EXISTENCE AND QUALITATIVE THEORY FOR STRATIFIED SOLITARY WATER WAVES

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Abstract. This paper considers two-dimensional gravity solitary waves moving through a body of density stratified water lying below vacuum. The fluid domain is assumed to lie above an impenetrable flat ocean bed, while the interface between the water and vacuum is a free boundary where the pressure is constant. We prove that, for any smooth choice of upstream velocity field and density function, there exists a continuous curve of such solutions that includes large-amplitude surface waves. Furthermore, following this solution curve, one encounters waves that come arbitrarily close to possessing points of horizontal stagnation.

We also provide a number of results characterizing the qualitative features of solitary stratified waves. In part, these include bounds on the wave speed from above and below, some of which are new even for constant density flow; an a priori bound on the velocity field and lower bound on the pressure; a proof of the nonexistence of monotone bores in this physical regime; and a theorem ensuring that all supercritical solitary waves of elevation have an axis of even symmetry.

Contents

1. Introduction

Water in the depths of the ocean has a lower temperature and higher salinity than that found near the surface. The resulting density distribution is thus *heterogeneous* or *stratified*, which creates the potential for types of wavelike motion not possible in the constant density regime. Indeed, the density strata in the bulk are themselves free surfaces along which waves may propagate. This can lead to a remarkable phenomenon wherein large-amplitude waves steal through the interior of the fluid while leaving the upper surface nearly undisturbed. Field observations have revealed that these internal waves are a common feature of coastal flows, and they are believed to play a

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central role in the dynamics of ocean mixing (cf., e.g., [43]). Stratified waves can be truly immense while traveling over vast distances: they include the largest waves ever recorded, with amplitude measuring up to 500 meters in some instances [3].

In this paper, we investigate two-dimensional solitary waves moving through a heterogeneous body of water. These are a type of traveling wave: they consist of a spatially localized disturbance riding at a constant velocity along an underlying current without changing shape. Solitary water waves have a long and rich mathematical history, stretching back to their discovery by Russell in 1834 (cf. [74]). Most of the work on this subject has been devoted to the homogeneous and irrotational regime, that is, the density is assumed to be constant and the curl of the velocity is assumed to vanish identically. This choice permits the use of many powerful tools from complex analysis such as conformal mappings and nonlocal formulations on the boundary. However, stratification generically creates vorticity, and thus investigations of heterogeneous waves are most naturally made in the rotational setting. The first rigorous existence theory for traveling waves with stratification was provided by Dubreil-Jacotin [32] over a century after Russell's discovery. She specifically studied the small-amplitude and periodic regime by means of a new formulation of the problem that did not rely on conformal transformations.

Solitary waves are typically more difficult to analyze than periodic waves due to compactness issues that we will elaborate below. Stratification further complicates matters by allowing for a wealth of possible qualitative structures. In [83, 85], Ter-Krikorov proved the existence of small-amplitude solitary waves with non-constant density. The first large-amplitude existence result for stratified solitary waves was given by Amick [4] and Amick–Turner [8]. This was the culmination of a burst of activity in the 1980s devoted largely to *channel flows*, that is, stratified waves in an infinite strip bounded above and below by rigid walls (see also [17, 88, 89]). Remarkably, these works came two decades before the development of an existence theory for large-amplitude periodic water waves with vorticity by Constantin and Strauss [24]. What explains this seeming discrepancy is that Amick, Turner, and their contemporaries restricted their attention to waves whose velocity is constant upstream and downstream, and hence are asymptotically irrotational. This assumption enabled them to handle stratification without confronting the effects of vorticity in their full generality. For further discussion of the literature, see Section 1.2.

One of the main contributions of the present work is an existence theory for large-amplitude surface solitary waves with density stratification. We are able to allow an arbitrary smooth density distribution and horizontal velocity profile at infinity. Moreover, the families we construct continue up to the appearance of an "extreme wave" that has a stagnation point. This is connected to the famous Stokes conjecture, which originally pertained to periodic irrotational waves (cf., [80, 5]) but has since been extended to other regimes. For example, in the setting of irrotational solitary waves, Amick and Toland proved the existence of a continuum that limited to stagnation [7]. For stratified solitary waves which are asymptotically irrotational, Amick [4] constructed a family of solutions and proved that either it contains an extreme wave in its closure, or a certain alternative occurs that he deemed highly unusual and conjectured never happens (see [4, Theorem 7.4]). Here, we are able to state without qualification that our continuum limits to stagnation. This is the first such result for rotational solitary waves, even in the constant density case (see [96] and [97, Section 6]).

Our existence theory is built upon a host of new theorems concerning the qualitative properties of water waves with stratification. We first construct a family of small-amplitude waves via center manifold reduction methods. The full continuum is then obtained using a new global bifurcation scheme that abstracts and extends the ideas of [96, 98]. This analysis hinges critically on having a thorough understanding of the possible structures that may arise as one moves away from the small-amplitude regime, and hence the qualitative theory plays an essential role.

It is worth mentioning that a great deal of recent research has centered on the Cauchy problem for water waves in various physical regimes. At present, a number of authors have proved results concerning the global in time well-posedness for irrotational waves with small data (see, e.g., [35, 100, 36, 46, 1]), or local in time existence for rotational waves or interfacial flows (see, [26, 76, 77]). Yet the stratified waves we wish to study are both colossal and long-lived. They are also fundamentally rotational due to the baroclinic generation of vorticity. In short, by considering the steady regime, we are able to treat waves that lie far beyond the current limitations of the time-dependent theory.

Now, let us describe the setting of the problem more precisely. We are interested in two-dimensional solitary waves with heterogeneous density ϱ which travel with constant speed c under the influence of gravity. Changing to a moving reference frame enables us to eliminate time dependence from the system. The wave then occupies a steady fluid domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 : -d < y < \eta(x)\},$$

where the a priori unknown function η is the free surface profile and $\{y = -d\}$ is an impermeable flat bed. We suppose that $\varrho > 0$ in $\overline{\Omega}$, and also that the fluid is continuously stratified in the sense that ϱ is smooth. Moreover, the fluid is taken to be stably stratified in that heavier fluid elements lie below lighter elements, which translates to $y \mapsto \varrho(\cdot, y)$ being non-increasing.

A stratified water wave is described mathematically by the fluid domain Ω, density $\varrho\colon \Omega \to \mathbb{R}_+$, velocity field $(u,v)\colon \Omega \to \mathbb{R}^2$, and pressure $P\colon \Omega \to \mathbb{R}$. The governing equations are the incompressible steady Euler system, which consists of the conservation of mass

$$(u-c)\varrho_x + v\varrho_y = 0 \qquad \text{in } \Omega, \tag{1.1a}$$

conservation of momentum

$$\begin{cases} \varrho(u-c)u_x + \varrho v u_y = -P_x \\ \varrho(u-c)v_x + \varrho v v_y = -P_y - g\varrho \end{cases} \qquad \text{in } \Omega, \tag{1.1b}$$

and incompressibility

$$u_x + v_y = 0 \qquad \text{in } \Omega. \tag{1.1c}$$

Here $g > 0$ is the gravitational constant of acceleration.

The free surface is assumed to be a material line, which results in the kinematic boundary condition

$$v = (u-c)\eta_x \qquad \text{on } y = \eta(x). \tag{1.2a}$$

The pressure is required to be continuous over the interface

$$P = P_{\mathrm{atm}} \qquad \text{on } y = \eta(x), \tag{1.2b}$$

where P_{atm} is the (constant) atmospheric pressure. Finally, the ocean bed is taken to be impermeable and thus

$$v = 0 \qquad \text{on } y = -d. \tag{1.2c}$$

It will be important for our later reformulations to require that there is no horizontal stagnation in the flow:

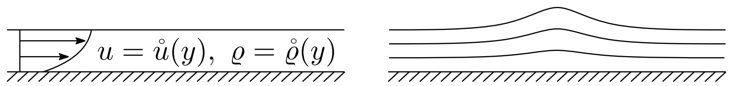
$$u - c < 0 \qquad \text{in } \overline{\Omega}. \tag{1.3}$$

One consequence of this assumption is that the streamlines, which are the integral curves of the relative velocity field $(u-c, v)$, extend from $x = -\infty$ to $x = +\infty$; see Figure 1(b). Indeed, a simple application of the implicit function theorem shows that each streamline is the graph of a function of x.

In this work we are concerned with *solitary waves*, which are traveling wave solutions satisfying the asymptotic conditions

$$(u,v) \to (\mathring{u}, 0), \quad \varrho \to \mathring{\varrho}, \quad \eta \to 0 \qquad \text{as } |x| \to \infty \tag{1.4}$$

uniformly in y. Here $\mathring{u} = \mathring{u}(y)$ is a given far-field velocity profile, and $\mathring{\varrho} = \mathring{\varrho}(y)$ is a given density function. We point out again that one of the primary contributions of our result is that $\mathring{u}$ is allowed



(a) laminar flow (b) monotone solitary wave

FIGURE 1.

to be *arbitrary*. This is in marked contrast to the existing literature which requires that the velocity is constant upstream and downstream.

Rather than using $\mathring{u}$, however, it will prove more convenient to fix a (scaled) asymptotic relative velocity $u^*\colon [-d,0] \to \mathbb{R}_+$ and consider the family

$$\mathring{u}(y) = c - Fu^*(y), \tag{1.5}$$

where $F > 0$ is a dimensionless parameter which we will call the *Froude number* (cf., (2.2) and (2.5) for the complete definition). The positivity of u^* is consistent with the lack of horizontal stagnation (1.3). In Section 2.1, we switch to dimensionless variables so that F is the only parameter appearing in the problem; we think of it as a dimensionless wave speed. It will later be proved that there exists a critical Froude number, denoted F_{cr}, that plays an important role in determining the structure of solutions. For constant density irrotational solitary waves, $F_{\mathrm{cr}} = 1$, but in the present context its definition is given in (3.4). We say that a solution with $F > F_{\mathrm{cr}}$ is *supercritical.*

Observe that the conservation of mass (1.1a) implies that the density is transported by the flow. By fixing $\mathring{\varrho}$, we therefore determine the density throughout the fluid region once the streamlines are known.

Finally, let us introduce some terminology for describing the qualitative features of these waves. A traveling wave is called *laminar* or *shear* if all of its streamlines are parallel to the bed. We say that a solitary wave is a *wave of elevation* provided that at each vertical cross section of the domain, the height of every streamline (except the one corresponding to the bed) lies above its limiting height as $|x| \to \infty$. In particular, this means that η is strictly positive. A traveling wave is said to be *symmetric* provided u and η are even in x while v is odd. We say a symmetric wave of elevation is *monotone* if the height of every streamline (except the bed) is strictly decreasing on either side of the crest line $\{x = 0\}$; see Figure 1.

1.1. **Statement of results.**

Existence theory. The contributions of this paper come in two parts. The first of these is a complete large-amplitude existence theory for stratified solitary waves with an arbitrary (smooth and stable) density and smooth upstream velocity.

Theorem 1.1 (Existence of large-amplitude solitary waves)**.** *Fix a Hölder exponent $\alpha \in (0, 1/2]$, wave speed $c > 0$, gravitational constant $g > 0$, asymptotic depth $d > 0$, density function $\mathring{\varrho} \in C^{2+\alpha}([-d, 0], \mathbb{R}_+)$, and positive asymptotic relative velocity $u^* \in C^{2+\alpha}([-d, 0], \mathbb{R}_+)$. There exists a continuous curve*

$$\mathscr{C} = \{(u(s), v(s), \eta(s), F(s)) : s \in (0, \infty)\}$$

of solitary wave solutions to (1.1)–(1.4) *with the regularity*

$$(u(s), v(s), \eta(s)) \in C^{2+\alpha}(\overline{\Omega(s)}) \times C^{2+\alpha}(\overline{\Omega(s)}) \times C^{3+\alpha}(\mathbb{R}), \tag{1.6}$$

where $\Omega(s)$ denotes the fluid domain corresponding to $\eta(s)$. The solution curve $\mathscr{C}$ has the following properties.

(a) (Extreme wave limit) *Following $\mathscr{C}$, we encounter waves that are arbitrarily close to having points of (horizontal) stagnation:*

$$\lim_{s\to\infty} \inf_{\Omega(s)} |c - u(s)| = 0. \tag{1.7}$$

(b) (Critical laminar flow) *The left endpoint of $\mathscr{C}$ is a critical laminar flow,*

$$\lim_{s\to 0}(u(s), v(s), \eta(s), F(s)) = (c - F_{\mathrm{cr}}u^*, 0, 0, F_{\mathrm{cr}}).$$

Here F_{cr} is the critical Froude number defined in (3.4)*.*

(c) (Symmetry and monotonicity) *Every solution in $\mathscr{C}$ is a wave of elevation that is symmetric, monotone, and supercritical.*

Remark 1.2. (i) When we say that $\mathscr{C}$ is continuous, we in particular mean that the mapping $(0,\infty) \ni s \mapsto \eta(s) \in C^{3+\alpha}(\mathbb{R})$ is continuous. In fact, we will show in Section 6 that, near any parameter value s_0, we can reparameterize $\mathscr{C}$ so that $s \mapsto \eta(s)$ is locally real-analytic.

The situation is more subtle for the velocity field $(u(s), v(s))$, which is defined on the s-dependent domain $\Omega(s)$. In Section 2.3, we will introduce a diffeomorphism (the Dubreil-Jacotin transform) depending on $(u(s), v(s), \eta(s))$ that sends $\Omega(s)$ to a fixed rectangular strip R. Composing the velocity field with the inverse of this diffeomorphism gives a continuous map from the interval $(0,\infty)$ to the fixed function space $C^{2+\alpha}(\overline{R}) \times C^{2+\alpha}(\overline{R})$. One consequence of this regularity is that $u(s)(x,y)$ and $v(s)(x,y)$ are jointly continuous in (x,y,s) on the domain $\{(x,y,s) \in \mathbb{R}^3 : s > 0,\ -d \le y \le \eta(s)(x)\}$.

(ii) In the special case of constant density, Theorem 1.1 recovers and improves upon the main result of [96, 97]. For some choices of u^*, those earlier papers were forced to include the possibility that $F(s) \to \infty$ while $c - u(s)$ remains uniformly bounded away from 0. Here, using the qualitative results described below, we are able to give the definitive statement that (1.7) holds for any u^*.

Qualitative theory. The second part of our results concerns the qualitative properties of solitary stratified waves. These are used at critical junctures in the argument leading to Theorem 1.1, but are of considerable interest in their own right. Several of them, in fact, improve substantially on the state-of-the-art for homogeneous flows. For the time being, we give slightly weaker statements than what is eventually proved because the optimal versions are best made using a reformulation of the problem introduced in the next section.

First, we establish a lower bound on P and an upper bound on (u,v) in terms of the given quantities F, u^*, $\mathring{\varrho}$, g, and d. To our knowledge, these are the only estimates of this type for stratified steady waves; for constant density rotational waves, analogous bounds were obtained by Varvaruca in [91].

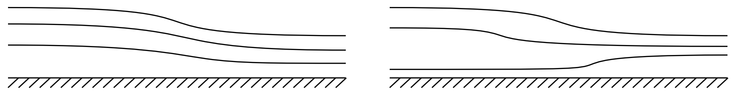
Proposition 1.3 (Bounds on velocity and pressure)**.** *The pressure and velocity fields for any solitary wave satisfy the bounds:*

$$P - P_{\mathrm{atm}} + MF\psi \ge 0 \quad \text{and} \quad (u-c)^2 + v^2 \le CF^2 \qquad \text{in } \overline{\Omega},$$

where ψ is the pseudo stream function defined uniquely by $\nabla^\perp \psi = \sqrt{\varrho}(u-c, v)$ and $\psi|_{y=\eta} = 0$, and the constants C and M depend only on u^*, $\mathring{\varrho}$, g, d, and a lower bound for F.

See Proposition 4.1, where this is proved in a non-dimensional form. For more on the pseudo stream function ψ, see Section 2.2.

Second, we provide estimates for the Froude number from above and below. Results of this type have a long history, going back at least to Starr [78], who found sharp bounds for F in the setting of homogeneous irrotational solitary waves; we refer the reader to the introduction to [97] for a detailed discussion and further references.



(a) monotone bore (b) non-monotone bore

Figure 2.

Theorem 1.4 (Upper bound on F)**.** *Let* (u,v,η,F) *be a solution of* (1.1)–(1.4). *Then the Froude number satisfies the bound*

$$F \le \frac{2}{\pi}\frac{gd}{\min(u^*)^2}\,\frac{\max\varrho}{\min\varrho}\,\frac{\sqrt{gd}}{\min_{\{x=0\}}(c-u)}.$$

Eliminating F *and* u^* *in favor of* $\mathring{u}$ *yields*

$$\frac{\min[\sqrt{\mathring{\varrho}}(c-\mathring{u})]^2}{gd\min\varrho}\;\frac{\displaystyle\min_{\{x=0\}}[\sqrt{\varrho}(c-u)]}{\displaystyle\frac{1}{d}\int_{-d}^{0}\sqrt{\mathring{\varrho}}(c-\mathring{u})\,dy} \le \frac{2}{\pi}\frac{\max\varrho}{\min\varrho}. \tag{1.8}$$

See Theorem 4.7. Observe that the first two factors on the left-hand side of (1.8) are dimensionless measures of how close the flow comes to horizontal stagnation at $\pm\infty$ and along the crest line $\{x=0\}$, respectively. This is the first such estimate for waves with vorticity (with or without density stratification) that makes no additional assumptions on the shear profile u^*; in [97], Wheeler established upper bounds for F that are independent of $\inf_{\{x=0\}}(c-u)$, but impose further requirements on u^*. Thanks to Theorem 4.7, we can avoid making similar restrictions in Theorem 1.1. Under different hypotheses on u^* than in [97], Kozlov, Kuznetsov, and Lokharu also prove estimates which imply upper bounds for F for constant density waves [55]. Their bounds involve the amplitude $\max\eta$, and apply not just to solitary waves but also to periodic waves as well as waves which are neither periodic nor solitary.

