OSCILLATION-INDUCED BLOW-UP TO THE MODIFIED CAMASSA–HOLM EQUATION WITH LINEAR DISPERSION

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Abstract. In this paper, we provide a blow-up mechanism to the modified Camassa-Holm equation with varying linear dispersion. We first consider the case when linear dispersion is absent and derive a finite-time blow-up result with an initial data having a region of mild oscillation. A key feature of the analysis is the development of the Burgers-type inequalities with focusing property on characteristics, which can be deduced from tracing the ratio between solution and its gradient. Using the continuity and monotonicity of the solutions, we then extend this blow-up criterion to the case of negative linear dispersion, and determine that the finite time blow-up can still occur if the initial momentum density is bounded below by the magnitude of the linear dispersion and the initial datum has a local mild-oscillation region. Finally, we demonstrate that in the case of non-negative linear dispersion the formation of singularities can be induced by an initial datum with a sufficiently steep profile. In contrast to the Camassa-Holm equation with linear dispersion, the effect of linear dispersion of the modified Camassa-Holm equation on the blow-up phenomena is rather delicate.

Keywords: Modified Camassa-Holm equation, oscillation-induced blow-up, integrable system, breaking wave, sign-changing momentum.

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1. Introduction

Wave motion is ubiquitous in nature and is one of the broadest subjects. Most of wave phenomena might be described by mathematical models that are based on certain hyperbolic-type, nonlinear dispersive partial differential equations. A fundamental question in the study of those models is when and how a singularity can form, or more precisely, whether a wave breaks (a wave profile remains bounded while its slope becomes unbounded in finite time) in certain time. Breaking waves, both whitecaps and surf, are commonly observed in the ocean, which provide a source of turbulent energy to mix the upper layers of the ocean [25].

One common ingredient in all the current wave breaking techniques is to seek an appropriate quantity depending on the information of the solution and to obtain a differential inequality for it which can generate finite-time blow-up with well chosen data. However the way of finding such a quantity is very much case-dependent [1, 7]. For quasilinear equations, in many cases the solution u remains well-defined at each point, but its gradient ∇u may become infinite in finite time. Typical examples of this situation can be found in the case of scalar conservation laws and some non-local transport type equations, where blow-ups are due to focusing of characteristics.

In this paper, we would like to investigate the blow-up mechanism of a cubic quasilinear dispersive equation, namely the modified Camassa-Holm (mCH) equation:

$$\begin{cases} m_t + \left((u^2 - u_x^2)m\right)_x + \gamma u_x = 0, \\ u(0,x) = u_0(x), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where

$$m = (1 - \partial_x^2)u = u - u_{xx} \tag{1.2}$$

represents *momentum density* of the system, and $\gamma \in \mathbb{R}$ characterizes the effect of the linear dispersion.

The mCH equation (1.1) can be derived by applying the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the focusing modified Korteweg-de Vries (mKdV) equation [13, 15, 21]. To see this, consider the focusing mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x.$$

In the bi-Hamiltonian form it can be written as

$$u_t = J_1 \frac{\delta H_2}{\delta u} = J_2 \frac{\delta H_1}{\delta u},$$

where the Hamiltonian operators J_1 and J_2 are expressed as

$$J_1 = \partial_x, \quad \text{and} \quad J_2 = \partial_x^3 + \partial_x u \partial_x^{-1} u \partial_x$$

with the Hamiltonians

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} u^2 dx, \quad \text{and} \quad H_2 = \frac{1}{2}\int_{\mathbb{R}} \left(\frac{1}{4}u^4 - u_x^2\right) dx.$$

Now define two new Hamiltonian operators

$$\hat{J}_1 = \partial_x - \partial_x^3, \qquad \text{and} \quad \hat{J}_2 = \alpha_1 \partial_x + \alpha_2 \partial_x^3 + \alpha_3 \partial_x u \partial_x^{-1} u \partial_x.$$

Applying the recursion operator $R = \hat{J}_2 \hat{J}_1^{-1}$ to the seed equation $u_t = u_x$ we deduce a hierarchy of an integrable system, namely,

$$u_t = R^n u_x = (\hat{J}_2 \hat{J}_1^{-1})^n u_x.$$

Let $u = (1 - \partial_x^2)v$ and choose $n = 1$. Then the above system becomes

$$u_t = \hat{J}_2 \hat{J}_1^{-1} u_x = \hat{J}_2 v = \frac{\alpha_3}{2}\left[u(v^2 - v_x^2)\right]_x + \alpha_1 v_x + \alpha_2 v_{xxx}.$$

The mCH equation (1.1) is then obtained from the above equation by letting $\alpha_1 = -\gamma$, $\alpha_2 = 0$, $\alpha_3 = -2$. Therefore the mCH equation is formally integrable possessing a bi-Hamiltonian structure. A Lax representation of (1.1) was later constructed [22, 24], making the mCH equation amenable to the method of inverse scattering. Physically, the mCH equation (1.1) models the unidirectional propagation of surface waves in shallow water over a flat bottom [12], where $u(t,x)$ represents the free surface elevation in dimensionless variables.

The mCH equation can be viewed as a cubic extension of the well-known Camassa-Holm (CH) equation

$$m_t + u m_x + 2 m u_x = 0, \qquad m = u - u_{xx}, \tag{1.3}$$

which was proposed as a model describing the uni-directional propagation of shallow water waves [5, 10, 16] and axially symmetric waves in hyperelastic rods [9, 11]. The CH equation was originally constructed by using the recursion operator method [16], and can also be derived by applying tri-Hamiltonian duality to the bi-Hamiltonian structure of the KdV equation [21]. One of the remarkable distinctive properties that the CH equation has, in contrast to the KdV equation, is the wave-breaking phenomenon: the solution u remains bounded while its slope u_x becomes infinite in finite time. It has been shown that the wave-breaking can be triggered by an initial sign-changing momentum density

$$m_0(x) = (1 - \partial_x^2)u_0 = u_0(x) - u_0''(x).$$

On the other hand, it can be prevented if m_0 does not change sign, c.f. [7, 8, 18, 20, 26]. In deriving such a blow-up, the global information of the data (such as conservation laws, integrability, antisymmetry, etc.) is needed due to the non-local feature of the equation.

The geometric formulation, integrability, local well-posedness, blow-up criteria and singularity formation, existence of peaked solitons (peakons), and the stability of single peakons and periodic peakons to the mCH equation (1.1) were studied recently in [14, 17, 18, 23]. It is shown that even if the initial momentum density $m_0(x)$ does not change sign, the solutions to the Cauchy problem (1.1) can still blow up in finite time, in contrast to the CH equation.

It is noticed that a new type of blow-up criteria in the sense of the so called "local-in-space" is recently established in [2, 3, 4], which illustrates that the blow-up condition is merely imposed on a small neighborhood of a single point in space variable, and hence local perturbation of data around that point does not prevent the singularity formation. The main idea used there is to track the dynamics of two linear combinations $u \pm \beta u_x$ (for the CH equation, $\beta = 1$) of u and u_x along the characteristics and to make use of some convolution estimates to bound the non-local terms by the local quantities. The blow-up result then follows from the continuity and monotonicity argument.

The goal of the present paper is two fold. First we want to understand how the local structure of the initial profile can affect the singularity formation. Similar to the case of the CH equation, where the blow-up is indicated from the loss of the lower bound on u_x, the blow-up criterion of the mCH equation asserts that blow-up occurs if and only if mu_x becomes unbounded from below (c.f. [17]). Applying the characteristics method we find that the evolution of mu_x along the trajectories of the flow map involves competition between u and u_x, along with some non-local convolutions. If any kind of convolution estimates as in the case of the CH equation is available, one may expect to produce a local-in-space blow-up. Unfortunately, the cubic nonlinearity makes it hard to control the non-local convolution terms, and hence the problem becomes more subtle and it is difficult to employ the approach in [2, 3, 4] directly. However in the case when $\gamma \leq 0$, the lower bound of the momentum density m is preserved under the flow, c.f. (2.5) and (4.5), which can be used to bound the non-local terms in terms of the local quantities. The resulting estimates become an interplay between u and u_x, which can in fact be measured by u_x/u. Physically, this quantity is related to the local oscillation of the datum. We are able to show that when the momentum density is bounded below by the weak dispersive index, blow-up can be induced by "local mild oscillations", independent of the size of the oscillation region. Our main results on the oscillation-induced blow-up can be formulated in the following two theorems.

