Onsager's energy conservation for inhomogeneous Euler equations

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Abstract

This paper addresses the problem of energy conservation for the two- and three-dimensional density-dependent Euler equations. Two types of sufficient conditions on the regularity of solutions are provided to ensure the conservation of total kinetic energy on the entire time interval including the initial time. The first class of data assumes integrability on the spatial gradient of the density, and hence covers the classical result of Constantin-E-Titi [6] for the homogeneous Euler equations. The other type of data imposes extra time Besov regularity on the velocity profile, and the corresponding result can be applied to deal with a wide class of rough density profiles.

Résumé Cette article étudie la conservation d'énergie pour les equations d'Euler avec densités variables en dimension deux et trois. Nous présentons deux types de conditions sur la régularité des solutions, assurant la conservation de l'énergie cinétique sur l'intervalle de temps considéré, y compris le temps initial. La première condition est une propriété d'intégrabilité sur le gradient de densité. Elle correspond à la condition du résultat de Constantin-E-Titi [6] pour le cas homogène. La deuxième famille de conditions impose de la régularité Sobolev en temps sur la vitesse, et permet de considérer une large classe de profiles de densité peu réguliers.

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1. Introduction

The theory of hydrodynamics is a fascinating subject, with a long history in both pure and applied mathematics. The first comprehensive model was proposed by Euler in the 1750s, with the most notable variant introduced almost a century later by Navier and Stokes to allow for viscous effects. Besides being able to describe smooth flow motion, the Euler equations also model a wide range of fluids with singularities or limited smoothness, for instance flows with point vortices, vortex sheets, and turbulent flows in the limit of vanishing viscosity. Such configurations naturally lead to considering the Euler equations in a *weak* sense.

Given that the solution to the Euler equations is sufficiently smooth, say, for e.g., C^1, it is easy to see that the total kinetic energy of the flow is conserved. On the other hand it had long been observed experimentally (but mathematically still open) that anomalous dissipation of energy – energy dissipation independent of viscosity – persists in fully developed turbulent flow. Therefore it is reasonable to expect the existence of weak solutions to the Euler equations which do not conserve energy; see Scheffer [30],

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Shnirelman [33] and De Lellis–Székelyhidi [9]. This possibility goes by the name of the "Onsager conjecture" [28]: non-conservation of energy in the three-dimensional Euler equations would be related to the loss of regularity. Specifically, Onsager conjectured that every weak solution to the Euler equations with Hölder continuity exponent $\alpha > \frac{1}{3}$ conserves energy; and anomalous dissipation of energy occurs when $\alpha < \frac{1}{3}$. See [10, 16, 20, 29] for reviews and further discussions.

The first part of the conjecture was proved by Eyink [15], Constantin-E-Titi [6], Duchon-Robert [13], Cheskidov et al. [4], among others. The sharpest result is given in [4], where the conservation of energy was proved in the Besov space setting, and the results allow for possible failure of energy conservation in the endpoint $\alpha = \frac{1}{3}$ case.

The development toward the other direction of the conjecture is more recent. The rigorous mathematical work establishing existence of dissipative weak Euler solutions of the type conjectured by Onsager began with the pioneering work of DeLellis-Székelyhidi [11, 12] based on the convex integration approach, and has since culminated in constructions of solutions up to the critical 1/3 regularity [3, 21, 22].

In this paper we consider the energy conservation for the weak solutions of the *inhomogeneous* Euler equations, namely

$$\begin{aligned} &\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \\ &(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = 0 \\ &\operatorname{div}\mathbf{u} = 0, \end{aligned} \tag{1.1}$$

with initial data

$$\rho|_{t=0} = \rho_0(x), \qquad \rho\mathbf{u}|_{t=0} = \mathbf{m}_0(x) = \rho_0\mathbf{u}_0, \tag{1.2}$$

where P denotes the pressure, $\rho \geq 0$ is the density of fluid, $\mathbf{u}$ stands for the velocity of fluid. For the sake of simplicity we will consider the periodic setting $x \in \mathbb{T}^N$ with $N = 2, 3$. Here we define $\mathbf{u}_0 = 0$ on the set $\{x : \ \rho_0(x) = 0\}$.

Density fluctuations are widely present in turbulent flows. They may arise from non-uniform species concentrations, temperature variation or pressure. In geo-fluids, for instance, where external forces are present (e.g. gravity), density stratification can be caused by temperature and salinity gradients (ocean) or moisture effects (atmosphere). Also, a strong density inhomogeneity can be induced by the mixing of different-density species. A canonical example is the Rayleigh-Taylor instability of an interface between two fluids of different densities.

Mathematically, having a variable density changes the regularity structure substantially, and poses additional challenges compared to the much more extensively studied homogeneous flows. Nevertheless, some results on the local well-posedness of (1.1) – (1.2) can be found in, for e.g., [7, 8, 27].

When density is not a constant throughout the fluid, the vorticity equation becomes

$$\omega_t + \mathbf{u}\cdot\omega = (\omega\cdot\nabla)\mathbf{u} + \frac{\nabla\rho\times\nabla P}{\rho^2},$$

where in 2D, the cross product is understood as $\nabla^{\perp}\rho \cdot \nabla P$. One sees that fluid particles representing different instantaneous density respond very differently to pressure gradients. Vorticity can be generated from non-aligned density and pressure gradients (the baroclinic torque). Moreover, the regularity of the density gradient does not propagate from the initial data. All of these indicate possible energy fluctuation coming from the loss of smoothness of the density.

On the other hand, the roughness of the density can be "traded off" by assuming more regularity of the velocity field. The general strategy to prove energy conservation is to mollify the momentum equation and then test it against some suitable velocity-type test function. When passing to the limit as the mollification parameter tends to zero, the commutator estimates are required for treating the nonlinear terms. Compared with the homogeneous equations, the momentum equation in (1.1) contains a nonlinear term $\rho\mathbf{u}$ in the time derivative, and hence needs a commutator estimate (in time).

Leslie-Shvydkoy [23] managed to avoid this time commutator estimate by using the momentum $\mathbf{m} = \rho\mathbf{u}$ as the unknown variable instead of the velocity $\mathbf{u}$. This way to obtain the energy conservation one needs to choose the test function to be $\frac{\mathbf{m}}{\rho}$ in a regularized form. The price to pay is that (i) the divergence-free structure of the test function is no longer valid, and hence additional assumptions on the pressure has to be made; (ii) the density must be bounded away from zero.

