# EXISTENCE, NONEXISTENCE, AND ASYMPTOTICS OF DEEP WATER SOLITARY WAVES WITH LOCALIZED VORTICITY

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Abstract. In this paper, we study solitary waves propagating along the surface of an infinitely deep body of water in two or three dimensions. The waves are acted upon by gravity and capillary effects are allowed — but not required — on the interface. We assume that the vorticity is localized in the sense that it satisfies certain moment conditions, and we permit there to be finitely many point vortices in the bulk of the fluid in two dimensions. We also consider a two-fluid model with a vortex sheet.

Under mild decay assumptions, we obtain precise asymptotics for the velocity field and free surface, and relate this to global properties of the wave. For instance, we rule out the existence of waves whose free surface elevations have a single sign and of vortex sheets with finite angular momentum. Building on the work of Shatah, Walsh, and Zeng [26], we also prove the existence of families of two-dimensional capillary-gravity waves with compactly supported vorticity satisfying the above assumptions. For these waves, we further show that the free surface is positive in a neighborhood of infinity, and that the asymptotics at infinity are linked to the net vorticity.

## Contents



## 1. Introduction

Consider a traveling wave moving through an infinitely deep body of water in dimension $n = 2$ or 3. Mathematically, we model this as a solution of the free boundary incompressible Euler problem that evolves by translating with a constant wave velocity $c$. Through a

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change of variables, all time dependence in the system can then be removed, allowing us to say that the water occupies the domain

$$\Omega := \{x \in \mathbb{R}^n : x_n < \eta(x')\}, \tag{1.1}$$

where we are writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Here $S := \partial\Omega$ is free surface at the interface between the air and water. The fluid region $\Omega$ is unbounded in the (vertical) $x_n$ direction, as opposed to the finite depth case where $\Omega$ is bounded below by some hyperplane $\{x_n = -d\}$.

In the moving frame, the velocity field $u = u(x)$ satisfies the steady incompressible Euler equations

$$(u - c) \cdot \nabla u + \nabla P + g e_n = 0, \quad \nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{1.2a}$$

where $g > 0$ is the gravitational constant of acceleration, $P = P(x)$ is the pressure, and $c = (c', 0)$ is the (horizontal) wave velocity. For convenience, we normalize the density of the water to unity. On the free boundary, we impose the kinematic and dynamic conditions:

$$(u - c) \cdot N = 0, \quad P = \sigma \nabla \cdot N \qquad \text{on } S, \tag{1.2b}$$

where $\sigma \geq 0$ is the coefficient of surface tension and $N$ is the outward unit normal to $S$. This model neglects the dynamics in the atmosphere, supposing that it is a region of constant pressure that is normalized to 0. The dynamic condition therefore mandates that the pressure across the interface experiences a jump proportional to the mean curvature. We treat both capillary-gravity waves (for which $\sigma > 0$) and gravity waves (for which $\sigma = 0$).

*Solitary waves* are localized traveling waves whose free surface profiles $\eta$ vanish at infinity. They are among the oldest and most well-studied examples of nonlinear wave phenomena in mathematics. This paper is focused on the following fundamental questions: Can we classify the regimes that admit solitary waves? When such waves do exist, what can be said about their asymptotics and decay rates? Are there any natural constraints on their form?

Russell famously reported having observed a solitary wave moving through the relatively shallow waters of the Glasgow–Edinburgh canal in 1844 [24]. In the intervening century and a half, a well-developed rigorous theory has been established for solitary waves in the finite-depth setting (see, for instance, [23, 14, 2, 10]). The study of capillary-gravity solitary waves in infinite depth began much later with the numerical work of Longuet-Higgins [19, 20] and the rigorous construction of Iooss and Kirrman [17]. A three-dimensional existence theory which extends to infinite depth has recently been developed by Buffoni, Groves, and Wahlén [4]. For infinite-depth gravity solitary waves, there are instead a number of *nonexistence* results. Craig [9] showed that there are no two- or three-dimensional waves of pure elevation or depression in the sense that $\eta \geq 0$ or $\eta \leq 0$ implies that the wave is trivial. In two dimensions, without any assumptions on the sign of $\eta$, Hur [16] proved that waves with the algebraic decay $\eta = O(1/|x'|^{1+\varepsilon})$ as $|x'| \to \infty$ are trivial. For both capillary-gravity and gravity waves in two dimensions, Sun [30] showed that the decay $\eta = O(1/|x'|^{1+\varepsilon})$ automatically improves to $\eta = O(1/|x'|^2)$, and in this case ruled out the existence of waves of pure elevation or depression. Wheeler [35] obtained similar results in three dimensions, and also found leading-order expressions for the asymptotic form of the waves.

In this paper, we are interested in the role of *vorticity* $\omega$, which is the curl of the fluid velocity field $u$. As usual, for two-dimensional flows we understand this to mean the scalar vorticity

$$\omega := \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

All of the theory discussed above pertains exclusively to irrotational waves where $\omega \equiv 0$. Incoming currents, the wind in the air, or temperature gradients in the water can all generate vorticity. Much of the recent activity in water waves has been dedicated to proving the existence of rotational steady waves in various regimes. We direct the reader to the survey by Strauss [27] and monograph by Constantin [8] for an overview of these developments.

We will focus on solitary waves whose vorticity is localized, either in the sense that it vanishes at infinity or else is confined to an interior interface. The former describes the situation where there are vortical structures like eddies in the flow, resulting in a concentration of vorticity in the near field. These are physically significant middle-points between irrotational waves and waves with vorticity throughout the fluid. The latter is often called a *vortex sheet*; it can be viewed as a boundary separating two irrotational fluids.

Our main contribution is to determine the asymptotic form that the velocity field and free surface must take and to rule out solitary waves in a number of regimes. In particular, our results apply to the two families of two-dimensional capillary-gravity waves with compactly supported vorticity constructed recently by Shatah, Walsh, and Zeng [26]. Among other things, we confirm that the free surface $\eta$ must take on both positive and negative values, and is positive in a neighborhood of infinity. Our arguments are quite general in that they are independent of the dimension, the presence of surface tension, the near-field shape of $S$, and the specific form of $\omega$. We are therefore able to extend these results to a wide variety of physical settings and vorticity distributions; a major consequence is the nonexistence of waves of pure elevation and depression. More precisely, we consider $\omega$ falling into one of the three cases detailed below.

**Case I-A: Non-singular localized vorticity.** In dimension $n = 2$ or 3, we study weak solutions of (1.2) whose vorticity $\omega$ satisfies

$$\omega \in L^1(\Omega) \cap L^\infty(\Omega), \quad |x|^k \omega \in L^1(\Omega) \quad \text{for some } k > n^2. \tag{1.3a}$$

In three dimensions we assume also that the vorticity is tangential to the free surface:

$$\omega \cdot N = 0 \qquad \text{on } S. \tag{1.3b}$$

For a two-dimensional flow embedded in three dimensions, the vector vorticity $(0, 0, \omega)$ is automatically normal to $N = (N_1, N_2, 0)$. The condition (1.3b) appears for instance in [25, Chapter 3.7]. Note that $\omega$ is divergence free in the distributional sense, and thus its normal trace on $S$ is well-defined as an element of $H^{-1/2}(S)$.

Importantly, this class of vorticity distributions includes waves with *vortex patches*, where the support of $\omega$ is compact and positively separated from $S$. There are a wealth of results on vorticity of this type in the absence of a free surface; see for instance [21, Section 8.3]. As we will discuss further below, two-dimensional traveling water waves with a vortex patch were first constructed rigorously by Shatah, Walsh, and Zeng [26]; to the best of our knowledge, no rigorous existence results are currently available for $n = 3$.

**Case I-B: Localized vorticity with point vortices.** In two dimensions, we allow for the presence of finitely many point vortices. Denoting their positions by

$$\{\xi^1, \ldots, \xi^M\} =: \Xi \subset \Omega$$

and strengths by $\varpi^i \in \mathbb{R}$, this means that

$$\omega = \sum_{i=1}^{M} \varpi^i \delta_{\xi^i} + \omega_{\mathrm{ac}}, \tag{1.4}$$

where $\delta_{\xi^i}$ is the Dirac measure with unit mass centered at $\xi^i$ and the function $\omega_{\mathrm{ac}}$ satisfies the localization assumptions (1.3). The corresponding velocity field $u$ solves the incompressible Euler equations (1.2a) in a distributional sense on $\Omega \setminus \Xi$, as well as the boundary conditions (1.2b) on $S$. We note that $u$ fails to be $L^2$ on the neighborhood of any point in $\Xi$. As is customarily done, we assume that each vortex is advected by the vector field found by taking the full velocity field and subtracting its own singular contribution. For traveling waves, this results in the following condition linking the wave velocity to the flow:

$$c = \left( u - \frac{1}{2\pi} \varpi^i \nabla^\perp \log|\cdot - \xi^i| \right)\Big|_{\xi^i}, \qquad \text{for } i = 1, \ldots, M. \tag{1.5}$$

One can arrive at (1.5) by taking vortex patch solutions to the full Euler system then shrinking the diameter of the patch to 0; see [22, Theorems 4.1, 4.2]. Note that (1.5) severely constrains the possible arrangements of the point vortex centers.

Point vortices are widely used in applications as idealizations of highly concentrated regions of vorticity; see, for instance, [25]. The existence of traveling capillary-gravity waves with a single point vortex was proved by Shatah, Walsh, and Zeng in [26]. Earlier works by Ter-Krikorov [32] and Filippov [11, 12] study the case of solitary gravity waves in finite-depth with a single vortex. Recently, Varholm constructed finite-depth capillary-gravity waves with one or more point vortices [33].

**Case II: Vortex sheet.** Finally, we consider two-fluid models with a water region $\Omega_- := \{x : x_n < \eta(x')\}$ as well as an air region $\Omega_+ := \{x : x_n > \eta(x')\}$. We set $\Omega = \Omega_- \cup \Omega_+$ and $S = \partial\Omega_- = \partial\Omega_+$. The two regions have (possibly different) constant densities $\rho_\pm > 0$. Letting $u_\pm := u|_{\Omega_\pm}$ and $P_\pm := P|_{\Omega_\pm}$, we require that

$$(u_\pm - c) \cdot \nabla u_\pm + \frac{1}{\rho_\pm} \nabla P_\pm + g e_n = 0, \quad \nabla \cdot u_\pm = 0, \quad \nabla \times u_\pm = 0 \qquad \text{in } \Omega_\pm,$$

and, on the boundary,

$$N_\pm \cdot (u_\pm - c) = 0, \qquad [\![P]\!] = +\sigma \nabla \cdot N_+ = -\sigma \nabla \cdot N_- \qquad \text{on } S.$$

Here $N_\pm$ denote the outward unit normals to $\Omega_\pm$ on $S$, and $[\![\cdot]\!] := (\cdot)_+ - (\cdot)_-$ is the jump of a quantity over $S$. Notice that the fluid velocity is irrotational in each region, but has a jump discontinuity in its tangential component over the interface. Thus the vorticity $\omega$ is a singular continuous measure supported on $S$.

Vortex sheets have been studied extensively in both the applied and mathematical literature. The existence of two-dimensional capillary-gravity solitary waves with a vortex sheet was proved by Amick [1] and Sun [29], though both considered the situation where the water region is bounded below by a rigid ocean bed. In [31], Sun constructed two-dimensional periodic capillary-gravity waves where both layers are infinite.

1.1. **Notation.** Given a point $x \in \mathbb{R}^2$, we denote $x^\perp := (-x_2, x_1)$. Similarly, the perpendicular gradient $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. We also use the Japanese bracket notation $\langle x \rangle := \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^n$. Finally, we let $\gamma_n := 2\pi^{n/2}/\Gamma(n/2)$ denote the surface area of an $n$-dimensional unit ball. In particular $\gamma_2 = 2\pi$ and $\gamma_3 = 4\pi$.

Given an open set $U \subset \mathbb{R}^n$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, a weight $w \in C^0(\overline{U}; \mathbb{R}_+)$, and a function $f \in C^k(U;\mathbb{R})$, we define the weighted Hölder norm

$$\|f\|_{C_w^{k+\alpha}(U)} := \sum_{|\beta| \le k} \|w \partial^\beta f\|_{C^0(U)} + \sum_{|\beta|=k} \|w [\partial^\beta f]_\alpha\|_{C^0(U)},$$

where here $[f]_\alpha(x)$ is the local Hölder seminorm

$$[f]_\alpha(x) := \sup_{\substack{|y|<1 \\ x+y \in U}} \frac{|f(x+y)-f(x)|}{|y|^\alpha}.$$

We denote by

$$C_w^{k+\alpha}(\overline{U}) := \left\{ f \in C^{k+\alpha}(\overline{U}) : \|f\|_{C_w^{k+\alpha}(U)} < \infty \right\}.$$

Occasionally, we will also work with the space $C_{\mathrm{bdd}}^{k+\alpha}(\overline{U})$, which is defined as $C_w^{k+\alpha}(\overline{U})$ with $w \equiv 1$.

1.2. **Statement of results.** Our first theorem gives finer decay and asymptotic properties for the families of small-amplitude two-dimensional capillary-gravity waves with a point vortex and vortex patch constructed in [26].