Theorem 1.1 (Oscillation-induced blow-up: $\gamma = 0$)**.** *Suppose $\gamma = 0$. Let $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ for $s > 1/2$ and $m_0(x) \geq 0, \forall x \in \mathbb{R}$. Assume there exists a point $x_0 \in \mathbb{R}$ such that*

$$m_0(x_0) > 0, \qquad \text{and} \qquad u_{0,x}(x_0) \leq -\frac{1}{\sqrt{2}} u_0(x_0). \tag{1.4}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2m_0(x_0)u_{0,x}(x_0)}. \tag{1.5}$$

Remark 1.2. *(i) By the blow-up criterion in Lemma 2.2 and a symmetric argument one can also investigate the formation of singularity with moment density data $m_0 \leq 0$. More specifically, assume $m_0(x) \leq 0, \forall x \in \mathbb{R}$. If there exists $x_0 \in \mathbb{R}$ such that*

$$m_0(x_0) < 0, \qquad \text{and} \qquad u_{0,x}(x_0) \geq -\frac{1}{\sqrt{2}} u_0(x_0),$$

then the solution blows up in finite time and the estimate on the blow-up time T^* is given in the same form as (1.5)*.*

(ii) Our blow-up result may in the mean time hint on some local information about the global solutions. Suppose that u is a given global solution to (1.1) *with initial datum satisfying $m_0(x) > 0$ for all $x \in \mathbb{R}$. Then from the above theorem we know that at any time t it follows that*

$$u_x(t,x) > -\frac{1}{\sqrt{2}} u(t,x), \qquad \text{for all } x \in \mathbb{R}.$$

Hence define an auxiliary function $\phi(t,x) = e^{\frac{x}{\sqrt{2}}} u(t,x)$ and we can compute the x-derivative to get

$$\phi_x(t,x) = e^{\frac{x}{\sqrt{2}}} \left(u_x + \frac{1}{\sqrt{2}} u \right)(t,x) > 0.$$

Thus for each fixed t, $x \mapsto \phi(t,x)$ is strictly increasing, leading to the following one-sided estimate of $u(t,x)$ in terms of the information of u at $x = 0$:

$$u(t,x) > e^{-\frac{x}{\sqrt{2}}} u(t,0), \quad \text{for } x > 0; \qquad u(t,x) < e^{-\frac{x}{\sqrt{2}}} u(t,0), \quad \text{for } x < 0.$$

Moreover the sign condition $m(t,x) > 0$ implies $|u_x(t,x)| < u(t,x)$, and hence a similar argument applied to the function $\psi(t,x) = e^{-x}u(t,x)$ yields

$$u(t,x) < e^x u(t,0), \quad \text{for } x > 0; \qquad u(t,x) > e^x u(t,0), \quad \text{for } x < 0.$$

Therefore $u(t,x)$ can be bounded in terms of $u(t,0)$ as

$$\begin{aligned} &e^{-\frac{x}{\sqrt{2}}} u(t,0) < u(t,x) < e^x u(t,0), \quad \text{for } x > 0, \\ &e^x u(t,0) < u(t,x) < e^{-\frac{x}{\sqrt{2}}} u(t,0), \quad \text{for } x < 0. \end{aligned}$$

Blow-up issue in the case $\gamma < 0$ is more subtle. The sign-preservation property of m is no longer available. It is interesting to note that the quantity $n = m + \sqrt{-\gamma/2}$ preserves its sign along the characteristics, which makes the previous analysis of the $\gamma = 0$ case applicable. We now state the blow-up result with the negative linear dispersion index in the following.

Theorem 1.3 (Oscillation-induced blow-up: $\gamma < 0$)**.** *Assume $\gamma = -2\beta^2 < 0$. Let $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ for $s > 1/2$, $m_0(x) \geq -\beta, \forall x \in \mathbb{R}$, and $H_1[u_0] \geq 3\sqrt{2}\beta$, where $H_1[u]$ is defined in* (1.8)*. Suppose that there exists a point $x_0 \in \mathbb{R}$ such that*

$$
\begin{aligned}
&m_0(x_0) > \beta,\ u_{0,x}(x_0) \leq \min\left\{-4\beta, -\frac{3}{4}\left(\frac{1}{\sqrt{2}}H_1[u] + \beta\right)\right\},\ \text{and} \\
&u_{0,x}(x_0) \leq -\sqrt{\frac{3}{4}}\left(u_0(x_0) + \beta\right).
\end{aligned} \tag{1.6}
$$

Then the corresponding solution $u(t,x)$ blows up in finite time.

Remark 1.4. *(i) It is found that the above results are different from the ones in* [2, 3, 4] *for the CH equation. In the case of the CH equation, finite-time blow-up is induced by "local fast oscillations" in the sense that $u_{0,x}(x_0) < -|u_0(x_0)|$, which implies that $\frac{|u_{0,x}(x_0)|}{|u_0(x_0)|} > 1$. In the case of the mCH equation, however, we have to include the global information $m_0 \geq 0$ (or $n_0 \geq 0$ when $\gamma < 0$) to obtain estimates on the convolution terms. Such a requirement already excludes any local fast oscillation, i. e. $\frac{1}{\sqrt{2}} \leq \frac{|u_{0,x}(x_0)|}{|u_0(x_0)|} < 1$ if $m_0 \geq 0$, for example. In other words, our result asserts that finite time blow-up to the mCH equation can be triggered even by mild local oscillations.*

(ii) Our choice (1.6) *of the initial data is not optimal in the sense of providing any sort of a threshold between blow-up and global wellposedness. A more general condition on $u_{0,x}(x_0)$ can be derived from the discussion in Section 4.2 and formulated as*

$$
u_{0,x}(x_0) \leq \min\{-s\beta, -\delta K\}
$$

where s and δ satisfy (4.20) *and* (4.24)*, and the relation between $u_{0,x}(x_0)$ and $u_0(x_0) + \beta$ should satisfy* (4.21)*. The numbers we use in the theorem is simply for computational convenience.*

(iii) The main difference between the two cases $\gamma < 0$ and $\gamma = 0$, which is also the main difficulty, lies in the fact that $\gamma \neq 0$ introduces some extra nonlocal terms in the dynamic equations of the key blow-up quantities which fail to be controlled in terms of the local terms. Bounding such terms gives rise to the additional conditions on the initial data in Theorem 1.3.

Our second aim in the present paper is to seek initial datum with a sign-changing momentum density m which can generate finite time blow-up. This is discussed in the case of non-negative dispersion $\gamma \geq 0$. At this moment we do not have a clear picture of how to produce an oscillation-induced blow-up criterion when $\gamma > 0$. But with the help of an additional conservation law, the nonlocal terms can still be bounded by some globally conserved quantities, which hints for the blow-up data with sufficiently steep profiles. If we set $r = \sqrt{m^2 + \gamma/2}$, the mCH equation (1.1) can be rewritten as the form of conservation laws [19]

$$r_t + \left((u^2 - u_x^2)r\right)_x = 0. \tag{1.7}$$

The three basic conserved quantities

$$\begin{aligned} H_0[u] &= \int_{\mathbb{R}} u\, dx, \\ H_1[u] &= \int_{\mathbb{R}} mu\ dx = \int_{\mathbb{R}} (u^2 + u_x^2)\, dx, \\ H_2[u] &= \frac{1}{4}\int_{\mathbb{R}} (u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4 + 2\gamma u^2)\, dx, \end{aligned} \tag{1.8}$$

are well-known and play an important role in all analysis of the solutions. Suppose the linear dispersion parameter $\gamma \geq 0$. If we denote

$$w(t,x) = r(t,x) - \sqrt{\gamma/2} = \sqrt{m^2 + \gamma/2} - \sqrt{\gamma/2} \geq 0,$$

we may have the following new conservation law [19]

$$\int_{\mathbb{R}} w(t,x)\, dx = \int_{\mathbb{R}} w_0(x)\, dx, \tag{1.9}$$

which is crucial in controlling the solution u and its slope u_x. This observation allows us to improve the blow-up results in [17, 18] to include the case of a sign-changing initial momentum density and the effect of the linear dispersion with $\gamma \geq 0$.