Feireisl et al [19] took a more direct approach by assuming Besov regularity also in time to pertain the divergence-free property of the test function and to allow for the existence of vacuum in the system. Their method can also be applied to treat the case of isentropic compressible flows. What was proved there is a stronger result, namely a local energy equality in the sense of distribution, under a different set of regularity assumptions than those of [23]. In particular the integrability on the pressure is required, as in [23]. Such an assumption can be removed when the global energy conservation is considered. However what is proved in [19] is that the global energy is conserved also in the sense of distribution, and hence it is still conceivable that the energy might fluctuate in time in a set of times of measure zero, unless additional smoothness conditions are imposed.

The objective of this paper is to continue addressing the relation between the energy conservation and the degree of regularity of the solutions for system (1.1). In particular we provide sufficient conditions on the regularity of solutions to ensure the conservation of the total energy. Our approach is in the spirit of Constantin-E-Titi [6], with additional care to the density term ρ. We choose to work with the unknowns $\mathbf{u}$ and ρ, like in Feireisl et al [19]. The main differences between our method and the one in [19], which also constitutes the main contribution of this paper, are explained in the paragraphs below. Note that many of the ideas have been successfully developed to the case of *compressible* Navier-Stokes equations [34], wherein the main purpose was to derive a priori estimates rather than energy equality. Further development of the method does lead to the energy conservation of such flows [35]. Here we see that this tool also works well for incompressible flows.

Global energy conservation and continuity at $t = 0$

Unlike in the homogeneous flows, where a spatially regularized velocity $\mathbf{u}^\varepsilon$ can be used as a test function to generate the global energy, in the inhomogeneous case $\mathbf{u}^\varepsilon$ fails to possess enough temporal regularity to qualify for a test function. This is why a further space-time cutoff function is used in [19], and thus only a distributional energy equality can be obtained. Our approach makes strong use of the incompressibility of $\mathbf{u}^\varepsilon$ and mollifies it in time by a time cutoff function. This allows one to prove the global energy conservation in the strong sense on a time interval $[\delta, T]$ for any $\delta > 0$. Therefore the remaining issue is to obtain the continuity of the energy at the initial time $t = 0$.

In the homogeneous case, the velocity field $\mathbf{u}$ is continuous at time $t = 0$, and hence the energy conservation holds all the way up to the initial time. This is not necessarily true for inhomogeneous flows. However from the energy point of view, the analogous requirement would be that $\sqrt{\rho}\mathbf{u}$ is continuous in the strong topology at $t = 0$. This can be done by studying the continuity of $\rho\mathbf{u}$ and $\sqrt{\rho}$ at $t = 0$. Using the equations they satisfy, one can show that they are continuous at $t = 0$ in the weak topology, which, together with the initial regularity on $\mathbf{u}$, is enough to conclude the right continuity of $\sqrt{\rho}\mathbf{u}$ in the strong topology. Finally to continue the energy conservation up to the initial time, we introduce a special type of temporal cutoff function which originally vanishes near $t = 0$, but will later be extended past $t = 0$ into $t < 0$.

Regularity of ρ

The next difficulty comes from the fact that the commutator estimates for the nonlinear time derivative $(\rho\mathbf{u})_t$ naturally requires the time regularity of ρ. However from the mass conservation, this time regularity can be transferred to the spatial regularity of ρ, provided that one has an L^r control of $\mathrm{div}\mathbf{u}$; see Remark 1.1 (2). As is explained earlier, the regularity of $\nabla\rho$ does not propagate by the flow. Therefore imposing assumptions on $\nabla\rho$ seems to be a reasonable choice for the energy conservation. In particular, doing so allows one to avoid assuming additional time regularity on the velocity field $\mathbf{u}$, and hence can recover the classical result of Constantin-E-Titi [6]; see Remark 1.1 (1). Moreover, similar as in [19], working with ρ and $\mathbf{u}$ makes it possible to choose test functions with a divergence-free structure. As a consequence, our result applies to the case when vacuum is present.

Pressure

When the test function hits the pressure term in the momentum equation, it is natural to impose some conditions on the pressure in order to obtain the energy equality. In the constant density case, the pressure solves an elliptic equation

$$-\Delta P = \sum_{i,j} \partial_{x_i}\partial_{x_j}(u_i u_j),$$

and thus the regularity of pressure can be inferred from the velocity. However for inhomogeneous flows, such a conclusion does not hold unless the density is sufficiently regular. Therefore in the density-dependent case it is common to assume certain regularity condition on the pressure; see [19, 23]. Here we introduce a special test function that is divergence-free to remove the pressure term completely, so that we need no assumption on the pressure.

1.1. Main results

The weak solution we are interested in here is in the sense of Leray-Hopf. That is,

Definition 1.1. We say $(\rho, \mathbf{u})$ is a weak solution to (1.1)-(1.2) if it satisfies (1.1) in the sense of distributions with initial data given in (1.2), and that

$$E(t) \leq E(0), \quad \text{for all } t \geq 0, \tag{1.3}$$

where $E(t)$ is the total energy

$$E(t) = \int_{\mathbb{T}^N} \left(\rho|\mathbf{u}|^2\right)(t)\,dx.$$

We provide two types of sufficient regularity conditions on the weak solution of (1.1) that ensure the conservation of the energy. The first one requires the control of $\nabla\rho$ to avoid additional time regularity assumption on ρ. The definition of the Besov spaces $B_p^{\alpha,\infty}$ is given in Section 2.

Theorem 1.1. *Let* $(\rho, \mathbf{u})$ *be a weak solution of* (1.1)-(1.2).

$$\begin{aligned} &\rho \in L^\infty([0,T]\times\mathbb{T}^N) \cap L^p(0,T;W^{1,p}(\mathbb{T}^N)), \quad \rho_t \in L^r([0,T]\times\mathbb{T}^N),\\ &\mathbf{u} \in L^q(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N)) \end{aligned} \tag{1.4}$$

for any $\frac{1}{r} + \frac{2}{q} \leq 1$, $\frac{1}{p} + \frac{3}{q} \leq 1$ *and* $\alpha > \frac{1}{3}$,

$$\sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\mathbb{T}^N)) \tag{1.5}$$

and

$$\mathbf{u}_0 \in L^2(\mathbb{T}^N).$$

Then the energy is conserved, that is, $E(t) = E(0)$, *for any* $t \in [0,T]$.

Remark 1.1. (1) In the case of constant density, condition (1.4) recovers the classical result of Constantin-E-Titi [6] by taking $p = r = \infty$ and $q = 3$.

(2) From the equation of mass conservation we see that the time regularity of ρ can be replaced by some additional spatial regularity of $\mathbf{u}$, say, for e.g., $\mathrm{div}\mathbf{u} \in L^r([0,T]\times\mathbb{T}^N)$. The reason why we chose to impose the regularity on ρ_t is to recover the result of Constantin-E-Titi [6].