We also mention that, in the special case of homogeneous irrotational waves, the argument leading to Theorem 4.7 can be modified to prove the upper bounds obtained by Starr in [78].

As a lower bound on F, in part we prove the following.

Theorem 1.5 (Critical waves are laminar)**.** *Let* (u,v,η,F) *be a solitary wave solution of* (1.1)–(1.4). *If* $F=F_{\mathrm{cr}}$, *then* $(u,v,\eta)=(c-F_{\mathrm{cr}}u^*,0,0)$. *That is, there exist no nonlaminar solitary waves with critical Froude number.*

In addition, we provide a partial characterization of waves of elevation in terms of the Froude number; for the complete statement, see Theorem 4.4 and the remark afterward. Theorem 1.5 in particular implies that a continuous curve of supercritical waves cannot limit to a subcritical wave without passing through a laminar flow. The nonexistence of critical solitary waves with constant density was shown by Wheeler [97] and then generalized to arbitrary constant density waves by Kozlov, Kuznetsov, and Lokharu [55]. The bound $F>F_{\mathrm{cr}}$ for waves of elevation was obtained by Wheeler [97] under the assumption of constant density and Amick and Toland [7] and McLeod [68] under the further assumption of irrotationality. Very recently, Kozlov, Kuznetsov, and Lokharu have given an essentially complete description of all constant density waves with near-critical Bernoulli constant [56].

A *bore* is a traveling wave that limits to distinct laminar flows as $x\to\pm\infty$; see Figure 2. They are observed in nature (see, e.g., [79]) and have been computed numerically in various regimes (cf., e.g., [90, 59, 41, 73]). Rigorous existence results for bores in multi-fluid channel flows have been obtained by Amick [9], Makarenko [67], and Tuleuov [87]. Bores play a special role in the global bifurcation theory analysis leading to Theorem 1.1. In particular, many previous studies of stratified waves include the existence of bores as an alternative to the extreme wave limit (cf., e.g.,

[8] where they are referred to as "surges"); this should be viewed as an indication of the system's lack of compactness. However, with a free upper surface, no bores exist with the property that the asymptotic height of all streamlines upstream lie at or below their asymptotic height downstream. This is the content of the next theorem.

Theorem 1.6 (Nonexistence of monotone bores). *Suppose that (u, v, η) is a solution of* (1.1)–(1.3) *which is a bore in the sense that*

$$(u(x,\,\cdot\,), v(x,\,\cdot\,), \eta(x)) \to (\mathring{u}_\pm(\,\cdot\,), 0, \eta_\pm), \qquad \text{as } x \to \pm\infty$$

pointwise, where $\eta_\pm > -d$ are constants and $\mathring{u}_\pm \in C^1([-d, \eta_\pm])$. If the limiting height of each streamline at $x = -\infty$ is no greater (or no less) than the limiting height of the same streamline at $x = \infty$, then in fact $\eta_+ = \eta_-$ and $\mathring{u}_+ \equiv \mathring{u}_-$. In particular, for any bounded solution of (1.1)–(1.3)*, v must change signs unless it vanishes identically.*

A stronger version of Theorem 1.6 is given in Theorem 4.8; we also generalize it to include multiple fluid flows in Corollary 4.12. To the best of our knowledge, the nonexistence of monotone bores with a free upper surface has never been previously recorded, which is somewhat surprising in view of the large number of works devoted to studying bores in similar physical regimes. Even in the relatively simple case of two homogeneous irrotational layers with a free surface, the usual calculations seem highly intractable (cf., [30, Appendix A]). It was proved by Wheeler that bores (not necessarily monotone) do not exist for free surface solitary waves with constant density [98], but this argument seems to break down entirely for highly stratified waves. Moreover, even in the special case of constant density, our method is considerably more direct for monotone bores.

Finally, we prove the following theorem characterizing the monotonicity and symmetry properties of stratified waves of elevation.

Theorem 1.7 (Symmetry). *Let (u, v, η, F) be a supercritical wave of elevation that solves* (1.1)–(1.3) *with $\|u\|_{C^2(\Omega)}, \|v\|_{C^2(\Omega)}, \|\eta\|_{C^3(\mathbb{R})} < \infty$. Suppose that*

$$(u, v) \to (\mathring{u}, 0) \qquad \text{uniformly as } x \to +\infty \text{ (or as } x \to -\infty).$$

Then, after a translation, the wave is a symmetric and monotone solitary wave.

The symmetry of steady water waves has been a very active subject of research. Indeed, the reformulated problem will turn out to be an elliptic PDE, and hence Theorem 1.7 falls into the larger category of results on the symmetry and monotonicity of positive solutions to elliptic systems. More direct antecedents for water waves are given by Craig and Sternberg [27]; Maia [66]; Constantin and Escher [23]; Hur [45]; Constantin, Ehrnström and Wahlén [22]; and Walsh [93]. These papers are set in different physical regimes, but are all built around the method of moving planes (cf. [37, 63]).

Compared to this body of work, Theorem 1.7 has two distinctive features. First, we only impose asymptotic conditions upstream (or downstream), but nevertheless can conclude evenness and monotonicity. Typical moving-plane arguments begin with the far more restrictive assumption that the solutions decay in both directions. Moreover, due to the stratification, the elliptic problem that we are forced to consider has a zeroth order term with adverse sign, which significantly complicates the procedure. For the full statement and further discussion, see Theorem 4.13.

In addition to the above theorems, a number of propositions concerning more refined monotonicity properties of solitary stratified waves are given in Section 4.5. These may be of some broader interest, but are primarily important for their application in proving Theorem 1.1.

1.2. **History of the problem.** The first rigorous constructions of exact nonlinear steady water waves were made in the 1920s by Nekrasov [72] and Levi-Civita [62]. They considered the case of irrotational periodic water waves of infinite depth using conformal mappings and power series expansions. Their techniques rely heavily on both irrotationality and the smallness of the amplitude. Large-amplitude periodic irrotational waves were first constructed by Krasovskiĭ [57], and later by

Keady and Norbury [49], who used global bifurcation theory. These solutions were further studied by Toland [86] and McLeod [69], eventually leading to the proof of the Stokes conjecture ([80]) by Amick, Fraenkel, and Toland [5].

Solitary waves present a greater technical challenge to study. This is apparent even in the small-amplitude regime: the linearized operator at a critical laminar flow fails to be Fredholm for solitary waves, making the local bifurcation theory analysis much more subtle than in the periodic case. The existence theory for small-amplitude irrotational solitary waves begins in the mid 1950s and early 60s. Lavrentiev [61] and Ter-Krikorov [83] used a construction based on long-wavelength limits of periodic waves. Friedrichs and Hyers [34] introduced a more direct iteration method. Later Beale [10] provided an alternative proof using the Nash–Moser implicit function theorem. Mielke [70] subsequently used spatial dynamics methods, in particular the center manifold reduction technique. Compared to Nash–Moser iteration, and other similar approximation schemes, spatial dynamics gives a fuller qualitative description of the small-amplitude solutions. For that reason, we shall use it in this paper for the small-amplitude existence theory.

The construction of large-amplitude solitary waves is further complicated by the loss of compactness coming from the unbounded domain. Amick and Toland [6, 7] circumvent this difficulty by considering a sequence of approximate problems with better compactness properties, constructing global curves of solutions to these approximate problems, and then taking a limit using the Whyburn lemma [99]. Large-amplitude irrotational solitary waves are also constructed in [14].

Unlike the above works, our interest in this paper lies in stratified flows, which are typically rotational. This is a consequence of the fact that vorticity is generated by the component of the density gradient that is orthogonal to the pressure gradient. Let us first discuss the constant density rotational theory, which is already significantly more involved than the irrotational case. In the presence of non-constant vorticity, the complex analytic machinery underlying much of the irrotational theory is no longer applicable; instead, one is required to explore the dynamics inside the fluid domain. In 1934 Dubreil-Jacotin [31] was able to use a nonconformal coordinate transformation to construct small-amplitude periodic waves with vorticity. Much later, Constantin and Strauss obtained large-amplitude periodic solutions with a general vorticity distribution through a global-bifurcation-theoretic approach [24].

The first rigorous construction of small-amplitude rotational solitary waves is due to Ter-Krikorov [84, 85], followed by Hur [44], who generalized the method of Beale [10], and Groves and Wahlén [40] using spatial dynamics. Quite recently, Wheeler developed an existence theory for large-amplitude rotational solitary waves by starting from the small-amplitude solutions of Groves and Wahlén, and then continuing them globally [96, 97, 98]. In contrast to Amick and Toland [7, 6], this is done without recourse to approximate problems, relying instead on a new global-bifurcation-theoretic technique that permits a lack of compactness at the cost of additional alternatives for the structure of the solution set. These alternatives are then winnowed down using qualitative theory, specifically the symmetry and monotonicity properties of the waves and the nonexistence of monotone bores. We further generalize this result in the present paper, obtaining an abstract global bifurcation principle that allows us to construct large-amplitude waves. It is important to note that, compared to the homogeneous problem considered by Wheeler, the qualitative theory here presents significant additional difficulties. Despite this, we are able to get a result that, specialized to the constant density case, is in fact *stronger* than the main theorem of [96, 97].

Let us now discuss the literature on stratified steady waves. While this paper is interested in the free upper surface problem, a large part of the study of inhomogeneous waves has focused on channel flows where the fluid domain is confined between two impermeable horizontal boundaries. Two-layered systems of this type have been considered in detail in numerous studies, e.g., Amick and Turner [8] and Sun [81], while small-amplitude channel flows with continuous stratification were constructed by Ter-Krikorov [85], Turner [88], Kirchgässner [52], Kirchgässner and Lankers [54], James [47], and Sun [82]. Large-amplitude existence theory for continuously stratified channel flows was provided by Bona, Bose, and Turner [17], Amick [4], and Lankers and Friesecke [60].

The subject of the present work is the free surface problem where there is no rigid lid and the upper boundary is instead a surface of constant pressure. This has many implications for the qualitative properties of the waves, and complicates the governing equations by introducing a fully-nonlinear boundary condition (see (2.17)). The first rigorous small-amplitude existence results for the free surface periodic stratified water wave problem are due to Dubreil-Jacotin [32]; Yanowitch [101] used a different approach to obtain similar results. The existence of large-amplitude periodic stratified waves was proved much more recently by Walsh [94], adapting the ideas of Constantin and Strauss [24]. To our knowledge, there are no prior existence results for large-amplitude solitary stratified waves with a free upper surface.

We reiterate that all of the above existence theory for solitary stratified waves is restricted to the case where the flow is uniform upstream and downstream, corresponding to $\mathring{u}$ being a constant, which precludes many physically interesting situations that involve wave-current interactions in the far field. As far as we are aware, Theorem 1.1 is the first existence result of any kind for solitary stratified flows with a free upper surface and a general $\mathring{u}$.

1.3. **Plan of the article.** Let us now briefly explain the overall structure of the paper and the main challenges ahead.

The first of these is already apparent: we are considering a free boundary problem. In Section 2, therefore, we begin (after non-dimensionalizing) by performing a change of variables that fixes the domain. This is done using the Dubreil-Jacotin transformation, which recasts the Euler system as a scalar quasilinear elliptic PDE with fully nonlinear boundary conditions; we can write it abstractly as an operator equation

$$\mathscr{F}(w, F) = 0, \tag{1.9}$$

where w describes the deviation of the streamlines from their asymptotic heights, and F is the Froude number. In these new coordinates, the fluid domain Ω is mapped to a fixed infinite strip R. Recall that the presence of vorticity (and stratification) mean that one cannot simply consider a nonlocal problem posed on the boundary, as is typical for irrotational waves.

Density stratification is manifested in (1.9) as a zeroth order term whose sign violates the hypotheses of the maximum principle (cf. (2.28) and Theorem B.1). As we discuss below, maximum principle arguments are crucial to proving the existence of large-amplitude waves, and thus the "bad sign" is a serious issue. Curiously, this means that it is actually simpler to work with unstable densities rather than the more physically relevant kind we consider here (see also [54] where this observation is made).

The second major difficulty is the "singularity" of the bifurcation point, in particular that the linearized operator at the critical laminar flow $\mathscr{F}_w(0, F_{\mathrm{cr}})$ is not Fredholm (see [44]). This completely rules out using a standard Lyapunov–Schmidt reduction to construct small-amplitude waves as in the periodic case [94].

Perhaps the most serious obstacle, though, is a lack of compactness. For periodic waves, one can use Schauder estimates to show that bounded sets of solutions are compact (and even more, that $\mathscr{F}$ is "locally proper") [94, Section 4]. With our unbounded domain R, however, Schauder estimates are no longer sufficient. In particular, there is the possibility that there exists a sequence of solutions to (1.9) for which the crest becomes progressively flatter and longer, and which therefore has no convergent subsequences; see Figure 3.

Proving Theorem 1.1 therefore requires us to overcome the potential loss of the maximum principle due to the stratification, the singularity of the bifurcation point, and the loss of compactness. This process begins in Section 3, where we establish some important preparatory results concerning the properties of the linearized operator $\mathscr{F}_w(w, F)$. First, we investigate the linearized equation at a laminar flow (corresponding to $w \equiv 0$) and restricted to functions that are independent of x.

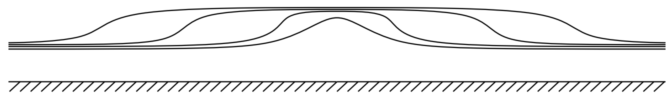


FIGURE 3. A sequence of waves with increasingly broad and flat crests.

This leads to a Sturm–Liouville problem that, at the critical Froude number F_{cr}, will have 0 as its lowest eigenvalue with the remainder of the spectrum lying on the positive real axis. Physically, F_{cr} divides the regimes of fast-moving (supercritical) waves that outrun all linear periodic waves and slow-moving (subcritical) waves that do not. We show that, in the supercritical regime, maximum principle arguments can be successfully carried out in many instances despite the adverse sign of the zeroth order term. More generally, we also prove that $\mathscr{F}_w(w, F)$ is a Fredholm operator with index 0 when $F > F_{\mathrm{cr}}$. This fact follows from the observation that the operator with coefficients evaluated at $x = \pm\infty$ is invertible (see [96, 92]).

Section 4 is devoted to the qualitative theory. A priori bounds on the pressure, velocity, and Froude number are established using maximum principle arguments and several new integral identities. Later, these will be key to winnowing the alternatives that arise in the global bifurcation argument. The nonexistence of monotone bores is proved using a conjugate flow analysis, while the symmetry follows from an adapted moving planes argument. Anticipating that these results may have interest beyond their applications to the existence theory, we have gathered them together in a single section rather than leaving them dispersed throughout the paper.

The next task, taken up in Section 5, is to construct a family of small-amplitude solitary waves bifurcating from the background laminar flow. As discussed above, the fact that $\mathscr{F}_w(0, F)$ is not Fredholm presents a serious obstruction. Various strategies have been devised to get around this; see the discussion in Section 1.2. In this paper, we choose to employ spatial dynamics and the center manifold reduction method. By treating the horizontal variable x as time-like, we are able to further reformulate problem (1.9) as an infinite-dimensional Hamiltonian system. As F increases past F_{cr}, the linearized operator has a pair of purely imaginary eigenvalues that collide at the origin and then become real. This gives rise to a two-dimensional center manifold and a corresponding reduced ODE system whose solutions can be lifted to bounded solutions of the full problem. For Froude numbers just above F_{cr}, the reduced problem is, to leading order, the ODE satisfied by the solitons of the Korteweg–de Vries equation. When F is slightly subcritical, an analogous argument gives the existence of periodic stratified waves with periods limiting to infinity as $F \nearrow F_{\mathrm{cr}}$; these are heterogeneous waves of cnoidal type. Here we are working very much in the spirit of Groves and Wahlén [39, 40], who considered homogeneous density solitary waves with vorticity, and also [96], where the construction in [40] is linearized step by step to show that the resulting solutions are nondegenerate.