Theorem 1.5 (Blow-up for a sign-changing momentum density)**.** *Let $\gamma \geq 0$, and suppose $m_0 \in H^s(\mathbb{R})$, $s > \frac{1}{2}$ and $w_0 \in L^1(\mathbb{R})$. Assume further that there exists an $x_1 \in \mathbb{R}$ such that*

$$m_0(x_1) \neq 0 \qquad \text{and} \qquad m_0(x_1)u_{0,x}(x_1) \leq -\sqrt{|m_0(x_1)|\left(2\gamma A_0 + \frac{11}{12}A_0^3\right)}, \tag{1.10}$$

where

$$A_0 = (\sqrt{1+\gamma/2} + \sqrt{\gamma/2})\|w_0\|_{L^1} + 2.$$

Then the solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq \frac{12}{24\gamma A_0 + 11A_0^3}\left(|u_{0,x}(x_1)| - \sqrt{u_{0,x}^2(x_1) - \frac{24\gamma A_0 + 11A_0^3}{12\,|m_0(x_1)|}}\right). \tag{1.11}$$

The rest of the paper is organized as follows. In Section 2, some preliminary estimates and results are recalled and presented. In Section 3 the evolution equations of various quantities related to finite time blow-up analysis along the characteristics are established. Last section, Section 4, is devoted to the proofs to our main blow-up results, Theorem 1.1, Theorem 1.3, and Theorem 1.5.

Notation. For convenience, in the following, given a Banach space X, we denote its norm by $\|\cdot\|_X$. If there is no ambiguity, we omit the domain of function spaces.

2. Preliminaries

Let us recall some basic results concerning the formation of singularities in the mCH equation (1.1). Note that the mCH equation (1.1) can be rewritten as a transport equation for the momentum density, that is,

$$m_t + (u^2 - u_x^2)m_x = -2m^2 u_x - \gamma u_x. \tag{2.1}$$

The theory of transport equations implies that the blow-up is determined by the slope of the transport velocity

$$(u^2 - u_x^2)_x = 2mu_x = 2u_x(u - u_{xx}) \tag{2.2}$$

In fact the local well-posedness and a blow-up criterion can be formulated as follows. The details of proof can be found in [17].

Lemma 2.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then there exists a time $T > 0$ such that the initial-value problem* (1.1) *has a unique strong solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$. Moreover, the map $u_0 \mapsto u$ is continuous from a neighborhood of the initial data u_0 in $H^s(\mathbb{R})$ into $C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$. In addition, if the corresponding solution u has maximum time T of existence with $0 < T < \infty$, then*

$$\int_0^T \|(m\, u_x)(t)\|_{L^\infty} dt = \infty.$$

Using the preceding criterion, one can derive the following precise blow-up condition [17].

Lemma 2.2. *Suppose that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then the corresponding solution u to the initial value problem* (1.1) *blows up in finite time $T > 0$ if and only if*

$$\lim_{t\to T} \inf_{x\in\mathbb{R}} \left\{ m(t,x)\, u_x(t,x) \right\} = -\infty.$$

Certain conservative properties of the momentum density m will play a key role in establishing our new blow-up criteria. First, note that an application of the method of characteristics to the transport equation (2.1) for m requires analyzing the flow governed by the effective wave speed $u^2 - u_x^2$, namely the solution $q(t,x)$ to the parametrized family of ordinary differential equations

$$\begin{cases} \dfrac{dq(t,x)}{dt} = u^2(t,q(t,x)) - u_x^2(t,q(t,x)) & x \in \mathbb{R}, \quad t \in [0,T). \\ q(0,x) = x, \qquad x \in \mathbb{R}. \end{cases} \tag{2.3}$$

One can easily check that

Proposition 2.1. *Suppose $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$, and let $T > 0$ be the maximal existence time of the strong solution u to the corresponding initial value problem* (1.1)*. Then* (2.3) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ such that $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(2\int_0^t m(s,q(s,x))\, u_x(s,q(s,x))\, ds\right) > 0 \tag{2.4}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$.

Furthermore, in the case when $\gamma = 0$, the potential $m = u - u_{xx}$ satisfies

$$m(t, q(t,x))q_x(t,x) = m_0(x), \qquad (t,x) \in [0,T) \times \mathbb{R}, \tag{2.5}$$

which implies that the zeros and the sign of m are preserved under the flow, whereas in the general case when $\gamma \neq 0$ one has

$$\frac{d}{dt}\left[m(t,q(t,x))q_x(t,x)\right] = -\gamma u_x q_x. \tag{2.6}$$

We are now in a position to present the following unusual conservation law which plays a crucial role in our proof of Theorem 1.5.

Proposition 2.2. *Assume $m_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s > \frac{1}{2}$ and $\gamma \geq 0$. Suppose u is the corresponding solution to* (1.1) *with the initial data u_0. Then*

$$\|w(t)\|_{L^1} = \|w_0\|_{L^1} = \int_{-\infty}^{\infty} \left(\sqrt{m_0^2(x) + \gamma/2} - \sqrt{\gamma/2}\right) dx, \qquad 0 \leq t < T. \tag{2.7}$$

Proof. Indeed, note that $w_t(t,x) = r_t(t,x)$. We can easily deduce from (1.7) that

$$w_t + \left((u^2 - u_x^2)(w + \sqrt{\gamma/2})\right)_x = 0,$$

from which (2.7) holds true. ◻

Denote $p(x) = \frac{1}{2}e^{-|x|}$, the fundamental solution of $1 - \partial_x^2$ on $\mathbb{R}$, that is, $(1 - \partial_x^2)^{-1} f = p * f$, and define the two convolution operators p_+, p_- as

$$\begin{aligned} p_+ * f(x) &= \frac{e^{-x}}{2} \int_{-\infty}^{x} e^y f(y)dy \\ p_- * f(x) &= \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} f(y)dy. \end{aligned} \tag{2.8}$$

Then we have the relation

$$p = p_+ + p_-, \qquad p_x = p_- - p_+. \tag{2.9}$$

3. Dynamics along the characteristics

It is known from most of the literatures that finite-time blow-up analysis is often carried out along the characteristics. The following lemmas give explicit information of how the blow-up quantities evolve along the characteristics.

Lemma 3.1. *Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$. Then $u(t,x)$ and $u_x(t,x)$ satisfy the following integro-differential equations*:

$$u_t + (u^2 - u_x^2)u_x = -\gamma p_x * u - \frac{2}{3}u_x^3 + \frac{1}{3}\left[p_+ * (u-u_x)^3 - p_- * (u+u_x)^3\right]. \tag{3.1}$$

$$\begin{aligned} &u_{tx} + (u^2 - u_x^2)u_{xx} \\ &\quad = \gamma(u - p*u) + \left(\frac{2}{3}u^3 - uu_x^2\right) - \frac{1}{3}p_+ * \left(2u^3 + 3uu_x^2 - u_x^3\right) \\ &\qquad - \frac{1}{3}p_- * \left(2u^3 + 3uu_x^2 + u_x^3\right) \\ &\quad = \gamma(u - p*u) + \left(\frac{1}{3}u^3 - uu_x^2\right) - \frac{1}{3}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right]. \end{aligned} \tag{3.2}$$

Proof. From (1.1), we have

$$
\begin{aligned}
&(1-\partial_x^2)\left(u_t+(u^2-u_x^2)u_x\right)\\
&\quad=-(u^2-u_x^2)m_x-2m^2u_x-\gamma u_x+(u^2-u_x^2)u_x-\partial_x^2\left((u^2-u_x^2)u_x\right)\\
&\quad=-(u^2-u_x^2)m_x-2m^2u_x-\gamma u_x+(u^2-u_x^2)u_x-6u_xu_{xx}m-2u_x^2m_x\\
&\qquad-(u^2-u_x^2)u_{xxx}\\
&\quad=-2m^2u_x-\gamma u_x-6u_xu_{xx}m-2u_x^2m_x,
\end{aligned}
\tag{3.3}
$$

which implies

$$
\begin{aligned}
u_t+(u^2-u_x^2)u_x&=-p*(2m^2u_x+\gamma u_x+6u_xu_{xx}m+2u_x^2m_x)\\
&=-p*(2muu_x+2(u_x^2m)_x+\gamma u_x).
\end{aligned}
\tag{3.4}
$$

Taking derivative to (3.4) with respect to x yields

$$
\begin{aligned}
&u_{tx}+(u^2-u_x^2)u_{xx}\\
&\quad=-2mu_x^2-p_x*(2muu_x+2(u_x^2m)_x+\gamma u_x)\\
&\quad=-2mu_x^2-2p_x*(muu_x)+2\left[u_x^2m-p*(u_x^2m)\right]+\gamma\,(u-p*u)\\
&\quad=\gamma\,(u-p*u)-2p_x*(muu_x)-2p*(u_x^2m).
\end{aligned}
\tag{3.5}
$$

