delta, \tag{4.4}$$

and therefore

$$F(\nabla w_\delta, |g(\nabla w_\delta, \nabla^2 w_\delta)|^{\alpha-1} g(\nabla w_\delta, \nabla^2 w_\delta)) \leq M_\delta^\alpha. \tag{4.5}$$

Hence by comparison principle for (4.1), we get

$$u^\alpha(x,t) \geq w_\delta(x,t) = w_0(x) - \delta - M_\delta^\alpha t \geq u_0(x) - 2\delta - M_\delta^\alpha t. \tag{4.6}$$

One may construct in a similar way a supersolution to show

$$u^\alpha(x,t) \leq u_0(x) + 2\delta + M_\delta^\alpha t. \tag{4.7}$$

We next fix any $\tau \geq 0$. It is clear that the function

$$(x,t) \mapsto u^\alpha(x, t+\tau) + 2\delta + M_\delta^\alpha \tau$$

satisfies the equation (4.1) and its initial value is not less than u_0 due to (4.6). By comparison principle, we have for any $x \in \mathbb{T}^n$ and $t \geq 0$,

$$u^\alpha(x, t+\tau) \geq u^\alpha(x,t) - M_\delta^\alpha \tau - 2\delta.$$

Analogously, we deduce

$$u^\alpha(x, t+\tau) \leq u^\alpha(x,t) + M_\delta^\alpha \tau + 2\delta,$$

which amounts to saying that for any $x \in \mathbb{T}^n$, $t \geq \tau$, and $\alpha > 0$,

$$u^\alpha(x, t-\tau) \geq u^\alpha(x,t) - M_\delta^\alpha \tau - 2\delta.$$

Passing to the half relaxed limits as $\alpha \to 0$, we obtain, by the continuity assumptions (H2) and (H4'),

$$\overline{u}(x,t) \geq \overline{u}(x,s) - M_\delta |t-s| - 2\delta,$$
$$\overline{u}(x,t) \leq \overline{u}(x,s) + M_\delta |t-s| + 2\delta,$$

where $M_\delta = \max\{|F(p,\theta)| : |p| \leq L+\delta, |\theta| \leq 1\}$. We finally let $\delta \to 0$ and get

$$|\overline{u}(x,t) - \overline{u}(x,s)| \leq M_F |t-s|,$$

where M_F is given as in (4.3). One may similarly get

$$|\underline{u}(x,t) - \underline{u}(x,s)| \leq M_F |t-s|.$$

□

Remark 4.1. It follows from (4.6) and (4.7) that for all $x \in \mathbb{T}^n$

$$\overline{u}(x,0) \leq u_0(x) \text{ and } \underline{u}(x,0) \geq u_0(x).$$

Remark 4.2. Note that our result for the Lipschitz time-continuity of solutions still holds even when the initial value u_0 is merely continuous, provided that in addition to the assumptions in Proposition 4.1, we also assume the local boundedness of F, that is, for any $R \leq 0$, there exists a C_R depending on R such that

$$|F(p,y)| \leq C_R, \quad \text{for any } p \in \mathbb{R}^n, |y| \leq R. \tag{4.8}$$

In this case, we still have (4.5) without using (4.4), and the rest of the proof goes the same way.

Proposition 4.2 (Existence of subsolutions and supersolutions)**.** *Assume (H1), (H2), (H3') and (H4'). Assume* (4.2)*. Let u^α be the solution of* (4.1) *with continuous initial value u_0. Then $\overline{u}$ and $\underline{u}$ are respectively a subsolution and a supersolution of* (1.1)*.*

Proof. Let us prove $\overline{u}$ is a subsolution. The proof for $\underline{u}$ being a supersolution is symmetric.

Step 1. We first verify the condition (1) in the definition of subsolutions. Suppose that $u - \phi$ attains a maximum at some $(x_0, t_0) \in \mathbb{T}^n \times (0, \infty)$. Then our verification follows the classical stability theorem of viscosity solutions. Indeed, suppose that $f(\nabla^2\phi) \geq 0$ at (x_0, t_0). Then we aim to show that

$$\phi_t(x_0,t_0) + F(\nabla\phi(x_0,t_0), 1) \leq 0. \tag{4.9}$$

To this end, we set

$$\tilde{\phi}(x,t) = \phi(x,t) + a|x-x_0|^2$$

for $a > 0$. Then $\overline{u} - \tilde{\phi}$ attains a strict maximum at (x_0, t_0) and

$$\text{sgn}\ f(\nabla^2\tilde{\phi}(x_0, t_0)) = \text{sgn}\ g\left(\nabla\tilde{\phi}(x_0, t_0), \nabla^2\tilde{\phi}(x_0, t_0)\right) = 1$$

due to (H2) and (4.2).