Ultimately, the center manifold analysis furnishes us with a family of small-amplitude solitary waves, $\mathscr{C}_{\mathrm{loc}}$. In Section 6, we complete the proof of Theorem 1.1 by continuing $\mathscr{C}_{\mathrm{loc}}$ to a global curve $\mathscr{C}$ using an adaptation of the method of Dancer [28, 29] and its generalization by Buffoni–Toland [20]. However, as mentioned above, one cannot directly apply this theory because it fundamentally requires that closed and bounded subsets of $\mathscr{F}^{-1}(0)$ be compact, and also that $\mathscr{F}_w$ be Fredholm of index 0 at the bifurcation point. Looking at the proofs in [20], we are first able to extract an abstract global bifurcation result, Theorem 6.1, that applies to a wider class of systems for which $\mathscr{F}_w$ may not be Fredholm at the bifurcation point and where $\mathscr{F}^{-1}(0)$ may not be locally compact (at the cost, of course, of a weaker conclusion). Following the strategy of [96, 98], we show that, for general elliptic problems in cylinders, compactness can fail for a sequence of asymptotically monotone solutions only if a translated subsequence is locally converging to a monotone bore-type solution with different limits as $x \to \pm\infty$ (see Lemma 6.3). By applying our qualitative theory, we are finally able to exclude most of the possibilities in Theorem 6.1 in our case, showing that $\mathscr{C}$ can be continued up to stagnation.

Two appendices are also included. Appendix A contains a number of proofs and calculations that are either largely standard or merely technical. To keep the presentation self-contained, in Appendix B we list several results from the literature that are used throughout the paper.

2. Formulation

2.1. **Non-dimensionalization.** In this subsection, we will choose characteristic length, velocity, and density scales in terms of the data $\mathring{\varrho}, u^*, d, g$.

The natural choice for the length scale is d. For the density scale we will use the density along the free surface $\{y = \eta(x)\}$:

$$\varrho_0 := \mathring{\varrho}(0). \tag{2.1}$$

To determine the velocity scale, we first normalize the u^* appearing in (1.5) so that it satisfies

$$\int_{-d}^{0} \sqrt{\mathring{\varrho}(y)}\, u^*(y)\, dy = \sqrt{g\varrho_0 d^3}. \tag{2.2}$$

Next we consider the *(relative) pseudo-volumetric mass flux* $m > 0$ defined by

$$m := \int_{-d}^{\eta(x)} \sqrt{\varrho(x,y)}\, [c - u(x,y)]\ dy, \tag{2.3}$$

which is a constant independent of x. Sending $|x| \to \infty$ in (2.3) and then using (1.4), we find that m is given in terms of $\mathring{\varrho}$, F, and u^* by

$$m = \int_{-d}^{0} \sqrt{\mathring{\varrho}(y)}[c - \mathring{u}(y)]\, dy = F \int_{-d}^{0} \sqrt{\mathring{\varrho}(y)} u^*(y)\, dy. \tag{2.4}$$

From (2.2) we see that F, m, and ϱ_0 are related by the simple formula

$$\frac{g\varrho_0 d^3}{m^2} = \frac{1}{F^2}, \tag{2.5}$$

and consequently the velocity scale can be chosen to be $m/(d\sqrt{\varrho_0})$.

Rescaling lengths we set

$$(\tilde{x}, \tilde{y}) := \frac{1}{d}(x,y), \qquad \tilde{\eta}(\tilde{x}) := \frac{1}{d}\eta(x). \tag{2.6a}$$

Rescaling the density we define

$$\tilde{\varrho}(\tilde{x}, \tilde{y}) := \frac{1}{\varrho_0}\varrho(x,y), \qquad \tilde{\mathring{\varrho}}(\tilde{y}) := \frac{1}{\varrho_0}\mathring{\varrho}(y), \tag{2.6b}$$

and rescaling velocities we set

$$\begin{gathered} \tilde{u}(\tilde{x}, \tilde{y}) := \frac{\sqrt{\varrho_0}d}{m} u(x,y), \qquad \tilde{v}(\tilde{x}, \tilde{y}) := \frac{\sqrt{\varrho_0}d}{m} v(x,y), \\ \tilde{c} := \frac{\sqrt{\varrho_0}d}{m} c, \qquad \tilde{\mathring{u}}(\tilde{y}) := \frac{\sqrt{\varrho_0}d}{m} \mathring{u}(y). \end{gathered} \tag{2.6c}$$

Finally, combining the length, density, and velocity scalings we set

$$\tilde{P}(\tilde{x}, \tilde{y}) := \frac{d^2}{m^2}(P(x,y) - P_{\mathrm{atm}}). \tag{2.6d}$$

In these variables, (1.1) becomes

$$\begin{cases} (\tilde{u}-\tilde{c})\tilde{\varrho}_{\tilde{x}} + \tilde{v}\tilde{\varrho}_{\tilde{y}} = 0 \\ \tilde{\varrho}(\tilde{u}-\tilde{c})\tilde{u}_{\tilde{x}} + \tilde{\varrho}\tilde{v}\tilde{u}_{\tilde{y}} = -\tilde{P}_{\tilde{x}} \\ \tilde{\varrho}(\tilde{u}-\tilde{c})\tilde{v}_{\tilde{x}} + \tilde{\varrho}\tilde{v}\tilde{v}_{\tilde{y}} = -\tilde{P}_{\tilde{y}} - \dfrac{1}{F^2}\tilde{\varrho} \\ \tilde{u}_{\tilde{x}} + \tilde{v}_{\tilde{y}} = 0 \end{cases} \qquad \text{in } \tilde{\Omega}, \tag{2.7}$$

where the rescaled fluid domain is

$$\tilde{\Omega} := \{(\tilde{x},\tilde{y}) \in \mathbb{R}^2 : -1 < \tilde{y} < \tilde{\eta}(\tilde{x})\}.$$

Here the particularly simple formula for the dimensionless parameter appearing in (2.7) is thanks to (2.5). The boundary conditions (1.2) take the dimensionless form

$$\begin{cases} \tilde{v} = (\tilde{u}-\tilde{c})\tilde{\eta}_{\tilde{x}} & \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}), \\ \tilde{v} = 0 & \text{on } \tilde{y} = -1, \\ \tilde{P} = 0 & \text{on } \tilde{y} = \tilde{\eta}(\tilde{x}), \end{cases} \tag{2.8}$$

and the asymptotic condition (1.4) becomes

$$(\tilde{u},\tilde{v}) \to (\mathring{\tilde{u}}, 0), \quad \tilde{\varrho} \to \mathring{\tilde{\varrho}}, \quad \tilde{\eta} \to 0 \qquad \text{as } |\tilde{x}| \to \infty. \tag{2.9}$$

For the solutions we construct, analogous convergence in fact holds for the first derivatives of the velocity field and density, and up to the second derivative for the surface profile. Combining (2.5) with (1.5), we see that

$$\mathring{\tilde{u}}(\tilde{y}) - \tilde{c} = -\frac{1}{\sqrt{gd}} u^*(y), \tag{2.10}$$

so the asymptotic conditions (2.9) are independent of F.

To simplify notation we will from now on *drop the tildes* on the dimensionless variables defined in (2.6) and the fluid domain $\tilde{\Omega}$.

2.2. **Stream function formulation.** The dimensionless Euler system (2.7) can be turned into a scalar equation by introducing the *pseudo (relative) stream function* ψ defined up to a constant by

$$\psi_x = -\sqrt{\varrho}v, \qquad \psi_y = \sqrt{\varrho}(u-c). \tag{2.11}$$

It is apparent from these equations that the streamlines are level sets of ψ. The kinematic condition and impermeability condition in (2.8) then imply that ψ is constant on the free surface and bed; we normalize ψ by setting $\psi = 0$ on $\{y = \eta(x)\}$. Thanks to the definition of m in (2.3) and the scaling (2.6), we then have $\psi = 1$ on the bed $\{y = -1\}$. Note also that the no stagnation condition (1.3) translates to

$$\psi_y < 0 \qquad \text{in } \overline{\Omega}. \tag{2.12}$$

Conservation of mass (1.1a) implies that the density is constant on each streamline. This permits us to introduce a *streamline density function* $\rho\colon [-1,0] \to \mathbb{R}^+$ such that

$$\varrho(x,y) = \rho(-\psi(x,y)) \qquad \text{in } \Omega. \tag{2.13}$$

For solitary waves, ρ is determined by the flow upstream (or downstream) and hence, and soon as u^* is fixed, we can recover ρ from $\mathring{\varrho}$ and conversely. In the remainder of the paper, we will use ρ in place of $\mathring{\varrho}$, as it is more convenient when working in the new coordinates described below in Section 2.3. Notice that stable stratification corresponds to assuming that $\rho' \le 0$.

Bernoulli's theorem states that the quantity

$$E := P + \frac{\varrho}{2}\left((u-c)^2 + v^2\right) + \frac{1}{F^2}\varrho y \tag{2.14}$$

is constant along streamlines, as can be verified by differentiation. This and the absence of stagnation allows us to define the so-called *Bernoulli function* $\beta\colon [0,1] \to \mathbb{R}$ by

$$\frac{dE}{d\psi}(x,y) = -\beta(\psi(x,y)) \qquad \text{in } \Omega. \tag{2.15}$$

It can be shown (see, for instance [21, Lemma A.2]) that the Euler system (1.1) under the assumption of no stagnation (1.3) is equivalent to Yih's equation (cf., [102]):

$$\Delta\psi - \frac{1}{F^2} y \rho'(-\psi) + \beta(\psi) = 0 \qquad \text{in } \Omega. \tag{2.16}$$

Earlier versions of this equation were found by Dubreil-Jacotin [32] and Long [65]. The elegant form of (2.16) is the main argument for using the pseudo-stream function defined by (2.11) instead of the usual stream function where the factors of $\sqrt{\varrho}$ are absent. If ϱ is constant, then (2.16) reduces to the well-known semilinear equation $\Delta\psi = -\beta(\psi)$ for steady rotational water waves (with $\beta \equiv 0$ for irrotational waves).

Evaluating E on the free surface, and recalling the normalization (2.6c), the boundary conditions (1.2) become

$$\begin{cases} |\nabla\psi|^2 + \dfrac{2}{F^2}\varrho(y+1) = Q & \text{on } y = \eta(x), \\ \psi = 0 & \text{on } y = \eta(x), \\ \psi = 1 & \text{on } y = -1, \end{cases} \tag{2.17}$$

where the constant

$$Q := 2\left(E + \frac{1}{F^2}\varrho\right)\Big|_{y=\eta(x)} = \left(\varrho(\mathring{u} - c)^2 + \frac{2}{F^2}\varrho\right)\Big|_{y=0}. \tag{2.18}$$

Finally, the asymptotic conditions (1.4) become

$$\nabla\psi \to \left(0, \sqrt{\mathring{\varrho}}(\mathring{u} - c)\right), \quad \eta \to 0, \quad \varrho \to \mathring{\varrho}, \qquad \text{as } |x| \to \infty. \tag{2.19}$$

Remark 2.1. For solitary waves, the behavior at infinity strongly constrains the near-field structure as well. In particular, the Bernoulli function β can be explicitly computed in terms of the asymptotic data at $x = \pm\infty$. Here, to compare our results to previous works in the literature, specifically those for periodic solutions, we derive an expression for the Bernoulli function β in terms of $\mathring{u}$ and ρ.

Let $\mathring{y}(p)$ be the limiting y-coordinate of the streamline $\{\psi = -p\}$, and define

$$\mathring{U}(p) := \mathring{u}(\mathring{y}(p)). \tag{2.20}$$

From (2.19), we see that $\mathring{y}$ and $\mathring{U}$ are related by

$$\mathring{y}(p) = \int_{-1}^{p} \frac{1}{\sqrt{\rho(s)}(c - \mathring{U}(s))}\, ds - 1. \tag{2.21}$$

Sending $|x| \to \infty$ in (2.16) and applying (2.19), we see that β is given by

$$\beta(-p) = \frac{1}{F^2}\mathring{y}\rho_p - \partial_y[\sqrt{\rho}(\mathring{U} - c)] = \left(\frac{1}{F^2}\mathring{y} - \frac{1}{2}(\mathring{U} - c)^2\right)\rho_p + \rho(\mathring{U} - c)\mathring{U}_p.$$

2.3. **Height function formulation.** Although Yih's equation is scalar, it is still posed on an a priori unknown domain. This presents a serious technical challenge for the existence theory (though not necessarily the qualitative theory) and thus we will make a change of variables that fixes the domain. Specifically, we use the Dubreil-Jacotin transformation [32]

$$(x,y) \longmapsto (x, -\psi(x,y)) =: (q,p), \tag{2.22}$$

which sends the fluid domain Ω to a rectangular strip

$$R := \{(q,p) \in \mathbb{R}^2 : \ p \in (-1,0)\}.$$

The free ("top") surface $\{y = \eta(x)\}$ is mapped to $T := \{p = 0\}$ and the image of the bed is $B := \{p = -1\}$. The new coordinates (q, p) are often called *semi-Lagrangian variables*.

It is also convenient to work with the new unknown $h = h(q, p)$ which gives the height above the bed of the point $(x, y) \in \overline{\Omega}$ with $x = q$ and lying on the streamline $\{\psi = -p\}$,

$$h(q, p) := y + 1. \tag{2.23}$$

We call h the *height function*. From (2.23), it is clear that h must be positive in $R \cup T$ and vanish on the bed B. In terms of h, the no stagnation condition now reads

$$h_p > 0. \tag{2.24}$$

Likewise, the asymptotic conditions for h have the form

$$h(q, p) \to H(p) \quad h_q(q, p) \to 0, \quad h_p(q, p) \to H_p(p), \quad \text{as } |q| \to \infty. \tag{2.25}$$

A simple calculation shows that we have the change-of-variables identities

$$h_q = \frac{v}{u - c}, \quad h_p = \frac{1}{\sqrt{\varrho}(c - u)}, \tag{2.26}$$

where the left-hand side is evaluated at (q, p) while the right-hand side is evaluated at $(x, y) = (q, h - 1)$. Combining (2.21) with (2.20) and (2.10), we see that the upstream (and downstream) height function $H(p)$ is the solution to the ODE

$$\begin{cases} H_p(p) = \left. \dfrac{1}{\sqrt{\mathring{\varrho}}(c - \mathring{u})} \right|_{y = H(p) - 1} & \text{in } -1 < p < 0, \\ H(-1) = 0, \qquad H(0) = 1. \end{cases} \tag{2.27}$$

One can show that Yih's equation (2.16) and the boundary conditions (2.17) are equivalent to the following quasilinear PDE for h:

$$\begin{cases} \left(-\dfrac{1 + h_q^2}{2h_p^2} + \dfrac{1}{2H_p^2} \right)_p + \left(\dfrac{h_q}{h_p} \right)_q - \dfrac{1}{F^2} \rho_p (h - H) = 0 & \text{in } R, \\ \dfrac{1 + h_q^2}{2h_p^2} - \dfrac{1}{2H_p^2} + \dfrac{1}{F^2} \rho (h - 1) = 0 & \text{on } T, \\ h = 0 & \text{on } B, \end{cases} \tag{2.28}$$

see [21]. One can check that (2.28) is a uniformly elliptic PDE when $\inf_R h_p > 0$, with a uniformly oblique boundary condition. The sign of the zeroth order coefficient in the first equation means that (2.28) does not satisfy the hypotheses of the maximum principle (Theorem B.1).

2.4. **Flow force.** The *flow force*

$$\mathscr{S}(x) := \int_{-1}^{\eta(x)} \left[P + \varrho (u - c)^2 \right] \, dy \tag{2.29}$$

is an important quantity to consider for steady waves. It is straightforward to check that $\mathscr{S}$ is independent of x. In semi-Lagrangian variables, we can equivalently write

$$\mathscr{S} := \int_{-1}^{0} \left[\frac{1 - h_q^2}{2h_p^2} + \frac{1}{2H_p^2} - \frac{1}{F^2} \rho (h - H) - \frac{1}{F^2} \int_0^p \rho H_p \, dp' \right] h_p \, dp. \tag{2.30}$$

Since H is fixed throughout the paper, we will view $\mathscr{S} = \mathscr{S}(h)$ as a functional acting on h.

In the spatial-dynamics formulation presented in Section 5, $\mathscr{S}$ essentially serves as the Hamiltonian. Later, in Section 4.3, we will be interested in q-independent height functions $h = K(p)$ which solve (2.28) and have $\mathscr{S}(K) = \mathscr{S}(H)$. In this case we will say that K and H are *conjugate* (cf., [12, 48, 50]).

2.5. **Function spaces and the operator equation.** Finally, let us introduce the precise formulation of the problem and in particular the function spaces we will be working in. For a (possibly unbounded) domain $D \subset \mathbb{R}^n$, k a nonnegative integer and $\alpha \in [0,1)$, we denote

$$C_{\mathrm{c}}^\infty(D) := \{\phi \in C^\infty(D) : \text{ the support of } \phi \text{ is a compact subset of } D\},$$
$$C^{k+\alpha}(D) := \left\{u \in C^k(D) : \ \|\phi u\|_{C^{k+\alpha}} < \infty \ \text{ for all } \phi \in C_{\mathrm{c}}^\infty(D)\right\}.$$

The spaces $C_{\mathrm{c}}^\infty(\overline{D})$ and $C^{k+\alpha}(\overline{D})$ are defined analogously. Furthermore, let

$$C_{\mathrm{b}}^{k+\alpha}(\overline{D}) := \left\{u \in C^k(D) : \ \|u\|_{C^{k+\alpha}} < \infty\right\},$$

which is a Banach space when equipped with the obvious norm. We also consider the closed subspace

$$C_0^{k+\alpha}(\overline{D}) := \left\{u \in C_{\mathrm{b}}^{k+\alpha}(\overline{D}) : \ \lim_{r\to\infty} \sup_{|x|=r} |D^j u(x)| = 0 \text{ for } 0 \le j \le k\right\},$$

of $C_{\mathrm{b}}^{k+\alpha}(\overline{D})$, which is a Banach space under the $C_{\mathrm{b}}^{k+\alpha}(\overline{D})$ norm. Finally, we write

$$u_n \to u \text{ in } C_{\mathrm{loc}}^{k+\alpha}(\overline{D}) \iff \|\phi(u_n - u)\|_{C^{k+\alpha}(D)} \to 0 \text{ for all } \phi \in C_{\mathrm{c}}^\infty(\overline{D}).$$

Remark 2.2. Note that, if $u_n \to u$ in $C_{\mathrm{loc}}^k(\overline{D})$ and $\|u_n\|_{C^{k+\alpha}}$ is uniformly bounded, then in fact $u \in C_{\mathrm{b}}^{k+\alpha}(\overline{D})$.