**Theorem 1.1** (Existence). *Let $w(x) := \langle x \rangle^2 / \langle x_2 \rangle$ and fix $\sigma > 0$ and $\alpha \in (0,1)$. Then there are $\varpi_0, \rho_0, \tau_0 > 0$ such that:*

(a) *There exists a family of two-dimensional capillary-gravity water waves*

$$\mathscr{C}_{\mathrm{loc}} = \{(\eta, u, c)(\varpi) : |\varpi| < \varpi_0\}$$

*with vorticity $\omega(\varpi) = \varpi \delta_{(0,-1)}$ bifurcating from the trivial state $(\eta, u, c)(0) = (0,0,0)$ and having the regularity*

$$(\eta, u)(\varpi) \in C_w^{3+\alpha}(\mathbb{R}) \times C_w^{2+\alpha}\big(\overline{\Omega(\varpi)} \setminus \{(0,-1)\}\big).$$

(b) *There exists a family of two-dimensional capillary-gravity water waves*

$$\mathscr{S}_{\mathrm{loc}} = \{(\eta, u, c)(\varpi, \rho, \tau) : |\varpi| < \varpi_0,\ 0 < \rho < \rho_0,\ |\tau| < \tau_0\},$$

*bifurcating from the trivial state with* $\operatorname{supp}\omega(\varpi, \rho, \tau) =: D(\varpi, \rho, \tau) \subset\subset \Omega(\varpi, \rho, \tau)$. *Each of these waves lies in the space*

$$(\eta, u)(\varpi, \rho, \tau) \in C^{3+\alpha}_w(\mathbb{R}) \times C^{2+\alpha}_w(\overline{\Omega(\varpi, \rho, \tau)} \setminus D(\varpi, \rho, \tau)).$$

(c) *For both families* $\mathscr{C}_{\mathrm{loc}}$ *and* $\mathscr{S}_{\mathrm{loc}}$*, the free surface profile and velocity have the asymptotic form:*

$$\begin{aligned} \eta &= \frac{1}{2g}\frac{\varpi^2}{\gamma_2}\left(1 + O(\varpi^2)\right)\frac{1}{x_1^2} + O\left(\frac{1}{|x_1|^{2+\varepsilon}}\right), &&\text{as } |x_1| \to \infty,\\ u &= \frac{2\varpi}{\gamma_2}\nabla\left((e_1 + O(\varpi))\cdot\frac{x}{|x|^2}\right) + O\left(\frac{1}{|x|^{2+\varepsilon}}\right), &&\text{as } |x| \to \infty, \end{aligned} \tag{1.7}$$

*for any* $\varepsilon \in (0, 1/3)$*. In particular,* $\eta > 0$ *in a neighborhood of infinity.*

Parts (a) and (b) build upon [26, Theorem 2.1, Theorem 2.3], where an existence theory is carried out in Sobolev spaces $H^k$ with $k$ arbitrarily large. The main improvement is the use of weighted Hölder spaces, which give considerably more information about asymptotic behavior. The proof is an application of the implicit function theorem. It relies on weighted estimates for Poisson equations on unbounded domains, which have played an important role in previous works on the decay properties of water waves by Craig and Sternberg [10], Amick [1], Sun [28, 30], and Hur [15]. In part (c) we go further and extract the leading-order form of $\eta$ and $u$ at infinity. This is a consequence of Theorems 1.5 and Theorem 1.7 presented below. See Section 2 for further properties of $\mathscr{C}_{\mathrm{loc}}$ and $\mathscr{S}_{\mathrm{loc}}$.

As mentioned above, the techniques that we use to study the asymptotic properties of localized vorticity waves extend to a much more general setting: vorticities in Case I-A, I-B, and II, dimensions $n = 2$ or 3, gravity or capillary-gravity waves, and large and small amplitude. The main new assumption that we require is that $u$ (or $u_\pm$ in the case of vortex sheet) and $\eta$ exhibit the decay:

$$\eta \in C^2_{\mathrm{bdd}}(\mathbb{R}^{n-1}), \quad \partial^\beta\eta = O\left(\frac{1}{|x'|^{n-1+|\beta|+4\varepsilon}}\right) \text{ as } |x'| \to \infty,\ 0 \le |\beta| \le 1 \tag{1.8a}$$

and

$$u = O\left(\frac{1}{|x|^{n-1+\varepsilon}}\right) \qquad \text{as } |x| \to \infty, \tag{1.8b}$$

for some $\varepsilon \in (0, 1/4)$. Related decay assumptions on $\eta$ appear in [30, 16] in two-dimensions, where they are used to initiate a linear bootstrapping argument which then improves the decay rate. An initial decay assumption is indeed required for such linear bootstrapping arguments to work, except when the coefficient of surface tension $\sigma$ is large in that $\sigma > |c|^4/4g$. We note that (1.8b) can be replaced by the weaker condition $\varphi = o(1/|x|^{n-2})$, where $\varphi$ is the velocity potential for the irrotational part of the flow introduced in Section 2.1. We make the non-optimal assumption on the velocity field as it is a more physical quantity; see Remark 3.4.

Our second result states that there are no waves of pure depression or pure elevation satisfying the above hypotheses.

**Theorem 1.2** (Nonexistence). *Consider a solitary wave with localized vorticity in Case I or Case II and suppose that and $u$ and $\eta$ have the decay* (1.8). *If $\eta \geq 0$ or $\eta \leq 0$, then $\eta \equiv 0$. Moreover, there is no excess mass:*

$$\int_{\mathbb{R}^{n-1}} \eta \, dx' = 0.$$

*Remark* 1.3. The decay assumption on $\eta$ in (1.8a) implies (just barely) that $\eta \in L^1$.

*Remark* 1.4. Note that in Theorem 1.2 we make no claim that $u \equiv 0$. Indeed, Constantin [7] has constructed two-dimensional stationary waves ($c = 0$) where $\eta \equiv 0$ and $u \equiv 0$ outside a perfectly circular region where the vorticity is non-constant. One can show that, in two dimensions, waves with $\eta \equiv 0$ and localized vorticity are necessarily stationary.

For two-dimensional irrotational solitary waves and vortex sheets in infinite depth, this was proved by Sun [30], and in the three-dimensional irrotational case it appears in [35]. With strong surface tension ($\sigma > |c|^4/4g$), Sun is able to assume even weaker decay than (1.8). In the two- and three-dimensional irrotational cases, the nonexistence of pure waves of elevation or depression is a celebrated result due to Craig [9], who obtained it using a maximum principle argument that avoids making decay assumptions as in (1.8) but does not yield the stronger fact that $\int \eta \, dx' = 0$. More recently, Hur [16] proved that there are no nontrivial two-dimensional gravity waves of any kind having $\eta = O(1/|x'|^{1+\varepsilon})$. To the best of our knowledge, the only prior nonexistence results for rotational water waves are due to Wahlén [34], who completely ruled out the possibility of three-dimensional solitary waves with constant vorticity in finite depth.

In our next theorem, we show that any solitary wave exhibiting the localization (1.8) necessarily decays even faster and has a specific asymptotic form. Indeed, we find that $u$ must tend to a dipole velocity field, where the dipole moment is purely horizontal.

**Theorem 1.5** (Asymptotic form). *Consider a solitary wave as in Theorem 1.2.*

(a) *For vorticity of Case I, there exists a dipole moment $p = (p', 0) \in \mathbb{R}^n$ such that*

$$\eta = \frac{1}{g} p' \cdot \nabla \left( \frac{c' \cdot x'}{|x'|^n} \right) + O\left( \frac{1}{|x'|^{n+\varepsilon}} \right), \qquad \text{as } |x'| \to \infty, \tag{1.9a}$$

*and*

$$u = \nabla \left( \frac{p \cdot x}{|x|^n} \right) + O\left( \frac{1}{|x|^{n+\varepsilon}} \right), \qquad \text{as } |x| \to \infty. \tag{1.9b}$$

(b) *For vorticity of Case II, there exists $p_\pm = (p'_\pm, 0) \in \mathbb{R}^n$ such that*

$$\eta = \frac{1}{g\,[\![\rho]\!]} [\![\rho p']\!] \cdot \nabla \left( \frac{c' \cdot x'}{|x'|^n} \right) + O\left( \frac{1}{|x'|^{n+\varepsilon}} \right), \qquad \text{as } |x'| \to \infty, \tag{1.10a}$$

*and*

$$u_\pm = \nabla \left( \frac{p_\pm \cdot x}{|x|^n} \right) + O\left( \frac{1}{|x|^{n+\varepsilon}} \right), \qquad \text{as } |x| \to \infty. \tag{1.10b}$$

*Remark* 1.6. In two dimensions, (1.9a) simplifies to

$$\eta = -\frac{1}{g}\frac{p'c'}{|x'|^2} + O\left(\frac{1}{|x'|^{2+\varepsilon}}\right), \qquad \text{as } |x'| \to \infty.$$

Thus $\eta$ is strictly positive in a neighborhood of infinity whenever $p'c' < 0$.

This generalizes the recent work of Wheeler [35] on the irrotational case. In the two-dimensional irrotational and vortex sheet cases, Sun [30] established similar decay rates (but not asymptotics) under analogous assumptions. Such asymptotic behavior is assumed by Longuet-Higgins in the mostly numerical paper [20], and is also considered by Benjamin and Olver [3, Section 6.5].

An important quantity describing the vorticity distribution in a water wave is the so-called *vortex impulse* defined by

$$m := \begin{cases} -\displaystyle\int_\Omega \omega(x-\xi^*)^\perp\,dx & \text{if } n=2, \\ -\dfrac{1}{2}\displaystyle\int_\Omega \omega \times x\,dx & \text{if } n=3. \end{cases} \tag{1.11}$$

See also [25, Chapter 3.2, 3.7]. The following identity reveals a link between the vortex impulse $m$ and the dipole moment $p$ appearing in Theorem 1.5. It is, in particular, essential to the proof of Theorem 1.1(c).

**Theorem 1.7** (Dipole moment formula)**.** *Consider a solitary wave as in Theorem 1.2.*

(a) *For vorticity of Case I,*

$$\int_\Omega \left[|u|^2 - (u-c)\cdot\left(V + \frac{1}{\gamma_n}\nabla\left(\frac{m\cdot x}{|x|^n}\right)\right)\right]dx = -\frac{\gamma_n}{2}\,c\cdot p, \tag{1.12}$$

*where $V$ is the vortical part of the velocity defined in* (2.2)*.*

(b) *For vorticity of Case II,*

$$\int_\Omega \rho|u|^2\,dx = \frac{\gamma_n}{2}c\cdot [\![\rho p]\!]\,. \tag{1.13}$$

Formulas (1.12) and (1.13) agree in the limit where $\omega$ and $\rho_+$ vanish. In this case they recover the formula obtained by Wheeler in [35]; also see [20]. With nontrivial vorticity in the bulk, it becomes considerably more difficult to find clean expressions for $p$, as the definition of $V$ is not in any way related to the location of the interface $S$.

The asymptotics in Theorem 1.5 and the identity in Theorem 1.7(b) have the following corollary.

**Corollary 1.8** (Angular momentum)**.** *Assume there exists a solitary wave with localized vorticity satisfying the assumptions of Theorem 1.2. If the total angular momentum*

$$\begin{cases} \displaystyle\int_\Omega x^\perp \cdot \rho u \, dx & \text{if } n = 2, \\ \displaystyle\int_\Omega x \times \rho u \, dx & \text{if } n = 3 \end{cases}$$

*is finite then* $p = 0$*. (Here* $\rho \equiv 1$ *in Case I.) Furthermore, for a vortex sheet every nontrivial wave has infinite angular momentum.*

In Case I, owing to the more complicated formula (1.12) for $p$, we cannot conclude that waves with finite angular momentum are trivial. However, the fact that $p = 0$ does imply that the leading-order terms for $u$ and $\eta$ in the asymptotics (1.9a) vanish.

While we do not pursue this here, all of the above results hold when the free surface $S$ is overhanging. It is only necessary that $S$ is smooth, does not self-intersect, and is asymptotically flat in the sense that, outside of some bounded neighborhood of 0, it is given as the graph of a function $\eta$ satisfying (1.8a).

## 2. Existence of waves with algebraic decay

In this section we prove Theorem 1.1 on the existence of steady water waves with localized vorticity and show that these waves exhibit the algebraic decay rates required by our asymptotic theory.

### 2.1. Splitting of $u$.

When analyzing rotational water waves, it is often useful to decompose the velocity field into an irrotational part, given as the gradient of a potential $\varphi$, and a vortical part $V$:

$$u = \nabla\varphi + V \qquad \text{in } \Omega, \tag{2.1}$$

where $\varphi$ is harmonic and $V$ is divergence free. How we choose to define $V$ will depend on the dimension. Consider first the case $n = 2$. Then the fact that $u$ is divergence free implies that there exists a stream function $\psi_V$ with $V = \nabla^\perp \psi_V$. In light of (2.1), it must also be true that $\Delta\psi_V = \omega$, and hence $\psi_V$ can be expressed as the Newtonian potential of $\omega$ up to a harmonic function. With that in mind, we define

$$V := \nabla^\perp \left[ \left( \frac{1}{\gamma_2} \log|\cdot| \right) * \omega - \frac{1}{\gamma_2} \varpi \log|\cdot - \xi^*| \right] \qquad \text{if } n = 2, \tag{2.2a}$$

where

$$\varpi := \int_\Omega \omega_{\mathrm{ac}} \, dx + \sum_{i=1}^M \varpi^i$$

is the (total) vortex strength and $\xi^*$ is a fixed point in $\Omega^c$. The second term in (2.2a) can be thought of as a "phantom vortex" in the air region that counterbalances the total vorticity in the water. By including it, we ensure that the splitting (2.1) decomposes $u$ as the sum of two $L^2$ functions on any closed subset of $\Omega \setminus \Xi$.

In the three-dimensional case, the Green's function for the Laplacian enjoys better decay at infinity, and thus we have this desirable splitting property without needing to introduce phantom vortices. Indeed, we define $V$ directly using the Biot–Savart law,

$$V(x) := \frac{1}{\gamma_3} \int_\Omega \omega(y) \times \frac{x-y}{|x-y|^3} \, dy \qquad \text{if } n = 3. \tag{2.2b}$$

Thanks to (1.3b), we show in Appendix A that $\int_\Omega \omega \, dx = 0$; see (A.7).