(3) As is noted before, the notable differences between our results and the ones in [19, 23] are that we are able to establish the global energy conservation on the whole time interval $[0,T]$, while in [19]

it is in the distributional sense; and we do allow the presence of vacuum state, whereas in [23] it is excluded. Moreover, we do not need any assumption on the pressure. In terms of the method, one of the new ingredients is the construction of a suitable class of test functions that maintain the divergence-free structure, which help eliminate the pressure term. Another new ingredient lies in the proof for the conservation up to the initial time, where our approach combines the weak continuity of $\sqrt{\rho}$ and ρu to ensure the strong right temporal continuity of $\sqrt{\rho}u$.

(4) Notice that imposing regularity assumption on the vorticity ω in order to obtain energy conservation for the two-dimensional homogeneous Euler system has been considered by Cheskidov et. al. [5]. This is due to the fact that some additional spatial regularity of $\mathbf{u}$ can be recovered from ω. However this is not true when only assuming spatial integrability on div$\mathbf{u}$. It would be interesting to try to understand how an L^p control of the vorticity could affect the energy conservation for density-dependent incompressible Euler system. This will be addressed in a forthcoming paper.

Our second theorem treats the case when density can experience large fluctuation. This covers a rather wide range of situations including flows with mixing layers and vortex sheets. In this case we need to impose extra time regularity on the velocity field $\mathbf{u}$ to compensate for the roughness of ρ.

Theorem 1.2. *Let* $(\rho, \mathbf{u})$ *be a weak solution of* (1.1)-(1.2) *in the sense of distributions. Assume*

$$\rho \in L^\infty([0,T]\times\mathbb{T}^N), \quad \mathbf{u} \in B_p^{\beta,\infty}(0,T; B_q^{\alpha,\infty}(\mathbb{T}^N)), \tag{1.6}$$

where $p, q \geq 3$ *and* $\alpha, \beta > 1/2$,

$$\sqrt{\rho}\mathbf{u} \in L^\infty(0,T; L^2(\mathbb{T}^N))$$

and

$$\mathbf{u}_0 \in L^2(\mathbb{T}^N).$$

Then the energy is conserved, that is, $E(t) = E(0)$, *for any* $t \in [0,T]$.

Remark 1.2. (1) In this theorem we do not pose any spatial and temporal regularity of ρ in the Besov spaces. To see this, note that when dealing with the commutator estimates for $(\rho\mathbf{u})_t$, an integration by parts places the time derivative on the test function which involves only the velocity $\mathbf{u}$. Hence assuming more time regularity on $\mathbf{u}$ would lead to the desired convergence.

(2) In [19] a similar situation is considered with $\rho \in L^1([0,T]\times\mathbb{T}^N)$, and the same regularity condition on the velocity is required. Here the difference between our result and the one in [19], like in the previous theorem, is that we are able to obtain the global energy conservation up to the initial time.

2. Besov Space and a commutator lemma

In this section we briefly recall some properties of the Besov space $B_p^{\alpha,\infty}(\Omega)$, and prove a key estimate which is similar to the commutator lemma in [18, 25].

For $0 < \alpha < 1$ and $1 \leq p \leq \infty$, we define the Besov space to be the set of all functions equipped with the following norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{\xi\in\Omega} \frac{\|w(\cdot+\xi) - w\|_{L^p(\Omega\cap(\Omega-\xi))}}{|\xi|^\alpha}, \tag{2.1}$$

where $\Omega - \xi = \{x - \xi : x \in \Omega\}$.

Next we define

$$f^\varepsilon(t,x) := \eta_\varepsilon * f(t,x), \quad t > \varepsilon,$$

where $\eta_\varepsilon = \frac{1}{\varepsilon^{N+1}}\eta(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$, and $\eta(t,x) \geq 0$ is a smooth even function compactly supported in the space-time ball of radius 1, and with integral equal to 1.

Now we are ready to recall some classical properties of the Besov space as follows:

$$\|w^{\varepsilon} - w\|_{L^p(\mathbb{R}^+, L^q(\Omega))} \le C\varepsilon^{\alpha} \|w\|_{B_p^{\beta,\infty}(\mathbb{R}^+, B_q^{\alpha,\infty}(\Omega))} \tag{2.2}$$

and

$$\|\nabla w^{\varepsilon}\|_{L^p(\mathbb{R}^+, L^q(\Omega))} \le C\varepsilon^{\alpha-1} \|w\|_{B_p^{\beta,\infty}(\mathbb{R}^+, B_q^{\alpha,\infty}(\Omega))}, \tag{2.3}$$

where we assume that $\beta \ge \alpha > 0$. We will rely on the following lemma which was proved in [18, 25]. The statement of the result we adopt here is in the spirit of [24].

Lemma 2.1. *Let ∂ be a partial derivative in space or time. Let $f,\ \partial f \in L^p(\mathbb{R}^+ \times \Omega)$, $g \in L^q(\mathbb{R}^+ \times \Omega)$ with $1 \le p,\ q \le \infty$, and $\frac{1}{p} + \frac{1}{q} \le 1$. Then, we have*

$$\left\| [\partial(fg)]^{\varepsilon} - \partial(f g^{\varepsilon}) \right\|_{L^r(\mathbb{R}^+ \times \Omega)} \le C(\|f_t\|_{L^p(\mathbb{R}^+ \times \Omega)} + \|\nabla f\|_{L^p(\mathbb{R}^+ \times \Omega)}) \|g\|_{L^q(\mathbb{R}^+ \times \Omega)}$$

for some constant $C > 0$ independent of ε, f and g, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$[\partial(fg)]^{\varepsilon} - \partial(f g^{\varepsilon}) \to 0 \quad \text{in } L^r(\mathbb{R}^+ \times \Omega)$$

as $\varepsilon \to 0$ if $r < \infty$.

For the purpose of this paper, we need to extend the above lemma to the Besov space.

Lemma 2.2. *Let $f \in B_{p_1}^{\beta,\infty}(\mathbb{R}^+, B_{p_2}^{\alpha,\infty}(\Omega))$, $g \in L^{q_1}(\mathbb{R}^+; L^{q_2}(\Omega))$ with $\beta \ge \alpha$, $1 \le p_1,\ p_2,\ q_1,\ q_2 \le \infty$. Then, we have*

$$\|(fg)^{\varepsilon} - f g^{\varepsilon}\|_{L^{r_1}(\mathbb{R}^+, L^{r_2}(\Omega))} \le C \|g\|_{L^{q_1}(\mathbb{R}^+, L^{q_2}(\Omega))} \|f\|_{B_{p_1}^{\beta,\infty}(\mathbb{R}^+, B_{p_2}^{\alpha,\infty}(\Omega))},$$

for some constant $C > 0$ independent of f and g, and with $\frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i}$, $i = 1, 2$. In addition,

$$\|(fg)^{\varepsilon} - f g^{\varepsilon}\|_{L^{r_1}(\mathbb{R}^+, L^{r_2}(\Omega))} \le C\varepsilon^{\alpha} \to 0$$

as $\varepsilon \to 0$ if $r_i < \infty$.