Introducing the difference

$$w(q,p) := h(q,p) - H(p)$$

between the height function h and its asymptotic value H at $|q| = \infty$, the asymptotics conditions (2.25) become simply $w \in C_0^1(\overline{R})$. Note also that $\eta(q) = w(q,0)$. In terms of w, the height equation (2.28) becomes

$$\begin{cases} \left(-\dfrac{1+w_q^2}{2(H_p+w_p)^2} + \dfrac{1}{2H_p^2}\right)_p + \left(\dfrac{w_q}{H_p+w_p}\right)_q - \dfrac{1}{F^2}\rho_p w = 0 & \text{in } R, \\ \dfrac{1+w_q^2}{2(H_p+w_p)^2} - \dfrac{1}{2H_p^2} + \dfrac{1}{F^2}\rho w = 0 & \text{on } T, \\ w = 0 & \text{on } B. \end{cases} \tag{2.31a}$$

The no stagnation assumption (1.3) and the boundedness of u translate to

$$0 < \inf_R (H_p + w_p) < \infty. \tag{2.31b}$$

Define the Banach spaces X and $Y = Y_1 \times Y_2$ by

$$\begin{aligned} X &:= \left\{w \in C_{\mathrm{b,e}}^{3+\alpha}(\overline{R}) \cap C_0^2(\overline{R}) : w = 0 \text{ on } B\right\}, \\ Y_1 &:= C_{\mathrm{b,e}}^{1+\alpha}(\overline{R}) \cap C_0^0(\overline{R}), \qquad Y_2 := C_{\mathrm{b,e}}^{2+\alpha}(T) \cap C_0^1(\overline{R}), \end{aligned} \tag{2.32}$$

where the subscript "e" denotes evenness in q. From (2.26) it is clear that the evenness of w in q is equivalent to the symmetry discussed in the paragraph before Section 1.1. We write (2.31) as an operator equation

$$\mathscr{F}(w,F) = 0, \tag{2.33a}$$

where

$$\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \colon U \subset X \times \mathbb{R} \longrightarrow Y$$

is given by

$$\mathscr{F}_1(w,F) := \left(-\frac{1+w_q^2}{2(H_p+w_p)^2}+\frac{1}{2H_p^2}\right)_p + \left(\frac{w_q}{H_p+w_p}\right)_q - \frac{1}{F^2}\rho_p w, \tag{2.33b}$$

$$\mathscr{F}_2(w,F) := \left(\frac{1+w_q^2}{2(H_p+w_p)^2}-\frac{1}{2H_p^2}+\frac{1}{F^2}\rho w\right)\bigg|_T. \tag{2.33c}$$

Here we seek solutions in the open subset

$$U := \left\{(w,F) \in X \times \mathbb{R} : \inf_R (H_p+w_p) > 0,\ F > F_{\mathrm{cr}}\right\} \subset X \times \mathbb{R}, \tag{2.34}$$

where F_{cr}, the *critical Froude number*, will be defined in (3.4). Since $\mathscr{F}$ is a rational function of w and its derivatives, one easily checks that it is (real-)analytic with domain U and values in Y. All of this remains true when F_{cr} in the definition of U is replaced by 0; we will occasionally use this extended definition of $\mathscr{F}$.

3. Linearized operators

In this section we prove several lemmas about the linearized operators $\mathscr{F}_w(w,F)$. The results in Section 3.1 will be needed in Section 4, while the work done in Section 3.2 lays the foundation for the existence theory in Sections 5 and 6.

3.1. **Sturm–Liouville problems.** We begin by studying the eigenvalue problem for the linearized operator $\mathscr{F}_w(0,F)$ around a laminar flow. Restricting to q-independent functions, this becomes the Sturm–Liouville problem

$$\begin{cases} \left(\dfrac{\dot w_p}{H_p^3}\right)_p - \mu\rho_p \dot w = -\nu \dfrac{\dot w}{H_p} & \text{in } -1<p<0, \\ \dot w = 0 & \text{on } p=-1, \\ -\dfrac{\dot w_p}{H_p^3} + \mu\rho\dot w = 0 & \text{on } p=0, \end{cases} \tag{3.1}$$

where we have introduced the shorthand $\mu := 1/F^2$ and ν is the eigenvalue.

First, we consider the situation for $\nu = 0$, characterizing the smallest positive value of μ for which the system

$$\begin{cases} \left(\dfrac{\dot w_p}{H_p^3}\right)_p - \mu\rho_p \dot w = 0 & \text{in } -1<p<0, \\ \dot w = 0 & \text{on } p=-1, \\ -\dfrac{\dot w_p}{H_p^3} + \mu\rho\dot w = 0 & \text{on } p=0, \end{cases} \tag{3.2}$$

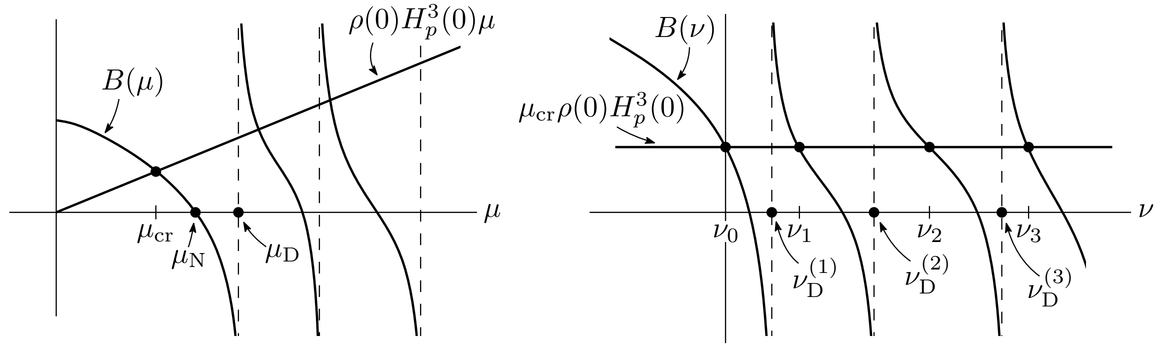
has a nontrivial solution $\dot w$. One can approach this task using variational methods, but for later convenience we instead give an argument in terms of the (unique) solution $\Phi = \Phi(p;\mu)$ of the initial value problem

$$\left(\frac{\Phi_p}{H_p^3}\right)_p - \mu\rho_p\Phi = 0 \quad \text{in } -1<p<0, \qquad \Phi(-1)=0, \quad \Phi_p(-1)=1. \tag{3.3}$$

together with the related function

$$A(\mu) := -\frac{\Phi_p(0;\mu)}{H_p^3(0)} + \mu\rho(0)\Phi(0;\mu),$$

which is defined so that Φ solves (3.2) for a given μ if and only if $A(\mu)=0$. Note that in the constant density case where $\rho_p \equiv 0$, Φ is independent of μ and hence A is simply an affine function.



(a) (b)

FIGURE 4. (a) The function $B = B(\mu)$ from the proof of Lemma 3.1. (b) The function $B = B(\nu)$ from the proof of Lemma 3.3.

Lemma 3.1 (Existence of critical Froude number)**.** *There exists a unique $\mu_{\mathrm{cr}} > 0$ such that the following holds.*

(a) *The Sturm–Liouville problem* (3.2) *has a nontrivial solution $\dot{w}$ for $\mu = \mu_{\mathrm{cr}}$. Moreover, we can take $\dot{w} = \Phi_{\mathrm{cr}}(p) := \Phi(p; \mu_{\mathrm{cr}})$.*
(b) *If $0 \le \mu \le \mu_{\mathrm{cr}}$, then $\Phi(p;\mu) > 0$ for $-1 < p \le 0$ and $\Phi_p(p;\mu) > 0$ for $-1 \le p \le 0$.*
(c) *For $0 \le \mu < \mu_{\mathrm{cr}}$, $A(\mu) < 0$.*

Recalling the definition of μ, this gives us at last the definition of the critical Froude number

$$F_{\mathrm{cr}} := \sqrt{\frac{1}{\mu_{\mathrm{cr}}}}. \tag{3.4}$$

A simple calculation confirms that in the case of constant density irrotational flow, $F_{\mathrm{cr}} = 1$, as in the classical theory.

Proof of Lemma 3.1. Observe that (3.2) has a nontrivial solution $\dot{w} \not\equiv 0$ if and only if $B(\mu) = \mu\rho(0)H_p^3(0)$, where

$$B(\mu) := \frac{\Phi_p(0;\mu)}{\Phi(0;\mu)}, \tag{3.5}$$

and that in this case we can take $\dot{w} = \Phi(\,\cdot\,;\mu)$. (Note that, by uniqueness for the initial value problem, the numerator and denominator in (3.5) cannot vanish simultaneously.) For $\mu = 0$ we compute

$$\Phi(p;0) = \frac{1}{H_p(-1)^3}\int_{-1}^{p} H_p(s)^3\,ds \tag{3.6}$$

and hence

$$B(0) = \frac{H_p(0)^3}{\int_{-1}^0 H_p^3\,dp} > 0.$$

Clearly, Φ depends smoothly on μ. Differentiating (3.3) we see that its derivative Φ_μ solves the inhomogeneous initial value problem

$$\left(\frac{\Phi_{\mu p}}{H_p^3}\right)_p - \mu\rho_p\Phi_\mu = \rho_p\Phi \quad \text{in } -1 < p < 0, \qquad \Phi_\mu(-1) = \Phi_{p\mu}(-1) = 0. \tag{3.7}$$

Together (3.7) and (3.3) lead to the Green's identity

$$\left(\frac{\Phi_{\mu p}\Phi}{H_p^3} - \frac{\Phi_p\Phi_\mu}{H_p^3}\right)\bigg|_{p=0} = \int_{-1}^0 \rho_p \Phi^2\, dp. \tag{3.8}$$

Differentiating (3.5) and using (3.8), we obtain

$$B_\mu = \left(\frac{\Phi_{\mu p}\Phi - \Phi_p\Phi_\mu}{\Phi^2}\right)\bigg|_{p=0} = \frac{H_p(0)^3}{\Phi(0)^2}\int_{-1}^0 \rho_p\Phi^2\, dp < 0,$$

provided, of course, that $\Phi(0) = \Phi(0;\mu) \neq 0$. From this and the positivity of $B(0)$ it follows that there is a unique smallest $\mu = \mu_{\mathrm{cr}} > 0$ such that $B(\mu) = \mu\rho(0)H_p^3(0)$ (see Figure 4(a)), and hence that (3.2) has a nontrivial solution $\dot{w} = \Phi_{\mathrm{cr}}$. The inequality $A(\mu) < 0$ is a consequence of the inequality $B(\mu) > \mu\rho(0)H_p^3(0)$ for $0 \le \mu < \mu_{\mathrm{cr}}$. We observe that $B(\mu)$, $\Phi(0;\mu)$, and $\Phi_p(0;\mu)$ are all strictly positive for $0 \le \mu \le \mu_{\mathrm{cr}}$.

Next we will show that

$$\Phi(p;\mu) > 0 \text{ for } -1 < p \le 0 \text{ and } 0 \le \mu \le \mu_{\mathrm{cr}}. \tag{3.9}$$

Toward that purpose, define the set

$$\mathcal{N} := \{\mu \in [0,\mu_{\mathrm{cr}}] : \Phi(p;\mu) > 0 \text{ for } -1 < p < 0\}.$$

From (3.6) it is clear that $0 \in \mathcal{N}$. The continuous dependence of Φ on μ, together with the positivity $\Phi_p > 0$ for $p = -1, 0$ and $0 \le \mu \le \mu_{\mathrm{cr}}$, show that $\mathcal{N}$ is a relatively open subset of $[0,\mu_{\mathrm{cr}}]$. Seeking a contradiction, suppose that it is not closed. Then there exists a limit point $\bar{\mu} \in [0,\mu_{\mathrm{cr}}] \setminus \mathcal{N}$. By definition, this means that there must be a point $\bar{p} \in (-1,0)$ where $\Phi(\bar{p},\bar{\mu}) = 0$. On the other hand, as $\bar{\mu}$ is a limit point, we have $\Phi(p;\bar{\mu}) \ge 0$ for $-1 \le p \le 0$. This further implies that Φ attains a local minimum at $\bar{p}$ and hence $\Phi_p(\bar{p};\bar{\mu}) = 0$. As before, $\Phi(\bar{p};\bar{\mu}) = \Phi_p(\bar{p};\bar{\mu}) = 0$ forces $\Phi(\,\cdot\,;\bar{\mu}) \equiv 0$ by uniqueness for the initial value problem, a contradiction. Thus we have proved that $\mathcal{N}$ is both relatively open and closed as a subset of $[0,\mu_{\mathrm{cr}}]$. It follows that $\mathcal{N} = [0,\mu_{\mathrm{cr}}]$, which, unraveling notation, gives the bound (3.9).

Fix $\mu \in [0,\mu_{\mathrm{cr}}]$. It remains to prove the second inequality in (b), that is, $\Phi_p(p;\mu) > 0$ for $-1 \le p \le 0$. Consider the function

$$f(p) := \frac{\Phi_p(p;\mu)}{H_p^3(p)}.$$

By construction $f(-1) > 0$, and we have shown that $f(0) > 0$ as well. By (3.3) and (3.9), $f_p(p) = \mu\rho_p\Phi(p;\mu) \le 0$ so that f is monotonically decreasing. Thus $f > 0$ and hence $\Phi_p(p;\mu) > 0$ for $-1 \le p \le 0$, as desired. ☐

An easy extension of the proof of Lemma 3.1 gives the following corollary; see Figure 4(a).

Corollary 3.2. *There exist $\mu_{\mathrm{D}}, \mu_{\mathrm{N}}$ with $\mu_{\mathrm{cr}} < \mu_{\mathrm{N}} \le \mu_{\mathrm{D}} \le +\infty$ such that the following hold.*

(a) *If $0 \le \mu < \mu_{\mathrm{N}}$, then $\Phi(p;\mu) > 0$ for $-1 < p \le 0$ while $\Phi_p(p;\mu) > 0$ for $-1 \le p \le 0$. Moreover, $\Phi_p(0;\mu_{\mathrm{N}}) = 0$.*

(b) *$\Phi(0;\mu) > 0$ for $0 \le \mu < \mu_{\mathrm{D}}$, while $\Phi(0;\mu_{\mathrm{D}}) = 0$.*

(c) *For $\mu_{\mathrm{cr}} < \mu \le \mu_{\mathrm{N}}$, $A(\mu) > 0$.*

We define the associated Froude numbers $F_{\mathrm{D}} := 1/\sqrt{\mu_{\mathrm{D}}}$, $F_{\mathrm{N}} := 1/\sqrt{\mu_{\mathrm{N}}}$. Their significance for the qualitative theory is discussed in Corollary 4.5. The first, F_{D}, is the critical Froude number for channel flow.

Now we return to the full eigenvalue problem (3.1), but with $\mu = \mu_{\mathrm{cr}}$ fixed.

Lemma 3.3 (Spectrum). *Let Σ denote the set of eigenvalues ν for the Sturm–Liouville problem (3.1) with $\mu = \mu_{\mathrm{cr}}$.*

(a) $\Sigma = \{\nu_j\}_{j=0}^{\infty}$, *where* $\nu_j \to \infty$ *as* $j \to \infty$, *and* $\nu_j < \nu_{j+1}$ *for all* $j \geq 0$;
(b) $\nu_0 = 0$; *and*
(c) *each* $\nu \in \Sigma$ *has geometric and algebraic multiplicity* 1.

Proof. As with (3.2), we will analyze (3.1) by introducing the solution $M = M(p;\nu)$ to the initial value problem

$$\left(\frac{M_p}{H_p^3}\right)_p - \mu_{\mathrm{cr}}\rho_p M = -\nu\frac{M}{H_p} \quad \text{in } -1 < p < 0, \qquad M(-1) = 0, \quad M_p(-1) = 1, \tag{3.10}$$

and the associated function

$$B(\nu) := \frac{M_p(0;\nu)}{M(0;\nu)}.$$

Note that $w = M(\,\cdot\,;\nu)$ solves (3.1) for ν provided $B(\nu) = \mu_{\mathrm{cr}}\rho(0)H_p(0)^3$. Also, B will have a pole at each ν_{D} that is an eigenvalue of the Dirichlet problem corresponding to (3.1):

$$-\left(\frac{\dot{w}_p}{H_p^3}\right)_p + \mu_{\mathrm{cr}}\rho_p\dot{w} = \nu_{\mathrm{D}}\frac{\dot{w}}{H_p}, \qquad \dot{w}(-1) = \dot{w}(0) = 0, \qquad \dot{w} \not\equiv 0. \tag{3.11}$$

By classical theory, the set of Dirichlet eigenvalues Σ_{D} is countably infinite, contains a sequence limiting to $+\infty$, and has no finite accumulation points. Moreover, each $\nu_{\mathrm{D}} \in \Sigma_{\mathrm{D}}$ is simple. We can therefore enumerate $\Sigma_{\mathrm{D}} = \{\nu_{\mathrm{D}}^{(j)}\}_{j=1}^{\infty}$ where $\nu_{\mathrm{D}}^{(j)} < \nu_{\mathrm{D}}^{(j+1)}$ and $\nu_{\mathrm{D}}^{(j)} \to \infty$ as $j \to \infty$. Due to the sign of ρ_p, a priori it is possible that several of the Dirichlet eigenvalues are negative. However, the definition of μ_{cr} ensures that this does not occur, as the following simple argument demonstrates.