2.2. **Existence theory in Sobolev spaces.** Let us recall two results from [26] in some detail. For each $k > 3/2$, there exists a one-parameter family of two-dimensional capillary-gravity solitary waves with a point vortex

$$\mathscr{C}_{\mathrm{loc}} = \{(\eta(\varpi), u(\varpi), c(\varpi)) : |\varpi| < \varpi_0\}$$

such that

$$\eta(\varpi) \in H^k_{\mathrm{e}}(\mathbb{R}), \qquad u(\varpi) - V(\varpi) \in H^{k-1}_{\mathrm{e}}(\Omega(\varpi)) \times H^{k-1}_{\mathrm{o}}(\Omega(\varpi)), \qquad \omega(\varpi) = \varpi \delta_{(0,-1)},$$

where $\Omega(\varpi)$ is the fluid domain corresponding to $\eta(\varpi)$, $\omega(\varpi)$ is the vorticity for the velocity field $u(\varpi)$, and $V(\varpi)$ is the function given by (2.2). The subscripts of 'e' and 'o' indicate evenness or oddness with respect to $x_1$, respectively. This curve bifurcates from the absolutely trivial state of no motion: $(\eta(0), u(0), c(0)) = (0, 0, 0)$, and to leading order the solutions have the form

$$\begin{aligned} c(\varpi) &= \left( -\frac{\varpi}{2\gamma_2} + o(\varpi^2) \right) e_1, \qquad \|u(\varpi) - V(\varpi)\|_{H^{k-1}(\Omega(\varpi))} = O(|\varpi|^3), \\ &\left\| \eta(\varpi) - \frac{\varpi^2}{4\pi^2} (g - \sigma \partial_{x_1}^2)^{-1} \left( \frac{x_1^2 - 1}{(1 + x_1^2)^2} \right) \right\|_{H^k(\mathbb{R})} = O(|\varpi|^3). \end{aligned} \tag{2.3}$$

In the same paper, the authors construct a three-parameter family of traveling capillary-gravity waves with a vortex patch:

$$\mathscr{S}_{\mathrm{loc}} = \left\{ (\eta, u, c)(\varpi, \rho, \tau) : |\varpi| < \varpi_0, \ 0 < \rho < \rho_0, \ |\tau| < \tau_0 \right\}.$$

Here, $\varpi$ is the total vorticity, $\rho$ measures the approximate radius of $\operatorname{supp} \omega(\varpi)$, and $\tau$ arises due to a certain degeneracy in the linearized problem at the trivial state. From now on we suppress the dependence on $\rho$ and $\tau$, which we hold fixed. These waves lie in $L^2$-based Sobolev spaces: the free surface profile $\eta(\varpi) \in H^k_{\mathrm{e}}(\mathbb{R})$, and the velocity field $u(\varpi) \in L^2_{\mathrm{e}}(\Omega(\varpi)) \times L^2_{\mathrm{o}}(\Omega(\varpi))$. The vorticity $\omega(\varpi)$ has compact support $D(\varpi)$ that is a perturbation of a ball centered at $(0, -1)$:

$$\partial D(\varpi, \rho, \tau) = \left\{ \rho \left( \cos\theta + \tau \sin(2\theta), \, \sin\theta - \tau \cos(2\theta) \right) + O(\rho^2(\rho + \varpi)) : \theta \in [0, 2\pi) \right\}.$$

Moreover, $u(\varpi) \in H^k(\Omega \setminus D)$ and $\omega(\varpi) \in H^1(D)$. As in the point vortex case, this family bifurcates from the absolutely trivial state. One can show that these waves have the same leading order form as in (2.3).

It is worth noting that, in fact, many families of the form $\mathscr{S}_{\mathrm{loc}}$ exist: the argument in [26] allows one to select at the outset the value of the vorticity on each streamline in the patch $D$ from a fairly generic class of distributions. For any such choice, there exists a corresponding $\mathscr{S}_{\text{loc}}$; see [26, Remark 2.2(b)].

The proofs in [26] work by applying implicit function-type arguments to nonlinear operators between Sobolev spaces. For point vorticies, the unknowns are $\eta, \varphi, c$ with $\varpi$ as a parameter, while for vortex patches there is an additional variable representing the shape of the patch, as well as two additional parameters $\rho, \tau$. In both cases the relevant linearized operators have an upper triangular structure which implies that $\eta, \varphi$ are (locally) uniquely determined by the remaining variables and parameters [26, Lemma 4.1 and the proof of Theorem 2.1].

While the above existence theory furnishes a leading-order description of these waves, it unfortunately does not say much about their pointwise behavior at infinity; knowing $\eta$ up to $O(\varpi^3)$ in $H^k(\mathbb{R})$ is not sufficient to infer an explicit decay rate, much less the specific algebraic asymptotics claimed in (1.7).

Our strategy in proving Theorem 1.1 is to reconsider the local uniqueness of $\eta, \varphi$ in weighted Hölder spaces, again taking advantage of the upper triangular structure of the linearized equations. We treat $c$ and $\omega$ as given functions of the parameters, coming from the solutions in Sobolev spaces, and neglect the Euler equations (1.2a) and advection condition (1.5) inside the fluid. Since our weighted Hölder spaces are contained in the relevant Sobolev spaces, the local uniqueness (in both spaces) implies that the Sobolev and weighted Hölder solutions coincide.

2.3. **Change of variables.** Fix any Sobolev regularity index $k \geq 5$ and take either the family of waves with a point vortex $\mathscr{C}_{\text{loc}}$ or the family of solitary waves with a vortex patch $\mathscr{S}_{\text{loc}}$. In the latter case, fix some (nonzero) values for $(\rho, \tau)$ and consider the corresponding curve lying in $\mathscr{S}_{\text{loc}}$ parameterized by $\varpi$. Note that by Morrey's inequality, $H^k(\mathbb{R}) \subset C^{3+\alpha}_{\text{bdd}}(\mathbb{R})$, for any $\alpha \in (0,1)$.

Let $\eta \in H^k_{\text{e}}(\mathbb{R})$ with $\Omega = \{x \in \mathbb{R}^2 : x_2 < \eta(x_1)\}$ the corresponding fluid domain. We wish to study the vector fields $u : \Omega \to \mathbb{R}^2$ that can be written as

$$u = \nabla^\perp \psi + V(\varpi) \tag{2.4}$$

for some harmonic function $\psi \in \dot{H}^k_{\text{e}}(\Omega)$, where $V(\varpi)$ is the divergence free vector field defined according to (2.2) using the vorticity $\omega(\varpi)$ given by the family that we have selected and with $\xi^* = (1,0)$. Note that the local uniqueness of solutions implies that $u$ is the velocity field for a solitary wave with wave velocity $c(\varpi)$ and vorticity $\omega(\varpi)$ if and only if $\eta = \eta(\varpi)$ and $\psi$ is a harmonic conjugate of $\varphi(\varpi)$. Plugging the ansatz (2.4) for $u$ into the boundary conditions (1.2b) and using Bernoulli's law, we see that $\psi$ must satisfy the following elliptic system:

$$\Delta\psi = 0 \qquad \text{in } \Omega, \tag{2.5a}$$

$$\frac{1}{2}|\nabla^\perp\psi + \nabla^\perp\psi_V(\varpi) + c(\varpi)|^2 + g\eta - \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = 0 \qquad \text{on } S, \tag{2.5b}$$

$$\psi + \psi_V(\varpi) - c_1(\varpi)\eta = 0 \qquad \text{on } S, \tag{2.5c}$$

where $\psi_V(\varpi)$ is the stream function for $V(\varpi)$ in the sense that $\nabla^\perp \psi_V(\varpi) := V(\varpi)$. Note also that (2.5c) results from taking the kinematic boundary condition in (1.2b), re-expressing it in terms of $\psi$, and integrating once.

Our first task is to make a change of coordinates to flatten the domain $\Omega$. With that in mind, fix an even function $a \in C_c^\infty(\mathbb{R})$ supported in $[-1,1]$ and with $a(0)=1$. We will work in the variables $(X,Y)$ defined implicitly by

$$x_1 = X, \qquad x_2 = x_2(X,Y) = Y + \eta(X)a(Y).$$

This is valid whenever $\|\eta\|_{L^\infty}\|a'\|_{L^\infty} < 1$, and so it can be done for small-amplitude waves like those in $\mathscr{C}_{\mathrm{loc}}$ or $\mathscr{S}_{\mathrm{loc}}$. For finite-amplitude waves, we can simply extend the support of $a$, but this makes the notation more cumbersome and so we will not pursue that here. It is easily seen that the mapping $(x_1,x_2) \mapsto (X,Y)$ sends the fluid domain $\Omega$ to the half-space

$$R := \{(X,Y) \in \mathbb{R}^2 : Y < 0\}.$$

In particular, the image of the free surface $S$ is just $T := \{(X,0) : X \in \mathbb{R}\}$.

It is tedious but elementary to show that (2.5) is transformed to the following system:

$$\psi_{XX} + \psi_{YY} - f_1(\psi,\eta) = 0 \qquad \text{in } R, \tag{2.6a}$$
$$-c_1(\varpi)\psi_Y + g\eta - \sigma\eta_{XX} - f_2(\psi,\eta,\varpi) = 0 \qquad \text{on } T, \tag{2.6b}$$
$$\psi - c_1(\varpi)\eta + \psi_V(\eta,\varpi) = 0 \qquad \text{on } T, \tag{2.6c}$$

where the nonlinear terms in (2.6) are described by the mappings

$$\begin{aligned}
f_1(\psi,\eta) := &-(a_Y\eta\psi_{YY} - a\eta_{XX}\psi_Y - a_{YY}\eta\psi_Y + 3a_Y\eta\psi_{XX} - 2a\eta_X\psi_{XY})\\
&- (a^2\eta_X^2\psi_{YY} - 2aa_Y\eta\eta_{XX}\psi_Y + 2aa_Y\eta_X^2\psi_Y + 3a_Y^2\eta^2\psi_{XX} - 4aa_Y\eta\eta_X\psi_{XY})\\
&- \eta(a^2a_Y\eta_X^2\psi_{YY} - aa_Y^2\eta\eta_{XX}\psi_Y - a^2a_{YY}\eta_X^2\psi_Y + 2aa_Y^2\eta_X^2\psi_Y\\
&\quad + a_Y^3\eta^2\psi_{XX} - 2aa_Y^2\eta\eta_X\psi_{XY}),\\
f_2(\psi,\eta,\varpi) := &-\tfrac{1}{2}\left(V_1(\eta,\varpi)^2 - 2\psi_Y V_1(\eta,\varpi) + V_2(\eta,\varpi)^2 + 2\psi_X V_2(\eta,\varpi) + \psi_Y^2 + \psi_X^2\right)\\
&+ \eta_X\psi_Y(V_2(\eta,\varpi) + \psi_X) - \tfrac{1}{2}\eta_X^2\psi_Y^2 + \sigma\left((1+\eta_X^2)^{3/2} - 1\right)\eta_{XX}.
\end{aligned}$$

We are abusing notation somewhat by writing

$$\psi_V(\eta,\varpi) := \psi_V(\varpi)(\cdot,\eta(\cdot)), \qquad V(\eta,\varpi) := V(\varpi)(\cdot,\eta(\cdot)).$$

It is also important to note that $f_1(\psi,\eta)$ is supported in the slab $\{-1 < Y < 0\}$ for any $(\psi,\eta)$. In the next subsection, we will describe more precisely the domains and codomains of $f_1$ and $f_2$.

2.4. **Functional-analytic setting.** As mentioned above, we wish to reexamine the existence problem in weighted Hölder spaces introduced in Section 1.1. Specifically, for the weight we will always take

$$w = w(X,Y) := \frac{X^2 + (1-Y)^2}{1-Y}.$$

Note that $1/w$ is related to the Poisson kernel for a half-plane. It is easily confirmed that this weight is equivalent to $\langle x\rangle^2/\langle x_2\rangle$ in the original variables.

Observe that, since the vorticity has compact support, asymptotically it behaves like a point vortex. In particular, looking at the explicit formula for $V = \nabla^\perp \psi_V$ in (2.2a), one can deduce that $\psi_V|_T$ decays like $\eta$. Thus, in what follows, we think of $\psi_V \in C_w^{3+\alpha}(T)$. Another useful feature of these weighted spaces is that, if $f, g \in C_w^{\ell+\alpha}$, then $fg \in C_{w^2}^{\ell+\alpha}$.

Finally, it will be convenient in the coming analysis to have notation for slab subdomains of $R$. For any $k > 0$, we denote by

$$R_k := \big\{(X,Y) \in \mathbb{R}^2 : -k < Y < 0\big\} \subset R$$

the strip of height $k$ having upper boundary $T$ and lower boundary $B_k := \{Y = -k\}$.

With this notation in place, our objective is to find solutions $(\psi,\eta) \in C_w^{3+\alpha}(\overline{R}) \times C_w^{3+\alpha}(\mathbb{R})$ of the flattened system (2.6) with $0 < |\varpi| \ll 1$. First, for each $\varpi \in \mathbb{R}$, $\eta \in C_w^{3+\alpha}(T)$, and $f \in C_w^{1+\alpha}(\overline{R})$ supported in $R_1$, we define $\Psi = K(\eta,\varpi)f$ to be the unique bounded solution to the Dirichlet problem

$$\Delta\Psi = f \text{ in } R, \qquad \psi = c_1(\varpi)\eta - \psi_V(\varpi) \text{ on } T.$$

Then (2.6) can be written as the operator equation $\mathcal{F}(\psi,\eta,\varpi) = 0$ for $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ with

$$\begin{aligned} \mathcal{F}_1(\psi,\eta,\varpi) &:= \psi - K(\eta,\varpi) f_1(\psi,\eta,\varpi), \\ \mathcal{F}_2(\psi,\eta,\varpi) &:= g\eta - \sigma\eta_{XX} - c_1(\varpi)\psi_V(\eta,\varpi) - f_2(\psi,\eta,\varpi). \end{aligned} \tag{2.7}$$

For the time being, this must be understood in a purely formal sense.

2.5. **Smoothness of $\mathcal{F}$.** We begin by proving that $\mathcal{F}$ is a well-defined $C^1$ mapping from $C_w^{3+\alpha}(\overline{R}) \times C_w^{3+\alpha}(T) \times \mathbb{R} \to C_w^{3+\alpha}(\overline{R}) \times C_w^{1+\alpha}(T)$. Inspecting the form of $f_1$, we see that the following lemma is sufficient.

**Lemma 2.1** (Smoothness)**.** *For any function $b \in C^\infty(\mathbb{R})$ with support in $[-1,0]$, the mapping*

$$K_b \colon C_w^{1+\alpha}(\overline{R}) \times C_w^{3+\alpha}(T) \times \mathbb{R} \to C_w^{3+\alpha}(\overline{R}), \qquad K_b(f,\eta,\varpi) := K(\eta,\varpi)(bf)$$

*is of class $C^1$.*

The proof of Lemma 2.1 relies on the following two lemmas on the decay properties of solutions to Dirichlet problems on the halfspace $R$.

**Lemma 2.2.** *Let $h \in C_w^0(T)$. Then for any $k > 0$ the unique bounded solution $\Psi$ to*

$$\Delta\Psi = 0 \text{ in } R, \qquad \Psi = h \text{ on } T$$

*obeys the estimate*

$$\|\Psi\|_{C_w^4(R\setminus R_k)} \le C_k \|h\|_{C_w^0(T)}.$$

*Proof.* This follows from [30, Lemma 3.1]. Note that there is a minor misprint in the statement of that lemma: $(1 \pm \psi)^{|\beta|-1}$ should be $|\psi|^{|\beta|-1}$. ☐

**Lemma 2.3.** *Let $f \in C_w^{1+\alpha}(\overline{R})$ with support contained in $R_1$. Then for any $k > 1$ the unique bounded solution $\Psi$ to*

$$\Delta\Psi = f \text{ in } R, \qquad \Psi = 0 \text{ on } T$$

*satisfies*

$$\|\Psi\|_{C_w^{3+\alpha}(R_k)} \leq C_k \|f\|_{C_w^{1+\alpha}(R)}.$$

*Proof.* Defining $F \in C_w^{2+\alpha}(\overline{R})$ by

$$F(X,Y) := \int_{-1}^{Y} f(X,Z)\,dZ,$$

we have that $F$ is supported in $R_1$ and satisfies $\partial_Y F = f$.