In view of proof in Lemma 3.1 [18], we know

$$
\begin{aligned}
p_+*(mu_x^2)&=-\frac{1}{6}u_x^3+\frac{1}{3}p_+*(3uu_x^2+u_x^3),\\
p_-*(mu_x^2)&=\frac{1}{6}u_x^3+\frac{1}{3}p_-*(3uu_x^2-u_x^3),\\
p_+*(muu_x)&=\frac{1}{6}u^3-\frac{1}{3}p_+*u^3-\frac{1}{4}uu_x^2+\frac{1}{2}p_+*(uu_x^2+u_x^3),\\
p_-*(muu_x)&=-\frac{1}{6}u^3+\frac{1}{3}p_-*u^3+\frac{1}{4}uu_x^2-\frac{1}{2}p_-*(uu_x^2-u_x^3).
\end{aligned}
$$

It then follows from (2.9) that

$$
\begin{aligned}
&2p{*}(muu_x)+2p_x*(u_x^2m)\\
&\quad=\frac{2}{3}u_x^3-\frac{1}{3}p_+*\left(2u^3+3uu_x^2-u_x^3\right)+\frac{1}{3}p_-*\left(2u^3+3uu_x^2+u_x^3\right)\\
&\quad=\frac{2}{3}u_x^3-\frac{1}{3}\left[p_+*(u-u_x)^3-p_-*(u+u_x)^3\right],
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&2p_x{*}(muu_x)+2p*(u_x^2m)\\
&\quad=uu_x^2-\frac{2}{3}u^3+\frac{1}{3}p_+*\left(2u^3+3uu_x^2-u_x^3\right)+\frac{1}{3}p_-*\left(2u^3+3uu_x^2+u_x^3\right)\\
&\quad=uu_x^2-\frac{1}{3}u^3+\frac{1}{3}\left[p_+*(u-u_x)^3+p_-*(u+u_x)^3\right].
\end{aligned}
\tag{3.7}
$$

Combining (3.4) with (3.6), and (3.5) with (3.7), we complete the proofs of (3.1) and (3.2). □

Next we focus on the dynamics of $M:=mu_x$ along the characteristics.

Lemma 3.2. *Assume that the initial data $m_0 \in H^s(\mathbb{R})$, $s \geq 3$. Let $T > 0$ be the maximal existence time of the resulting solution $m(t,x)$ to the initial value problem (1.1). Then $M := mu_x$ satisfies*

$$\begin{aligned} M_t + (u^2 - u_x^2)M_x &= -2M^2 - \gamma u_x^2 + \gamma m(u - p * u) \\ &\quad - 2mp_x * (muu_x) - 2mp * (u_x^2 m) \\ &= -2M^2 - \gamma u_x^2 + \gamma m(u - p * u) \\ &\quad + m\left(\frac{1}{3}u^3 - uu_x^2\right) - \frac{1}{3}m\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right]. \end{aligned} \tag{3.8}$$

Proof. By simple computation, we know that

$$M_t + (u^2 - u_x^2)M_x = m\big(u_{tx} + (u^2 - u_x^2)u_{xx}\big) + \big(m_t + (u^2 - u_x^2)m_x\big)u_x.$$

Now, multiplying (3.5) and (2.1) with m and u_x respectively, then combining them together with (3.2), we finish the proof of (3.8). □

4. Finite-time blow-up data

In this section we will investigate the blow-up conditions for the initial data to the mCH equation (1.1) in various cases corresponding to different sign of γ. We emphasize two different blow-up mechanisms. The first one stems from the idea of local-in-space blow-up [2, 3, 4] in the study of Camassa-Holm and hyperelastic rod equations, where the blow-up criteria assert that local perturbation of data around a single point does not prevent the singularity formation. Such type of blow-up can also be viewed as induced by "local fast oscillations" which is independent of the size of the oscillation region. The second mechanism involves the global information of the data, which relies on controlling the non-local terms in the transport formulation of the PDE by certain conservation laws.

In view of Lemma 2.2, an approach toward the the blow-up of (1.1) is to track the evolution of M, which is given in (3.8). Apparently, the convolution terms in (3.8) contain non-local information, and do not seem to be easily controlled by any local quantity. However in the case when $\gamma = 0$, one has the sign-preservation property of the momentum $m(t,x)$ (c.f. (2.5)). When m does not change sign, the convolution terms have a determined sign, making it possible to obtain the oscillation-induced blow-up. In fact, the oscillation mechanism can be measured by u_x/u. When $\gamma = 0$ and the sign of m remains fixed, fast oscillation is automatically excluded. But it is possible to preserve mild oscillations along the characteristics, which is enough to lead to a blow-up.

For the case of general γ, one loses the sign preservation of $m(t,x)$. Therefore it becomes difficult to control the nonlocal terms in equations (3.1), (3.2) and (3.8) in terms of the local terms directly. When $\gamma < 0$, however, we find a transformation that "absorbs" this term, making it fit in a similar structure as of the $\gamma = 0$ case, and hence an oscillation-induced blow-up follows. At this moment we don't have a clear picture of how to generate the oscillation-induced blow-up criterion when $\gamma > 0$. But with the help of the additional conservation law (2.7), the nonlocal terms can still be bounded by some globally conserved quantities, which makes it possible to obtain a finite time blow-up for data with sufficiently steep profiles.

4.1. **Oscillation-induced blow-up when $\gamma = 0$.** We are now in a position to prove the oscillation-induced blow-up result for $\gamma = 0$.

Proof of Theorem 1.1. As is explained before, we will trace the dynamics along the characteristics emanating from x_0. Denote

$$\begin{aligned} &\widehat{u}(t) = u(t, q(t, x_0)), \quad \widehat{u_x}(t) = u_x(t, q(t, x_0)), \\ &\widehat{m}(t) = m(t, q(t, x_0)), \quad \widehat{M}(t) = (mu_x)(t, q(t, x_0)), \end{aligned} \tag{4.1}$$

and let " $'$ " denote the derivative along the characteristics $q(t, x_0)$. Then from (3.1), (3.2), and (3.8) we have the following ODE system for $\widehat{u}$, $\widehat{u_x}$, and $\widehat{M}$.

$$\begin{aligned} \widehat{u}'(t) &= -\frac{2}{3}\widehat{u_x}^3(t) + \frac{1}{3}\left[p_+ * (u-u_x)^3 - p_- * (u+u_x)^3\right](t, q(t, x_0)), \\ \widehat{u_x}'(t) &= \left(\frac{1}{3}\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right)(t) - \frac{1}{3}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right](t, q(t, x_0)), \\ \widehat{M}'(t) &= -2\widehat{M}^2(t) + \widehat{m}\left(\frac{1}{3}\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right)(t) \\ &\quad - \frac{1}{3}\widehat{m}(t)\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right](t, q(t, x_0)). \end{aligned}$$

Since we know that $m_0 \geq 0, \forall x \in \mathbb{R}$, in particular, $m_0(x_0) > 0$. So from (2.5), we know $m(t, x) \geq 0, \forall x \in \mathbb{R}$ and $m(t, q(t, x_0)) > 0$. Therefore from the fact that

$$u(t, x) = p * m(t, x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}m(y)\, dy, \quad u_x(t, x) = p_x * m(t, x).$$

we have $u(t, x) \geq 0$ and $\widehat{u}(t) = u(t, q(t, x_0)) > 0$. Moreover

$$u - u_x = 2p_+ * m \geq 0, \quad u + u_x = 2p_- * m \geq 0.$$

Hence we know $|u_x(x, t)| \leq u(x, t)$. Since $\widehat{u}(t) > 0$, we can track the dynamics of $\widehat{u_x}/\widehat{u}$ along the characteristics.

$$\begin{aligned} \left(\frac{\widehat{u_x}}{\widehat{u}}\right)'(t) = &\ \frac{(\widehat{u}^2 - \widehat{u_x}^2)(\widehat{u}^2 - 2\widehat{u_x}^2)}{3\widehat{u}^2} - \frac{\widehat{u} + \widehat{u_x}}{3\widehat{u}^2}p_+ * (u-u_x)^3(t, q(t, x_0)) \\ &- \frac{\widehat{u} - \widehat{u_x}}{3\widehat{u}^2}p_- * (u+u_x)^3(t, q(t, x_0)) \\ < &\ \frac{(\widehat{u}^2 - \widehat{u_x}^2)(\widehat{u}^2 - 2\widehat{u_x}^2)}{3\widehat{u}^2}. \end{aligned} \tag{4.2}$$