Then there exist $(x_\alpha, t_\alpha) \to (x_0, t_0)$ as $\alpha \to 0$ such that $u^\alpha - \tilde{\phi}$ attains a maximum at (x_α, t_α). It is obvious that there exist α_0 and $\sigma > 0$ such that

$$g(\nabla\tilde{\phi}(x_\alpha, t_\alpha), \nabla^2\tilde{\phi}(x_\alpha, t_\alpha)) > \sigma$$

for all $0 < \alpha < \alpha_0$. Since u^α is a subsolution of (4.1), we have

$$\tilde{\phi}_t + F(\nabla\tilde{\phi}, |g(\nabla\tilde{\phi}, \nabla^2\tilde{\phi})|^{\alpha-1} g(\nabla\tilde{\phi}, \nabla^2\tilde{\phi})) \leq 0,$$

at (x_α, t_α). Passing to the limit $\alpha \to 0$, we get at (x_0, t_0)

$$\tilde{\phi}_t + F(\nabla\tilde{\phi}, \text{sgn}(g(\nabla\tilde{\phi}, \nabla^2\tilde{\phi}))) \leq 0,$$

which is essentially (4.9), since $\nabla\tilde{\phi}(x_0, t_0) = \nabla\phi(x_0, t_0)$ and

$$\text{sgn}\, g(\nabla\tilde{\phi}, \nabla^2\tilde{\phi}) = 1 \text{ at } (x_0, t_0).$$

Step 2. We next show that whenever there exists a $t_0 \geq 0$ such that $\overline{u}(\cdot, t_0)$ is a constant, then

$$\overline{u}(x, t) \leq \overline{u}(x, t_0) - C_F(t - t_0) \text{ for any } x \in \mathbb{T}^n \text{ and } t \geq t_0. \tag{4.10}$$

In fact, $\overline{u}(\cdot, t_0)$ being a constant implies, due to compactness of $\mathbb{T}^n$, that for any $\delta > 0$, there is an $\alpha_\delta > 0$ such that $u^\alpha(\cdot, t_0) \leq \overline{u}(\cdot, t_0) + \delta$ for any $\alpha \in (0, \alpha_\delta)$. It is clear that $w(x, t) = \overline{u}(x, t_0) - C_F(t - t_0) + \delta$ is a solution of (4.1) in $\mathbb{T}^n \times (T, \infty)$ with $w(x, t_0) \geq u^\alpha(x, t_0)$ when $\alpha < \alpha_\delta$. Then, by comparison, we have

$$u^\alpha(x, t) \leq \overline{u}(x, t_0) - C_F(t - t_0) + \delta$$

for any $(x, t) \in \mathbb{T} \times [t_0, \infty)$ and $\alpha \in (0, \alpha_\delta)$, which yields

$$\overline{u}(x, t) \leq \overline{u}(x, t_0) - C_F(t - t_0) + \delta.$$

By sending $\delta \to 0$, we get (4.10). □

Theorem 4.3 (Convergence)**.** *Assume that F, f, g satisfy (H1)–(H5), (H3'), (H4') and (4.2). Let u^α be the unique solution of (4.1) with Lipschitz initial value u_0. Then $u^\alpha \to u$ locally uniformly in $\mathbb{T}^n \times [0, \infty)$ as $\alpha \to 0+$, where u is the unique Lipschitz continuous solution of (1.1) with $u(x, 0) = u_0(x)$ for all $x \in \mathbb{T}^n$.*

Remark 4.3. As stated in the following corollary, the assumptions (H3')–(H4') and (4.2) are not needed if one chooses $g(p, X) = f(X)$. It is clear that they are implied by (H3)–(H4).