Proof. Considering

$$\begin{aligned}
(fg)^{\varepsilon} - f g^{\varepsilon} &= \int_{\mathbb{R}^+ \times \mathbb{R}^d} (f(s,y)g(s,y) - f(t,x)g(s,y))\, \eta_{\varepsilon}(|t-s|, |x-y|)\, ds\, dy \\
&= \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(s,y)\, (f(s,y) - f(t,x))\, \eta_{\varepsilon}(|t-s|, |x-y|)\, ds\, dy \\
&= \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(t-r, x-z)(f(t-r, x-z) - f(t, x-z) + f(t, x-z) - f(t,x)) \eta_{\varepsilon}(|r|, |z|)\, dr\, dz \\
&= \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(t-r, x-z)\Big(\frac{(f(t-r, x-z) - f(t, x-z))}{r^{\beta}} r^{\beta} + \frac{(f(t, x-z) - f(t, x-z))}{|z|^{\alpha}} |z|^{\alpha}\Big) \eta_{\varepsilon}(|r|, |z|)\, dr\, dz \\
&\le C\varepsilon^{\alpha} \int_{\mathbb{R}^+ \times \mathbb{R}^d} g(t-r, x-z)\Big(\frac{(f(t-r, x-z) - f(t, x-z))}{r^{\beta}} + \frac{(f(t, x-z) - f(t, x-z))}{|z|^{\alpha}}\Big) \eta_{\varepsilon}(|r|, |z|)\, dr\, dz
\end{aligned}$$

for any (t, x), thus we have

$$\|(fg)^{\varepsilon} - f g^{\varepsilon}\|_{L^{r_1}(\mathbb{R}^+, L^{r_2}(\Omega))} \le C \|g\|_{L^{q_1}(\mathbb{R}^+, L^{q_2}(\Omega))} \|f\|_{B_{p_1}^{\beta,\infty}(\mathbb{R}^+, B_{p_2}^{\alpha,\infty}(\Omega))} \varepsilon^{\alpha}. \qquad \square$$

3. Proof of Theorem 1.1

We introduce a function $\Phi(t,x) = (\psi(t)\mathbf{u}^\varepsilon)^\varepsilon$ as a test function for deriving the energy equality, where $\psi(t) \in \mathfrak{D}(0,+\infty)$ is a test function, with $\mathfrak{D}(0,+\infty)$ being the class of all smooth compactly supported functions on $(0,+\infty)$. Here we further remark that this function vanishes near $t = 0$. However, later it is needed to extend the result for $\psi(t) \in \mathfrak{D}(-1,+\infty)$ in order to recover the initial value of the energy.

Note that $\psi(t)$ is compactly supported in $(0,\infty)$. Hence Φ is a well-defined test function for $t \in (0,\infty)$ and for ε small enough. Multiplying Φ on both sides of the second equation in (1.1), one obtains

$$\int_0^T \int_{\mathbb{T}^N} \Phi \left[(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P \right] dx\,dt = 0,$$

which in turn yields

$$\int_0^T \int_{\mathbb{T}^N} \psi(t)\mathbf{u}^\varepsilon \left[(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P \right]^\varepsilon dx\,dt = 0, \tag{3.1}$$

where we have used the fact that $\eta(-t,-x) = \eta(t,x)$.

The first term in (3.1) can be computed as

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \psi(t)\mathbf{u}^\varepsilon \left((\rho\mathbf{u})_t\right)^\varepsilon dx \\
&\quad= \int_0^T \int_{\mathbb{T}^N} \psi(t) \left[((\rho\mathbf{u})_t)^\varepsilon - (\rho\mathbf{u}^\varepsilon)_t \right] \mathbf{u}^\varepsilon \,dx\,dt + \int_0^T \int_{\mathbb{T}^N} \psi(t)(\rho\mathbf{u}^\varepsilon)_t\mathbf{u}^\varepsilon\,dx\,dt \\
&\quad=: A_\varepsilon + \int_0^T \int_{\mathbb{T}^N} \psi(t)\rho\partial_t \frac{|\mathbf{u}^\varepsilon|^2}{2}\,dx\,dt + \int_0^T \int_{\mathbb{T}^N} \psi(t)\rho_t|\mathbf{u}^\varepsilon|^2\,dx\,dt.
\end{aligned} \tag{3.2}$$

Similarly, the second term in (3.1) can be treated as

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^N} \psi(t)\mathbf{u}^\varepsilon \left(\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})\right)^\varepsilon dx\,dt \\
&\quad= \int_0^T \int_{\mathbb{T}^N} \psi(t) \left[(\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}))^\varepsilon - \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}^\varepsilon) \right] \mathbf{u}^\varepsilon\,dx\,dt \\
&\qquad+ \int_0^T \int_{\mathbb{T}^N} \psi(t)\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon\,dx\,dt \\
&\quad=: B_\varepsilon + \int_0^T \int_{\mathbb{T}^N} \psi(t)\rho\mathbf{u}\cdot\nabla\frac{|\mathbf{u}^\varepsilon|^2}{2}\,dx + \int_0^T \int_{\mathbb{T}^N} \psi(t)\operatorname{div}(\rho\mathbf{u})|\mathbf{u}^\varepsilon|^2\,dx\,dt \\
&\quad= B_\varepsilon - \int_0^T \int_{\mathbb{T}^N} \psi(t)\rho_t\frac{|\mathbf{u}^\varepsilon|^2}{2}\,dx,
\end{aligned} \tag{3.3}$$

where an integration by parts and the first equation of (1.1) are used to obtain the last equality.

Note that from (1.4), $\rho_t \in L^r([0,T]\times\mathbb{T}^N)$. Therefore the last term of the right-hand side in (3.2) and the second term of the right-hand side in (3.3) are well-defined. Thanks to (3.1), (3.2) and (3.3), we have

$$-\int_0^T \int_{\mathbb{T}^N} \psi_t \frac{1}{2}\rho|\mathbf{u}^\varepsilon|^2\,dx\,dt + A_\varepsilon + B_\varepsilon = 0. \tag{3.4}$$

From (1.4) and (2.2), one obtains

$$\int_0^T \int_{\mathbb{T}^N} \frac{1}{2}\rho|\mathbf{u}^\varepsilon|^2\psi_t\,dx\,dt \longrightarrow \int_0^T \int_{\mathbb{T}^N} \frac{1}{2}\rho|\mathbf{u}|^2\psi_t\,dx\,dt \quad \text{as} \quad \varepsilon \to 0. \tag{3.5}$$

To derive the energy equality from (3.4) in the distributional sense, we will apply Lemma 2.1 to prove

$$A_{\varepsilon}(t,x) \to 0 \tag{3.6}$$

as ε goes to zero.