So if initially $\widehat{u_x}(0) \leq -\frac{1}{\sqrt{2}}\widehat{u}(0)$, we see that the right-hand side of the above estimate is non-positive. Hence $\widehat{u_x}/\widehat{u}$ decreases, and thus

$$\left(\frac{\widehat{u_x}}{\widehat{u}}\right)(t) < \frac{\widehat{u_x}}{\widehat{u}}(0) \leq -\frac{1}{\sqrt{2}},$$

which implies that $\widehat{u}(t) + \sqrt{2}\widehat{u_x}(t) < 0$. It is then inferred that $\widehat{u}(t) + \sqrt{3}\widehat{u_x}(t) < 0$. Meanwhile it is also found that $\widehat{u}(t) - \sqrt{3}\widehat{u_x}(t) > 0$. Hence it is concluded that

$$\widehat{u}^2(t) - 3\widehat{u_x}^2(t) < 0.$$

Plugging this into the ODE for $\widehat{M}$ we have

$$\widehat{M}'(t) \leq -2\widehat{M}^2(t) + \widehat{m}\left(\frac{1}{3}\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right)(t) = -2\widehat{M}^2(t) + \frac{1}{3}\widehat{m}\widehat{u}(\widehat{u}^2 - 3\widehat{u_x}^2)(t) < -2\widehat{M}^2(t),$$

which generates blow-up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2\widehat{M}(0)} = -\frac{1}{2m_0(x_0)u_{0,x}(x_0)}.$$

□

4.2. **Oscillation-induced blow-up when $\gamma < 0$.** As is discussed at the beginning of this section, we look for a transformation that eliminates the γu_x term in equation (1.1). Consider the following change-of-unknown

$$v(t,x) = u(t,x) + \beta \tag{4.3}$$

for some constant β to be determined. Plugging (4.3) into (1.1) and denoting $n = v - v_{xx}$, it follows

$$n_t + \left[((v-\beta)^2 - v_x^2)n\right]_x - 2\beta v_x n + (2\beta^2 + \gamma)v_x = 0.$$

Thus choosing $\beta = \sqrt{-\gamma/2}$ we obtain the equation for v as

$$n_t + \left[((v-\beta)^2 - v_x^2)n\right]_x - 2\beta v_x n = 0. \tag{4.4}$$

Note that we have the following relation between the two momenta m and n

$$m(t,x) = n(t,x) - \beta,$$

and $(v-\beta)^2 - v_x^2 = u^2 - u_x^2$. This implies that the new equation for n has the same characteristics as defined in (2.3) for the equation (1.1). Moreover along the characteristic $q(t,x)$ we have

$$\frac{d}{dt}(nq_x) = 2\beta v_x(nq_x),$$

which implies that

$$(nq_x)(t, q(t,x)) = n(0,x)\exp\left(2\int_0^t \beta v_x(s, q(s,x))\, ds\right). \tag{4.5}$$

Therefore the zeros and the sign of n are preserved under the flow.

Let $N = nv_x$. Then from definition of n, v we know that $N = M + \beta v_x$. From (3.8) and (3.2) and using the relation $\gamma = -2\beta^2$ we deduce

$$\begin{aligned} N_t + (u^2 - u_x^2)N_x &= M_t + (u^2 - u_x^2)M_x + u_{xt} + (u^2 - u_x^2)u_{xx} \\ &= -2N^2 + 4\beta N u_x + \gamma n(u - p * u) \\ &\quad + n\left(\frac{1}{3}u^3 - uu_x^2\right) - \frac{1}{3}n\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right]. \end{aligned} \tag{4.6}$$

This suggests us to look for the blow-up of $N(t,x)$. We still want to consider the blow-up induced by oscillations. Thus from the previous argument in Section 4.1, we would like to check the dynamics of u_x/u along the characteristics. However we no longer have the sign-preservation for u now. But instead, the sign of v can be preserved, provided that n does not change sign. Therefore we can compute the dynamics of $u_x/v = v_x/v$. Recall $\widehat{u}(t), \widehat{u_x}(t)$ as in (4.1). We further denote

$$\widehat{v}(t) = v(t, q(t, x_0)), \quad \widehat{v_x}(t) = v_x(t, q(t, x_0)),$$
$$\widehat{n}(t) = n(t, q(t, x_0)), \quad \widehat{N}(t) = N(t, q(t, x_0)),$$

and let the notation '$\,'\,$' denote the derivative with respect to the time t along the characteristics $q(t, x_0)$. Note that the evolution of $\widehat{v_x}$ is the same as the one for $\widehat{u_x}$:

$$\begin{aligned}\widehat{v_x}'(t) = \;& \gamma\widehat{u}(t) - \gamma(p * u)(t, q(t, x_0)) + \left(\frac{1}{3}\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right)(t) \\ & - \frac{1}{3}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right](t, q(t, x_0)) \\ = \;& -2\beta^2\widehat{v}(t) + 2\beta^2(p * v)(t, q(t, x_0)) + \left(\frac{1}{3}(\widehat{v} - \beta)^3 - (\widehat{v} - \beta)\widehat{v_x}^2\right)(t) \\ & - \frac{1}{3}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right](t, q(t, x_0)).\end{aligned} \tag{4.7}$$

Using (4.2) and the following identity

$$\left(\frac{\widehat{v_x}}{\widehat{v}}\right)'(t) = \left(\frac{\widehat{u_x}}{\widehat{u}}\right)' \cdot \frac{\widehat{u}^2}{\widehat{v}^2} + \frac{\beta\widehat{u_x}'}{\widehat{v}^2},$$

we can calculate

$$\begin{aligned}\left(\frac{\widehat{v_x}}{\widehat{v}}\right)'(t) = \;& \frac{1}{\widehat{v}^2}\Bigg[\gamma\widehat{u}^2 + \gamma\beta(\widehat{u} + \widehat{v}) + \frac{1}{3}\widehat{u}^4 - \widehat{u}^2\widehat{u_x}^2 + \frac{2}{3}\widehat{u_x}^4 \\ & - \gamma\left[(\widehat{v} + \widehat{v_x})p_+ * v + (\widehat{v} - \widehat{v_x})p_- * v\right] \\ & + \beta\left(\frac{\widehat{u}^3}{3} - \widehat{u}\widehat{u_x}^2\right) - \frac{\widehat{v} + \widehat{v_x}}{3}p_+ * (u - u_x)^3(t, q(t, x_0)) \\ & - \frac{\widehat{v} - \widehat{v_x}}{3}p_- * (u + u_x)^3(t, q(t, x_0))\Bigg] \\ = \;& \frac{1}{3\widehat{v}^2}\Big[2\widehat{v_x}^4 - 3\widehat{v}(\widehat{v} - \beta)\widehat{v_x}^2 + \widehat{v}(\widehat{v} - \beta)^3 - 6\beta^2\widehat{v}^2 \\ & + 6\beta^2\left[(\widehat{v} + \widehat{v_x})p_+ * v + (\widehat{v} - \widehat{v_x})p_- * v\right] \\ & - (\widehat{v} + \widehat{v_x})p_+ * (u - u_x)^3(t, q(t, x_0)) \\ & - (\widehat{v} - \widehat{v_x})p_- * u + u_x)^3(t, q(t, x_0))\Big].\end{aligned} \tag{4.8}$$

From the assumption that $n_0(x) \geq 0$, $n_0(x_0) > 0$ we know that $n(t, x) \geq 0$ and $\widehat{n}(t) > 0$. This implies that

$$v(t, x) \geq 0, \quad |v_x(t, x)| \leq v(t, x), \quad \widehat{v}(t) > 0, \tag{4.9}$$

which, in terms of u and u_x, reads

$$u(t, x) \geq -\beta, \quad u(t, x) \pm u_x(t, x) \geq -\beta, \quad \widehat{u}(t) > -\beta.$$

Using these estimates together with the fact that $\gamma < 0$ we can deduce that

$$\begin{aligned} &-\frac{1}{3}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right] \le \frac{1}{3}\beta^3. \\ &-(\widehat{v}+\widehat{v_x})p_+ * (u-u_x)^3 - (\widehat{v}-\widehat{v_x})p_- * (u+u_x)^3 \le \beta^3\widehat{v}. \end{aligned} \tag{4.10}$$