Now, we may express $\Psi$ in terms of $f$ via the Poisson kernel representation formula

$$\Psi(X,Y) = \int_R f(W,Z) G(X,Y,W,Z)\,dW\,dZ,$$

where

$$G(X,Y,W,Z) := -\frac{1}{4\pi}\left(\log\left(|W-X|^2 + |Z-Y|^2\right) - \log\left(|W-X|^2 + |Z+Y|^2\right)\right). \quad (2.8)$$

Integrating this by parts once in $Z$ yields

$$\Psi(X,Y) = -\int_{R_1} F(W,Z) G_Z(X,Y,W,Z)\,dW\,dZ.$$

Note that the integration domain is $R_1$ since this is where $F$ is supported. We easily check that

$$\partial_X^j G_Z,\ [\partial_X^j G_Z]_\alpha \leq \frac{C_k}{|W-X|^2+1} \qquad \text{for } (W,Z) \in R_1,\ Y = -k, \text{ and } j \leq 3,$$

which is enough to prove that

$$\|\Psi\|_{C_w^{3+\alpha}(B_k)} \leq C_k \|F\|_{C^0(R_1)} \leq C_k \|f\|_{C^0(R_1)}.$$

We conclude by applying [10, Lemma 5.2], which gives the following weighted estimate for the Poisson equation in a slab:

$$\begin{aligned}\|\Psi\|_{C_w^{3+\alpha}(R_k)} &\leq C_k \left(\|\Psi\|_{C_w^{3+\alpha}(T)} + \|\Psi\|_{C_w^{3+\alpha}(B_k)} + \|\Delta\Psi\|_{C_w^{1+\alpha}(R_k)}\right) \\ &\leq C_k \|f\|_{C_w^{1+\alpha}(R_1)}.\end{aligned} \quad (2.9)$$

□

*Proof of Lemma 2.1.* For $f \in C_w^{1+\alpha}(\overline{R})$ and $\eta \in C_w^{3+\alpha}(T)$, we know that $K_b(f,\eta,\varpi)$ exists as an element of $C^{3+\alpha}(\overline{R})$. It remains to show that it is also in $C_w^{3+\alpha}(\overline{R})$ and that it depends smoothly on its arguments.

Towards that end, we decompose

$$K_b(f,\eta,\varpi) = K_{b,1}f + K_{b,2}(\eta,\varpi),$$

where $K_{b,1}f := \Psi_1$ is the unique bounded $C^{3+\alpha}(\overline{R})$ solution to

$$\Delta\Psi_1 = bf \text{ in } R, \qquad \Psi_1 = 0 \text{ on } T,$$

and $K_{b,2}(\eta,\varpi) := \Psi_2$ is the unique bounded $C^{3+\alpha}(\overline{R})$ solution to

$$\Delta\Psi_2 = 0 \text{ in } R, \qquad \Psi_2 = c_1(\varpi)\eta - \psi_V(\eta,\varpi) \text{ on } T.$$

Applying Lemma 2.3 to $\Psi_1$ yields the estimate

$$\|\Psi_1\|_{C_w^{3+\alpha}(R_4)} \le C\|bf\|_{C_w^{1+\alpha}(R)} \le C\|f\|_{C_w^{1+\alpha}(R)}.$$

In particular, this gives us control of $\Psi_1$ restricted to the line $B_2 \subset R_4$, and hence Lemma 2.2 can be used to bound $\Psi_1$ on the half-space $R \setminus R_3$:

$$\|\Psi_1\|_{C_w^4(R\setminus R_3)} \le C\|\Psi_1\|_{C_w^0(B_2)} \le C\|\Psi_1\|_{C_w^{3+\alpha}(R_4)} \le C\|f\|_{C_w^{1+\alpha}(R)}.$$

Combining these two estimates yields

$$\|\Psi_1\|_{C_w^{3+\alpha}(R)} \le C\|f\|_{C_w^{1+\alpha}(R)}.$$

On the other hand, we obtain from Lemma 2.2 the following bound for $\Psi_2$:

$$\|\Psi_2\|_{C_w^4(R\setminus R_1)} \le C\left(\|\eta\|_{C_w^0(T)} + \|\psi_V(\eta,\varpi)\|_{C_w^0(T)}\right).$$

Thus $\Psi_2$ is controlled in a half-space positively separated from $T$. To control it on the remainder of $R$, we again use the estimate (2.9) from [10, Lemma 5.2] to obtain

$$\begin{aligned}\|\Psi_2\|_{C_w^{3+\alpha}(R_2)} &\le C\left(\|\Psi_2\|_{C_w^{3+\alpha}(T)} + \|\Psi_2\|_{C_w^{3+\alpha}(B_2)}\right)\\ &\le C\left(\|\eta\|_{C_w^{3+\alpha}(T)} + \|\psi_V(\eta,\varpi)\|_{C_w^{3+\alpha}(T)} + \|\Psi_2\|_{C_w^{3+\alpha}(B_2)}\right).\end{aligned}$$

As $B_2 \subset R \setminus R_1$, the preceding two bounds can be combined to show

$$\|\Psi_2\|_{C_w^{3+\alpha}(R)} \le C\left(\|\eta\|_{C_w^{3+\alpha}(T)} + \|\psi_V(\eta,\varpi)\|_{C_w^{3+\alpha}(T)}\right).$$

Thus $K_b$ is indeed well-defined as a mapping $C_w^{1+\alpha}(\overline{R}) \times C_w^{3+\alpha}(T) \times \mathbb{R} \to C_w^{3+\alpha}(\overline{R})$. Since $K_{b,1}$ is linear, the above estimates show that it is also smooth. Similarly $K_{b,2}$ is the composition of a linear map and the smooth map $C_w^{3+\alpha}(T) \times \mathbb{R} \to C_w^{3+\alpha}(T)$ given by $(\eta,\varpi) \mapsto c_1(\varpi)\eta - \psi_V(\eta,\varpi)$. □

2.6. **Improved decay for $\eta_X$.** So far we have been working with $\eta \in C_w^{3+\alpha}(T)$, which only implies $\eta, \eta_X = O(1/X^2)$. In this subsection we show that $\eta_X = O(1/|X|^{2+\varepsilon})$ so that (1.8a) holds. The proof will rely on the fact that $c = O(\varpi)$ implies that $\sigma > |c|^4/4g$ for $0 < |\varpi| \ll 1$.

**Lemma 2.4.** *Suppose that there exists a two-dimensional capillary-gravity solitary wave with localized vorticity and $\sigma > |c|^4/4g$ such that the velocity field has the decomposition*

$$u = \nabla^\perp\psi + V, \qquad \psi \in C_w^{3+\alpha}(\Omega),$$

*and the free surface* $\eta \in C_w^{3+\alpha}(\mathbb{R})$*. Then the derivatives of the free surface profile* $\eta$ *enjoy the improved decay*

$$\partial_{x'}\eta = O\left(\frac{1}{|x'|^{2+\varepsilon}}\right), \qquad \text{as } |x'| \to \infty. \tag{2.10}$$

*Proof.* To obtain a closed equation for $\eta$, we begin with (2.6b) and write $\psi_Y$ as a Dirichlet–Neumann operator applied to $\psi|_T$. Because $\psi$ is not harmonic in $R$, this will introduce terms involving the kernel $G$ defined in (2.8):

$$-\sigma\eta_{XX} - c_1^2|\partial_X|\eta + g\eta = f_2 + c_1\partial_Y G * f_1 - c_1|\partial_X|\psi_V =: F.$$

Note from the structure of $f_1,\ f_2$, Lemma 2.3, and the definition of $\psi_V$ that $F \in C^0_{w^2}(\mathbb{R})$.

Inverting the above equation, we may write

$$\eta = \tilde{G} * F,$$

where $\tilde{G} = \tilde{G}(X)$ has the Fourier transform

$$\widehat{\tilde{G}}(k) = \frac{1}{\sigma k^2 - |c|^2|k| + g}.$$

We subtract off from $\tilde{G}$ the Green's kernel for $g - \sigma\partial_X^2$, and write

$$\eta = G_1 * F + (g - \sigma\partial_X^2)^{-1}F,$$

where

$$\begin{aligned}
(g - \sigma\partial_X^2)^{-1}F &= \frac{1}{2\sqrt{\sigma g}}e^{-\sqrt{g/\sigma}|\cdot|} * F =: G_2 * F, \\
\widehat{G}_1(k) &= \frac{|c|^2|k|}{(\sigma k^2 - |c|^2|k| + g)(\sigma k^2 + g)}.
\end{aligned}$$

Since convolution with $G_2$ and $G_2'$ preserves the algebraic decay of any function it acts on, we know that

$$G_2 * F, G_2' * F \in C^0_{w^2}(\mathbb{R}). \tag{2.11}$$

Examining the form of $G_1$, we find

$$\begin{cases} G_1(X) = O(1), & G_1'(X) = O(1) \qquad \text{for } X \text{ small}, \\ G_1(X) = O(1/|X|^2), & G_1'(X) = O(1/|X|^3) \qquad \text{for } X \text{ large}. \end{cases}$$

In particular,

$$\langle X\rangle^{2+\varepsilon}\partial_X G_1 = O(1/|X|^{1-\varepsilon}) \quad \text{for } X \text{ large}.$$

Denote $r(X) := \langle X\rangle^{2+\varepsilon}$. Then we see that

$$\begin{aligned}\left|rG_1' * F\right| &= \left|\int_{\mathbb{R}} r(X)G_1'(W)F(X-W)\,dW\right| \\ &\leq \int_{\mathbb{R}} \left|G_1'(W)\frac{r(X)}{r(X-W)}\right| |F(X-W)r(X-W)|\,dW \\ &\leq \sup_{X,W}\frac{r(X)}{r(X-W)r(W)}\|rG_1'\|_{L^p}\|rF\|_{L^q} \\ &\lesssim \|rG_1'\|_{L^p}\|F^{q-2}r^q\|_{L^\infty}^{1/q}\|F\|_{L^2}^{2/q}.\end{aligned}$$

Note that now

$$rG_1' \in L^p(\mathbb{R}) \quad \text{for any } p > \frac{1}{1-\varepsilon}.$$

On the other hand, $F \in L^2$ and

$$\|r^q F^{q-2}\|_{L^\infty}^{1/q} = \left\|r^{q/(q-2)}F\right\|_{L^\infty}^{(q-2)/q} \leq \|F\|_{C^0_{w^2}}^{(q-2)/q}, \qquad \text{for} \qquad \frac{q}{q-2} \leq \frac{4}{3+\varepsilon}.$$

Therefore we have

$$\varepsilon < \frac{1}{q} \leq \frac{1-\varepsilon}{8}, \tag{2.12}$$

which implies that $\varepsilon < \frac{1}{9}$. This way choosing $p, q$ that satisfy (2.12) we have

$$rG_1' = \langle X\rangle^{2+\varepsilon}G_1' \in L^\infty. \tag{2.13}$$

Writing

$$\eta = G_1 * F + G_2 * F, \qquad \eta_X = G_1' * F + G_2' * F.$$

and using (2.11) and (2.13) we conclude that

$$\langle X\rangle^{2+\varepsilon}\eta_X \in C_{\mathrm{bdd}}(\mathbb{R}).$$

□

2.7. **Proof of existence.** Having shown that $\mathcal{F}$ is smooth and well-defined, we can at last complete the argument leading to Theorem 1.1.

*Proof of Theorem 1.1.* Following our discussion above, we will in fact prove statements (a) and (b) simultaneously. From Lemma 2.1, we have seen that the mapping $\mathcal{F} : C_w^{3+\alpha}(\overline{R}) \times C_w^{3+\alpha}(T) \times \mathbb{R} \to C_w^{3+\alpha}(\overline{R}) \times C_w^{1+\alpha}(T)$ defined in (2.7) is $C^1$.

It is also evident that the Fréchet derivative $D_{(\psi,\eta)}\mathcal{F}(0,0,0)$ is a linear isomorphism from $C_w^{3+\alpha}(\overline{R}) \times C_w^{3+\alpha}(T)$ to $C_w^{3+\alpha}(\overline{R}) \times C_w^{1+\alpha}(T)$. Indeed, the nonlinear terms in $f_1$ and $f_2$ are quadratic or higher, and so

$$D_{(\psi,\eta)}\mathcal{F}(0,0,0) = \begin{pmatrix}\mathrm{id} & 0 \\ 0 & g - \sigma\partial_X^2\end{pmatrix}.$$

Because our weight $w$ is algebraic, $g - \sigma\partial_X^2$ is invertible as a mapping $C_w^{3+\alpha}(T) \to C_w^{1+\alpha}(T)$, with its inverse given as convolution with the exponentially decaying kernel $G_2$ from Lemma 2.4. Thus the full operator matrix $D_{(\psi,\eta)}\mathcal{F}(0,0,0)$ is invertible.

We can now apply the implicit function theorem to $\mathcal{F}$ at the trivial solution $(0,0,0)$ to deduce the existence of a local curve of solutions in $C_w^{3+\alpha}(\overline{R}) \times C_w^{3+\alpha}(T) \times \mathbb{R}$. These weighted Hölder spaces lie inside the Sobolev spaces from Section 2.2 with $k=3$, and so the local uniqueness for $\eta, \psi$ in Sobolev spaces implies that they agree with those constructed in [26]. In particular, the corresponding $(\eta, u, c)$ solve the full problem (1.2) and, in the point vortex case, the advection condition (1.5).

Next, Lemma 2.4 ensures that, for each $(\eta, u, c)$ in this family, $\eta_X$ decays faster as stated in (1.8a). Since $\eta$ is small in $C_w^{3+\alpha}(\mathbb{R})$, Remark 3.4 implies that $\varphi = o(1)$ and hence by Theorem 1.5 that $\eta$ has the improved decay (2.10). Lastly, the leading-order expressions for $\eta$ and $u$ at infinity given in (1.7) are obtained by using (1.12) to compute the dipole moment $p$, and inserting that into (1.9). This last straightforward calculation is left to Section 4.3. □

## 3. Asymptotic properties

The purpose of this section is to show that the weaker localization assumption (1.8) implies the improved the decay and asymptotic form claimed in Theorem 1.5. Our strategy is to split the velocity field as in Section 2.1 and then separately investigate $\varphi$ and $V$.

### 3.1. Asymptotic form of $V$.

We begin with an elementary lemma that establishes the asymptotics for $V$ under the vorticity localization assumptions in (1.3).