In fact, note from (1.4) that $\rho_t \in L^r([0,T]\times\mathbb{T}^N)$. Thus, Lemma 2.1 gives

$$\begin{aligned}|A_{\varepsilon}| &\le \|\psi(t)\|_{L^\infty(0,T)}\int_0^T\int_{\mathbb{T}^N}\left|\mathbf{u}^{\varepsilon}[((\rho\mathbf{u})_t)^{\varepsilon} - (\rho\mathbf{u}^{\varepsilon})_t]\right|dx\,dt\\ &\le C\|\psi(t)\|_{L^\infty(0,T)}(\|\rho_t\|_{L^r([0,T]\times\mathbb{T}^N)} + \|\nabla\rho\|_{L^p([0,T]\times\mathbb{T}^N)})\|\mathbf{u}\|^2_{L^q(0,T;L^q(\mathbb{T}^N))},\end{aligned}$$

provided that $\frac{1}{r}+\frac{2}{q}\le 1$ and $\frac{1}{p}+\frac{2}{q}\le 1$. Moreover, as ε tends to zero, we have

$$A_{\varepsilon}\to 0.$$

We are not able to apply Lemma 2.1 to control B_ϵ directly because there is no estimate on $\nabla(\rho\mathbf{u})$ in L^p. Instead, we will use Lemma 2.2. In fact,

$$\begin{aligned}B_\epsilon &= \int_0^T\int_{\mathbb{T}^N}\psi(t)\,[(\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}))^{\varepsilon} - \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}^{\varepsilon})]\,\mathbf{u}^{\varepsilon}\,dx\,dt\\ &= -\int_0^T\int_{\mathbb{T}^N}\psi(t)\,[(\rho\mathbf{u}\otimes\mathbf{u})^{\varepsilon} - (\rho\mathbf{u}\otimes\mathbf{u}^{\varepsilon})]\,\nabla\mathbf{u}^{\varepsilon}\,dx\,dt\\ &= \int_0^T\int_{\mathbb{T}^N}\psi(t)\,(\mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u})^{\varepsilon} - \mathrm{div}[\rho(\mathbf{u}\otimes\mathbf{u})^{\varepsilon}])\,\mathbf{u}^{\varepsilon}\,dx\,dt-\\ &\int_0^T\int_{\mathbb{T}^N}\psi(t)\rho[(\mathbf{u}\otimes\mathbf{u})^{\varepsilon} - \mathbf{u}\otimes\mathbf{u}^{\varepsilon}]\nabla\mathbf{u}^{\varepsilon}\,dx\,dt\\ &= B_{1\varepsilon} + B_{2\varepsilon}.\end{aligned}$$

Note that $\nabla\rho$ is bounded in $L^p(0,T;L^p(\mathbb{T}^N))$, we are able to apply Lemma 2.1 to have

$$|B_{1\varepsilon}| \le C\|\mathbf{u}\|^3_{L^q(0,T;L^q(\mathbb{T}^N))}\|\nabla\rho\|_{L^p(0,T;L^p(\mathbb{T}^N))}$$

provided that $\frac{3}{q}+\frac{1}{p}\le 1$. This implies that $B_{1\varepsilon}$ tends to zero as ε goes to zero. This convergence does not depend on the value of α.

Meanwhile, Note that

$$(\mathbf{u}\otimes\mathbf{u})^{\varepsilon} - \mathbf{u}\otimes\mathbf{u}^{\varepsilon} = ((\mathbf{u}\otimes\mathbf{u})^{\varepsilon} - \mathbf{u}^{\varepsilon}\otimes\mathbf{u}^{\varepsilon}) + (\mathbf{u}^{\varepsilon}\otimes\mathbf{u}^{\varepsilon} - \mathbf{u}\otimes\mathbf{u}^{\varepsilon}).$$

Using the same argument as in [6], we can show that

$$\left|\int_0^T\int_{\mathbb{T}^N}\psi(t)\rho(\mathbf{u}\otimes\mathbf{u})^{\varepsilon} - \mathbf{u}^{\varepsilon}\otimes\mathbf{u}^{\varepsilon})\nabla\mathbf{u}^{\varepsilon}\,dx\,dt\right| \le C\|\mathbf{u}\|^3_{L^q(0,T;B^{\alpha,\infty}_q(\mathbb{T}^N))}\varepsilon^{3\alpha-1}\to 0,$$

as $\varepsilon\to 0$ for $\alpha > \frac{1}{3}$. We calculate

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^N}\psi(t)\rho(\mathbf{u}^{\varepsilon}\otimes\mathbf{u}^{\varepsilon} - \mathbf{u}\otimes\mathbf{u}^{\varepsilon})\nabla\mathbf{u}^{\varepsilon}\,dx\,dt\\ &= \frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\psi(t)\rho(\mathbf{u}^{\varepsilon} - \mathbf{u})\nabla|\mathbf{u}^{\varepsilon}|^2\,dx\,dt\\ &= -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\psi(t)\mathrm{div}(\rho\mathbf{u}^{\varepsilon} - \rho\mathbf{u})|\mathbf{u}^{\varepsilon}|^2\,dx\,dt\\ &= -\frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\psi(t)\mathrm{div}(\rho\mathbf{u}^{\varepsilon} - (\rho\mathbf{u})^{\varepsilon} + (\rho\mathbf{u})^{\varepsilon} - \rho\mathbf{u})|\mathbf{u}^{\varepsilon}|^2\,dx\,dt\\ &= R_{\varepsilon} + \frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\psi(t)(\rho^{\varepsilon}_t - \rho_t)|\mathbf{u}^{\varepsilon}|^2\,dx\,dt,\end{aligned}$$

By Lemma 2.1, we have $R_\varepsilon \to 0$ as $\varepsilon \to 0$. Again since ρ_t is bounded in $L^r([0,T]\times\mathbb{T}^N)$, we have

$$\frac{1}{2}\int_0^T\int_{\mathbb{T}^N}\psi(t)(\rho_t^\varepsilon-\rho_t)|\mathbf{u}^\varepsilon|^2\,dx\,dt\to 0$$

as $\varepsilon \to 0$ for any $0<\alpha\le 1$.