Corollary 4.4 (Convergence and Existence)**.** *Assume that F and f satisfy (H1)–(H5). Let u^α be the unique solution of (4.1) with Lipschitz initial value u_0 and $g(p, X) = f(X)$. Then $u^\alpha \to u$ locally uniformly in $\mathbb{T}^n \times [0, \infty)$ as $\alpha \to 0+$, where u is the unique Lipschitz continuous solution of (1.1) with $u(x, 0) = u_0(x)$ for all $x \in \mathbb{T}^n$.*

Proof of Theorem 4.3. By Proposition 4.2, the semi relaxed limits $\underline{u}$ and $\overline{u}$ are respectively a supersolution and a subsolution of (1.1). Moreover it follows from Proposition 4.1 that $\underline{u}$ and $\overline{u}$ are continuous in time.

Since $\overline{u}(x,0) \leq u_0(x,0) \leq \underline{u}(x,0)$, it follows from Theorem 3.1, that $\overline{u} \leq \underline{u}$ in $\mathbb{T}^n \times [0,T_0)$, where $T_0 = \max\{T(\underline{u}), T(\overline{u})\}$. The proof is completed if $T_0 = \infty$.

If instead $T_0 \neq \infty$, we have $\underline{u} = \overline{u}$ in $\mathbb{T}^n \times [0,T_0]$ due to the continuity of $\overline{u}$ and $\underline{u}$ in time. This implies that $\overline{u} = \underline{u}$ is a continuous solution in $\mathbb{T}^n \times [0,T_0)$. It is also clear that $T(\underline{u}) = T(\overline{u}) = T_0$, which implies that

$$\overline{u}(\cdot, T_0) = \underline{u}(\cdot, T_0) \equiv C \text{ in } \mathbb{T}^n$$

for some $C \in \mathbb{R}$. By applying Definition 2.1(2) to $\overline{u}$ and $\underline{u}$ respectively, we obtain that

$$\overline{u}(x,t) = \underline{u}(x,t) \equiv C - C_F(t - T_0)$$

for all $x \in \mathbb{T}^n$ and $t \geq T_0$. It is easily seen that $u = \overline{u} = \underline{u}$ is now the unique solution.

□

Remark 4.4. As mentioned in Remark 4.2, by adding the assumption (4.8), we are able to construct viscosity solutions which are Lipschitz continuous in time and continuous in space, with continuous initial data.

Example 4.5. *One important example is the motion of graph by the sign of mean curvature in one space dimension:*

$$\begin{cases} u_t - \sqrt{1+u_x^2}\,\mathrm{sgn}(u_{xx}) = 0 & \text{in } \mathbb{T} \times (0,\infty), \quad (4.11) \\ u(x,0) = u_0(x) & \text{in } \mathbb{T}. \quad (4.12) \end{cases}$$

In order to apply our general results above, we let

$$F(p,y) = -\sqrt{1+|p|^2}y$$

and

$$f(X) = \mathrm{tr}\, X = X$$

for any $p \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $X \in \mathbb{R}$. It is not difficult to verify that such F and f satisfy (H1)–(H5). Indeed,

$$F(p,y_1) - F(p,y_2) \leq -\sqrt{1+|p|^2}(y_1 - y_2) \leq 0$$

if $y_1 \geq y_2$ and

$$f(X_1) - f(X_2) \geq \mathrm{tr}(X_1 - X_2)$$

for all $X_1 \geq X_2$. Therefore Theorem 3.1 and Theorem 3.5 hold in this special case. In order to get the existence of a unique continuous solution, we need to approximate sgn f *and then apply Theorem 4.3. There are multiple choices of the approximation function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We take the one most consistent with our motivation:*

$$g(p,X) = \frac{X}{(1+p^2)^{\frac{3}{2}}}, \tag{4.13}$$

because the approximating equation (4.1) in this case describes the motion of graph by curvature raised to the power α:

$$u_t - \sqrt{1+u_x^2}\left(\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right)^{\alpha} = 0.$$

(Rigorously speaking, the power function x^α should be understood as $|x|^{\alpha-1}x$ to maintain the parabolicity.)

Note that (4.2) is fulfilled, since $\operatorname{sgn} g(p, X) = \operatorname{sgn} X = \operatorname{sgn} f(X)$. *It is also easily seen that (H3') and (H4') are satisfied as well. We may apply Theorem 4.3 to get the existence as well. We conclude this example by presenting the following theorem:*

Theorem 4.6. *Assume that u_0 is Lipschitz continuous in $\mathbb{T}$. Then there exists a unique Lipschitz continuous solution u of (4.11)–(4.12). Moreover, if u^α is the solution of (4.1) with g given in (4.13), then $u^\alpha \to u$ locally uniformly in $\mathbb{T}\times[0,\infty)$ as $\alpha \to 0$.*

5. Stability of Solutions

Let us consider the stability of the Lipschitz solution of (1.1)–(1.2) with respect to Lipschitz continuous initial data u_0.