**Lemma 3.1** (Asymptotics for $V$ with localized vorticity)**.** *Suppose that $\omega$ satisfies* (1.3)*, and let $V$ be defined as in* (2.2)*. Then*

$$V = -\frac{1}{\gamma_n}\nabla\left(\frac{m\cdot x}{|x|^n}\right) + O\left(\frac{1}{|x|^{n+\varepsilon}}\right) \qquad \text{as } |x| \to \infty, \tag{3.1}$$

*where $\gamma_n$ is the surface area of the unit sphere in $\mathbb{R}^n$ and $m$ is the vortex impulse* (1.11)*.*

Statements like (3.1) are well known for compactly supported vorticity, see for instance the formal proof in [25, Chapter 3.2]. Similar computations are commonplace in electrostatics where the term dipole originates. The main point of Lemma 3.1, therefore, is that the asymptotics are the same under the weaker localization assumptions in (1.3). Because it is somewhat technical, the proof is deferred to Appendix A.

As a simple corollary, we detail the decay rates of two quantities that are important for the analysis in the next section; cf. (3.4b) and (3.4c).

**Corollary 3.2.** *Under the hypotheses of Lemma 3.1, if $\eta$ has the decay* (1.8a)*, it follows that*

$$c\cdot V|_S = -\frac{1}{\gamma_n} c'\cdot\nabla\left(\frac{m'\cdot x'}{|x'|^n}\right) + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right) \tag{3.2}$$

*and*

$$|x|^n N\cdot V|_S = -\frac{m\cdot e_n}{\gamma_n} + O\left(\frac{1}{|x'|^{\varepsilon}}\right), \tag{3.3}$$

*as $|x'| \to \infty$.*

*Proof.* The first equation (3.2) follows immediately from Lemma 3.1 and the decay assumptions on $\eta$ in (1.8a). For the second, we compute that

$$\begin{aligned}|x|^n N\cdot V|_S &= -\frac{1}{\gamma_n}\left(|x'|^n + O\left(\frac{1}{|x'|^{n-1+\varepsilon}}\right)\right)\left(e_n + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right)\right)\\ &\qquad\cdot\left(\frac{m}{|x'|^n} - n\frac{(m'\cdot x')x'}{|x'|^{n+2}} + O\left(\frac{1}{|x'|^{n(n+1+\varepsilon)}}\right)\right)\\ &= -\frac{m\cdot e_n}{\gamma_n} + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right),\end{aligned}$$

where we have again used the decay assumptions (1.8a). □

3.2. **Asymptotic form of $\varphi$.** Now we turn to the potential for the irrotational part of the flow. Consider first localized vorticity of Case I. Recall that we split the velocity $u = \nabla\varphi + V$ where the vortical part $V$ is defined by (2.2). The velocity potential $\varphi$ can be recovered from $\eta$ and $V$ as the unique solution to the elliptic problem

$$\Delta\varphi = 0 \qquad \text{in } \Omega \tag{3.4a}$$

$$N\cdot(\nabla\varphi + V) = c\cdot N \qquad \text{on } S, \tag{3.4b}$$

which vanishes as $|x|\to\infty$. From the Euler equations (1.2), we see that $\varphi$ must in addition satisfy the nonlinear boundary condition

$$\frac{1}{2}|\nabla\varphi + V|^2 - c\cdot(\nabla\varphi+V) + g\eta = -\sigma\nabla\cdot N \qquad \text{on } S. \tag{3.4c}$$

Here we have applied Bernoulli's law on $S$ and evaluated the pressure using the dynamic boundary condition from (1.2b). This is justified by our assumption in (1.3b) that $\omega$ is tangential to the free surface.

As it is harmonic, we clearly have $\varphi\in C^\infty(\Omega)$. To confirm the regularity of $\varphi$ up to the boundary, we first notice from (1.3a) and classical potential theory that $V\in C^\alpha(\overline{\Omega})$ for any $\alpha\in(0,1)$; see, for example, [13, Chapter 4]. Now, under the assumption that $\eta\in C^2_{\mathrm{bdd}}(\mathbb{R}^{n-1})$ and decays according to (1.8a), the Neumann data for $\varphi$ in (3.4b) is of class $C^\alpha(S)$, for all $\alpha\in(0,1)$. Applying elliptic regularity theory, these deductions at last lead us to the conclusion that

$$\varphi\in C^{1+\alpha}(\overline{\Omega}), \qquad \text{for all } \alpha\in(0,1). \tag{3.5}$$

Finally, we observe that the decay rate assumed for $u$ in (1.8b) together with the asymptotics (3.1) for $V$ imply that the potential satisfies

$$\varphi = o\left(\frac{1}{|x|^{n-2}}\right), \qquad \text{as } |x|\to\infty. \tag{3.6}$$

Indeed, in two dimensions, this asserts only that $\varphi$ vanishes at infinity, which follows from the discussion above. For the three-dimensional case, (3.6) would be a direct consequence of the fundamental theorem of calculus if $\eta$ were identically zero; the modification of this argument to allow for nontrivial $\eta$ only requires that $\eta$ and $\nabla\eta$ be uniformly bounded.

**Lemma 3.3.** *Suppose that there exists a solitary wave with Case I vorticity and the decay* (1.8)*. Then there exists a dipole moment $q = (q', m_n/\gamma_n)$ such that*

$$\varphi(x) = \frac{q \cdot x}{|x|^n} + O\left(\frac{1}{|x|^{n-1+\varepsilon}}\right), \qquad \nabla\varphi(x) = \nabla\left(\frac{q \cdot x}{|x|^n}\right) + O\left(\frac{1}{|x|^{n+\varepsilon}}\right), \tag{3.7}$$

*as $|x| \to \infty$.*

*Proof.* The main idea behind this argument is that the asymptotic properties (3.7) can be determined via elliptic theory from the kinematic boundary condition, the equation satisfied by $\varphi$, and the decay assumption (1.8). In comparison to the irrotational case considered in [35], there are additional terms coming from the vortical contribution $V$ that must be understood using the asymptotic information contained in Lemma 3.1 and Corollary 3.2.

With that in mind, recall that the Kelvin transform $\tilde{\varphi} = \tilde{\varphi}(\tilde{x})$ of $\varphi = \varphi(x)$ is defined by

$$\tilde{x} := \frac{x}{|x|^2}, \qquad \tilde{\varphi}(\tilde{x}) := \frac{1}{|\tilde{x}|^{n-2}}\varphi\left(\frac{\tilde{x}}{|\tilde{x}|^2}\right).$$

Denote by $\Omega^\sim$ the image of $\Omega \setminus B_1(0)$ under the map $x \mapsto \tilde{x}$:

$$\Omega^\sim := \left\{\tilde{x} \in \mathbb{R}^n : \frac{\tilde{x}}{|\tilde{x}|^2} \in \Omega \setminus B_1(0)\right\}.$$

The asymptotic behavior of $\varphi$ as $|x| \to \infty$ is determined by the behavior of $\tilde{\varphi}$ in a neighborhood of the origin in the $\tilde{x}$-variables. Notice that by (1.8a), we have $0 \in \partial\Omega^\sim$, and (3.6) ensures that $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}$ can be extended to the boundary as a $C^0(\overline{\Omega^\sim})$ class function.

Similarly as in [35, Appendix A], we find that the decay assumptions on $\eta$ we made in (1.8a) guarantee that $\partial\Omega^\sim$ is $C^2$ in a neighborhood of 0; let $S^\sim \subset \partial\Omega^\sim$ be a small portion of the boundary containing $\tilde{x} = 0$.

As the Kelvin transform of a harmonic function is harmonic, we know that $\Delta\tilde{\varphi} = 0$ in $\Omega^\sim$. We now show that the kinematic equation (3.4b) leads to an oblique boundary condition for $\tilde{\varphi}$. A simple calculation shows that $\tilde{N}$, the normal vector to $\Omega^\sim$ at $\tilde{x} \in S^\sim$, is related to $N$ by the formula

$$\tilde{N}(\tilde{x}) = N(x) - 2\left(\frac{N(x) \cdot x}{|x|^2}\right)x.$$

We then find that

$$N \cdot (c - V) = N \cdot \nabla\varphi = -(n-2)\frac{x \cdot N}{|x|^n}\tilde{\varphi} + \frac{1}{|x|^n}\tilde{N} \cdot \nabla\tilde{\varphi},$$

or, equivalently,

$$\tilde{N} \cdot \nabla\tilde{\varphi} + \tilde{a}\tilde{\varphi} = \tilde{b} \qquad \text{on } S^\sim \setminus \{0\}, \tag{3.8}$$

where $\tilde{a} = \tilde{a}(\tilde{x})$ and $\tilde{b} = b(\tilde{x})$ are given by

$$\tilde{a}(\tilde{x}) := -(n-2)\,(N \cdot x)\,, \qquad \tilde{b}(\tilde{x}) := |x|^n N \cdot (c - V). \tag{3.9}$$

Our objective is to use elliptic theory to infer that $\tilde{\varphi}$ has the desired Hölder regularity in a neighborhood of the origin in the Kelvin transform variables. Naturally, this requires us to establish the $C^\varepsilon(S^\sim)$ Hölder continuity of the coefficients $\tilde{a}$ and $\tilde{b}$ above. This can be achieved from the (stronger) Hölder regularity and the decay properties of $\tilde{a}, \tilde{b}$ as functions of the untransformed variable $x$. In particular, we apply Lemma B.1 with $\alpha = \varepsilon$, $\beta = 2\varepsilon$, and $k = 4\varepsilon$. Moreover from the decay of $\eta$ in (1.8a) and asymptotics of $V \cdot N|_S$ obtained in (3.3) it follows that

$$\tilde{a}(0) = 0, \quad \tilde{b}(0) = \frac{m_n}{\gamma_n}.$$

The proof of the lemma is now essentially complete. Using the regularity of $\varphi$ in (3.5), we can argue as in [35, Appendix A] to show that $\tilde{\varphi}$ is an $H^1(\Omega^\sim)$ weak solution of the Laplace equation with oblique boundary condition (3.8). Elliptic regularity theory then implies that $\tilde{\varphi} \in C^{1+\varepsilon}(\Omega^\sim \cup S^\sim)$ (see, for example, [18, Theorem 5.51]), and therefore that it admits the expansion

$$\tilde{\varphi}(\tilde{x}) = q \cdot \tilde{x} + O(|\tilde{x}|^{1+\varepsilon}), \qquad \nabla\tilde{\varphi}(\tilde{x}) = q + O(|\tilde{x}|^{\varepsilon}),$$

where $q := \nabla\tilde{\varphi}(0)$. Finally, evaluating the transformed boundary condition (3.8) at $\tilde{x} = 0$, we infer that

$$\frac{m_n}{\gamma_n} = \tilde{b} = \tilde{N} \cdot \nabla\tilde{\varphi} + \tilde{a}\tilde{\varphi} = e_n \cdot q$$

and hence $q = (q', m_n/\gamma_n)$. Returning to the original variables, this confirms that $\varphi$ has the asymptotic form (3.7). ☐

*Remark* 3.4. Examining the above argument, it is clear that the localization assumption on $u$ in (1.8) is needed only insofar as it implies that $\varphi$ decays according to (3.6). In two dimensions, this can be removed altogether when $\eta$ has sufficiently small Lipschitz constant. This is a consequence of the fact that, in this setting, the unique solution of the elliptic system (3.4) is given by the single-layer potential for the Neumann data $(c-V)\cdot N$. But an easily calculation shows that $\int_S (c-V)\cdot N\, dS = 0$, which implies that $\varphi = o(1)$.

Next, consider the situation for the vortex sheet. As the flow is irrotational in the air and water, we have $u_\pm = \nabla\varphi_\pm$, where $\varphi_\pm$ will then satisfy

$$\Delta\varphi_\pm = 0 \qquad \text{in } \Omega_\pm \tag{3.10a}$$

$$N_\pm \cdot \nabla\varphi_\pm = c \cdot N_\pm \qquad \text{on } S \tag{3.10b}$$

$$\frac{1}{2}\,[\![\rho|\nabla\varphi|^2]\!] - [\![\rho c \cdot \nabla\varphi]\!] + g\,[\![\rho]\!]\,\eta = \pm\sigma\nabla\cdot N_\pm \qquad \text{on } S. \tag{3.10c}$$

Again, the assumptions on $u$ and $\eta$ in (1.8) ensure that $\varphi_\pm$ have the decay rate (3.6).

**Corollary 3.5.** *Suppose that there exists a traveling wave solution with Case II vorticity and the decay* (1.8)*. Then there exist dipole moments $p_\pm = (p'_\pm, 0)$ such that*

$$\varphi_\pm(x) = \frac{p_\pm \cdot x}{|x|^n} + O\left(\frac{1}{|x|^{n-1+\varepsilon}}\right), \qquad \nabla\varphi_\pm(x) = \nabla\left(\frac{p_\pm \cdot x}{|x|^n}\right) + O\left(\frac{1}{|x|^{n+\varepsilon}}\right), \tag{3.11}$$

*as $|x| \to \infty$.*

*Proof.* This proof works almost identically to that of Lemma 3.3 but without the difficulties related to $V$. Let $\Omega_\pm^\sim$ denote the image under $x \mapsto \tilde{x}$ of $\Omega_\pm \setminus B_1(0)$. Likewise, let $\tilde{\varphi}_\pm$ be the Kelvin transform of $\varphi_\pm$. Once again, the vanishing of $\eta$ at infinity (1.8a) implies that $0 \in \partial\Omega_\pm^\sim$, and the decay of $\varphi_\pm$ gives $\tilde{\varphi}_\pm(0) = 0$. It follows that each $\tilde{\varphi}_\pm$ can be extended to the boundary as a $C^0(\overline{\Omega_\pm^\sim})$ class function.