Thus, B_ε tends to zero as $\varepsilon\to 0$ for any $\frac{1}{r}+\frac{2}{q}\le 1$, $\frac{1}{p}+\frac{3}{q}\le 1$ and $\alpha>\frac{1}{3}$.

We are ready to pass to the limits in (3.4). Letting ε go to zero and using (3.5)-(3.6), what we have proved is that in the limit,

$$-\int_0^T\int_{\mathbb{T}^N}\psi_t\frac{1}{2}\rho|\mathbf{u}|^2\,dx\,dt=0 \tag{3.7}$$

for any test function $\psi\in\mathfrak{D}(0,\infty)$. This in turn implies that

$$\frac{dE(t)}{dt}=0 \quad \text{in the sense of distribution.} \tag{3.8}$$

The goal now is to obtain the exact energy conservation up to the initial time. An approximation argument shows that (3.7) remains valid for functions ψ_τ belonging only to $W^{1,\infty}$. First we prove the continuity of $(\sqrt{\rho}\mathbf{u})(t)$ in the strong topology as $t\to 0^+$. For this we compute

$$\begin{aligned}
&\operatorname*{ess\,lim\,sup}_{t\to 0^+}\int_{\mathbb{T}^N}|\sqrt{\rho}\mathbf{u}-\sqrt{\rho_0}\mathbf{u}_0|^2\,dx\\
&\le \operatorname*{ess\,lim\,sup}_{t\to 0^+}\left(\int_{\mathbb{T}^N}\rho|\mathbf{u}|^2\,dx-\int_{\mathbb{T}^N}\rho_0|\mathbf{u}_0|^2\,dx\right)\\
&+\operatorname*{ess\,lim\,sup}_{t\to 0^+}\left(2\int_{\mathbb{T}^N}\sqrt{\rho_0}\mathbf{u}_0(\sqrt{\rho_0}\mathbf{u}_0-\sqrt{\rho}\mathbf{u})\,dx\right).
\end{aligned} \tag{3.9}$$

This together with the (1.3) yields

$$\begin{aligned}
&\operatorname*{ess\,lim\,sup}_{t\to 0^+}\int_{\mathbb{T}^N}|\sqrt{\rho}\mathbf{u}-\sqrt{\rho_0}\mathbf{u}_0|^2\,dx\\
&\le 2\operatorname*{ess\,lim\,sup}_{t\to 0^+}\int_{\mathbb{T}^N}\sqrt{\rho_0}\mathbf{u}_0(\sqrt{\rho_0}\mathbf{u}_0-\sqrt{\rho}\mathbf{u})\,dx=:W.
\end{aligned} \tag{3.10}$$

To show the continuity of $(\sqrt{\rho}\mathbf{u})(t)$ in the strong topology as $t\to 0^+$, we need $W=0$. To this end, for any fixed $\phi\in\mathfrak{D}(\mathbb{T}^N)$ satisfying $\operatorname{div}\phi=0$, we define the function f on $[0,T]$ as

$$f(t)=\int_{\mathbb{T}^N}(\rho\mathbf{u})(t,x)\cdot\phi(x)\,dx.$$

Note from (1.4) that the function

$$\int_{\mathbb{T}^N}(\rho\mathbf{u})(t,x)\cdot\phi(x)\,dx$$

is continuous function with respect to $t\in[0,T]$. On the other hand, note from (1.4) and (1.5) that

$$\rho\in L^\infty(0,T;L^\infty(\mathbb{T}^N)) \quad \text{and} \quad \sqrt{\rho}\mathbf{u}\in L^\infty(0,T;L^2(\mathbb{T}^N)),$$

and hence we obtain that

$$\rho\mathbf{u}\in L^\infty(0,T;L^2(\mathbb{T}^N)).$$

From equation (1.1) we further know that

$$\frac{d}{dt}\int_{\mathbb{T}^N}(\rho\mathbf{u})(t,x)\cdot\phi(x)\,dx=\int_{\mathbb{T}^N}\rho\mathbf{u}\otimes\mathbf{u}:\nabla\phi\,dx, \tag{3.11}$$

which is bounded due to (1.5). Therefore from [18, Corollary 2.1] it follows that

$$\rho\mathbf{u} \in C([0,T]; L^2_{\text{weak}}(\mathbb{T}^N)). \tag{3.12}$$

Applying the same argument to $\rho_t = -\text{div}(\rho\mathbf{u})$ with $\text{div}\mathbf{u} = 0$, we have

$$\sqrt{\rho} \in C([0,T]; L^b_{\text{weak}}(\mathbb{T}^N)) \tag{3.13}$$

for any $b > 1$.

We now estimate W as follows

$$\begin{aligned} W &:= 2\text{ess}\limsup_{t\to 0^+} \int_{\mathbb{T}^N} \mathbf{u}_0(\rho_0\mathbf{u}_0 - \sqrt{\rho_0\rho}\mathbf{u})\,dx \\ &\le 2\text{ess}\limsup_{t\to 0^+} \int_{\mathbb{T}^N} \mathbf{u}_0(\rho_0\mathbf{u}_0 - \rho\mathbf{u})\,dx \\ &\quad + 2\text{ess}\limsup_{t\to 0^+} \int_{\mathbb{T}^N} \mathbf{u}_0(\sqrt{\rho} - \sqrt{\rho_0})\sqrt{\rho}\mathbf{u}\,dx. \end{aligned} \tag{3.14}$$

Meanwhile, by (1.4), we have

$$\sqrt{\rho}\mathbf{u} \in L^q(0,T; L^q(\mathbb{T}^N)) \quad \text{ for any } q > 3. \tag{3.15}$$

Using (3.12), (3.13) and (3.15) in (3.14), one deduces $W = 0$ provided that $\mathbf{u}_0 \in L^2(\mathbb{T}^N)$. Thus, we have

$$\text{ess}\limsup_{t\to 0^+} \int_{\mathbb{T}^N} |\sqrt{\rho}\mathbf{u} - \sqrt{\rho_0}\mathbf{u}_0|^2\,dx = 0,$$

which gives us

$$(\sqrt{\rho}u)(t) \to (\sqrt{\rho}u)(0) \quad \text{ strongly in } L^2(\mathbb{T}^N) \text{ as } t \to 0^+. \tag{3.16}$$

Similarly, one has the right temporal continuity of $\sqrt{\rho}u$ in L^2, that is, for any $t_0 \ge 0$,

$$(\sqrt{\rho}u)(t) \to (\sqrt{\rho}u)(t_0) \quad \text{ strongly in } L^2(\mathbb{T}^N) \text{ as } t \to t_0^+. \tag{3.17}$$