Theorem 5.1 (Stability with respect to Lipschitz initial value)**.** *Assume that F and f satisfy (H1)–(H5). Let u^ε be the unique solution of (1.1) with initial value $u(\cdot, 0) = u_0^\varepsilon$ in $\mathbb{T}^n$. Assume that*

$$|u_0^\varepsilon(x) - u_0^\varepsilon(y)| \leq L_\varepsilon |x-y| \tag{5.1}$$

for all $x, y \in \mathbb{T}^n$ with $L_\varepsilon > 0$ bounded in $\varepsilon > 0$. If $u_0^\varepsilon \to u_0$ uniformly in $\mathbb{T}^n$ as $\varepsilon \to 0$, then $u^\varepsilon \to u$ locally uniformly in $\mathbb{T}^n \times [0,\infty)$ as $\varepsilon \to 0$, where u is the unique Lipschitz continuous solution of (1.1) with $u(x,0) = u_0(x)$ for all $x \in \mathbb{T}^n$.

Proof. We take the relaxed limits

$$\overline{u} = \limsup_{\varepsilon\to 0}{}^{*}\, u^\varepsilon \quad \text{and} \quad \underline{u} = \liminf_{\varepsilon\to 0}{}_{*}\, u^\varepsilon.$$

By Proposition 4.1, we obtain

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \leq L_\varepsilon|x-y| + M_F^\varepsilon|t-s|$$

for all $x, y \in \mathbb{T}^n$ and $t, s \in [0,\infty)$, where

$$M_F^\varepsilon = \max\{|F(p,\theta)| : |p| \leq L_\varepsilon, |\theta| \leq 1\}.$$

It is then clear that $M_F^\varepsilon > 0$ is uniformly bounded in $\varepsilon > 0$, which implies that $\overline{u}$ and $\underline{u}$ are both Lipschitz continuous in $\mathbb{T}^n \times [0,\infty)$ and $\overline{u}(\cdot,0) = \underline{u}(\cdot,0) = u_0$ in $\mathbb{T}^n$.

We next show that $\overline{u}$ is a subsolution of (1.1) and $\underline{u}$ is a supersolution of (1.1). We again only prove the former. Note that it is standard to show that $\overline{u}$ satisfies (1) in Definition 2.1, since under the assumptions on F and f, we have

$$F_*(p, \operatorname{sgn}(f(X))) = F(p, \operatorname{sgn}^*(f(X)))$$

for all $p \in \mathbb{R}^n$ and $X \in \mathbf{S}^n$. The verification of (2) in Definition 2.1 is essentially the same as Step 2 in the proof of Proposition 4.2.

Following the proof of Theorem 4.3, we immediately obtain that $\overline{u} = \underline{u}$ in $\mathbb{T}^n \times [0,\infty)$, which implies the locally uniform convergence $u^\varepsilon \to u$ as $\varepsilon \to 0$. $\square$

6. Diffusion Driven by Sign in 1D

Our analysis applies to the diffusion by the sign of second derivative in one space dimension:

$$\begin{cases} u_t - \operatorname{sgn}(u_{xx}) = 0 & \text{in } \mathbb{T}\times(0,\infty), \quad (6.1)\\ u(x,0) = u_0(x) & \text{for all } x\in\mathbb{T}. \quad (6.2)\end{cases}$$

Uniqueness and existence of solutions are immediate consequences of Theorem 3.1 and Theorem 4.3.

Theorem 6.1. *Let u^α be the solution of the following equation*

$$\begin{cases} u_t - |u_{xx}|^{\alpha-1}u_{xx} = 0 & \textit{in } \mathbb{T}\times(0,\infty), \quad (6.3)\\ u(x,0) = u_0(x) & \textit{for all } x\in\mathbb{T} \quad (6.4)\end{cases}$$

with u_0 Lipschitz continuous. Then u^α converges uniformly, as $\alpha\to 0+$, to the unique solution u of (6.1)–(6.2).