Furthermore, we can easily confirm that $\partial\Omega_\pm^\sim$ is $C^2$ in a neighborhood of 0. Let $S^\sim \subset \partial\Omega_\pm^\sim$ be a portion of the mutual boundary containing $\tilde{x} = 0$. The transformed potentials $\tilde{\varphi}_\pm$ are harmonic in $\Omega_\pm^\sim$ and satisfy the oblique boundary conditions

$$\tilde{N}_\pm \cdot \nabla\tilde{\varphi}_\pm + \tilde{a}_\pm\tilde{\varphi}_\pm = \tilde{b}_\pm \qquad \text{on } S^\sim \setminus \{0\},$$

where $\tilde{N}_\pm$ are the images of $N_\pm$ under the Kelvin transform, and

$$\tilde{a}_\pm(\tilde{x}) := -(n-2)\left(\frac{N_\pm \cdot \tilde{x}}{|\tilde{x}|^2}\right), \qquad \tilde{b}_\pm(\tilde{x}) := \frac{1}{|\tilde{x}|^n} N_\pm \cdot c_\pm.$$

Observe that the elliptic problems for $\tilde{\varphi}_\pm$ are essentially decoupled. Arguing exactly as in the irrotational case, we can show that $\tilde{a}_\pm$ and $\tilde{b}_\pm$ are uniformly $C^\varepsilon$ in a neighborhood of the origin. Elliptic regularity then implies the existence of the dipole moments $p_\pm =: \tilde{\varphi}_\pm(0)$, and the fact that $p_{n\pm} = 0$ follows once more from evaluating the boundary condition at $\tilde{x} = 0$ but noting that $\tilde{b}(0) = 0$ in this case. □

3.3. **Asymptotic forms of $\eta$ and $u$.** Having determined that $\varphi$ and $V$ are dipoles at infinity, we are now prepared to prove our theorem characterizing the asymptotic forms of $\eta$ and $u$.

*Proof of Theorem 1.5.* First consider the statement in part (a). Solving for $\eta$ in the Bernoulli condition (3.4c), we find that

$$\begin{aligned}\eta(x') &= \frac{1}{g}\left(c\cdot\nabla\varphi + \frac{1}{2}|\nabla\varphi|^2 - \sigma\nabla\cdot N + c\cdot V + \nabla\varphi\cdot V + \frac{1}{2}|V|^2\right)\\ &= \frac{1}{g|x'|^n}\left(c\cdot q - n\frac{(c'\cdot x')(q'\cdot x')}{|x'|^2}\right) + \frac{1}{g}\left(c\cdot V + \nabla\varphi\cdot V + \frac{1}{2}|V|^2\right) + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right),\end{aligned}$$

where the second line follows from the decay assumed on $\eta$ in (1.8a) and our estimate of $\varphi$ in (3.7). Now, from (3.1) we know that the $\nabla\varphi\cdot V$ and $|V|^2$ are $O(1/|x'|^{n+\varepsilon})$. Inserting the leading-order formula for $c\cdot V$ derived in (3.2) then yields

$$\eta = \frac{1}{g|x'|^n}\left(c'\cdot\left(q' - \frac{m'}{\gamma_n}\right) - n\frac{(c'\cdot x')\left((q' - \frac{m'}{\gamma_n})\cdot x'\right)}{|x'|^2}\right) + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right).$$

Defining $p := q - m/\gamma_n$, this is exactly the claimed asymptotic expression for $\eta$ in (1.9a). Note that because $q_n = m_n/\gamma_n$, it is indeed true that $p_n = 0$. Likewise, the asymptotic form of $u$ stated in (1.9b) simply follows from writing $u = \nabla\varphi + V$ and using the dipole formula for $V$ in (3.1) and for $\nabla\varphi$ in (3.7).

The argument for (b) is similar. From (3.10c), we find that

$$\begin{aligned}\eta(x') &= \frac{1}{g\,[\![\rho]\!]}\left(\left[\!\!\left[\rho c\cdot\nabla\varphi+\frac{1}{2}\rho|\nabla\varphi|^2\right]\!\!\right]-\sigma\nabla\cdot N\right)\\ &= \frac{1}{g\,[\![\rho]\!]\,|x'|^n}\left(c\cdot[\![\rho p]\!]-n\frac{(c'\cdot x')([\![\rho p']\!]\cdot x')}{|x'|^2}\right)+O\left(\frac{1}{|x'|^{n+\varepsilon}}\right),\end{aligned}$$

which implies (1.10a). The asymptotics for $u_\pm$ asserted in (1.10b) were already proved in Corollary 3.5, since $u_\pm = \nabla\varphi_\pm$. □

## 4. Nonexistence and the dipole moment formula

4.1. **Nonexistence.** First, we establish that there exist no waves with localized vorticity in Case I or Case II having a single-signed free surface profile. In fact, we prove the stronger statement that all such waves must have no excess mass in the sense that $\int \eta\,dx = 0$.

In the two-dimensional setting, we will need the following result on the configuration of the streamlines in a neighborhood of a point vortex.

**Lemma 4.1** (Streamlines)**.** *Suppose that $n = 2$ and that there exists a solitary wave with Case I-B vorticity and the decay (1.8). Fix $\alpha \in (0,1)$. For each vortex center $\xi^i \in \Xi$, and $\delta > 0$ sufficiently small, there exists an open connected set $\tilde{B}^i_\delta \subset \Omega$ with $\xi^i \in \tilde{B}^i_\delta$, and $\partial\tilde{B}^i_\delta$ is a closed integral curve of $u - c$ that admits the global parameterization*

$$\partial\tilde{B}^i_\delta = \{(\tilde{r}^i_\delta(\theta)\cos(\theta),\, \tilde{r}^i_\delta(\theta)\sin(\theta)) : \theta \in [0,2\pi)\}, \tag{4.1}$$

*where $\tilde{r}^i_\delta \in C^{1+\alpha}$ is a $2\pi$-periodic function with $\tilde{r}^i_\delta(0) = \delta$, $\partial_\theta \tilde{r}^i_\delta = O(\delta^3)$.*

*Proof.* As we are only concerned with local properties of the flow, we may without loss of generality suppose that $\xi^i = 0$. In light of (1.5) and (2.2a), we know that

$$c = \nabla\varphi(0) + V_{\mathrm{ac}}(0) + \sum_{\substack{j=1\\ j\neq i}}^{M} V^j(0) + V_{\mathrm{p}}(0),$$

where $V_{\mathrm{ac}}$, $V^j$, and $V_{\mathrm{p}}$ are the contributions of the absolutely continuous part of the vorticity, the $j$-th point vortex, and the phantom vortex, respectively. Note that $\varphi$, $V^j$, and $V_{\mathrm{p}}$ are each harmonic near the origin, whereas $V_{\mathrm{ac}} \in C^\alpha$, since $\omega_{\mathrm{ac}} \in L^\infty(\Omega)$. Then we may write

$$u(x) - c =: \frac{1}{2\pi}\varpi^i\nabla^\perp \log|x| + G(x), \tag{4.2}$$

where $G(0) = 0$, and in a neighborhood of the origin, $G$ belongs to $C^\alpha$ and is divergence free in the distributional sense. We can therefore introduce a function $\Psi$ of class $C^{1+\alpha}$ near the origin such that $\nabla^\perp\Psi = G$ and $\Psi(0) = 0$. It follows that the level sets of the function

$$H(x) := \frac{1}{2\pi}\varpi^i \log|x| + \Psi(x) \tag{4.3}$$

coincide locally with the integral curves of $u-c$. Let $\delta_0 > 0$ be sufficiently small so that

$$\frac{x}{|x|} \cdot \nabla H(x) = \frac{\varpi^i}{2\pi|x|} + \frac{x^\perp}{|x|} \cdot G(x) > 0 \qquad \text{in } B_{\delta_0}(0) \setminus \{0\}.$$

Then $H$ is strictly increasing in the radial direction on this punctured ball. We define the neighborhoods $\tilde{B}^i_\delta$ to be the super level sets of $H$. Writing (4.3) in polar coordinates and applying the implicit function theorem then yields the parameterization function $\tilde{r}^i_\delta$. □

In the next lemma, we establish a key integral identity that is a consequence of Bernoulli's theorem and the localization of the vorticity (1.3).

**Lemma 4.2.** *Suppose that there exists a solitary wave with Case I-A vorticity and the decay* (1.8)*. Then, in the the two-dimensional case,*

$$\int_\Omega (u-c)\omega \, dx = 0, \tag{4.4}$$

*and in the three-dimensional setting*

$$\int_\Omega (u-c) \times \omega \, dx = 0. \tag{4.5}$$

*For localized vorticity in Case I-B,* (4.4) *holds with $\omega_{\mathrm{ac}}$ in place of $\omega$.*

*Proof.* First observe that, working in three dimensions, the Euler equations lead to

$$(u-c) \times \omega = \nabla \left( \frac{1}{2}|u-c|^2 + P + gx_3 - \frac{1}{2}|c|^2 \right), \tag{4.6}$$

which holds in the sense of distributions on $\Omega$. As the left-hand side above is in $L^\infty(\Omega)$, we have that

$$B(x) := \frac{1}{2}|u-c|^2 + P + gx_n - \frac{1}{2}|c|^2 \in W^{1,\infty}(\Omega).$$

The identity (4.6) ensures that the weak tangential derivative of $B$ vanishes on any smooth integral curve of $u-c$. In particular, this applies to the free surface, and from (1.2b) and (1.8a) we infer that $B$ vanishes identically on $S$. Now, taking $R > 0$ large and integrating over $\Omega \cap B_R(0)$ using (4.6), we find that

$$\int_{\Omega \cap B_R(0)} (u-c) \times \omega \, dx = \int_{\partial B_R(0) \cap \Omega} BN \, dS. \tag{4.7}$$

An analogous identity can be derived in two dimensions. Suppose that there are point vortices in the flow, as this can be easily adapted to the case of non-singular localized vorticity. The Euler equations once again imply that

$$(c-u)^\perp \omega = \nabla \left( \frac{1}{2}|u-c|^2 + P + gx_2 - \frac{1}{2}|c|^2 \right), \tag{4.8}$$

in the distributional sense on $\Omega \setminus \Xi$. Note that we are free to replace $\omega$ by $\omega_{\mathrm{ac}}$ above, as they agree away from $\Xi$. For $R > 0$ and $\delta > 0$ sufficiently small, define the domain

$$\Omega_{R,\delta} := (\Omega \cap B_R(0)) \setminus \bigcup_{i=1}^{M} \tilde{B}_\delta^i, \tag{4.9}$$

where the sets $\tilde{B}_\delta^i$ are those described in Lemma 4.1. From (4.8), we have that

$$\int_{\Omega_{R,\delta}} (u-c)^\perp \omega_{\mathrm{ac}} \, dx = \int_{\Omega \cap \partial B_R(0)} BN \, dS. \tag{4.10}$$

Observe that there are no boundary integral terms over the sets $\partial \tilde{B}_\delta^i$ because they are smooth closed integral curves of $u - c$, and hence $B$ is constant along them according to the above discussion. Likewise, $B$ vanishes on the free surface and thus there is no integral over $S$ occurring in (4.10).

To finish the argument, we will show that $B \in L^1(\Omega)$, which guarantees that there exists a sequence of radii $\{R_j\}$ with $R_j \to \infty$ and such that the integrals on the right-hand sides of (4.7) and (4.10) vanish as $j \to \infty$. Observe that the identities (4.6) and (4.8), together with the localization assumption (1.3), imply that

$$\nabla B \in L^1(\Omega) \cap L^\infty(\Omega), \quad |x|^k \nabla B \in L^1(\Omega),$$

where recall that $k > n^2$. Our main tool for translating estimates in weighted Sobolev spaces to $L^1$ is the Caffarelli–Kohn–Nirenberg inequality [5, 6], which states that

$$\left( \int_\Omega |x|^{-bs} |B|^s \, dx \right)^{2/s} \lesssim \int_\Omega |x|^{-2a} |\nabla B|^2 \, dx \tag{4.11}$$

for any $a$, $b$, and $s$ satisfying the relations

$$\begin{aligned} a \in (-\infty, 0),\ b \in (a, a+1],\ s = \frac{2}{b-a} \qquad & \text{if } n = 2, \\ a \in (-\infty, 1/2),\ b \in [a, a+1],\ s = 6 + 2(b-a) \qquad & \text{if } n = 3. \end{aligned}$$

In three dimensions, this gives

$$\left( \int_\Omega |x|^{3k} |B|^6 \, dx \right)^{1/3} \lesssim \int_\Omega |x|^k |\nabla B|^2 \, dx \leq \|\nabla B\|_{L^\infty} \int_\Omega |x|^k |\nabla B| \, dx.$$

Thus, from Hölder's inequality and the above estimate we see that

$$\|B\|_{L^1(\Omega)} \lesssim \left( \int_\Omega (1 + |x|^{3k}) |B|^6 \, dx \right)^{1/6} < \infty,$$

where we have applied the Gagliardo–Nirenberg–Sobolev inequality to control $B$ in $L^6(\Omega)$.

The argument in two dimensions is similar. Taking $a = -k/2$, $b = -(k-1)/2$, and $s = 4$, we infer from (4.11) that

$$\left( \int_\Omega |x|^{2(k-1)} |B|^4 \, dx \right)^{1/2} \lesssim \|\nabla B\|_{L^\infty} \int_\Omega |x|^k |\nabla B| \, dx < \infty.$$

On the other hand, choosing $a = -1/2$, $b = 0$, and $s = 4$ yields

$$\left( \int_\Omega |B|^4 \, dx \right)^{1/2} \lesssim \int_\Omega |x| |\nabla B|^2 \, dx \leq \left\| |\nabla B|^{2-\frac{1}{k}} \right\|_{L^{\frac{k}{k-1}}} \int_\Omega |x|^k |\nabla B| \, dx,$$

and so together these estimates furnish the bound

$$\|B\|_{L^1(\Omega)} \lesssim \left( \int_\Omega \left( 1 + |x|^{2(k-1)} \right) |B|^4 \, dx \right)^{1/4} < \infty.$$

Here we have used Hölder's inequality and relied on the fact that $k > 4$. □

With the additional understanding of the streamlines near a point vortex given by Lemma 4.1, and the identities (4.4)–(4.5), we can now prove the nonexistence theorem.