Now for $t_0 > 0$, we choose some positive τ and α such that $\tau + \alpha < t_0$ and define the following test function

$$\psi_\tau(t) = \begin{cases} 0, & 0 \le t \le \tau, \\ \dfrac{t-\tau}{\alpha}, & \tau \le t \le \tau+\alpha, \\ 1, & \tau+\alpha \le t \le t_0, \\ \dfrac{t_0 - t}{\alpha}, & t_0 \le t \le t_0 + \alpha, \\ 0, & t_0 + \alpha \le t. \end{cases}$$

Substituting the above test function into (3.7) we have that

$$\frac{1}{\alpha}\int_\tau^{\tau+\alpha}\int_{\mathbb{T}^N} \frac{1}{2}\rho|u|^2\,dxds - \frac{1}{\alpha}\int_{t_0}^{t_0+\alpha}\int_{\mathbb{T}^N} \frac{1}{2}\rho|u|^2\,dxds = 0.$$

Sending $\alpha \to 0$ and using the right continuity of $\sqrt{\rho}u$ in L^2 yields

$$E(\tau) - E(t_0) = 0.$$

Finally sending $\tau \to 0$, from (3.16) we have $E(t_0) = E(0)$, completing the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Following the previous section, we have

$$-\int_0^T\int_{\mathbb{T}^N}\psi_t\frac{1}{2}\rho^\varepsilon|\mathbf{u}^\varepsilon|^2\,dx\,dt+A_\varepsilon+B_\varepsilon=0, \tag{4.1}$$

with

$$A_\varepsilon=\int_0^T\int_{\mathbb{T}^N}\psi(t)\left(((\rho\mathbf{u})_t)^\varepsilon-(\rho^\varepsilon\mathbf{u}^\varepsilon)_t\right)\mathbf{u}^\varepsilon\,dx\,dt \tag{4.2}$$

and

$$B_\varepsilon=\int_0^T\int_{\mathbb{T}^N}\psi(t)\left((\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}))^\varepsilon-\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}^\varepsilon)\right)\mathbf{u}^\varepsilon\,dx\,dt. \tag{4.3}$$

To prove Theorem 1.2, we need to show

$$A_\varepsilon\to 0,\quad \text{and } B_\varepsilon\to 0\quad \text{as } \varepsilon\to 0.$$

We handle the term A_ε as follows

$$\begin{aligned}A_\varepsilon&=-\int_0^T\int_{\mathbb{T}^N}\psi_t\left((\rho\mathbf{u})^\varepsilon-\rho^\varepsilon\mathbf{u}^\varepsilon\right)\mathbf{u}^\varepsilon\,dx\,dt-\int_0^T\int_{\mathbb{T}^N}\psi\left((\rho\mathbf{u})^\varepsilon-\rho^\varepsilon\mathbf{u}^\varepsilon\right)\mathbf{u}^\varepsilon_t\,dx\,dt\\&=:A_{1\varepsilon}+A_{2\varepsilon}.\end{aligned}$$

The first term can be estimated as

$$|A_{1\varepsilon}|\le C(\psi_t)\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u})^\varepsilon-\rho^\varepsilon\mathbf{u}\right||\mathbf{u}^\varepsilon|\,dx\,dt+C(\psi_t)\int_0^T\int_{\mathbb{T}^N}\left|\rho^\varepsilon\mathbf{u}-\rho^\varepsilon\mathbf{u}^\varepsilon\right||\mathbf{u}^\varepsilon|\,dx\,dt.$$

We can then use Lemma 2.2 to control the first term of the right-hand side of the above as follows

$$\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u})^\varepsilon-\rho^\varepsilon\mathbf{u}\right||\mathbf{u}^\varepsilon|\,dx\,dt\le C\varepsilon^\alpha\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}^N))}\|\mathbf{u}\|^2_{B_p^{\beta,\infty}(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\to 0;$$

and the second term as

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^N}\left|\rho^\varepsilon\mathbf{u}-\rho^\varepsilon\mathbf{u}^\varepsilon\right||\mathbf{u}^\varepsilon|\,dx\,dt\\&\qquad\le C\|\rho^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^N))}\|\mathbf{u}^\varepsilon-\mathbf{u}\|_{L^p(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\|\mathbf{u}^\varepsilon\|_{L^p(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\\&\qquad\le C\varepsilon^\alpha\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}^N))}\|\mathbf{u}\|^2_{L^p(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\to 0\end{aligned}$$

as $\varepsilon\to 0$, where $p,q\ge 2$. In this section, we assume that $\alpha\le\beta$.

Similar argument applying to $A_{2\varepsilon}$ yields

$$\begin{aligned}|A_{2\varepsilon}|&\le C\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u})^\varepsilon-\rho^\varepsilon\mathbf{u}^\varepsilon\right||\mathbf{u}^\varepsilon_t|\,dx\,dt\\&\le C\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u})^\varepsilon-\rho^\varepsilon\mathbf{u}\right||\mathbf{u}^\varepsilon_t|\,dx\,dt+C\int_0^T\int_{\mathbb{T}^N}\left|(\rho^\varepsilon\mathbf{u})-\rho^\varepsilon\mathbf{u}^\varepsilon\right||\mathbf{u}^\varepsilon_t|\,dx\,dt\\&\le 2C\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{T}^N))}\|\mathbf{u}\|_{B_p^{\beta,\infty}(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\varepsilon^{2\alpha-1},\end{aligned}$$

which converges to zero as ε goes to zero when $\alpha>1/2$. Thus, we have $A_\varepsilon\to 0$ as ε goes to zero.

To handle B_ε,

$$\begin{aligned}|B_\varepsilon| &\le C\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\rho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon\right||\nabla\mathbf{u}^\varepsilon|\,dx\,dt\\ &\le C\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u}\otimes\mathbf{u})^\varepsilon-(\rho\mathbf{u})^\varepsilon\otimes\mathbf{u}\right||\nabla\mathbf{u}^\varepsilon|\,dx\,dt\\ &\quad+C\int_0^T\int_{\mathbb{T}^N}\left|(\rho\mathbf{u})^\varepsilon\otimes\mathbf{u}-(\rho\mathbf{u})^\varepsilon\otimes\mathbf{u}^\varepsilon\right||\nabla\mathbf{u}^\varepsilon|\,dx\,dt\\ &=: B_{1\varepsilon}+B_{2\varepsilon}.\end{aligned}$$

Note that $\mathbf{u}\in B_p^{\beta,\infty}(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))$. From Lemma 2.2 we conclude that

$$B_{1\varepsilon}\le C\|\mathbf{u}\|^3_{B_p^{\beta,\infty}(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\varepsilon^{2\alpha-1}\to 0,$$

and

$$B_{2\varepsilon}\le C\|\mathbf{u}\|^3_{B_p^{\beta,\infty}(0,T;B_q^{\alpha,\infty}(\mathbb{T}^N))}\varepsilon^{2\alpha-1}\to 0,$$

for any $\alpha>1/2$ and $p,q\ge 3$.