Suppose that $\nu_{\mathrm{D}} \leq 0$ and let $\dot{w}$ be a corresponding solution of (3.11). For $\delta > 0$, consider a new function

$$\dot{v}^\delta := \frac{\dot{w}}{\delta + \Phi_{\mathrm{cr}}}, \tag{3.12}$$

which will then satisfy

$$-\left(\frac{\delta + \Phi_{\mathrm{cr}}}{H_p^3}\dot{v}_p^\delta\right)_p - \frac{(\Phi_{\mathrm{cr}})_p}{H_p^3}\dot{v}_p^\delta + \left(\mu_{\mathrm{cr}}\rho_p\delta - \nu_{\mathrm{D}}\frac{\delta + \Phi_{\mathrm{cr}}}{H_p}\right)\dot{v}^\delta = 0, \tag{3.13}$$

where we have used the equation (3.3) solved by Φ_{cr} to simplify some terms.

For $\nu_{\mathrm{D}} < 0$, taking δ small enough ensures that the coefficient of the zeroth order term is positive, while for $\nu_{\mathrm{D}} = 0$, this coefficient vanishes as $\delta \searrow 0$. We may therefore apply the maximum principle to the above equation if δ is sufficiently small. But then, because $\dot{v}^\delta = 0$ on the top and bottom, it follows that $\dot{v}^\delta \equiv 0$ and hence $\dot{w} \equiv 0$. Thus $\Sigma_{\mathrm{D}} \subset (0,\infty)$ as desired.

Next, we differentiate (3.10) with respect to ν to find

$$-\left(\frac{M_{\nu p}}{H_p^3}\right)_p + \mu_{\mathrm{cr}}\rho_p M_\nu = \frac{M}{H_p} + \nu\frac{M_\nu}{H_p}, \qquad M_\nu(-1;\nu) = M_{\nu p}(-1;\nu) = 0. \tag{3.14}$$

Using this and the original equation (3.10), we obtain the Green's identity

$$\left(\frac{M_p M_\nu}{H_p^3} - \frac{M M_{\nu p}}{H_p^3}\right)\bigg|_{p=-1}^{0} = \int_{-1}^{0} \frac{M^2}{H^p}\,dp.$$

From this it follows that

$$B'(\nu) = \frac{M M_{\nu p} - M_p M_\nu}{M^2}\bigg|_{p=0} = -\frac{H_p^3(0)}{M^2(0)}\int_{-1}^{0}\frac{M^2}{H_p}\,dp < 0 \tag{3.15}$$

whenever $M(0;\mu) \neq 0$. Thus B is a strictly decreasing function of ν.

The presence of the poles at the Dirichlet eigenvalues then implies $B(\nu) \to \pm\infty$ as $\nu \to \nu_{\mathrm{D}}^{(j)}\pm$ for all j. Thus, for each $j \geq 1$, there is a unique $\nu_j \in (\nu_{\mathrm{D}}^{(j)}, \nu_{\mathrm{D}}^{(j+1)})$ with $B(\nu_j) = \mu_{\mathrm{cr}}\rho(0)H_p(0)^3$;

see Figure 4(b). Moreover, since B is strictly decreasing and smooth on $(-\infty, \nu_{\mathrm{D}}^{(1)})$, there exists at most one $\nu_0 \in (-\infty, \nu_{\mathrm{D}}^{(1)})$ for which the same holds true. In fact, taking $\nu = 0$, (3.1) becomes (3.2), and hence has the nontrivial solution Φ_{cr} by Lemma 3.1. We infer that ν_0 exists and is simply 0. Recalling the definition of B, we see that $\{\nu_j\}_{j=0}^{\infty}$ are precisely the eigenvalues of (3.1). This proves parts (a) and (b).

Finally, the simplicity of these eigenvalues is derived from the fact that the Sturm–Liouville problem (3.1) is formally self-adjoint. □

Remark 3.4. The critical Froude numbers $F_{\mathrm{D}}, F_{\mathrm{N}}, F_{\mathrm{cr}}$ can also be defined as follows. Set

$$L\dot{w} := -\left(\frac{\dot{w}_p}{H_p^3}\right)_p + \frac{1}{F^2}\rho_p \dot{w}$$

and let $L_{\mathrm{D}}, L_{\mathrm{N}}, L_{\mathrm{R}}$ denote L with domain

$$\begin{aligned}
\mathcal{D}(L_{\mathrm{D}}) &:= \left\{\dot{w} \in C^2([-1,0]) : \dot{w}(-1) = \dot{w}(0) = 0\right\},\\
\mathcal{D}(L_{\mathrm{N}}) &:= \left\{\dot{w} \in C^2([-1,0]) : \dot{w}(-1) = \dot{w}_p(0) = 0\right\},\\
\mathcal{D}(L_{\mathrm{R}}) &:= \left\{\dot{w} \in C^2([-1,0]) : \dot{w}(-1) = -\frac{\dot{w}_p(0)}{H_p^3(0)} + \frac{1}{F^2}\rho(0)\dot{w}(0) = 0\right\}.
\end{aligned}$$

It is easy to see that the eigenvalues of $L_{\mathrm{D}}, L_{\mathrm{N}}, L_{\mathrm{R}}$ are strictly positive when F is sufficiently large. We can therefore define

$$F_{\mathrm{D}} := \inf\{F > 0 : \text{eigenvalues of } L_{\mathrm{D}} \text{ all positive}\},$$

and likewise for F_{N} and F_{cr}. Note that if ρ is a constant, then $F_{\mathrm{D}} = F_{\mathrm{N}} = 0$.

As a simple concrete example, consider the situation when H_p and ρ_p are both constants, in which case our normalization (2.2) together with (2.27) forces $H_p \equiv 1$. Using this, we easily calculate

$$F_{\mathrm{N}} = \frac{2}{\pi}\sqrt{|\rho_p|}, \qquad F_{\mathrm{D}} = \frac{1}{2}F_{\mathrm{N}} = \frac{1}{\pi}\sqrt{|\rho_p|},$$

while $F_{\mathrm{cr}} = 1/\sqrt{\mu_{\mathrm{cr}}}$ where $\mu_{\mathrm{cr}} > 0$ is the smallest positive solution of

$$\frac{\sqrt{|\rho_p|}}{\sqrt{\mu_{\mathrm{cr}}}} = \tan\left(\sqrt{\mu_{\mathrm{cr}}|\rho_p|}\right).$$

In the limit as $\rho_p \to 0$, these formulas simplify to $F_{\mathrm{D}} = F_{\mathrm{N}} = 0$ and $F_{\mathrm{cr}} = 1$. Likewise, the eigenvalues $\nu \geq 0$ of the related Sturm–Liouville problem (3.1) are solutions of the algebraic equation

$$\sqrt{\nu + \mu_{\mathrm{cr}}|\rho_p|} = \mu_{\mathrm{cr}} \tan\left(\sqrt{\nu + \mu_{\mathrm{cr}}|\rho_p|}\right).$$

3.2. **Fredholm and invertibility properties.** We now move on to the linearized operators $\mathscr{F}_w(w, F)$ for $(w, F) \in U$. Not surprisingly, the operators $\mathscr{F}_w(0, F)$ obtained by linearizing about the trivial solution $w = 0$ play a special role; here it is crucial that $F > F_{\mathrm{cr}}$ so that we can use Lemma 3.1.

Consider the problem $\mathscr{F}_w(0, F)\dot{w} = (f_1, f_2)$, i.e.,

$$\begin{cases}
\left(\dfrac{\dot{w}_p}{H_p^3}\right)_p + \left(\dfrac{\dot{w}_q}{H_p}\right)_q - \dfrac{1}{F^2}\rho_p \dot{w} = f_1 & \text{in } R,\\
-\dfrac{\dot{w}_p}{H_p^3} + \dfrac{1}{F^2}\rho\dot{w} = f_2 & \text{on } T,\\
\dot{w} = 0 & \text{on } B.
\end{cases} \tag{3.16}$$

We will view $\mathscr{F}_w(0, F)$ as a map $X \to Y$ but also as a map $X_{\mathrm{b}} \to Y_{\mathrm{b}}$, where the spaces

$$X_{\mathrm{b}} := \left\{u \in C_{\mathrm{b}}^{3+\alpha}(\overline{R}) : \ u|_B = 0\right\}, \quad Y_{\mathrm{b}} := C_{\mathrm{b}}^{1+\alpha}(\overline{R}) \times C_{\mathrm{b}}^{2+\alpha}(T).$$

are like X and Y but without evenness or decay at infinity.

Both of the coefficients in front of $\dot{w}$ in (3.16) have the "bad" sign in the sense that they do not satisfy the hypotheses of the maximum principle (cf. Theorem B.1). We can get around this, however, by using a slight variation of the function Φ from Lemma 3.1, namely the function $\tilde{\Phi}$ defined by

$$\left(\frac{\tilde{\Phi}_p}{H_p^3}\right)_p - \frac{1}{F^2}\rho_p\tilde{\Phi} = 0 \quad \text{in } -1 < p < 0, \qquad \tilde{\Phi}(-1) = \epsilon, \quad \tilde{\Phi}_p(-1) = 1, \tag{3.17}$$

where $0 < \epsilon \ll 1$ is a constant (depending on F) to be determined.

Lemma 3.5. *If $F > F_{\mathrm{cr}}$, then, for $\epsilon > 0$ sufficiently small,*

$$\tilde{\Phi} > 0 \text{ for } -1 < p \leq 0, \qquad \tilde{\Phi}_p > 0 \text{ for } -1 \leq p \leq 0, \tag{3.18}$$

$$-\frac{\tilde{\Phi}_p}{H_p^3} + \frac{1}{F^2}\rho\tilde{\Phi} < 0 \text{ on } p = 0. \tag{3.19}$$

Proof. Comparing (3.17) to (3.3), we see that $\tilde{\Phi} = \Phi$ when $\epsilon = 0$. Thus, from Lemma 3.1 we know that (3.19) and the second inequality in (3.18) hold for ϵ sufficiently small. Indeed, since $\Phi_p > 0$ for $-1 \leq p \leq 0$, we in fact have $\Phi_p \geq \delta > 0$ for some constant δ, and therefore $\tilde{\Phi}_p \geq \delta/2$, say, for sufficiently small ϵ. Because $\tilde{\Phi}(-1) = 0$, the first inequality in (3.18) follows by integrating the second. □

Fix $F > F_{\mathrm{cr}}$ and let $\tilde{\Phi}$ be the function whose existence is guaranteed by Lemma 3.5. Making the change of dependent variable $\dot{w} =: \tilde{\Phi}v$, a calculation shows that $\mathscr{F}_w(0, F)\dot{w} = (f_1, f_2)$ is equivalent to

$$\begin{cases} \left(\dfrac{v_p}{H_p^3}\right)_p + \left(\dfrac{v_q}{H_p}\right)_q + \dfrac{2\tilde{\Phi}_p v_p}{\tilde{\Phi}H_p^3} = \dfrac{f_1}{\tilde{\Phi}} & \text{in } R, \\ -\dfrac{v_p}{H_p^3} + \dfrac{1}{\tilde{\Phi}}\left(-\dfrac{\tilde{\Phi}_p}{H_p^3} + \dfrac{1}{F^2}\rho\tilde{\Phi}\right)v = \dfrac{f_2}{\tilde{\Phi}} & \text{on } T, \\ v = 0 & \text{on } B. \end{cases} \tag{3.20}$$

The elliptic operator in (3.20) has no zeroth order term, and from (3.19) the coefficient in front of v in the boundary condition on T has the "good" sign.

Lemma 3.6 (Fredholm index 0)**.** *For $(w, F) \in U$, $\mathscr{F}_w(w, F)$ is Fredholm with index 0 and $\mathscr{F}_w(0, F)$ is invertible.*

Proof. First consider $\mathscr{F}_w(0, F)$ for $F > F_{\mathrm{cr}}$. Because of the "good" signs in (3.20), [96, Lemma A.5] and [98, Corollary A.11] imply that $\mathscr{F}_w(0, F)$ is invertible $X_{\mathrm{b}} \to Y_{\mathrm{b}}$ and $X \to Y$. For a general $(w, F) \in U$, $\mathscr{F}_w(w, F)$ is then Fredholm with index 0 by [98, Lemma A.12], which is proved using a translation argument together with the continuity of the index; see [92]. □

4. Qualitative properties

4.1. **Bounds on the velocity and pressure.** Very little is known about the distribution of pressure in traveling waves with vorticity, even less so for those with density stratification. Our first result gives an a priori lower bound on the pressure in a stratified steady wave. This generalizes the best currently available lower bound for constant density rotational waves, due to Varvaruca [91]. Bernoulli's law then furnishes an upper estimate on the magnitude of the relative velocity.

Proposition 4.1 (Bounds on velocity and pressure)**.** *The pressure and velocity fields for any solitary wave satisfy the bounds*

$$P + M\psi \geq 0 \qquad \text{and} \qquad (u - c)^2 + v^2 \leq C \qquad \text{in } \overline{\Omega}, \tag{4.1}$$

where the constants C and M depend only on u^*, $\mathring{\varrho}$, g, d, and a lower bound for F.

Converting to dimensional variables we obtain Proposition 1.3.

Proof of Proposition 4.1. Recall from Remark 2.1 that β and hence E are completely determined by u^*, ρ, g, and d.

Let $f := P + M\psi$, where M is a constant to be determined, and assume that $F > F_0$ for some fixed lower bound F_0. A tedious calculation using Yih's equation (2.16) shows that f satisfies the elliptic equation

$$\Delta f - b_1 f_x - b_2 f_y = \frac{2F^{-2}\rho(2M+\Delta\psi)\psi_y - 2F^{-4}\rho^2}{|\nabla\psi|^2} - \left[(2M+\Delta\psi)M - \frac{1}{F^2}\rho_p\psi_y\right], \tag{4.2}$$

where the coefficients b_1 and b_2 are given by

$$b_1 := 2\frac{\psi_x(2M+\Delta\psi)}{|\nabla\psi|^2}, \qquad b_2 := 2\frac{\psi_y\,(2M+\Delta\psi) - 2F^{-2}\rho}{|\nabla\psi|^2}. \tag{4.3}$$

From Bernoulli's law (2.17) and (2.18), we see that

$$\frac{1}{F^2}\eta < \frac{(\mathring{u}(0)-c)^2}{2} = \frac{u^*(0)^2}{2gd}. \tag{4.4}$$

Yih's equation (2.16) then gives the estimate

$$\Delta\psi = -\beta + \frac{1}{F^2}y\rho_p \geq -\|\beta_+\|_{L^\infty} - \frac{u^*(0)^2}{2gd}\|\rho_p\|_{L^\infty}. \tag{4.5}$$

Thus, for

$$M > M_1 := \|\beta_+\|_{L^\infty} + \frac{u^*(0)^2}{2gd}\|\rho_p\|_{L^\infty}, \tag{4.6}$$

we have $\Delta\psi + M \geq 0$. Using this fact in (4.2) and estimating quite crudely yields the inequality

$$\Delta f - b_1 f_x - b_2 f_y \leq -\left[M^2 - \frac{1}{F^2}\rho_p\psi_y\right]. \tag{4.7}$$

Putting

$$M_2 := \frac{1}{F_0}\|\psi_y\|_{L^\infty}^{1/2}\|\rho_p\|_{L^\infty}^{1/2}, \tag{4.8}$$

and taking $M := \max\{M_1, M_2\}$ therefore guarantees that

$$\Delta f - b_1 f_x - b_2 f_y \leq 0 \qquad \text{in } \Omega. \tag{4.9}$$

We claim that, with this choice of M, $f \geq 0$. At $x = \pm\infty$, the pressure is hydrostatic and hence positive, so $\liminf_{|x|\to\infty} f(x,\cdot) \geq 0$. On the surface, $f = 0$, and on the bed $\{y = -1\}$,

$$f_y = P_y + M\psi_y = -\frac{1}{F^2}\rho + M\psi_y < 0 \qquad \text{on } y = -1,$$

so that f cannot be minimized there. As f is a supersolution of the elliptic problem (4.9), the claim follows from the maximum principle.

Using Bernoulli's law again, the inequality $f \geq 0$ means

$$-\frac{1}{2}|\nabla\psi|^2 - \frac{1}{F^2}\rho y + E = P \geq -M\psi \geq -M,$$

and hence, after rearranging,

$$|\nabla\psi|^2 \leq 2M - \frac{2}{F^2}\rho y + 2E \leq 2M + \frac{2}{F_0^2} + 2E, \tag{4.10}$$

where we have used the inequalities $y > -1$ and $0 < \rho \leq 1$ satisfied by the dimensionless variables ρ and y.