*Proof of Theorem 1.2.* We begin with Case I vorticity. Consider the vector field $A$ defined by

$$A := u_n(c - u) + \left( \frac{1}{2}|u|^2 - c \cdot u \right) e_n,$$

with domain $\Omega$ for Case I-A and $\Omega \setminus \Xi$ for Case I-B. It is easy to compute that

$$\nabla \cdot A = \begin{cases} (c_1 - u_1)\omega & \text{if } n = 2 \\ e_n \cdot (\omega \times (c - u)) & \text{if } n = 3, \end{cases}$$

in the distributional sense, while the Bernoulli condition and (1.2b) together imply that

$$N \cdot A = \left( \frac{1}{2}|u|^2 - c \cdot u \right) N \cdot e_n = \frac{1}{\langle \nabla \eta \rangle} (-g\eta - \sigma \nabla \cdot N) \qquad \text{on } S.$$

Our plan will be to apply the divergence theorem to $A$. The most sensitive argument is needed for Case I-B, so we treat that scenario first. Let $\Omega_{R,\delta}$ be the domain defined in (4.9). Integrating $\nabla \cdot A$ over $\Omega_{R,\delta}$ furnishes the identity

$$\begin{aligned} \int_{\Omega_{R,\delta}} (c_1 - u_1) \, \omega \, dx = g \int_{B_R(0) \cap S} \eta \, dx' + \sigma \int_{\partial B_R(0) \cap S} \nu \cdot N \, ds \\ + \sum_i \int_{\partial \tilde{B}^i_\delta} A \cdot N \, dS + \int_{\partial B_R(0) \cap \Omega} A \cdot N \, dS, \end{aligned} \tag{4.12}$$

where in the second integral on the right-hand side we have used (1.2b) and then integrated by parts; $\nu$ and $ds$ refer to the normal vector and arc-length element with respect to the projection of $\partial B_R(0) \cap S$ onto $\mathbb{R}^{n-1} \times \{0\}$, respectively. We know from Lemma 4.2 that as $R \to \infty$ and $\delta \to 0$, the left-hand side will vanish. On the other hand, from Lemma 3.1 and Lemma 3.3, we see that

$$A = O(|u|) = O\left( |\nabla \varphi| + |V| \right) = O\left( \frac{1}{|x|^n} \right), \qquad \text{as } |x| \to \infty,$$

and hence that the integral over $\partial B_R(0) \cap \Omega$ on the right-hand side of (4.12) vanishes in the limit $R \to \infty$. Similarly, our assumptions on the decay of $\eta$ in (1.8a) guarantee that the integral over $\partial B_R(0) \cap S$ vanishes as $R \to \infty$.

Consider now the third term on the right-hand side in (4.12). Without loss of generality, let us take $\Xi = \{0\}$. By construction, $u - c$ is tangent to $\partial \tilde{B}^i_\delta$, and hence

$$\begin{aligned}\int_{\partial \tilde{B}^i_\delta} A \cdot N \, dS = \int_{\partial \tilde{B}^i_\delta} \left[ \frac{1}{2}|u|^2 - c \cdot u \right] N_n \, dS &= \frac{1}{2} \int_{\partial \tilde{B}^i_\delta} \left( |\nabla^\perp H|^2 - |c|^2 \right) N_n \, dS \\ &= \frac{1}{2} \int_{\partial \tilde{B}^i_\delta} \left| \frac{\varpi^i}{2\pi} \frac{x^\perp}{|x|^2} + G \right|^2 N_n \, dS \\ &= \frac{1}{2} \int_{\partial \tilde{B}^i_\delta} \left[ \left( \frac{\varpi^i}{2\pi} \right)^2 \frac{1}{|x|^2} + \frac{\varpi^i}{\pi} \frac{x^\perp \cdot G}{|x|^2} + |G|^2 \right] N_n \, dS.\end{aligned}$$

Recall that the function $G \in C^\alpha(\tilde{B}^i_\delta)$, for any $\alpha \in (0,1)$, and satisfies $G(0) = 0$, so in particular $G(x) = O(|x|^\alpha)$. The explicit parameterization (4.1) shows that $\operatorname{diam} \tilde{B}^i_\delta = O(\delta)$, and therefore

$$\int_{\partial \tilde{B}^i_\delta} \left( \frac{\varpi^i}{\pi} \frac{x^\perp \cdot G}{|x|^2} + |G|^2 \right) N_n \, dS \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

From Lemma 4.1, we know that $|x| = \tilde{r}^i_\delta = \delta + O(\delta^3)$ on $\partial \tilde{B}^i_\delta$, and so a simple calculation reveals that

$$\int_{\partial \tilde{B}^i_\delta} \frac{N_n}{|x|^2} \, dS = \int_{\partial \tilde{B}^i_\delta} N_n \left( \frac{1}{\delta^2} + O(1) \right) dS = O(\delta).$$

Putting together all of the above deductions, we conclude that

$$\int_{\partial \tilde{B}^i_\delta} A \cdot N \, dS \longrightarrow 0 \quad \text{as } \delta \to 0.$$

Finally, returning to (4.12) and taking $\delta \to 0$ and $R \to \infty$, we find

$$\lim_{R \to \infty} \int_{B_R(0) \cap S} \eta \, dx' = \int_{\mathbb{R}} \eta \, dx' = 0,$$

which completes the argument for the two-dimensional setting.

Next, consider non-singular localized vorticity in $\mathbb{R}^3$. Applying the divergence theorem to $A$ on $\Omega \cap B_R(0)$, we obtain

$$\begin{aligned}\int_{B_R(0) \cap \Omega} e_n \cdot (\omega \times (c - u)) \, dx = g \int_{B_R(0) \cap S} \eta \, dx' &+ \sigma \int_{\partial B_R \cap S} N \cdot \nu \, ds \\ &+ \int_{\partial B_R(0) \cap \Omega} A \cdot N \, dS.\end{aligned} \tag{4.13}$$

In view of Lemma 4.2, this implies that

$$g \int_{B_R \cap S} \eta \, dx' + \sigma \int_{\partial B_R(0) \cap S} N \cdot \nu \, ds + \int_{\partial B_R(0) \cap \Omega} A \cdot N \, dS \longrightarrow 0 \text{ as } R \to \infty.$$

Thanks again to (1.9b), we have $A = O(|u|) = O(1/|x|^n)$, and so the remaining integral over $\partial B_R \cap \Omega$ also vanishes as $R \to \infty$, leaving us with $\int \eta \, dx' = 0$.

Lastly, the argument for the case of a vortex sheet is a simpler version of that given above. The vector field $A$ is divergence free (in the classical sense) in both the air and water regions, and its normal trace is continuous over $S$. An application of the divergence theorem as above yields $\int \eta \, dx' = 0$. □

4.2. **Dipole moment formula.** The objective of this section is to derive the formula (1.12) relating the dipole moment to the vortex impulse. Following Wheeler [35], our strategy is based on identifying a vector field whose divergence gives this energy-like quantity and which decays at infinity in such a way that we can recover $p$ upon integrating by parts. The presence of vorticity significantly complicates this task.

*Proof of Theorem 1.7.* First consider part (a). Let $A$ be the vector field

$$A := \left( \varphi - \frac{m \cdot x}{\gamma_n |x|^n} \right) (u - c) + (c \cdot x) u, \tag{4.14}$$

with domain $\Omega$ for the non-singular localized vorticity case, or $\Omega \setminus \Xi$ if there are point vortices. It is easy to compute that

$$\begin{aligned} \nabla \cdot A &= u \cdot \nabla \varphi + c \cdot V - \frac{u - c}{\gamma_n} \cdot \nabla \left( \frac{m \cdot x}{|x|^n} \right) \\ &= |u|^2 - (u - c) \cdot \left( V + \frac{1}{\gamma_n} \nabla \left( \frac{m \cdot x}{|x|^n} \right) \right). \end{aligned} \tag{4.15}$$

Fix $\delta > 0$ and $R > 0$ and let $\Omega_{R,\delta}$ be given as in (4.9). Applying the divergence theorem to $A$ on $\Omega_{R,\delta}$ leads to the identity

$$\begin{aligned} \int_{\Omega_{R,\delta}} (u \cdot \nabla \varphi + c \cdot V) \, dx &= \int_{B_R(0) \cap S} A \cdot N \, dS + \sum_i \int_{\partial \tilde{B}^i_\delta} A \cdot N \, dS \\ &\quad + \int_{\partial B_R(0) \cap \Omega} A \cdot N \, dS =: \mathbf{I} + \mathbf{II} + \mathbf{III}. \end{aligned}$$

Because $u \cdot N = c \cdot N$ on $S$, we have that

$$A \cdot N = (c \cdot N)(c \cdot x) \qquad \text{on } S. \tag{4.16}$$

Therefore,

$$\mathbf{I} = -\int_{B_R(0) \cap S} \frac{c' \cdot \nabla \eta}{\langle \nabla \eta \rangle} (c \cdot x) \, dS = \int_{B_R(0) \cap S} \left( \frac{|c|^2}{\langle \nabla \eta \rangle} \eta - \nabla_S \cdot (\eta (c \cdot x) c) \right) dS,$$

where $\nabla_S \cdot$ denotes the surface divergence on $S$. The second term in the integrand is a total derivatives of quantities vanishing at infinity, and thus

$$\mathbf{I} \to |c|^2 \int_{\mathbb{R}^{n-1}} \eta(x') \, dx' \qquad \text{as } R \to \infty.$$

We already established in Theorem 1.2 that waves of this type have no excess mass, and hence $\mathbf{I} \to 0$ as $R \to \infty$.

Next, consider $\mathbf{II}$. For Case I-A in either the two- or three-dimensional setting, the asymptotic information contained in Lemma 3.1 in particular guarantees that $V \in L^2(\Omega)$, and hence **II** will simply vanish in the limit as $\delta \to 0$. Likewise, in Case I-B, the same will be true for $V_{\mathrm{ac}}$. To understand the contributions of the point vortices, note that we have from (4.2), and (4.3) that $u - c = \nabla^\perp H$ around each point vortex, and thus $(u-c)\cdot N|_{\partial \tilde{B}_\delta^i} = T \cdot \nabla H = 0$, which leads to

$$\mathbf{II} = \sum_i \int_{\partial \tilde{B}_\delta^i} (c\cdot x)(c \cdot N)\, dS = O(\delta).$$

Finally, to compute **III**, we note that from the asymptotic formulas (3.1) and (3.7),

$$V = O(1/R^n), \quad \varphi = O(1/R^{n-1}), \quad \nabla\varphi = O(1/R^n), \quad \text{on } \partial B_R(0) \cap \Omega.$$

Thus

$$\mathbf{III} = -\int_{\partial B_R(0)\cap\Omega} \left[ \left( \varphi - \frac{m\cdot x}{\gamma_n |x|^n} \right)(c\cdot N) - (c\cdot x) N \cdot (\nabla\varphi + V) \right] dS + O\left(\frac{1}{R^n}\right).$$

Hence, as $R \to \infty$, **III** approaches the constant value

$$\begin{aligned}
\mathbf{III} \to & \int_{\partial B_R(0)\cap\{x_n<0\}} \left( -\left( \frac{q\cdot x}{|x|^n} - \frac{m\cdot x}{\gamma_n|x|^n} \right) \frac{c\cdot x}{|x|} + (c\cdot x)\frac{x}{|x|}\cdot \nabla \left( \frac{q\cdot x}{|x|^n} - \frac{m\cdot x}{\gamma_n |x|^n} \right) \right) dS \\
&= \int_{\partial B_R(0)\cap\{x_n<0\}} \left( -n \frac{(c\cdot x)(q\cdot x)}{|x|^{n+1}} + n \frac{(c\cdot x)(m\cdot x)}{\gamma_n |x|^{n+1}} \right) dS \\
&= -n \int_{\partial B_1(0)\cap\{x_n<0\}} (c\cdot x) \left( q - \frac{m}{\gamma_n} \right) \cdot x \, dS = -\frac{\gamma_n}{2} c \cdot \left( q - \frac{m}{\gamma_n} \right) = -\frac{\gamma_n}{2} c \cdot p.
\end{aligned}$$

The argument for (b) is a slight variation of that given above. Let us redefine $A$ to be the vector field

$$A := \rho\varphi(\nabla\varphi - c) + \rho(c\cdot x)\nabla\varphi,$$

which is smooth in $\Omega = \Omega_+ \cup \Omega_-$. Applying the divergence theorem to $A$ on the set $B_R(0) \setminus S$ then gives

$$\begin{aligned}
\int_{B_R(0)\setminus S} \rho |\nabla\varphi|^2 \, dx &= \int_{B_R(0)\cap S} (A_+ \cdot N_+ + A_- \cdot N_-)\, dS + \int_{\partial B_R(0)\setminus S} A\cdot N \, dS \\
&=: \mathbf{I} + \mathbf{II}.
\end{aligned} \tag{4.17}$$

To evaluate **I**, we use the kinematic boundary condition (3.10b) to infer that

$$A_\pm \cdot N_\pm = \rho_\pm (\nabla\varphi_\pm - c)\cdot N_\pm + \rho_\pm (c\cdot x)\nabla\varphi_\pm \cdot N_\pm = \rho_\pm (c\cdot x) c\cdot N_\pm \qquad \text{on } S,$$

hence

$$\begin{aligned}
\mathbf{I} &= - [\![\rho]\!] \int_{S\cap B_R(0)} (c\cdot x) c \cdot N_- \, dS \\
&= - [\![\rho]\!] \int_{S\cap B_R(0)} \left( \frac{|c|^2}{\langle \nabla \eta\rangle} \eta - \nabla_S \cdot (\eta (c\cdot x) c) \right) dS,
\end{aligned}$$

which vanishes in the limit as $R \to \infty$ in view of Theorem 1.2. On the other hand,

$$\mathbf{II} = \int_{\partial B_R(0)\cap\Omega_+} A \cdot N \, dS + \int_{\partial B_R(0)\cap\Omega_-} A \cdot N \, dS \to \frac{\gamma_n}{2}\rho_+ p_+ \cdot c - \frac{\gamma_n}{2}\rho_- p_- \cdot c,$$

as $R \to \infty$. Combining this with (4.17) gives the vortex sheet dipole formula in (1.13), completing the proof. □

With Theorems 1.5 and 1.7 in hand, we can now prove Corollary 1.8 on the angular momentum.

*Proof of Corollary 1.8.* Since the differences between the $n = 2$ and $n = 3$ are merely notational, we only give the three-dimensional argument. From the asymptotic expansion of $u$ in (1.9b), we conclude that as $R \to \infty$,

$$\int_{\partial B_R(0)\cap\Omega} x \times u \, dS = \int_{\partial B_R(0)\cap\{x_n<0\}} x \times \nabla\left(\frac{p \cdot x}{|x|^n}\right) dS + O\left(\frac{1}{R^{1+\varepsilon}}\right).$$

It follows that

$$\int_{\partial B_R(0)\cap\Omega} x \times u \, dS \to -p \times \int_{\partial B_1(0)\cap\{x_n<0\}} x \, dS = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} p \times e_n,$$

as $R \to \infty$. Clearly, if the right-hand side above does not vanish, then the integral $\int_\Omega x \times u \, dx$ describing the total angular momentum will be divergent. It follows that $p' = 0$ is a necessary condition for the angular momentum to be finite. On the other hand, we already know from Lemma 3.3 that $p_n = 0$, and hence $p$ must vanish identically. The proof for Case I vorticity in two dimensions is identical, and hence omitted.