Letting $\varepsilon\to 0$ in (4.1), one obtains

$$-\int_0^T\int_{\mathbb{T}^N}\psi_t\frac{1}{2}\rho|\mathbf{u}|^2\,dx\,dt=0.$$

Because the regularity of ρ and $\mathbf{u}$ allow us to have

$$\sqrt{\rho}\in C([0,T];L^2_{\text{weak}}(\mathbb{T}^N))$$

and

$$\sqrt{\rho}\mathbf{u}\in C([0,T];L^2(\mathbb{T}^N)).$$

Thus, we can repeat the same argument in Section 3 to show

$$\int_{\mathbb{T}^N}\frac{1}{2}\rho|\mathbf{u}|^2(t)\,dx=\int_{\mathbb{T}^N}\frac{1}{2}\rho_0|\mathbf{u}_0|^2\,dx$$

for any $t\in[0,T]$.

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[1] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi, Jr., *Anomalous dissipation for 1/5-Holder Euler flows.* Ann. of Math. **182** (2015), 127-172.

[2] T. Buckmaster, C. De Lellis, and L. Székelyhidi, Jr., *Dissipative Euler flows with Onsager-critical spatial regularity.* Comm. Pure and Appl. Math. **69** (2016), 1613-1670.

[3] T. Buckmaster, C. De Lellis, L. Székelyhidi, Jr., and V. Vicol, *Onsager's conjecture for admissible weak solutions.* to appear on Comm. Pure Appl. Math.

[4] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy, *Energy conservation and Onsager's conjecture for the Euler equations.* Nonlinearity **21** (2008) 1233-1252.

[5] A. Cheskidov, M. C. Lopes Filho, H. J. Nussenzveig Lopes, and R. Shvydkoy, *Energy conservation in two-dimensional incompressible ideal fluids*, Comm. Math. Phys. **348** (2016), 129–143.

[6] P. Constantin, W. E and E. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation.* Comm. Math. Phys. **165** (1994), no. 1, 207–209.

[7] R. Danchin, *On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework.* J. Differ. Equ., **248** (2010), 2130–2170.

[8] R. Danchin and F. Fanelli, *The well-posedness issue for the density-dependent Euler equations in endpoint Besov spaces.* J. Math. Pures Appl. **96** (2011), 253–278.

[9] C. De Lellis, and L. Székelyhidi, Jr., *The Euler equations as a differential inclusion.* Ann. of Math. **170** (2009), 1417-1436.

[10] C. De Lellis, and L. Székelyhidi, Jr., *Continuous dissipative Euler flows and a conjecture of Onsager.* European Congress of Mathematics, Eur. Math. Soc., Zürich (2013), 13-29.

[11] C. De Lellis, and L. Székelyhidi, Jr., *Dissipative continuous Euler flows.* Invent. Math. **193** (2013), 377-407.

[12] C. De Lellis, and L. Székelyhidi, Jr., *Dissipative Euler flows and Onsager's conjecture.* J. Eur. Math. Soc. (JEMS) **16** (2014), 1467-1505.

[13] J. Duchon and R. Robert, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations.* Nonlinearity **13** (2000), 249-255.

[14] L. C. Evans, *Partial differential equations*. Second edition, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[15] G. L. Eyink. *Energy dissipation without viscosity in ideal hydrodynamics: I. Fourier analysis and local energy transfer.* Phys. D **78** (1994), 222-240.

[16] G. L. Eyink and K. R. Sreenivasan, *Onsager and the theory of hydrodynamic turbulence.* Reviews of Modern Physics **78** (2006), 1-46.

[17] E. Feireisl, A. Novotný, H. Petzeltová, *On the existence of globally defined weak solutions to the Navier-Stokes equations.* J. Math. Fluid Mech. **3** (2001), 358-392.

[18] E. Feireisl, *Dynamics of viscous compressible fluids.* Oxford Lecture Series in Mathematics and its Applications, 26. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2004.

[19] E. Feireisl, P. Gwiazda, A. Swierczewska-Gwiazda, and E. Wiedemann, *Regularity and energy conservation for the compressible Euler equations.* Arch. Ration. Mech. Anal. **223** (2017), 1375-1395.

[20] U. Frisch, *Turbulence: The legacy of A. N. Kolmogorov.* Cambridge University Press, Cambridge, 1995.

[21] P. Isett, *A proof of Onsager's conjecture.* Ann. of Math. (2) 188 (2018), no. 3, 871-963.

[22] P. Isett, *On the endpoint regularity in Onsager's conjecture*. Preprint, 2017. arXiv:1706.01549.

[23] T.M. Leslie, R. Shvydkoy, *The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations.* J. Differ. Equ. **261** (2016), 3719-3733.

[24] I. Lacroix-Violet, A. Vasseur, *Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit*. J. Math. Pures Appl. (9) 114 (2018), 191-210.

[25] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models.* Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[26] P.-L. Lions, *Mathematical topics in fluid mechanics.* Vol. 2. Compressible models. Oxford Lecture Series in Mathematics and its Applications, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[27] J. E. Marsden, *Well-posedness of the equations of a non-homogeneous perfect fluid.* Comm. Partial Differ. Equ., **1** (1976), 215–230.

[28] L. Onsager, *Statistical Hydrodynamics.* Nuovo Cimento (Supplemento), **6** (1949), 279-287.

[29] R. Robert, *Statistical hydrodynamics (Onsager revisited).* Handbook of Mathematical Fluid Dynamics vol 2 ed Friedlander and Serre, Elsevier, Amsterdam, (2003), 1-55.

[30] V. Scheffer, *An inviscid flow with compact support in space-time.* J. Geom. Anal. **3** (1993), 343-401.

[31] J. Serrin, *The initial value problem for the Navier-Stokes equations.* 1963 Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962) pp. 69-98 Univ. of Wisconsin Press, Madison, Wis.

[32] M. Shinbrot, *The energy equation for the Navier-Stokes system.* SIAM J. Math. Anal. **5** (1974), 948-954.

[33] A. Shnirelman, *On the nonuniqueness of weak solution of the Euler equation.* Commun. Pure Appl. Math. **50** (1997), 1261-1286.

[34] A. Vasseur, C. Yu, *Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations.* Invent. Math. 206 (2016), no. 3, 935-974.

[35] C.Yu, *Energy conservation for the weak solutions of the compressible Navier-Stokes equations.* Arch. Ration. Mech. Anal. 225 (2017), no. 3, 1073-1087.