First suppose that $M_2 \geq M_1$ so that $M = M_2$. Then taking the supremum of the left-hand side of (4.10) and dropping the ψ_x term, we find

$$\|\psi_y\|_{L^\infty}^2 \leq \left(\frac{1}{2}\|\psi_y\|_{L^\infty}^2 + \frac{3}{2F_0^{4/3}}\|\rho_p\|_{L^\infty}^{2/3}\right) + \frac{2}{F_0^2} + 2\|E\|_{L^\infty}$$

and hence

$$\|\psi_y\|_{L^\infty}^2 < \frac{3}{F_0^{4/3}}\|\rho_p\|_{L^\infty}^{2/3} + \frac{4}{F_0^2} + 4\|E\|_{L^\infty}. \tag{4.11}$$

In particular, M_2 is controlled by

$$M_2 \leq \frac{1}{F_0}\|\rho_p\|_{L^\infty}^{1/2}\left(\frac{3}{F_0^{4/3}}\|\rho_p\|_{L^\infty}^{2/3} + \frac{4}{F_0^2} + 4\|E\|_{L^\infty}\right)^{1/4}.$$

Plugging $M = M_2$ back into (4.10) then yields a bound on the full gradient:

$$\|\nabla\psi\|_{L^\infty}^2 \leq \frac{2}{F_0}\|\rho_p\|_{L^\infty}^{1/2}\left(\frac{3}{F_0^{4/3}}\|\rho_p\|_{L^\infty}^{2/3} + \frac{4}{F_0^2} + 4\|E\|_{L^\infty}\right)^{1/4} + \frac{2}{F_0^2} + \|E\|_{L^\infty}, \tag{4.12}$$

On the other hand, if $M_1 \geq M_2$ so that $M = M_1$, (4.10) gives immediately that

$$\|\nabla\psi\|_{L^\infty}^2 \leq 2M_1 + \frac{2}{F_0^2} + 2\|E\|_{L^\infty}. \qquad \square$$

The estimates (4.1) can be translated into bounds on the semi-Lagrangian quantities.

Corollary 4.2 (Bounds on w and ∇w). *There exist positive constants C_* and δ_* so that any solitary wave with $F \geq F_{\mathrm{cr}}$ satisfies*

$$\inf_R (w_p + H_p) > \delta_* \qquad \text{and} \qquad \|w\|_{C^1(R)} < C_*(1 + \|w_p\|_{C^0(R)}). \tag{4.13}$$

Proof. Letting $F_0 = F_{\mathrm{cr}}$ in Proposition 4.1 and using (2.26), we have

$$\frac{1}{h_p^2} + \frac{h_q^2}{h_p^2} = \rho\left[(u-c)^2 + v^2\right] < C,$$

for some constant C. Dropping the second term on the left-hand side yields

$$\inf_R (H_p + w_p) = \inf_R h_p \geq \frac{1}{\sqrt{C}} =: \delta_*,$$

while dropping the first yields

$$|w_q| = |h_q| < \sqrt{C} h_p = \sqrt{C}(H_p + w_p) \leq C_2(1 + |w_p|). \tag{4.14}$$

Taking the supremum over R of (4.14), we conclude that $\|w_q\|_{C^0(R)} \leq C_2(1 + \|w_p\|_{C^0(R)})$. The full bound on $\|w\|_{C^1(R)}$ then follows from writing $w(q,p) = \int_{-1}^p w_p(q,p')\,dp'$. $\square$

4.2. **Bounds on the Froude number.** The objective of this subsection is to derive a priori estimates from above and below for the Froude number. These are of general interest to studies of solitary waves, but are of special importance to our arguments in Section 6. The earliest work on this topic we are aware of was carried out by Starr [78], who formally derived sharp lower and upper bounds for homogeneous irrotational solitary waves. We refer to [97, Section 1.2] for a detailed historical discussion, but emphasize that our upper bound is new even for constant density waves with vorticity. In particular, while our estimate involves a measure of stagnation that does not appear in [97], it does not require any additional assumptions on u^*.

4.2.1. *Lower bound.* First we will prove that there are no nontrivial solutions (w, F) of (2.31) with critical Froude number $F = F_{\mathrm{cr}}$. We call this a lower bound because it will imply that the continuous curve of solutions that we construct in Section 6 can only reach a subcritical wave with $F < F_{\mathrm{cr}}$ by first passing through the critical laminar flow $(0, F_{\mathrm{cr}})$. In the special case of constant density, our argument reduces to that in [97], and further implies the inequality $F > F_{\mathrm{cr}}$ for waves of elevation. To get a similar result with non-constant density, we need to assume $F > F_{\mathrm{N}}$ where $F_{\mathrm{N}} < F_{\mathrm{cr}}$; see Corollary 4.5. This extra assumption is related to the additional complexity of the Sturm–Liouville problem studied in Section 3.1. We note that the proof that $F \neq F_{\mathrm{cr}}$ in [97] was extended in [56] to constant density waves which are not necessarily solitary or even periodic.

The main ingredient in our argument is the following integral identity involving the functions Φ and A defined at the start of Section 3.1, as well as the free surface profile $\eta(\,\cdot\,) = w(\,\cdot\,, 0)$. For two homogeneous and irrotational layers, this identity yields (B.8) in [30], at least formally.

Lemma 4.3. *For any solution $(w, F) \in X \times \mathbb{R}$ of the height equation* (2.31) *we have*

$$\int_{-M}^{M} \int_{-1}^{0} \frac{H_p^3 w_q^2 + (H_p + 2h_p) w_p^2}{2h_p^2 H_p^3} \Phi_p\Big(p; \frac{1}{F^2}\Big)\, dp\, dq + A\Big(\frac{1}{F^2}\Big) \int_{-M}^{M} \eta\, dx \longrightarrow 0 \tag{4.15}$$

as $M \to \infty$.

Proof. Multiplying (2.28) by Φ, integrating by parts, and using the equation (3.3) satisfied by Φ, we obtain

$$\begin{aligned} 0 &= \int_{-M}^{M} \int_{-1}^{0} \bigg[\Big(-\frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \Big)_p \Phi + \Big(\frac{h_q}{h_p} \Big)_q \Phi - \frac{1}{F^2} \rho_p (h - H) \Phi \bigg]\, dp\, dq \\ &= \int_{-M}^{M} \int_{-1}^{0} \bigg[\Big(\frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \Big) \Phi_p - \Big(\frac{\Phi_p}{H_p^3} \Big)_p (h - H) \bigg]\, dp\, dq \\ &\qquad + \int_{-M}^{M} \Big(-\frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \Big) \Phi \Big|_{p=0} dq + \int_{-1}^{0} \frac{h_q}{h_p} \Phi\, dp \bigg|_{q=-M}^{q=M}, \end{aligned}$$

so that integrating by parts once more yields

$$\begin{aligned} 0 &= \int_{-M}^{M} \int_{-1}^{0} \bigg[\Big(\frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \Big) \Phi_p + \frac{\Phi_p}{H_p^3} (h_p - H_p) \bigg]\, dp\, dq \\ &\quad + \int_{-M}^{M} \bigg[\Big(-\frac{1 + h_q^2}{2h_p^2} + \frac{1}{2H_p^2} \Big) \Phi - \frac{\Phi_p}{H_p^3} (h - H) \bigg]_{p=0} dq + \int_{-1}^{0} \frac{h_q}{h_p} \Phi\, dp \bigg|_{q=-M}^{q=M}. \end{aligned} \tag{4.16}$$

Rewriting the first integrand in (4.16) as

$$\bigg[\Big(\frac{1 + h_q^2}{2h_p^2} - \frac{1}{2H_p^2} \Big) + \frac{1}{H_p^3} (h_p - H_p) \bigg] \Phi_p = \frac{H_p^3 w_q^2 + (H_p + 2h_p) w_p^2}{2h_p^2 H_p^3} \Phi_p,$$

and applying the boundary conditions in (2.28) and (3.3), we are left with

$$\begin{aligned} \int_{-M}^{M} \int_{-1}^{0} & \frac{H_p^3 h_q^2 + (H_p + 2h_p) w_p^2}{2h_p^2 H_p^3} \Phi_p\, dp\, dq + A\Big(\frac{1}{F^2}\Big) \int_{-M}^{M} \eta\, dx \\ &= -\int_{-1}^{0} \frac{h_q}{h_p} \Phi\, dp \bigg|_{q=-M}^{q=M} \longrightarrow 0 \end{aligned}$$

as $M \to \infty$ as desired. □

Theorem 4.4. *Let $(w, F) \in X \times \mathbb{R}$ be a solution of the height equation* (2.31)*.*

(i) *If $w > 0$ on T, and if F is such that $\Phi_p \geq 0$, then $A(1/F^2) < 0$.*

(ii) *If $F = F_{\mathrm{cr}}$, then $w \equiv 0$. That is, there exist no nontrivial solitary waves with critical Froude number.*

Proof. To prove (i), suppose that $w > 0$ on T (i.e., $\eta > 0$) and that $\Phi_p > 0$. Then both of the integrals in (4.15) are strictly positive for all $M > 0$. Thus, in order for the limit in (4.15) to hold, the coefficient $A(1/F^2)$ must be negative.

To prove (ii), observe that $A(1/F_{\mathrm{cr}}^2) = 0$ by Lemma 3.1(a) so that, for $F = F_{\mathrm{cr}}$, (4.15) reduces to

$$\int_{-M}^{M} \int_{-1}^{0} \frac{H_p^3 w_q^2 + (H_p + 2h_p) w_p^2}{2h_p^2 H_p^3} (\Phi_{\mathrm{cr}})_p \, dp \, dq \longrightarrow 0 \tag{4.17}$$

as $M \to \infty$. Since $(\Phi_{\mathrm{cr}})_p > 0$ by Lemma 3.1(b), the left-hand side of (4.17) is a nonnegative, nondecreasing function of $M > 0$. Thus the limit in (4.17) forces the left-hand side to vanish for all M, which in turn forces $w_q, w_p \equiv 0$ and hence $w \equiv 0$. $\square$

Corollary 4.5. *Let $(w, F) \in X \times \mathbb{R}$ be a solution of the height equation* (2.31)*. If $w > 0$ on T, and if $F \geq F_{\mathrm{N}}$ then in fact $F > F_{\mathrm{cr}} > F_{\mathrm{N}}$. Here $F_{\mathrm{N}}^2 = 1/\mu_{\mathrm{N}}$ is defined in Corollary 3.2. Thus waves of elevation are either supercritical with $F > F_{\mathrm{cr}}$ or quite subcritical in that $F < F_{\mathrm{N}} < F_{\mathrm{cr}}$.*

Proof. This follows immediately from Theorem 4.4(i) and Corollary 3.2. $\square$

4.2.2. *Upper bound.* Next we will prove an upper bound on the Froude number. Unlike [97] in the case of constant density, our estimate does not require any additional assumptions on the asymptotic height function H. Our argument is more closely related to those given by Starr [78] and Keady and Pritchard [51]. While they are able to obtain an upper bound $F < \sqrt{2}$, the presence of vorticity seems to unavoidably introduce a term like $\|h_p(0, \cdot\,)\|_{L^\infty}$ which measures how close the wave is to stagnation on the line beneath the crest. This additional term also means that our bound is not a strict generalization of [97]. For constant density waves, Kozlov, Kuznetsov, and Lokharu [56] have obtained yet another distinct upper bound on F — really on the Bernoulli constant Q defined in (2.18) — in terms of the amplitude $\max \eta$. Their method involves a detailed characterization of one-parameter families of laminar flows with the same vorticity function, and hence seems particularly difficult to generalize to the stratified case.

Our upper bound is a consequence of the following integral identity.

Lemma 4.6. *For any solution $(w, F) \in X \times \mathbb{R}$ of the height equation* (2.31)*,*

$$\frac{1}{F^2} \left[\int_{-1}^{0} |\rho_p| w(0,p)^2 \, dp + \rho(0) \eta(0)^2 \right] = \int_{-1}^{0} \frac{w_p^2}{H_p^2 h_p}(0,p) \, dp. \tag{4.18}$$

Proof. Comparing the flow force $\mathscr{S}$ (2.30) at $q = 0$ and $q = \pm\infty$, we find

$$\begin{aligned} \mathscr{S}(h) &= \int_{-1}^{0} \left(\frac{1}{2h_p^2} + \frac{1}{2H_p^2} - \frac{1}{F^2} \rho (h - H) - \int_0^p \frac{1}{F^2} \rho H_p \, dp' \right) h_p \, dp \\ &= \int_{-1}^{0} \left(\frac{1}{2H_p^2} + \frac{1}{2H_p^2} - \int_0^p \frac{1}{F^2} \rho H_p \, dp' \right) H_p \, dp = \mathscr{S}(H), \end{aligned}$$

where, here and in what follows, all integrals are taken at $q = 0$. Grouping terms and integrating by parts, we get

$$\begin{aligned} 0 &= \int_{-1}^{0} \left[\left(\frac{1}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p + \left(\frac{1}{2h_p^2} + \frac{1}{2H_p^2} \right) w \right] dp \\ &\quad - \int_{-1}^{0} \left[\frac{1}{F^2} \rho w w_p + \frac{1}{F^2} \rho w H_p + \left(\int_0^p \frac{1}{F^2} \rho H_p \, dp' \right) w_p \right] dp \\ &= \int_{-1}^{0} \left[\left(\frac{1}{2h_p^2} - \frac{1}{2H_p^2} \right) H_p + \left(\frac{1}{2h_p^2} + \frac{1}{2H_p^2} \right) w_p \right] dp - \frac{1}{F^2} \int_{-1}^{0} \rho w w_p \, dp. \end{aligned} \tag{4.19}$$

Integrating by parts again, the second integral in (4.19) becomes

$$\frac{1}{F^2}\int_{-1}^{0} \rho w w_p \, dp = \frac{1}{2F^2}\int_{-1}^{0} |\rho_p| w^2 \, dp + \frac{1}{2F^2}\rho w^2\bigg|_{p=0},$$

while some algebra shows that the first integral collapses to

$$\int_{-1}^{0}\left[\left(\frac{1}{2h_p^2}-\frac{1}{2H_p^2}\right)H_p + \left(\frac{1}{2h_p^2}+\frac{1}{2H_p^2}\right)w_p\right] dp = -\int_{-1}^{0}\frac{w_p^2}{2H_p^2 h_p}\,dp,$$

leaving us with (4.18) as desired. □

Theorem 4.7 (Upper bound on F)**.** *Let $(w,F) \in X \times \mathbb{R}$ be a solution of the height equation* (2.31) *and set $h = w + H$. Then the Froude number F satisfies the bound*

$$F^2 \le \frac{2}{\pi}\|H_p\|_{L^\infty}^2 \|\rho\|_{L^\infty} \|h_p(0,\,\cdot\,)\|_{L^\infty}. \tag{4.20}$$

Proof. Rewriting (4.18) slightly as

$$\frac{1}{2F^2}\int_{-1}^{0}\rho w w_p\,dp = \int_{-1}^{0}\frac{w_p^2}{H_p^2 h_p}\,dp,$$

we crudely estimate the left-hand side by

$$\frac{1}{2F^2}\int_{-1}^{0}\rho w w_p\,dp \le \frac{1}{F^2}\|\rho\|_{L^\infty}\|w(0,\,\cdot\,)\|_{L^2}\|w_p(0,\,\cdot\,)\|_{L^2} \le \frac{2}{\pi F^2}\|\rho\|_{L^\infty}\|w_p(0,\,\cdot\,)\|_{L^2}^2$$

and the right-hand side by

$$\int_{-1}^{0}\frac{w_p^2}{H_p^2 h_p}\,dp \ge \left(\min_p \frac{1}{H_p^2}\right)\left(\min_{\{q=0\}}\frac{1}{h_p}\right)\|w_p(0,\,\cdot\,)\|_{L^2}^2.$$

Together, these estimates imply

$$\frac{2}{\pi F^2}\|\rho\|_{L^\infty}\|w_p(0,\,\cdot\,)\|_{L^2}^2 \ge \left(\min_p \frac{1}{H_p^2}\right)\left(\min_{\{q=0\}}\frac{1}{h_p}\right)\|w_p(0,\,\cdot\,)\|_{L^2}^2,$$

and hence

$$F^2 \le \frac{2}{\pi}\|H_p\|_{L^\infty}^2\|\rho\|_{L^\infty}\|h_p(0,\,\cdot\,)\|_{L^\infty},$$

which completes the proof. □

The identity (4.18) can also be used in the case of homogeneous and irrotational flow to recover Starr's and Keady–Pritchard's upper bound of $\sqrt{2}$ for the Froude number. This is possible simply because the right-hand side of (4.18) can be simplified for such waves. The presence of non-constant stratification forces us to do more crude estimates when proving Theorem 4.7.

4.3. **Nonexistence of monotone bores.** By a bore we mean a solution h of the height equation (2.28) satisfying

$$h(q,p) \to H_\pm(p) \text{ as } q \to \pm\infty, \tag{4.21}$$

pointwise in p, where H_- and H_+ represent distinct laminar flows. Outside of hydrodynamics, traveling waves of this type are often called fronts. If one adopts the spatial dynamics viewpoint, thinking of q as the time variable, this is equivalent to having a heteroclinic orbit connecting the rest points H_- and H_+.