Putting (4.6)-(4.10) together and plugging $\widehat{u} = \widehat{v} - \beta$ we can estimate the evolution of $\widehat{N}(t), \widehat{v_x}(t)$ and $\widehat{v_x}/\widehat{v}$ as

$$\widehat{N}'(t) \le -2\widehat{N}^2 + \gamma\widehat{n}(\widehat{u} - p*u) + \widehat{n}\left[\frac{1}{3}\left[(\widehat{v}-\beta)^3 + \beta^3\right] - (\widehat{v} - 5\beta)\widehat{v_x}^2\right], \tag{4.11}$$

$$\widehat{v_x}'(t) \le \frac{1}{3}\left[(\widehat{v}-\beta)^3 + \beta^3\right] - (\widehat{v}-\beta)\widehat{v_x}^2 - 2\beta^2(\widehat{v} - p*v), \tag{4.12}$$

$$\begin{aligned} \left(\frac{\widehat{v_x}}{\widehat{v}}\right)'(t) \le\ & \frac{\left[2\widehat{v_x}^4 - 3\widehat{v}(\widehat{v}-\beta)\widehat{v_x}^2 + \widehat{v}(\widehat{v}-\beta)^3 - 6\beta^2\widehat{v}^2 + \beta^3\widehat{v}\right]}{3\widehat{v}^2} \\ & + \frac{2\beta^2}{\widehat{v}^2}\left[(\widehat{v}+\widehat{v_x})p_+ * v + (\widehat{v}-\widehat{v_x})p_- * v\right]. \end{aligned} \tag{4.13}$$

Recall from (2.1) the evolution for $\widehat{n}$

$$\widehat{n}'(t) = \widehat{m}'(t) = -(2\widehat{m}^2 + \gamma)\widehat{v_x} = -2(\widehat{n} - 2\beta)\widehat{n}\widehat{v_x}. \tag{4.14}$$

It is observed from the above equation that when $\widehat{n} > 2\beta$ and $\widehat{v_x} < 0$, $\widehat{n}$ grows exponentially. Thus by the definition $\widehat{N} = \widehat{n}\widehat{v_x}$, it is conceivable that (4.11) can lead to a Burgers' type blow-up inequality, provided that $\widehat{v_x}$ is non-increasing. Therefore we would like to show that the right-hand side of (4.12) is nonpositive, that is

$$(\widehat{v}-\beta)\widehat{v_x}^2 \ge \frac{1}{3}\left[(\widehat{v}-\beta)^3 + \beta^3\right] - 2\beta^2(\widehat{v} - p*v), \tag{4.15}$$

from which we see that we need an upper bound for the non-local term $p*v$. But it seems difficult to obtain an estimates in terms of local quantities. Instead, we make use of the following global information.

$$p*v = |p*v| \le \|v\|_{L^\infty} \le \|u\|_{L^\infty} + \beta \le \frac{1}{\sqrt{2}}H_1[u] + \beta.$$

Let

$$K = \frac{1}{\sqrt{2}}H_1[u] + \beta.$$

In a similar way we can estimate

$$p_+ * v, p_- * v \le \frac{1}{2}\|v\|_{L^\infty} \le \frac{1}{2}K,$$

and hence

$$(\widehat{v}+\widehat{v_x})p_+ * v + (\widehat{v}-\widehat{v_x})p_- * v \le K\widehat{v}. \tag{4.16}$$

In the case when

$$\widehat{v} \ge s\beta, \quad \widehat{v} \ge \delta K, \qquad \text{for some } s > 1, 0 < \delta < 1, \tag{4.17}$$

a sufficient condition for (4.15) to hold is

$$\widehat{v_x}^2 \ge \frac{1}{3}(\widehat{v}-\beta)^2 + \frac{\beta^3 + 6\beta^2 K - 6\beta^2\widehat{v}}{3(\widehat{v}-\beta)}. \tag{4.18}$$

Controlling the right-hand side of the above leads to

$$\begin{aligned}\frac{1}{3}(\widehat{v}-\beta)^2+\frac{\beta^3+6\beta^2K-6\beta^2\widehat{v}}{3(\widehat{v}-\beta)} &\le \frac{1}{3}\widehat{v}^2+\frac{\beta^2}{3(s-1)}+\frac{2\beta(1-\delta)}{(s-1)\delta}\widehat{v}\\ &\le \left[\frac{1}{3}+\frac{1}{(s-1)s^2}+\frac{2(1-\delta)}{s(s-1)\delta}\right]\widehat{v}^2.\end{aligned}$$

Therefore a sufficient condition for (4.18) is

$$\widehat{v_x}^2 \ge \left[\frac{1}{3}+\frac{1}{(s-1)s^2}+\frac{2(1-\delta)}{s(s-1)\delta}\right]\widehat{v}^2, \tag{4.19}$$

where a constraint on s and δ must be imposed

$$\frac{1}{3}+\frac{1}{(s-1)s^2}+\frac{2(1-\delta)}{s(s-1)\delta}<1. \tag{4.20}$$

Next we use (4.13) to look for a sufficient condition to persist the local oscillation. Similarly as in the case of $\gamma=0$, one wants the right-hand side of (4.13) to be negative. Plugging (4.16) into (4.13) and solving the inequality yield

$$\frac{3\widehat{v}(\widehat{v}-\beta)-\sqrt{\Delta}}{4}\le \widehat{v_x}^2 \le \frac{3\widehat{v}(\widehat{v}-\beta)+\sqrt{\Delta}}{4}, \tag{4.21}$$

where

$$\Delta = \widehat{v}^4+6\beta\widehat{v}^3+33\beta^2\widehat{v}^2-48\beta^2K\widehat{v}. \tag{4.22}$$

Further plugging in (4.17) we have

$$\Delta \ge \left[s^2+6s-\left(\frac{48}{\delta}-33\right)\right]\beta^2\widehat{v}^2. \tag{4.23}$$

Solvability of (4.21) requires that the discriminant $\Delta\ge 0$, which holds when

$$s^2+6s\ge \frac{48}{\delta}-33. \tag{4.24}$$

Now we choose $\delta=\frac{3}{4}$. From condition (4.24) we may choose $s=4$. This way we have

$$\frac{1}{3}+\frac{1}{(s-1)s^2}+\frac{2(1-\delta)}{s(s-1)\delta}=\frac{59}{144},\quad s^2+6s-\left(\frac{48}{\delta}-33\right)=9.$$

Therefore

$$\frac{3\widehat{v}(\widehat{v}-\beta)-\sqrt{\Delta}}{4}\le \frac{3}{4}\widehat{v}^2 \le \frac{3\widehat{v}(\widehat{v}-\beta)+\sqrt{\Delta}}{4}. \tag{4.25}$$

Further plugging the values of s and δ into (4.19) yields another sufficient condition for $\widehat{v_x}'(t)\le 0$ as

$$\widehat{v_x}^2 \ge \frac{3}{4}\widehat{v}^2. \tag{4.26}$$

In fact the above condition implies that $\widehat{v_x}'(t)<0$.

We now illustrate our strategy of proving Theorem 1.3. We first choose an initial datum which satisfies (4.17) and (4.26), from which we know that $\widehat{v_x}'(0)\le 0$. Then $\widehat{v_x}$ decreases within a short time interval, which makes $\widehat{n}$ increase. To continue the monotonicity further in time, we need to persist (4.17) and (4.26). But the property (4.17) can be inferred from the upper bound of $\widehat{v_x}$, due to the sign-preservation of

n. To check (4.26) at later time, we will use the monotonicity of $\widehat{v_x}^2/\widehat{v}^2$ near 3/4. Now let us give the details of proof of Theorem 1.3.

Proof of Theorem 1.3. From the assumptions of Theorem 1.3 we know that

$$\begin{aligned} n_0(x) \geq 0,\ n_0(x_0) > 2\beta,\ v_{0,x}(x_0) \leq \min\left\{-4\beta, -\frac{3}{4}K\right\}, \\ \text{and}\ \ v_{0,x}(x_0) \leq -\sqrt{\frac{3}{4}}v_0(x_0). \end{aligned} \tag{4.27}$$

So from the above argument we know that

$$\widehat{v_x}'(0) < 0,$$

and hence $\widehat{v_x}$ decreases over a certain time interval $[0, t_0]$ with $t_0 > 0$. So

$$\widehat{v_x}(t) < \widehat{v_x}(0) = v_{0,x}(x_0) \leq \min\left\{-4\beta, -\frac{3}{4}K\right\}, \quad t \in [0, t_0], \tag{4.28}$$

Together with (4.9) we know

$$\widehat{v}(t) \geq |\widehat{v_x}(t)| > \max\left\{4\beta, \frac{3}{4}K\right\}, \tag{4.29}$$

which satisfies (4.17).