For Case II in either the two- or three-dimensional settings, the argument is the same. Note that if $[\![\rho p]\!] = 0$, however, then the dipole moment formula (1.13) implies that $u_\pm \equiv 0$, meaning that the wave must be trivial. □

4.3. **Dipole moment for $\mathscr{C}_{\mathrm{loc}}$ and $\mathscr{S}_{\mathrm{loc}}$.** Finally, as an example application of Theorem 1.7, in this subsection we determine the dipole moment $p$ to leading order for the families $\mathscr{C}_{\mathrm{loc}}$ and $\mathscr{S}_{\mathrm{loc}}$ constructed in Theorem 1.1.

First, observe that given the leading-order forms of $\eta(\varpi)$ and $c(\varpi)$ detailed in (2.3) for $\mathscr{C}_{\mathrm{loc}}$ and $\mathscr{S}_{\mathrm{loc}}$, it suffices to compute all integrals on the lower half-plane $\{x_2 < 0\}$. In fact, it is enough to simply consider $\mathscr{C}_{\mathrm{loc}}$, as the vortex patches in $\mathscr{S}_{\mathrm{loc}}$ limit to the point vortices as the radius of the patch $\rho$ is sent to 0. Looking at the left-hand side of (1.12), we anticipate that the highest-order term is

$$\begin{aligned}\mathbf{I} &:= \int_{\{x_2<0\}} \left(c \cdot V + \frac{1}{\gamma_2} c \cdot \nabla\left(\frac{m \cdot x}{|x|^2}\right)\right) dx \\ &= -\frac{c_1\varpi}{\gamma_2} \int_{\{x_2<0\}} \partial_{x_2}\left(\log|x+e_2| - \log|x-e_2|\right) dx + \frac{c_1\varpi}{\gamma_2} \int_{\{x_2<0\}} \partial_{x_1}\left(-\frac{2x_1}{|x|^2}\right) dx,\end{aligned}$$

where we are taking $\xi = (0,-1)$, $\xi^* = (0,1)$, so that $m = -2e_1$. After an elementary argument, we find that $\mathbf{I} = -c_1\varpi$.

Now, using (2.3), we know that

$$\|V\|_{L^2(\Omega)} = O(\varpi), \qquad \|\nabla\varphi\|_{L^2} = O(\varpi^3), \qquad c = \left(-\frac{\varpi}{2\gamma_2} + o(\varpi^2)\right) e_1,$$

and hence (1.12) becomes

$$\begin{aligned}
-\frac{\gamma_2}{2} c \cdot p &= \int_\Omega \left[ |u|^2 - u \cdot \left( V + \frac{1}{\gamma_2} \nabla \left( \frac{m \cdot x}{|x|^2} \right) \right) \right] dx + \mathbf{I} \\
&= \int_\Omega \left[ \nabla\varphi \cdot V + |\nabla\varphi|^2 - \frac{1}{\gamma_2} \nabla\varphi \cdot \nabla \left( \frac{m \cdot x}{|x|^2} \right) \right] dx + \mathbf{I} \\
&= \mathbf{I} + O(\varpi^4) = \frac{\varpi^2}{2\gamma_2} + O(\varpi^4).
\end{aligned}$$

Thus, for $|\varpi| \ll 1$, we find that

$$p = \frac{2\varpi}{\gamma_2} + O(\varpi^2).$$

Taking this value for $p$ in (1.9), we arrive at the asymptotic expressions in (1.7).

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## Appendix A. Asymptotics of $V$

In this appendix, we provide the proof of the asymptotics for $V$. We begin with the following elementary lemma.

**Lemma A.1.** *Let $U \subset \mathbb{R}^n$ and $\omega \in L^1(U) \cap L^p(U)$ for some $1 < p \leq \infty$. Then for any $0 < s < n(p-1)/p$, and $1 \leq q < \infty$ such that $1/p + 1/q = 1$, we have*

$$\sup_{x \in \mathbb{R}^n} \int_U \frac{|\omega(y)|}{|x-y|^s}\, dy \leq C \|\omega\|_{L^p(U)}^{qs/n} \|\omega\|_{L^1(U)}^{(n-qs)/n}, \tag{A.1}$$

*where $C = C(n, q, p, s) > 0$.*

*Proof.* For any $r > 0$, we may estimate

$$\begin{aligned}\int_U \frac{|\omega(y)|}{|x-y|^s}\,dy &= \int_{U\setminus B_r(x)} \frac{|\omega(y)|}{|x-y|^s}\,dy + \int_{U\cap B_r(x)} \frac{|\omega(y)|}{|x-y|^s}\,dy \\ &\le \frac{\|\omega\|_{L^1(U)}}{r^s} + \left(\frac{\gamma_n}{n-qs}\right)^{1/q} r^{(n-qs)/q}\|\omega\|_{L^p(U)}.\end{aligned}$$

Taking $r^{2/q} := \|\omega\|_{L^1(U)}/\|\omega\|_{L^p(U)}$, we obtain (A.1). □

We now prove the main result of the appendix.

*Proof of Lemma 3.1.* Denote

$$K_{\xi^*}(x,y) := f(x,y) - f(x,\xi^*), \qquad f(x,z) := \frac{x-z}{|x-z|^n}. \tag{A.2}$$

It is easy to compute that

$$\partial_{x_i} f(x,z) = -\partial_{z_i} f(x,z) = \frac{1}{|x-z|^n}e_i - n\frac{x_i - z_i}{|x-z|^{n+2}}(x-z). \tag{A.3}$$

We first consider the two-dimensional case. Note that $V$ can be written in the form

$$V(x) = \frac{1}{\gamma_2}\int_\Omega \omega(y) K_{\xi^*}(x,y)^\perp\,dy.$$

Now we divide the domain of integration into the regions

$$A := \{y :\ |y| \le |x|^{1-\varepsilon}\} \cap \Omega \quad \text{and} \quad B := \{y :\ |y| > |x|^{1-\varepsilon}\} \cap \Omega.$$

Using (A.3) we compute $K_{\xi^*}$ on $A$ as

$$K_{\xi^*}(x,y) = -\frac{y-\xi^*}{|x|^2} + 2\frac{(y-\xi^*)\cdot x}{|x|^4}x + O\left(\frac{|y-\xi^*|^2}{|x|^3}\right), \quad \text{as } |x| \to \infty. \tag{A.4}$$

It follows that

$$\int_A \omega(y) K_{\xi^*}(x,y)^\perp\,dy = \left(-\frac{m}{|x|^2} + 2\frac{(x\cdot m)}{|x|^4}x\right) + O\left(\frac{1}{|x|^3}\right), \qquad \text{as } |x| \to \infty,$$

where we have used the definition of $m$ in (1.11) and the fact that

$$\left|\int_B \omega(y)(y-\xi^*)\,dy\right| \le \int_B |y|^k|\omega(y)|\frac{|y-\xi^*|}{|y|^k}\,dy = O\left(\frac{1}{|x|^{(k-1)(1-\varepsilon)}}\right), \text{ as } |x| \to \infty,$$

which follows from moment condition in (1.3a).

On $B$, we apply Lemma A.1 to estimate

$$\begin{aligned}\left|\int_B \omega(y) K_{\xi^*}(x,y)^\perp\,dy\right| &\lesssim \int_B \frac{|\omega(y)|}{|x-y|}\,dy + \int_B \frac{|\omega(y)|}{|x-\xi^*|}\,dy \\ &\lesssim \|\omega\|_{L^1(B)}^{1/2} + \frac{1}{|x|^{k(1-\varepsilon)}|x-\xi^*|}\int_B |y|^k|\omega(y)|\,dy \lesssim \frac{1}{|x|^{2+\varepsilon}}.\end{aligned}$$

Putting together the above computation we obtain the asymptotics (3.1) for $n = 2$.

Now consider the three-dimensional case. Using the notation introduced in (A.2), we may write

$$V(x) = \frac{1}{\gamma_3}\int_\Omega \omega(y) \times [f(x,0) + K_0(x,y)]\, dy. \tag{A.5}$$

Notice that the first term involves the total vorticity and has the form $\frac{1}{\gamma_3}\left(\int_\Omega \omega(y)\, dy\right) \times \frac{x}{|x|^3}$. An application of the divergence theorem leads to the identity

$$\int_{\partial(\Omega\cap B_R(0))} y_i \omega \cdot N\, dS = \int_{\Omega\cap B_R(0)} \nabla\cdot(y_i\omega)\, dy = \int_{\Omega\cap B_R(0)} \omega_i\, dy. \tag{A.6}$$

Here additional care is needed due to the low regularity of $\omega$. In particular, the boundary integral is understood as an $H^{1/2}$-$H^{-1/2}$ duality pair, and the last equality holds because $\nabla\cdot\omega = 0$ in the sense of distributions.

From the finite moment assumption (1.3), we know that $|x|^9\omega \in L^1(\Omega)$. Therefore, there exists a sequence of radii $R_j \nearrow +\infty$ such that

$$\lim_{j\to\infty} \int_{\Omega\cap\partial B_{R_j}(0)} |y|^9 |\omega(y)|\, dy = 0.$$

Evaluating (A.6) with $R = R_j$, and recalling (1.3b), we have therefore proved that

$$\int_\Omega \omega\, dx = 0. \tag{A.7}$$

It is quite well-known that the total vorticity is 0 for three-dimensional solitary waves with compactly supported vorticity; the above argument shows that this remains the case in the more general setting of our localization assumptions (1.3).

The expansion of the second term in (A.5) can be treated similarly as in the two-dimensional case. We partition $\Omega$ into the regions $A$ and $B$ defined as before. From (A.3) we have on $A$ that

$$K_0(x,y) = -\frac{y}{|x|^3} + 3\frac{x\cdot y}{|x|^5}x + O\left(\frac{|y|^2}{|x|^4}\right), \qquad \text{as } |x|\to\infty.$$

Thus

$$\frac{1}{\gamma_3}\int_A \omega(y)\times K_0(x,y)\, dy = \frac{1}{\gamma_3}\int_\Omega \omega(y)\times\left[-\frac{y}{|x|^3} + 3\frac{x\cdot y}{|x|^5}x\right] dy + O\left(\frac{1}{|x|^4}\right).$$

On the other hand, for the integral over $B$ we estimate

$$\begin{aligned}\left|\int_B \omega(y)\times K_0(x,y)\, dy\right| &\le \int_B \frac{|\omega(y)|}{|x-y|^2}\, dy + \frac{1}{|x|^2}\int_B |\omega(y)|\, dy \\ &\lesssim \|\omega\|_{L^1(B)}^{1/3} + \frac{1}{|x|^{k+1-k\varepsilon}} \lesssim \frac{1}{|x|^{3+\varepsilon}},\end{aligned}$$

where the last two inequalities follow from Lemma A.1 and assumption (1.3). Together, these two computations give the asymptotics

$$V(x) = \frac{1}{4\pi}\int_\Omega \omega(y)\times\left[-\frac{y}{|x|^3} + 3\frac{x\cdot y}{|x|^5}x\right] dy + O\left(\frac{1}{|x|^{3+\varepsilon}}\right).$$

The final step is to show that the integral above at leading order involves the vortex impulse. For a fixed $1 \leq i, j \leq 3$, the vector field $y_i y_j \omega(y)$ is tangential to $S$ and in $L^1(\Omega)$. It follows that its divergence (in the distributional sense) must satisfy

$$\int_\Omega (y_j \omega_i + y_i \omega_j)\, dy = 0.$$

Multiplying by $x_i$ we find

$$0 = \int_\Omega \Big( y(\omega(y) \cdot x) + (y \cdot x)\omega(y) \Big)\, dy. \tag{A.8}$$

Now we rewrite

$$\omega(y)(x \cdot y) = (\omega(y) \cdot x) y - (\omega(y) \times y) \times x. \tag{A.9}$$

Integrating (A.9) and plugging in (A.8), we have

$$\begin{aligned} \int_\Omega \omega(y)(x \cdot y)\, dy &= -\int_\Omega \omega(y)(x \cdot y)\, dy - \int_\Omega (\omega(y) \times y) \times x\, dy \\ &= -\frac{1}{2} \left( \int_\Omega (\omega(y) \times y)\, dy \right) \times x. \end{aligned}$$

Recalling the definition of the vortex impulse $m$ (1.11), this leads to the formula

$$\begin{aligned} V(x) &= \frac{2m}{\gamma_3 |x|^3} + \frac{3(m \times x) \times x}{\gamma_3 |x|^5} + O\left( \frac{1}{|x|^{3+\varepsilon}} \right) \\ &= \frac{1}{\gamma_3} \left( -\frac{m}{|x|^3} + 3 \frac{(x \cdot m)}{|x|^5} x \right) + O\left( \frac{1}{|x|^{3+\varepsilon}} \right) = -\frac{1}{\gamma_3} (m \cdot \nabla) \frac{x}{|x|^3} + O\left( \frac{1}{|x|^{3+\varepsilon}} \right). \quad \square \end{aligned}$$

## Appendix B. Hölder regularity under inversion

In this appendix we provide a simple lemma that translates decay to Hölder continuity for the spherical inversion.

**Lemma B.1.** *Let $0 < \alpha < \beta < 1$ be given and set $k := 2\alpha\beta/(\beta - \alpha)$. Then, if $f \in C^\beta_{\mathrm{bdd}}(\mathbb{R}^n)$ satisfies $f(x) = O(1/|x|^k)$ as $|x| \to \infty$, the function $g(x) := f(\tilde{x})$ has a $C^\alpha$ extension to a neighborhood of $0$.*

*Proof.* Let $\alpha$, $\beta$, and $k$ be given as above, and consider the Hölder quotient

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha},$$

for $x$ and $y$ in a neighborhood of the origin. By symmetry, we can always assume that $|y| \leq |x|$.

Put $\theta := k/\alpha = 2\beta/(\beta - \alpha)$, and suppose first that $|x - y| \geq |x|^\theta$. Then

$$\frac{|g(x) - g(y)|}{|x - y|^\alpha} \lesssim \frac{|x|^k}{|x|^{\theta\alpha}} = 1.$$

On the other hand, if $|x-y| \leq |x|^\theta$, then $|y| \gtrsim |x|$ as $\theta > 1$. We may therefore estimate

$$\frac{|g(x)-g(y)|}{|x-y|^\alpha} \lesssim \frac{|\tilde{x}-\tilde{y}|^\beta}{|x-y|^\alpha} = \frac{|x-y|^{\beta-\alpha}}{|xy|^\beta} \lesssim \frac{|x-y|^{\beta-\alpha}}{|x|^{2\beta}} \lesssim \frac{|x|^{(\beta-\alpha)\theta}}{|x|^{2\beta}} = 1. \qquad \square$$

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