Bores are of particular importance to this work because they represent a barrier to proving that $\mathscr{F}$ has certain compactness properties. Observe that, a priori, there may be a sequence of even solitary waves of elevation where the crest flattens and expands into an infinite shelf; see Figure 3. Naturally, this scenario precludes the existence of a subsequential limit in $C_0^2(R)$, and hence would mean that $\mathscr{F}^{-1}(0)$ is not locally compact. Using a translation argument, it is shown in Section 6 that this would also imply the existence of a bore which is *monotone* in that $H \le H_+ \le H_-$ (cf. Lemmas 6.3 and 6.7); see Figure 2.

As mentioned in the introduction, while there are no constant density rotational bores (see [96, Lemma 2.10] or [98, Lemma 3.8]), it is well-known that bores can be found in many stratified regimes. However, as we prove in this subsection, one can completely rule out the existence of *monotone* bores for *free surface* stratified waves.

Theorem 4.8 (Nonexistence of monotone bores)**.** *Suppose that $h \in C^2_{\mathrm{b}}(\overline{R})$ is a bore solution of the height equation* (2.28) *with $\inf_R h_p > 0$, and let $H_\pm$ be as in* (4.21)*. If*

$$H_+ \ge H_- = H \text{ on } [-1,0] \qquad \text{or} \qquad H_+ \le H_- = H \text{ on } [-1,0],$$

then $H_+ = H_- = H$. The same result holds with the roles of H_- and H_+ reversed.

The proof of Theorem 4.8 relies on the following integral identity.

Lemma 4.9. *Suppose that $h = K(p)$ is a solution of the height equation* (2.28) *with $K \in C^2([-1,0])$ and $K_p > 0$. If H and K are conjugate in that $\mathscr{S}(K) = \mathscr{S}(H)$, then*

$$\int_{-1}^{0} \frac{(K_p - H_p)^3}{H_p^2 K_p^2}\, dp = 0. \tag{4.22}$$

Proof. Let K be given as above. Since K is a q-independent solution of the height equation, it satisfies the ODE

$$-\left(\frac{1}{2K_p^2}\right)_p + \left(\frac{1}{2H_p^2}\right)_p - \frac{1}{F^2}\rho_p(K-H) = 0 \qquad \text{in } (-1,0) \tag{4.23}$$

and vanishes on the bed, and on the top

$$-\frac{1}{2K_p(0)^2} + \frac{1}{2H_p(0)^2} = \frac{1}{F^2}\rho(0)(K(0)-H(0)). \tag{4.24}$$

Multiplying (4.23) by $J := K - H$ and integrating by parts, we find

$$\begin{aligned} 0 &= \int_{-1}^{0} \left[\left(-\frac{1}{2K_p^2} + \frac{1}{2H_p^2}\right)_p J - \frac{1}{F^2}\rho_p J^2\right] dp \\ &= \int_{-1}^{0} \left(\frac{1}{2K_p^2} - \frac{1}{2H_p^2}\right) J_p\, dp - \int_{-1}^{0} \frac{1}{F^2}\rho_p J^2\, dp + \left(-\frac{1}{2K_p^2} + \frac{1}{2H_p^2}\right) J\bigg|_{p=0}. \end{aligned}$$

Using the boundary conditions (4.24), this simplifies to

$$\int_{-1}^{0} \frac{1}{F^2}\rho_p J^2\, dp = \int_{-1}^{0} \left(\frac{1}{2K_p^2} - \frac{1}{2H_p^2}\right) J_p\, dp + \frac{1}{F^2}\rho J^2\bigg|_{p=0}. \tag{4.25}$$

Since K and H have the same flow force, we can argue as in the proof of Lemma 4.6 but with $h(q,\,\cdot\,)$ replaced by K to obtain an analogue of (4.19):

$$0 = \int_{-1}^{0} \left[\left(\frac{1}{2K_p^2} - \frac{1}{2K_p^2}\right) K_p + \left(\frac{1}{2K_p^2} + \frac{1}{2K_p^2}\right) J_p\right] dp - \frac{1}{F^2}\int_{-1}^{0} \rho J J_p\, dp.$$

Integrating by parts in the last integral we get

$$0 = \int_{-1}^{0} \left[\left(\frac{1}{2K_p^2} - \frac{1}{2H_p^2}\right) H_p + \left(\frac{1}{2K_p^2} + \frac{1}{2H_p^2}\right) J_p\right] dp + \frac{1}{2F^2}\int_{-1}^{0} \rho_p J^2\, dp - \frac{1}{2F^2}\rho J^2\bigg|_{p=0}.$$

Substituting (4.25), the boundary terms cancel, leaving us with

$$\begin{aligned} 0 &= \int_{-1}^{0} \left[\left(\frac{1}{2K_p^2} - \frac{1}{2H_p^2} \right) H_p + \left(\frac{1}{2K_p^2} + \frac{1}{2H_p^2} \right) J_p + \frac{1}{2} \left(\frac{1}{2K_p^2} - \frac{1}{2H_p^2} \right) J_p \right] dp \\ &= \frac{1}{4} \int_{-1}^{0} \frac{J_p^3}{H_p^2 K_p^2} \, dp \end{aligned} \tag{4.26}$$

as desired. □

In order to apply Lemma 4.9 to prove Theorem 4.8, we need to know that the asymptotic states $H_\pm$ in the statement of the theorem are themselves solutions to the height equation with the expected regularity and flow force, based solely on the pointwise limit in (4.21). This is the content of the following technical lemma.

Lemma 4.10. *Let $h \in C_{\mathrm{b}}^2(\overline{R})$ be a solution of the height equation* (2.28) *with* $\inf_R h_p > 0$ *which is a bore in that* (4.21) *holds for some functions $H_\pm$. Then $H_\pm \in C^2([-1,0])$ are q-independent solutions to the height equation* (2.28) *and $\mathscr{S}(H_+) = \mathscr{S}(H_-)$.*

Proof. Consider the left-translated sequence $\{h_n\}$ defined by $h_n := h(\,\cdot\, + n, \,\cdot\,)$. Via a standard argument, we can extract a subsequence so that $h_n \to h_+$ in $C_{\mathrm{loc}}^{1+1/2}(\overline{R})$. Letting $\inf_R h_p =: \delta > 0$, we easily check that $\inf_R (h_+)_p \geq \delta$ and $\mathscr{S}(h_+) = \mathscr{S}(H)$. Moreover, $h_+ \in C_{\mathrm{loc}}^{1+1/2}(\overline{R})$ solves the (divergence form) height equation (2.28) in the weak sense. By elliptic regularity, $h_+ \in C_{\mathrm{loc}}^2(\overline{R})$. Finally, comparing $h_n \to h_+$ in $C_{\mathrm{loc}}^{1+1/2}(\overline{R})$ with (4.21), we see that $h_+(q,p) = H_+(p)$, which completes the proof for H_+. Arguing similarly with right translations we obtain the same results for H_-. □

Proof of Theorem 4.8. Set $K = H_+$, and assume first that $K \geq H$. By Lemma 4.10, we know that $K \in C^2([-1,0])$ solves the height equation (2.28) with $K_p > 0$ and that $\mathscr{S}(K) = \mathscr{S}(H)$. In particular, as in the proof of Lemma 4.9, K satisfies (4.23) and (4.24). A simple consequence of (4.23) is that

$$\left(-\frac{1}{2K_p^2} + \frac{1}{2H_p^2} \right)_p = \frac{1}{F^2} \rho_p (K - H) \leq 0, \tag{4.27}$$

hence the quantity in parentheses on the left-hand side above is nonincreasing in p. From the boundary condition (4.24) on T, we then have

$$-\frac{1}{2K_p^2} + \frac{1}{2H_p^2} \geq \frac{1}{F^2} \rho(0)(K(0) - H(0)) \geq 0 \tag{4.28}$$

for all p and hence $K_p \geq H_p$ on $[-1,0]$. But now Lemma 4.9 implies that $K_p - H_p \equiv 0$. Since $H(-1) = K(-1) = 0$, it follows that $H \equiv K$. A similar argument shows that the same holds true if $K \leq H$. □

The above proof is built on two facts. First, since $K_p, H_p > 0$, the equation $J = K - H$ satisfies is elliptic. Thus if J is nonnegative (or nonpositive) it is a supersolution (or subsolution). For the free surface problem, the boundary condition on $\{p = 0\}$ then allows us to infer that J_p cannot change signs. By contrast, for a channel flow, the boundary condition at the top will be inhomogeneous Dirichlet, which does not permit us to draw the same conclusion. Indeed, by Rolle's theorem, J_p *must* change signs in the interior. The second building block is the integral identity (4.22). An analogous identity was discovered by Lamb and Wan [59, Appendix A] for solitary stratified flows in a channel with uniform velocity at infinity, and for general stratified solitary waves in a channel by Lamb [58, Appendix A]. Again, this does not lead to a contradiction because J_p is not single-signed in the regimes studied by these authors.

A look at the proof of Theorem 4.8 shows that we do not need to assume $H_- = H$, but only that $\mathscr{S}(H) = \mathscr{S}(H_-) = \mathscr{S}(H_+)$ and that either $H_+ \geq H$ or $H_+ \leq H$, and similarly for H_-. This leads to the following corollary, which we will need in Section 6.

Corollary 4.11. *Suppose that $h \in C^2_{\mathrm{b}}(\overline{R})$ is a bore solution of the height equation* (2.28) *with $\inf_R h_p > 0$, and let $H_\pm$ be as in* (4.21)*. If $\mathscr{S}(H_+) = \mathscr{S}(H_-) = \mathscr{S}(H)$, and if*

$$H_+ \geq H \text{ on } [-1,0] \quad \text{or} \quad H_+ \leq H \text{ on } [-1,0],$$

then $H_+ \equiv H$. The same result holds with H_+ replaced by H_-.

Theorem 4.8 can also be generalized to the case of a free surface flow with multiple layers. These are not the main subject of this paper, but given that many authors have performed conjugate flow analysis for multi-fluid flows, we wish to emphasize that the nonexistence result above is not a consequence of the smoothness of ρ.

Let us briefly recall the governing equations for the layer-wise continuous stratification; a more thorough discussion is given, for example, in [21]. The fluid domain is assumed to be partitioned into finitely many immiscible strata

$$\Omega = \bigcup_{i=1}^N \Omega_i, \quad \Omega_i := \{(x,y) \in \Omega : \eta_{i-1}(x) < y < \eta_i(x)\}$$

with the density ϱ smooth in each layer:

$$\varrho \in C^{1+\alpha}(\overline{\Omega_1}) \cap \cdots \cap C^{1+\alpha}(\overline{\Omega_N}). \tag{4.29}$$

The indexing convention is that $\eta_0 := -1$, $\eta_N := \eta$ and Ω_i lies beneath Ω_{i+1} for $i = 1, \ldots, N-1$. Euler's equation will hold in the strong sense in each interior layer and the pressure is assumed to be continuous in $\overline{\Omega}$.

It is possible even in this setting to perform the Dubreil-Jacotin transform. Letting $R_i := \{(q,p) \in \mathbb{R} \times (p_{i-1}, p_i)\}$ be the image of Ω_i, one finds that the height function h will have the regularity

$$h \in C^{0+\alpha}(\overline{R}) \cap C^{1+\alpha}(\overline{R_1}) \cap \cdots \cap C^{1+\alpha}(\overline{R_N}) \tag{4.30}$$

and solve the height equation (2.28) in the sense of distributions. Notice that ρ_p will include Dirac δ masses on the internal interfaces; the continuity of the pressure precisely guarantees that the quantity on the left-hand side of the height equation is equal to 0 in the distributional sense.

In particular, a laminar flow K will satisfy (4.23) in the strong sense on each $[p_i, p_{i+1}]$, along with top boundary condition (4.24). The continuity of the pressure translates to the requirement that

$$\left[\!\!\left[\frac{1}{2K_p^2} - \frac{1}{2H_p^2} \right]\!\!\right]_i + \frac{1}{F^2} [\![\rho]\!]_i (K - H) = 0 \qquad \text{on } \{p = p_i\}, \tag{4.31}$$

for $i = 1, \ldots, N-1$. Here, $[\![f]\!]_i := f(p_i+) - f(p_i-)$ denotes the jump across the i-th layer. One can see quite easily that (4.31) is equivalent to viewing (4.23) as being satisfied in the sense of distributions.

A bore in this context means that (4.21) holds for distinct laminar flows $H_\pm$ with regularity (4.30).

Corollary 4.12. *Suppose that h has the regularity* (4.30) *and is a distributional solution of the height equation* (2.31) *for an Eulerian density as in* (4.29)*. If h is a bore with $H_\pm$ as in* (4.21)*, $\inf_{R_i} h_p > 0$ for $i = 1, \ldots, N$, and*

$$H_+ \geq H_- = H \text{ on } [-1,0] \quad \text{or} \quad H_+ \leq H_- = H \text{ on } [-1,0], \tag{4.32}$$

then $H_+ = H_- = H$.

Proof. This follows from a straightforward adaptation of the proof of Theorem 4.8 and Lemma 4.9. Let $K = H_+$, and assume that $K \geq H$. The inequality $K_p \geq H_p$ holds by the same reasoning, as $K \geq H$ implies that $\rho_p(K - H)$ is nonpositive as a distribution. The computations of $\mathscr{S}$ can be carried out as before, only there will now be boundary terms on the interior interfaces when integrating by parts. However, the jump conditions (4.31) cause them to cancel out completely, and so we arrive again at (4.22). □

4.4. **Symmetry.** Next, we turn our attention to the question of even symmetry for solitary stratified water waves. The tool we use for this is the classical moving planes method introduced by Alexandroff [2] and then further developed by Serrin [75], Gidas, Nirenberg, and Ni [37], Berestycki and Nirenberg [15], and many others (see also [16] for a survey). Specifically, we use a version due to C. Li [63] that treats fully nonlinear problems on cylindrical domains, and base our approach on the work of Maia [66], who applied Li's ideas to study the case of multiple-layered channel flows with uniform velocity at infinity.

Theorem 4.13 (Symmetry)**.** *Let $(w, F) \in C^2_{\mathrm{b}}(\overline{R}) \times \mathbb{R}$ be a solution of equation* (2.31) *that is a wave of elevation*

$$w > 0 \qquad \text{in } R \cup T, \tag{4.33}$$

supercritical, and satisfies the upstream (or downstream) condition

$$w, Dw \to 0 \quad \textit{uniformly as} \quad q \to -\infty \ (\textit{or} \ +\infty). \tag{4.34}$$

Then, after a translation, w is a symmetric and monotone solitary; i.e., there exists $q_* \in \mathbb{R}$ such that $q \mapsto w(q, \cdot)$ is even about $\{q = q_*\}$ and

$$\pm w_q > 0 \qquad \textit{for } \pm(q_* - q) > 0,\ -1 < p \leq 0. \tag{4.35}$$

Remark 4.14. It is worth emphasizing that Theorem 4.13 only assumes that $w \to 0$ as $q \to -\infty$ instead of the much stronger condition $w \in C^2_0(\overline{R})$ which we impose elsewhere, and which is the standard hypothesis for moving planes arguments. This is in line with the hypotheses of Maia's symmetry result [66] for channel flows, which states that either there is an axis of symmetry or the solution is monotone in the entire strip. We are able to rule out the second possibility in our setting using Theorem 4.8 on the nonexistence of monotone bores.

Remark 4.15. One advantage of Li's approach is that it exploits the structure of the operator at infinity. In particular, this allows us to avoid constructing an explicit Green's function for the linearized problem in order to obtain precise decay estimates.

Remark 4.16. While Theorem 4.13 is of independent interest, we note that it is used in the proof of Theorem 1.1 only in so far as it permits us to infer that the small-amplitude waves of elevation constructed in Section 5 are in fact monotone.

To apply the moving plane method, we consider the usual reflected functions

$$h^\lambda(q, p) := h(2\lambda - q, p),$$

where $\{q = \lambda\}$ is the axis of reflection. We let v^λ denote the difference

$$v^\lambda := h^\lambda - h.$$

Thus $\{q = q_*\}$ is an axis of even symmetry if and only if v^{q_*} vanishes identically. We will work with the λ-dependent sets

$$R^\lambda := \{(q, p) \in R : q < \lambda\}, \quad T^\lambda := \{(q, 0) : q < \lambda\}, \quad B^\lambda := \{(q, -1) : q < \lambda\}. \tag{4.36}$$

If h is a solution of the height equation (2.28), then for each λ, v^λ solves the PDE

$$\mathscr{L} v^\lambda = 0 \quad \text{in } R^\lambda, \qquad \mathscr{B} v^\lambda = 0 \quad \text{on } T^\lambda, \qquad v^\lambda = 0 \quad \text{on } B^\lambda, \tag{4.37}$$

where $\mathscr{L}$ is defined as

$$\begin{aligned}\mathscr{L} :=& \frac{1}{h_p^\lambda}\partial_q^2 - \frac{2h_q^\lambda}{(h_p^\lambda)^2}\partial_q\partial_p + \frac{1+(h_q^\lambda)^2}{(h_p^\lambda)^3}\partial_p^2 + \frac{h_{pp}(h_q^\lambda + h_q) - 2h_p^\lambda h_{pq}}{(h_p^\lambda)^3}\partial_q \\ &+ \frac{h_{qq}(h_p^\lambda + h_p) - 2h_q^\lambda h_{pq} + \big(\beta(-p) - F^{-2}\rho_p h\big)\left[(h_p^\lambda)^2 + h_p^\lambda h_p + h_p^2\right]}{(h_p^\lambda)^3}\partial_p - \frac{1}{F^2}\rho_p,\end{aligned} \tag{4.38}$$

and $\mathscr{B}$ is given by

$$\mathscr{B} := \frac{h_q^\lambda + h_q}{2h_p^2}\partial_q - \frac{(h_p^\lambda + h_p)(1+(h_q^\lambda)^2)}{2h_p^2(h_p^\lambda)^2}\partial_p + \frac{1}{F^2}\rho. \tag{4.39}$$

See [93] for the details of the calculation. Indeed, from [93, Lemma 1] we know that $\mathscr{L}$ is uniformly elliptic.