Claim 4.1. *The ratio estimate* (4.26) *holds as long as the solution exists.*

To see this, since (4.26) is satisfied at initial time, it may propagate over a short time interval by continuity. If the claim is not true, then there exist times $t_2 > t_1 > 0$ such that

$$\begin{aligned} &\widehat{v_x}^2(t) \geq \frac{3}{4}\widehat{v}^2(t) \ \text{ for } 0 \leq t \leq t_1, \\ &\widehat{v_x}^2(t_1) = \frac{3}{4}\widehat{v}^2(t_1), \\ &\widehat{v_x}^2(t) < \frac{3}{4}\widehat{v}^2(t) \ \text{ for } t_1 < t \leq t_2. \end{aligned}$$

From (4.29) and the first estimate in the above we know that $\widehat{v_x}(t)$ is decreasing on $[0, t_1]$. Hence $\widehat{v_x}(t_1) < 0$. Another application of (4.29) to (4.23) implies that the inequality in (4.25) should be a strict inequality. Therefore from the last two estimates of the above we know by a continuity argument that there is a $t_3 > t_1$ such that condition (4.21) is satisfied for $t \in [t_1, t_3]$, which implies that $\widehat{v_x}/\widehat{v}$ is non-increasing over $[t_1, t_3]$. Together with the information of the sign of $\widehat{v}$ and $\widehat{v_x}$ we know that $\widehat{v_x}^2/\widehat{v}^2$ is non-decreasing over $[t_1, t_3]$. Hence $\widehat{v_x}^2/\widehat{v}^2 \geq 3/4$, which is a contradiction, This completes the proof of the claim.

In view of (4.29) and the above claim it follows that $\widehat{v_x}(t)$ always decreases, and that $\widehat{v}(t) > 4\beta$. This way from (4.14), the fact that $\widehat{v_x}(t) < -4\beta < 0$, and the initial condition on $\widehat{n}$ we know that $\widehat{n}$ is increasing, more precisely,

$$\widehat{n}'(t) > 8\beta(\widehat{n}(0) - 2\beta)\widehat{n}(t), \quad \Rightarrow \quad \widehat{n}(t) > \widehat{n}(0)e^{8\beta(\widehat{n}(0)-2\beta)t}.$$

This is then deduced that $\widehat{N}(t) = \widehat{n}(t)\widehat{v_x}(t) < 0$.

Comparing (4.11) and (4.12), it is deduced from (4.18) that

$$\frac{1}{3}\left[(\widehat{v}-\beta)^3+\beta^3\right]-(\widehat{v}-5\beta)\widehat{v_x}^2 \le 4\beta\widehat{v_x}^2.$$

Therefore

$$\begin{aligned}\widehat{N}'(t) &< -2\widehat{n}\left[(\widehat{n}-2\beta)\widehat{v_x}^2+\beta^2\left(\widehat{u}-p*u\right)\right]\\ &\le -2\widehat{n}\left[(\widehat{n}-2\beta)\widehat{v_x}^2-\sqrt{2}\beta^2 H_1[u]\right] \le -\widehat{N}^2,\end{aligned}$$

for t large enough, say,

$$t \ge t_* := \frac{1}{8\beta(\widehat{n}(0)-2\beta)}\log\frac{\sqrt{2}H_1[u]+32\beta}{8\widehat{n}(0)}.$$

Hence $\widehat{N}(t)$ blows up to $-\infty$ in finite-time T^* with

$$T^* \le t_* - \frac{1}{\widehat{N}(t_*)}.$$

Finally notice that $\widehat{N}=\widehat{M}+\beta\widehat{v_x}$, and from

$$|\widehat{v_x}| \le \widehat{v} = \widehat{u}+\beta \le \frac{1}{\sqrt{2}}H_1[u]+\beta,$$

we know that in this case

$$\lim_{t\to T^{*-}}\widehat{M}(t) = -\infty,$$

which completes the proof of Theorem 1.3. □

Remark 4.2. *As is pointed out in Remark 1.4, the choices for s and δ are not unique. All needed are* (4.20) *and* (4.24)*, and the ratio condition* (4.26) *should be accordingly adjusted. For instance taking $s=6$, $\delta=1/2$, and $v_{0,x}(x_0)\le-\sqrt{\frac{3}{4}}v_0(x_0)$ in* (4.27) *can also generate finite time blow-up.*

4.3. **Finite-time blow-up when $\gamma>0$.** In this case, we first introduce the following global estimates.

Lemma 4.3. *Assume that the initial data $m_0\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$, $s>\frac{1}{2}$. Let $T>0$ be the maximal existence time of the resulting solution $u(t,x)$ to the initial value problem* (1.1)*. Then we have*

$$|u(t,x)| \le \frac{A_0}{2}, \qquad |u_x(t,x)| \le \frac{A_0}{2}, \tag{4.30}$$

where

$$A_0 = (\sqrt{1+\gamma/2}+\sqrt{\gamma/2})\|w_0\|_{L^1}+2.$$

Furthermore, there holds

$$|p*u(t,x)| \le \frac{A_0}{2}. \tag{4.31}$$

Proof. We first estimate $|u(t,x)|$ from

$$u(t,x) = p * m(t,x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} m(t,y)\,dy. \tag{4.32}$$

Taking absolute value on (4.32), we have

$$\begin{aligned}
|u(t,x)| &\le \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}|m(t,y)|\,dy \\
&= \frac{1}{2}\int_{|m|>1} e^{-|x-y|}|m(t,y)|\,dy + \frac{1}{2}\int_{|m|\le 1} e^{-|x-y|}|m(t,y)|\,dy \\
&\le \frac{1}{2}\int_{|m|>1} |m(t,y)|\,dy + \frac{1}{2}\int_{|m|\le 1} e^{-|x-y|}\,dy.
\end{aligned} \tag{4.33}$$

Next we evaluate the first term on the right-hand side of (4.33). Since

$$\begin{aligned}
\|w(t)\|_{L^1} &= \int_{\mathbb{R}} \left(\sqrt{m^2(t,x)+\gamma/2} - \sqrt{\gamma/2}\right) dx = \int_{\mathbb{R}} \frac{m^2(t,x)}{\sqrt{m^2(t,x)+\gamma/2}+\sqrt{\gamma/2}}\,dx \\
&= \int_{|m|>1} \frac{m^2(t,x)}{\sqrt{m^2(t,x)+\gamma/2}+\sqrt{\gamma/2}}\,dx + \int_{|m|\le 1} \frac{m^2(t,x)}{\sqrt{m^2(t,x)+\gamma/2}+\sqrt{\gamma/2}}\,dx \\
&\ge \int_{|m|>1} \frac{m^2(t,x)}{\sqrt{m^2(t,x)+\gamma/2}+\sqrt{\gamma/2}}\,dx \\
&\ge \frac{1}{\sqrt{1+\gamma/2}+\sqrt{\gamma/2}} \int_{|m|>1} |m(t,x)|\,dx,
\end{aligned}$$

we get the following estimate by noticing Proposition 2.2

$$\int_{|m|>1} |m(t,x)|\,dx \le \left(\sqrt{1+\gamma/2}+\sqrt{\gamma/2}\right) \|w_0\|_{L^1}. \tag{4.34}$$

Substituting (4.34) back into (4.33) completes the proof of the first estimate of (4.30). By similar argument, we can get the above bound of $u_x(t,x)$.