Proposition 6.2. *Let u be the solution of (6.1)–(6.2) with u_0 Lipschitz continuous in $\mathbb{T}$. Then u is Lipschitz continuous with Lipschitz constant 1, i.e.,*

$$|u(x,t)-u(x,s)|\le|t-s|$$

for all $t,s\ge 0$.

Proposition 6.3. *Suppose that $u\in C(\mathbb{T}\times[0,\infty))$ is the solution of (6.1). Then*

$$\min_{x\in\mathbb{T}} u(x,t) - \min_{x\in\mathbb{T}} u(x,s) \ge t-s \tag{6.5}$$

for any $0\le s\le t\le T(u)$. Similarly, u satisfies

$$\max_{x\in\mathbb{T}} u(x,t) - \max_{x\in\mathbb{T}} u(x,s) \le -(t-s) \tag{6.6}$$

for any $0\le s\le t\le T(u)$.

Proof. Let $\psi(t)=\min_{x\in\mathbb{T}} u(x,t)$. If there exists $\phi\in C^1([0,\infty))$ such that $\psi(t)-\phi(t)$ attains a minimum at $t_0\in(0,T(u))$, then $u(x,t)-\phi(t)$ also attains a minimum at $I\times\{t_0\}$, where

$$I=\{x\in\mathbb{T} : u(x,t_0)=\min u(x,t_0)\}.$$

Since $I\neq\mathbb{T}$, we may take $\beta,h\in\mathbb{R}$ and $a>0$ such that

$$u(x,t)-\phi(t)+\beta(t-t_0)^2+h(x-x_0)-a(x-x_0)^2$$

attains a minimum at $(\hat{x},\hat{t})$, where

$$(\hat{x},\hat{t})\to(x_0,t_0) \text{ for some } x_0\in I \text{ as } \beta,h,a\to 0.$$

Since u is a supersolution of (6.1), we get

$$\phi_t(\hat{t})-\beta(\hat{t}-t_0)\ge 1.$$

Sending $\beta,h,a\to 0$, we end up with

$$\phi_t(t_0)\ge 1.$$

Our argument above shows that ψ is a viscosity supersolution of

$$\psi_t\ge 1 \text{ in } (0,T(u)).$$

Hence, it is clear that $\psi(t)-t$ is increasing, which completes the proof of (6.5). A symmetric argument works for (6.6) as well. ☐

Proposition 6.4. *Let u be the solution of (6.1)–(6.2) with $u(x,0)=u_0(x)$. Then*

$$\min_{x\in\mathbb{T}} u(x,t)=\min_{x\in\mathbb{T}} u_0(x)+t \text{ for all } t\in[0,T(u)) \tag{6.7}$$

and

$$\max_{x\in\mathbb{T}} u(x,t)=\max_{x\in\mathbb{T}} u_0(x)-t \text{ for all } t\in[0,T(u)). \tag{6.8}$$

Moreover, if $u(x_0,s)=\min_{x\in\mathbb{T}} u(x,s)$ (resp., $u(x_0,s)=\max_{x\in\mathbb{T}} u(x,s)$) for some $x_0\in\mathbb{T}$ and $s<T(\underline{u})$, then

$$u(x_0,t)=\min_{x\in\mathbb{T}} u(x,t) \text{ for all } t\in[s,T(u)).$$

(Resp.,

$$u(x_0,t)=\max_{x\in\mathbb{T}} u(x,t) \text{ for all } t\in[0,T(u)).)$$

Proof. By Proposition 6.3, we get

$$\min u(x,t)\geq \min u(x,s)+t-s$$

for all $s\leq t\leq T(u)$. On the other hand, by the Lipschitz continuity, we have

$$\min u(x,t)\leq \min u(x,s)+t-s$$

for all $t,s\geq 0$. Combining these two inequalities, we get

$$\min u(x,t)=\min u(x,s)+t-s$$

for all $t,s\geq 0$. Taking $s=0$, we get (6.7).