An application of Young's inequality infers that

$$|p * u(t,x)| \le \|p\|_{L^1}\|u\|_{L^\infty} = \|u\|_{L^\infty} \le \frac{A_0}{2}. \tag{4.35}$$

□

Proof of Theorem 1.5. Again we introduce the notation

$$\begin{aligned}
&\widehat{u}(t) := u(t,q(t,x_1)), && \widehat{u_x}(t) := u_x(t,q(t,x_1)), \\
&\widehat{m}(t) := m(t,q(t,x_1)), && \widehat{M}(t) := M(t,q(t,x_1)) = \widehat{m}(t)\,\widehat{u_x}(t),
\end{aligned} \tag{4.36}$$

where x_1 is given in (1.10), and let ' $'$ ' denote the derivative with respect to t along the characteristics $q(t,x_1)$. Then equations (1.1) and (2.3) imply that

$$\widehat{m}'(t) = -\left(2\,\widehat{m}(t)^2 + \gamma\right)\,\widehat{u_x}(t). \tag{4.37}$$

Moreover, combining (3.2) and (2.3) with Lemma 4.3, we find

$$\begin{aligned}\widehat{u_x}'(t) &\leq \left(\gamma A_0 + \frac{1}{3}A_0^3\right) + \widehat{u}(t)\left[\frac{1}{3}\widehat{u}(t)^2 - \widehat{u_x}(t)^2\right] \\ &\leq \left(\gamma A_0 + \frac{1}{3}A_0^3\right) + |\widehat{u}(t)| \max\left\{\frac{1}{3}\widehat{u}(t)^2,\ \widehat{u_x}(t)^2\right\} \leq \gamma A_0 + \frac{11}{24}A_0^3, \\ \widehat{u_x}'(t) &\geq -\left(\gamma A_0 + \frac{1}{3}A_0^3\right) + \frac{1}{3}\widehat{u}(t)^3 - \widehat{u_x}(t)^2\,\widehat{u}(t) \geq -\left(\gamma A_0 + \frac{11}{24}A_0^3\right).\end{aligned} \tag{4.38}$$

Integrating (4.38) from 0 to t produces

$$-\left(\gamma A_0 + \frac{11}{24}A_0^3\right)t + \widehat{u_x}(0) \leq \widehat{u_x}(t) \leq \left(\gamma A_0 + \frac{11}{24}A_0^3\right)t + \widehat{u_x}(0). \tag{4.39}$$

We will divide our argument into two cases, according to the sign of $m_0(x_1)$.

Case 1: $m_0(x_1) > 0$. Then we know from (4.39) and the initial conditions that

$$u_x(t, q(t,x)) = \widehat{u_x}(t) < 0 \tag{4.40}$$

for $0 < t \leq T_+$, where

$$T_+ = -\frac{24u_{0,x}(x_1)}{24\gamma A_0 + 11A_0^3}.$$

Next we claim that under assumption (1.10), it holds

$$\widehat{m}(t) = m(t, q(t,x_1)) > 0, \qquad \text{for } \ 0 < t \leq T_+. \tag{4.41}$$

Indeed, from (2.6), (2.4) and (4.45) we know that along the characteristics

$$\frac{d}{dt}\left(\widehat{m}(t)q_x(t,x_1)\right) = -\gamma\widehat{u_x}(t)q_x(t,x_1) > 0, \quad \text{for } \ 0 < t \leq T_+.$$

Thus

$$\widehat{m}(t)q_x(t,x_1) > \widehat{m}(0) > 0,$$

which in turn implies that $\widehat{m}(t) > 0$ on $[0, T_+]$. Moreover it follows from (2.4) that

$$q_x(t,x_1) = \exp\left(2\int_0^t \widehat{m}(s)\widehat{u_x}(s)\ ds\right) < 1.$$

Hence we have

$$\widehat{m}(t) > \widehat{m}(0) > 0 \quad \text{for } \ 0 < t \leq T_+.$$

Thus on $[0, T_+]$, $\frac{1}{\widehat{m}(t)}$ is well-defined and

$$\left(\frac{1}{\widehat{m}(t)}\right)' = -\frac{1}{\widehat{m}(t)^2}\widehat{m}' = \left(2 + \frac{\gamma}{\widehat{m}(t)^2}\right)\widehat{u_x}(t) \leq 2\widehat{u_x}(t). \tag{4.42}$$

Integrating (4.42) from 0 to t and using (4.39), we get

$$\begin{aligned}\frac{1}{\widehat{m}(t)} &\leq 2t\left[\left(\gamma A_0 + \frac{11}{24}A_0^3\right)\ t + \widehat{u_x}(0)\right] + \frac{1}{\widehat{m}(0)} \\ &= \left(2\gamma A_0 + \frac{11}{12}A_0^3\right)\ t^2 + 2\,u_{0,x}(x_1)\,t + \frac{1}{m_0(x_1)}.\end{aligned} \tag{4.43}$$

From the assumption on the initial data we know $u_{0,x}(x_1) < 0$ and $m_0(x_1) > 0$. So when

$$u_{0,x}(x_1) \le -\sqrt{\frac{24\gamma A_0 + 11A_0^3}{12m_0(x_1)}}$$

the right-hand side of (4.43) becomes zero at

$$T_1 = -\frac{12}{24\gamma A_0 + 11A_0^3}\left(u_{0,x}(x_1) + \sqrt{u_{0,x}^2(x_1) - \frac{24\gamma A_0 + 11A_0^3}{12m_0(x_1)}}\right) < T_+.$$

It is then deduced that there exists $0 < T \le T_1$ such that $\widehat{m}(t) \to +\infty$ as $t \to T$. On the other hand, (4.39) also implies that for $0 < t \le T_1$

$$\begin{aligned}
\widehat{M}(t) = \widehat{m}(t)\widehat{u_x}(t) &\le \widehat{m}(t)\left[\left(\gamma A_0 + \frac{11}{24}A_0^3\right) t + \widehat{u_x}(0)\right] \\
&\le \widehat{m}(t)\left[\left(\gamma A_0 + \frac{11}{24}A_0^3\right) T_1 + u_{0,x}(x_1)\right] \\
&= \frac{1}{2}\widehat{m}(t)\left(u_{0,x}(x_1) - \sqrt{u_{0,x}^2(x_1) - \frac{24\gamma A_0 + 11A_0^3}{12m_0(x_1)}}\right),
\end{aligned} \tag{4.44}$$

which thus implies that

$$\inf_{x\in\mathbb{R}} M(x,t) \le \widehat{M}(t) \to -\infty, \quad \text{as} \quad t \to T \le T_1.$$

Therefore the solution $u(t,x)$ blows up at a time $0 < T \le T_1$.

Case 2: $m_0(x_1) < 0$. The argument follows along the same line as in the previous case. For the sake of completeness we give the details as follows.

In view of (4.39) and the initial conditions, $\widehat{u_x}(t)$ satisfies

$$u_x(t, q(t,x)) = \widehat{u_x}(t) > 0 \tag{4.45}$$

for $0 < t \le T_-$, where

$$T_- = \frac{24u_{0,x}(x_1)}{24\gamma A_0 + 11A_0^3}.$$

Hence

$$\widehat{m}(t) < \widehat{m}(0) < 0, \quad \text{for} \;\; 0 < t < T_-,$$

which implies that $\frac{1}{\widehat{m}(t)}$ is well-defined on $[0, T_-]$, and we further compute to get

$$\left(\frac{1}{\widehat{m}(t)}\right)' = -\frac{1}{\widehat{m}(t)^2}\widehat{m}' = \left(2 + \frac{\gamma}{\widehat{m}(t)^2}\right)\widehat{u_x}(t) \ge 2\widehat{u_x}(t).$$

Integrating the above leads to

$$\frac{1}{\widehat{m}(t)} \ge -\left(2\gamma A_0 + \frac{11}{12}A_0^3\right) t^2 + 2\,u_{0,x}(x_1)\,t + \frac{1}{m_0(x_1)}.$$

When

$$u_{0,x}(x_1) \ge \sqrt{\frac{24\gamma A_0 + 11A_0^3}{-12m_0(x_1)}},$$

the right-hand side becomes zero at

$$T_2 = \frac{12}{24\gamma A_0 + 11A_0^3}\left(u_{0,x}(x_1) - \sqrt{u_{0,x}^2(x_1) + \frac{24\gamma A_0 + 11A_0^3}{12m_0(x_1)}}\right) < T_-.$$

Hence there exists some $T \in (0, T_2]$ such that $\widehat{m}(t) \to -\infty$ as $t \to T$. Therefore

$$\begin{aligned}\widehat{M}(t) = \widehat{m}(t)\widehat{u_x}(t) &\le \widehat{m}(t)\left[\widehat{u_x}(0) - \left(\gamma A_0 + \frac{11}{24}A_0^3\right)t\right]\\ &\le \widehat{m}(t)\left[u_{0,x}(x_1) - \left(\gamma A_0 + \frac{11}{24}A_0^3\right)T_2\right]\\ &= \frac{1}{2}\widehat{m}(t)\left(u_{0,x}(x_1) + \sqrt{u_{0,x}^2(x_1) + \frac{24\gamma A_0 + 11A_0^3}{12m_0(x_1)}}\right),\end{aligned}$$

which in turn implies that

$$\inf_{x\in\mathbb{R}} M(x,t) \le \widehat{M}(t) \to -\infty, \quad \text{as} \quad t \to T \le T_2.$$

Therefore in this case the solution $u(x,t)$ also blows up at a time $0 < T \le T_2$. This completes the proof of Theorem 1.5. □

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