The other statements in this proposition follow easily. Suppose that $u(x_0,s)=\min_{x\in\mathbb{T}} \underline{u}(x,s)$ for some $x_0\in\mathbb{T}$ and $s<T(u)$. Then by Proposition 6.2 again, we have

$$u(x_0,t)\leq u(x_0,s)+t-s=\min_{x\in\mathbb{T}} u(x,s)+t-s$$

for any $s\leq t<T(u)$, which by (6.7) implies

$$u(x_0,t)=\min_{x\in\mathbb{T}} u(x,t).$$

The proof for the remaining part follows in an analogous way. □

The proposition above shows very important properties of the limit u. It turns out that the solution $u\in C([0,T))$ satisfies the minimizer/maximizer increasing property: whenever $u(x_0,s)=\min_{x\in\mathbb{T}} u(x,s)$ (or $u(x_0,s)=\max_{x\in\mathbb{T}} u(x,s)$) for some $x_0\in\mathbb{T}$ and $0\leq s<T$, we have

$$u(x_0,t)=\min_{x\in\mathbb{T}} u(x,t) \text{ or } u(x_0,t)=\max_{x\in\mathbb{T}} u(x,t) \text{ for all } t\in[s,T).$$

Obviously, the minimizer/maximizer increasing property implies that $u(x,t)$ is constant in x for all $t\geq s$ provided that $u(x,s)$ is constant in x.

As mentioned in Introduction, the space derivative of the solution of (6.1)–(6.2) is formally the unique solution of the one dimensional total variation flow equation. We justify this observation in the following special case.

Proposition 6.5. *Let $u_0\in C^{1,1}(\mathbb{R})$ be periodic with the least period 1. Assume further that u_0 is of class piecewise C^2 in the following sense: there exist open intervals $I_1=(a_0,b_0)$ and $I_2=(c_0,d_0)$ such that u_0 is of C^2 in I_i with $i=1,2$ and in $I_3=[b_0,c_0]\cup[d_0,a_0+1]$, and satisfies*

(1) $u_0'' < 0$ *in* I_1,
(2) $u_0'' > 0$ *in* I_2,
(3) $u_0'' = 0$ *in* I_3.

Let u *be the solution of* (6.1)–(6.2). *Then* $v = u_x$ *is a Lipschitz function in* $\mathbb{R}\times(0,\infty)$ *and it is the unique solution of*

$$v_t = (\operatorname{sgn}(v_x))_x \text{ in } \mathbb{T}\times(0,\infty) \tag{6.9}$$

with $v(\cdot,0) = u_0'$.

Proof. The solution u can be explicitly expressed. Set $T = \frac{1}{2}(\max u_0 - \min u_0)$. By the implicit function theorem, there exist C^1 functions $a, b : [0,\infty) \to (a_0, b_0)$ and $c, d : [0,\infty) \to (c_0, d_0)$ with $a \le b$ and $c \le d$ satisfying, for any $t \in [0, T)$,

$$\begin{aligned} u_0(c(t)) - u_0(b(t)) + 2t &= u_0'(b(t))(c(t) - b(t)) \\ &= u_0'(c(t))(c(t) - b(t)); \\ u_0(a(t)+1) - u_0(d(t)) - 2t &= u_0'(d(t))(a(t) - d(t) + 1) \\ &= u_0'(a(t)+1)(a(t) - d(t) + 1). \end{aligned} \tag{6.10}$$

Also, we have $a(0) = a_0$, $b(0) = b_0$, $c(0) = c_0$, $d(0) = d_0$ and

$$a(t) - b(t) \to 0, \quad c(t) - d(t) \to 0 \text{ as } t \to T.$$

It is not difficult to find that the unique solution u in one period $[a(t), a(t)+1)$ is

$$u(x,t) = \begin{cases} u_0(x) - t, & \text{for } (x,t) \in (a(t), b(t)) \times [0,T), \\ u_0(x) + t, & \text{for } (x,t) \in (c(t), d(t)) \times [0,T), \\ u_0(b(t)) + u_0'(b(t))(x - b(t)) - t & \text{for } (x,t) \in [b(t), c(t)] \times [0,T), \\ u_0(d(t)) + u_0'(d(t))(x - d(t)) + t & \text{for } (x,t) \in [d(t), a(t)+1] \times [0,T), \\ \frac{1}{2}\max u_0 + \frac{1}{2}\min u_0 & \text{for } (x,t) \in [a(t), a(t)+1) \times [T,\infty). \end{cases}$$

Therefore

$$v(x,t) = u_x(x,t) = \begin{cases} v_0(x) & \text{for } x \in (a(t), b(t)) \cup (c(t), d(t)) \text{ and } t \in [0,T), \\ v_0(b(t)) & \text{for } x \in [b(t), c(t)] \text{ and } t \in [0,T), \\ v_0(d(t)) & \text{for } x \in [d(t), a(t)+1] \text{ and } t \in [0,T), \\ 0 & \text{for } x \in [a(t), a(t)+1] \text{ and } t \ge T, \end{cases}$$

where $v_0 = u_0'$ in $\mathbb{R}$. One can show that v is the unique viscosity solution of (6.9); see [15, Example 2.12]. Indeed, it suffices to show that

$$\int_{h_0}^{h(t)} \left(B^{-1}(\eta) - A^{-1}(\eta)\right) d\eta = 2t, \tag{6.11}$$

where $h(t) = v_0(b(t)) = v_0(c(t))$, $h_0 = h(0)$ and

$$\begin{aligned} A(x) &= v_0(x) \text{ for all } x \in [a_0, b_0]; \\ B(x) &= v_0(x) \text{ for all } x \in [c_0, d_0]. \end{aligned}$$

Note that $a(t), b(t), c(t)$ and $d(t)$ are all monotone in t. By change of variable $\eta = h(s) = v_0(c(s))$ and integration by parts, we have

$$\begin{aligned}\int_{h_0}^{h(t)} B^{-1}(\eta)\, d\eta =& v_0^{-1}(h(t))h(t) - v_0^{-1}(h_0)h_0 - \int_0^t u_0'(c(s))c'(s)\, ds\\ =& c(t)u_0'(c(t)) - c_0u_0'(c_0) - u_0(c(t)) + u_0(c_0).\end{aligned}$$

Similarly, we get

$$\int_{h_0}^{h(t)} A^{-1}(\eta)\, d\eta = b(t)u_0'(b(t)) - b_0u_0'(b_0) - u_0(b(t)) + u_0(b_0).$$

In view of (6.10), we obtain (6.11) by combining the equalities above.

□

Remark 6.1. The proof above shows that $C^{1,1}$ is the highest possible regularity of u that can be expected in general, even when u_0 is smooth.

We conjecture that the space derivative of the Lipschitz continuous solution u corresponds to a (discontinuous) solution of (6.9) also for general Lipschitz continuous function u_0. An example is given below in this case. We also refer the reader to [27] for a direct analysis on discontinuous solutions of this equation.

Example 6.6. *Let*

$$u_0(x) = \begin{cases} 4m + 1 - 2x & \text{if } x \in [2m, 2m+1) \text{ for all } m \in \mathbb{Z},\\ 2x - 4m - 3 & \text{if } x \in [2m+1, 2m+2] \text{ for all } m \in \mathbb{Z}.\end{cases}$$

One may easily verify that the function u below is a solution of (6.1)–(6.2):

$$u(x,t) = \begin{cases} (t-1)(2x - 4m - 1) & \text{if } (x,t) \in [2m, 2m+1) \times [0,1] \text{ for } m \in \mathbb{Z},\\ (1-t)(2x - 4m - 3) & \text{if } (x,t) \in [2m+1, 2m+2] \times [0,1] \text{ for } m \in \mathbb{Z},\\ 0 & \text{for } (x,t) \in \mathbb{R} \times (1,\infty).\end{cases} \tag{6.12}$$

By Theorem 3.5, u is also the unique Lipschitz continuous solution of (6.1)–(6.2). It is clear that $v = u_x$ exists almost everywhere in $\mathbb{R} \times (0,\infty)$ and

$$v(x,t) = \begin{cases} 2t - 2 & \text{if } (x,t) \in (2m, 2m+1) \times [0,1] \text{ for } m \in \mathbb{Z},\\ 2 - 2t & \text{if } (x,t) \in (2m+1, 2m+2) \times [0,1] \text{ for } m \in \mathbb{Z},\\ 0 & \text{for } (x,t) \in \mathbb{R} \times (1,\infty),\end{cases}$$

which coincides with the solution of (6.9) with

$$v(x,0) = \begin{cases} -2 & \text{if } x \in (2m, 2m+1) \text{ for } m \in \mathbb{Z},\\ 2 & \text{if } x \in (2m+1, 2m+2) \text{ for } m \in \mathbb{Z};\end{cases}$$

see, for example, the behavior of step-like solutions to (6.9) in [5].

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