As in Section 3.2, the coefficients in $\mathscr{L}$ and $\mathscr{B}$ may have "bad" signs which do not satisfy the hypotheses of the maximum principle (cf. Theorem B.1). Since the maximum principle is the main tool in classical moving planes arguments, this is a major obstacle. For supercritical waves, we can overcome this obstacle by using a slight modification of the function Φ defined in (3.3). Fix $0 < \varepsilon \ll 1$ and let $\Psi = \Psi(p; F, \varepsilon)$ be the solution of the initial value problem

$$\left(\frac{\Psi_p}{H_p^3}\right)_p - \frac{1}{F^2}(\rho_p - \varepsilon)\Psi = 0 \quad \text{in } -1 < p < 0, \qquad \Psi(-1) = \varepsilon,\ \Psi_p(-1) = 1. \tag{4.40}$$

Lemma 4.17. *If $F > F_{\mathrm{cr}}$, then, for $\varepsilon > 0$ sufficiently small,*

$$\begin{gathered}\Psi > 0 \text{ for } -1 \le p \le 0, \qquad \Psi_p > 0 \text{ for } -1 \le p \le 0, \\ -\frac{\Psi_p}{H_p^3} + \frac{1}{F^2}\rho\Psi < 0 \text{ on } p = 0.\end{gathered} \tag{4.41}$$

Proof. The proof is identical to the proof of Lemma 3.5. □

As in the discussion before Lemma 3.6, the change of variables $v =: \Psi u$ transforms the linear elliptic equation $\mathscr{F}_w(0,F)v = (f_1, f_2)$ into an equation for u whose zeroth order coefficients have the correct sign for the applications of the maximum principle and Hopf maximum principle (Theorem B.1). The change of variables $v =: \tilde{\Phi}u$ used in Section 3.2 makes the zeroth order coefficient in the elliptic operator vanish; here we define Ψ in a slightly more complicated way in order for this coefficient to be strictly negative.

Lemma 4.18. *Under the hypotheses of Theorem 4.13, there exists $K > 0$ such that*

$$v^\lambda \ge 0 \text{ in } R^\lambda \quad \text{for all } \lambda < -K, \tag{4.42}$$

and

$$h_q \ge 0 \text{ in } R^\lambda \quad \text{for all } \lambda < -K. \tag{4.43}$$

Proof. Let Ψ be defined as in (4.40). Recall that this means in particular that

$$\left(-\frac{1}{H_p^3}\Psi_p + \frac{1}{F^2}\rho\Psi\right)\bigg|_{p=0} < 0 \qquad \text{for } 0 \le \frac{1}{F^2} < \frac{1}{(F_{\mathrm{cr}})^2}. \tag{4.44}$$

Also, by taking $\varepsilon > 0$ sufficiently small, we may assume that $\Psi > \varepsilon$ on $(-1, 0)$.

We are therefore justified in defining u^λ by $v^\lambda = u^\lambda\Psi$. A calculation shows that u^λ satisfies

$$\tilde{\mathscr{L}}u^\lambda = 0 \quad \text{in } R^\lambda, \qquad \tilde{\mathscr{B}}u^\lambda = 0 \quad \text{on } T^\lambda, \qquad u^\lambda = 0 \quad \text{on } B^\lambda, \tag{4.45}$$

where

$$\tilde{\mathscr{L}}u^\lambda := \Psi(\mathscr{L}u^\lambda + \frac{1}{F^2}\rho_p u^\lambda) + \left[\frac{2(1+(h_q^\lambda)^2)}{(h_p^\lambda)^3}\Psi_p\right]u_p^\lambda - \left[\frac{2h_q^\lambda}{(h_p^\lambda)^2}\Psi_p\right]u_q^\lambda + Zu^\lambda,$$

$$\tilde{\mathscr{B}}u^\lambda := \Psi\mathscr{B}u^\lambda + (\mathscr{B}_{\mathrm{P}}\Psi)u^\lambda,$$

with the zeroth order coefficient Z given by

$$Z := \frac{1+(h_q^\lambda)^2}{(h_p^\lambda)^3}\Psi_{pp} - \frac{1}{F^2}\rho_p\Psi + \left(\beta(-p) - \frac{1}{F^2}\rho_p h\right)\left((h_p^\lambda)^2 + h_p^\lambda h_p + h_p^2\right)\frac{\Psi_p}{(h_p^\lambda)^3},$$

and $\mathscr{B}_{\mathrm{P}}$ is the principal parts of $\mathscr{B}$:

$$\mathscr{B}_{\mathrm{P}} := \frac{h_q^\lambda + h_q}{2h_p^2}\partial_q - \frac{(h_p^\lambda + h_p)(1+(h_q^\lambda)^2)}{2h_p^2(h_p^\lambda)^2}\partial_p.$$

We claim that there exists some $K > 0$ large enough so that $u^\lambda \geq 0$ in R^λ for all $\lambda \leq -K$, which then implies (4.42). Assume on the contrary that no matter how large K is, there exists some $\lambda_0 \leq -K$ such that u^{λ_0} takes on a negative value in R^{λ_0}. By hypothesis, h is a wave of elevation, and so clearly the same is true of h^λ for any λ. Now u^λ vanishes on the vertical line segment $\{q = \lambda\}$, and

$$u^\lambda = \frac{h^\lambda - h}{\Psi} > \frac{H-h}{\Psi}, \tag{4.46}$$

with the right-hand side of the above inequality limiting to 0 as $q \to -\infty$. Thus if u^{λ_0} is negative somewhere in R^{λ_0} then there must exist some $(q_0, p_0) \in R^{\lambda_0} \cup T^{\lambda_0}$ such that

$$u^{\lambda_0}(q_0, p_0) = \inf_{R^{\lambda_0}} u^{\lambda_0} < 0. \tag{4.47}$$

We consider separately two cases.

Case 1: $(q_0, p_0) \in R^{\lambda_0}$. Then u^{λ_0} attains its (global) minimum at an interior point, so

$$\nabla u^{\lambda_0}(q_0, p_0) = 0,$$

and hence

$$0 = \nabla u^{\lambda_0}(q_0, p_0) = \left[\frac{\nabla v^{\lambda_0}}{\Psi} - \frac{v^{\lambda_0}}{\Psi^2}(0, \Psi_p)\right](q_0, p_0). \tag{4.48}$$

In light of (4.34), for each $\delta > 0$, we may take K sufficiently large so that

$$\|w\|_{C^2(R^{-K})} = \|h - H\|_{C^2(R^{-K})} < \delta, \tag{4.49}$$

from which we have the chain of inequalities

$$H(p_0) < h^{\lambda_0}(q_0, p_0) < h(q_0, p_0) < H(p_0) + \delta. \tag{4.50}$$

Moreover (4.49) and (4.48) lead to the bounds on v^{λ_0} and ∇v^{λ_0} at (q_0, p_0):

$$|v^{\lambda_0}(q_0, p_0)| < \delta, \qquad |\nabla v^{\lambda_0}(q_0, p_0)| = \left|\frac{\Psi_p(p_0)}{\Psi(p_0)}v^{\lambda_0}(q_0, p_0)\right| < C\delta,$$

where C depends only on ε. Therefore in terms of h^{λ_0} we have that at (q_0, p_0),

$$\left|h^{\lambda_0} - H\right| < \delta, \quad \left|\nabla h^{\lambda_0} - \nabla H\right| < C\delta. \tag{4.51}$$

From (4.49) and (4.51) we conclude that

$$Z(p_0) = \left(\frac{1}{H_p^3}\Psi_{pp} - \frac{3H_{pp}}{H_p^4}\Psi_p - \frac{1}{F^2}\rho_p\Psi\right)(p_0) + \mathcal{O}(\delta).$$

But, recalling the ODE satisfied by Ψ (4.40), we know that

$$\frac{1}{H_p^3}\Psi_{pp} - \frac{3H_{pp}}{H_p^4}\Psi_p - \frac{1}{F^2}\rho_p\Psi = -\varepsilon\frac{1}{F^2}\Psi < 0.$$

Therefore, by taking K sufficiently large, and hence δ sufficiently small, we can guarantee that $Z < 0$ at p_0. Applying the maximum principle to (4.45) at (q_0, p_0) then leads to a contradiction.

Case 2: $(q_0, p_0) \in T^{\lambda_0}$, that is $p_0 = 0$. Applying the Hopf lemma, we see that

$$u_q^{\lambda_0}(q_0, 0) = 0, \quad u_p^{\lambda_0}(q_0, 0) < 0. \tag{4.52}$$

From the first of these equalities it follows that

$$h_q^{\lambda_0}(q_0, 0) = h_q(q_0, 0). \tag{4.53}$$

Moreover, arguing as in the previous case, (4.50) and (4.49) show that for K large enough,

$$\big|h(q_0, 0) - H(0)\big| < \delta, \quad \big|h_p(q_0, 0) - H_p(0)\big| < \delta, \quad \big|h^{\lambda_0}(q_0, 0) - H(0)\big| < \delta. \tag{4.54}$$

Since both h and h^{λ_0} solve the height equation (2.28), we can evaluate the boundary condition at $(q_0, 0)$ to obtain

$$\frac{1 + h_q^2}{2h_p^2} + \frac{1}{F^2}\rho h = \frac{1 + (h_q^\lambda)^2}{2(h_p^\lambda)^2} + \frac{1}{F^2}\rho h^\lambda.$$

Together, (4.53) and (4.54) furnish the estimate

$$\big|h_p^{\lambda_0}(q_0, 0) - h_p(q_0, 0)\big| < C\delta. \tag{4.55}$$

Combing (4.54) and (4.55), the boundary condition in (4.45) can then be written as

$$\tilde{\mathscr{B}}u^{\lambda_0} = \left(-\frac{(h_p^\lambda + h_p)(1 + (h_q^\lambda)^2)}{2h_p^2(h_p^\lambda)^2}u_p^{\lambda_0}\right)\Psi + \left[-\frac{1}{H_p^3}\Psi_p + \frac{1}{F^2}\rho\Psi + \mathcal{O}(\delta)\right]u^{\lambda_0} = 0 \tag{4.56}$$

on T^λ. In view of (4.41), this means that for K sufficiently large, the coefficient of u^{λ_0} above is negative, and therefore from (4.47) we know that $u_p^{\lambda_0}(q_0, 0) > 0$, which is a contradiction to (4.52).

Thus we have confirmed that there exists a K sufficiently large so that (4.42) is satisfied. The monotonicity property (4.43) then follows immediately: for any $\lambda \le -K$, we have $v^\lambda(\lambda, p) = 0$ and therefore $v_q^\lambda(\lambda, p) \le 0$, which implies that $h_q(\lambda, p) \ge 0$. □

Following the moving plane method, we complete the proof of Theorem 4.13 by employing a continuation argument. The general procedure is quite standard, having been worked out in similar contexts by Hur [45], in the case of constant density water, and Maia [66], for stratified channel flow. Our situation is different in that, unlike Hur, we allow a priori for there to be no decay in one direction, and, unlike Maia, we have at our disposal Theorem 4.8.

Proof of Theorem 4.13. Let λ_* be defined by

$$\lambda_* := \sup\Big\{\lambda_0 :\ v^\lambda > 0 \text{ in } R^\lambda \text{ for all } \lambda < \lambda_0\Big\},$$

where the above set is nonempty in light of the previous lemma.

Case 1. $\lambda_* < +\infty$. By continuity we know that $v^{\lambda_*} \ge 0$ on R^{λ_*}. Looking at the elliptic system (4.37) that v^{λ_*} satisfies in R^{λ_*}, we can therefore apply the strong maximum principle (see Theorem B.1(i)) to conclude that either $v^{\lambda_*} > 0$ or $v^{\lambda_*} \equiv 0$ in R^{λ_*}.

We claim that $v^{\lambda_*} \equiv 0$ in R^{λ_*}. Seeking a contradiction, assume instead that $v^{\lambda_*} > 0$ in R^{λ_*}. The maximality of λ_* implies that there exist sequences $\{\lambda_k\}$ and $\{(q_k, p_k)\}$ with $\lambda_k \searrow \lambda_*$ and $(q_k, p_k) \in \overline{R^{\lambda_k}}$ such that

$$v^{\lambda_k}(q_k, p_k) = \inf_{R^{\lambda_k}} v^{\lambda_k} < 0.$$

Since $v^{\lambda_k} = 0$ on B^{λ_k} and $\{q = \lambda_k\}$, the strong maximum principle forces $(q_k, p_k) \in T^{\lambda_k}$. This trivially implies

$$v_p^{\lambda_k}(q_k, 0) \le 0, \qquad \text{and} \qquad v_q^{\lambda_k}(q_k, 0) = 0. \tag{4.57}$$

Next, we prove that $\{q_k\}$ is bounded from below. Were this not true, then we have $q_k < -K$ for all k sufficiently large, where K is chosen as in (4.42). Consider once more the function $u^{\lambda_k} := v^{\lambda_k}/\Psi$ introduced in Lemma 4.18. Clearly u^{λ_k} satisfies (4.45) in R^{λ_k}, but from (4.56) we see that $\tilde{\mathscr{B}} u^{\lambda_k}(q_k, 0) > 0$, a contradiction.

Thus $\{q_k\}$ is indeed bounded from below by $-K$. It is also obviously bounded from above by λ_1. Up to a subsequence, therefore,

$$(q_k, 0) \to (q_*, 0) \in \overline{T^{\lambda_*}} \quad \text{as } k \to \infty \tag{4.58}$$

for some $q_* \in [-K, \lambda_*]$. The fact that $v^{\lambda_*} > 0$ in R^{λ_*} forces

$$\lim_{k\to\infty} v^{\lambda_k}(q_k, 0) = v^{\lambda_*}(q_*, 0) = 0.$$

If $q_* < \lambda_*$, then we have by continuity that

$$v^{\lambda_*}(q_*, 0) = v_q^{\lambda_*}(q_*, 0) = 0, \tag{4.59}$$

and, furthermore, from the Hopf lemma, $v_p^{\lambda_*}(q_*, 0) < 0$. Recalling the definition of the boundary operator $\mathscr{B}$ in (4.39), these inequalities show that $\mathscr{B} v^{\lambda_*}(q_*, 0) > 0$, which is impossible because v^{λ_*} solves (4.37).

Therefore $(q_*, 0)$ must be a corner point of R^{λ_*}, i.e., $q_* = \lambda_*$. From (4.59), we then see $h_q^{\lambda_*}(\lambda_*, 0) = 0$. We can rewrite the top boundary condition in (4.37) as

$$(h_p^\lambda)^2 (h_q^\lambda + h_q) v_q^\lambda - (h_p^\lambda + h_p)(1 + (h_q^\lambda)^2) v_p^\lambda + 2\frac{1}{F^2} \rho h_p^2 (h_p^\lambda)^2 v = 0. \tag{4.60}$$

Letting $\lambda = \lambda_*$, differentiating the above equality with respect to q, and then evaluating it at $(\lambda_*, 0)$ we obtain

$$2h_p(\lambda_*, 0) v_{qp}^{\lambda_*}(\lambda_*, 0) = 0,$$

where we have used the identities

$$h_q^{\lambda_*}(\lambda_*, 0) = -h_q(\lambda_*, 0), \quad h_p^{\lambda_*}(\lambda_*, 0) = h_p(\lambda_*, 0), \quad h_{qp}^{\lambda_*}(\lambda_*, 0) = -h_{qp}(\lambda_*, 0).$$

From the no stagnation assumption (2.24), we can therefore conclude

$$v_{qp}^{\lambda_*}(\lambda_*, 0) = 0.$$

Moreover, because $v^{\lambda_*}(\lambda_*, \cdot)$ vanishes identically, it follows that

$$v_p^{\lambda_*}(\lambda_*, 0) = v_{pp}^{\lambda_*}(\lambda_*, 0) = 0.$$

Finally, by using the PDE to express $v_{qq}^{\lambda_*}$ in terms of $v^{\lambda_*}, v_q^{\lambda_*}, v_p^{\lambda_*}, v_{qp}^{\lambda_*}$ and $v_{pp}^{\lambda_*}$, we see that

$$v_{qq}^{\lambda_*}(\lambda_*, 0) = 0.$$

The previous paragraph demonstrates that v^{λ_*}, together with all its derivatives up to second order, vanishes at the corner point $(\lambda_*, 0)$. Since v^{λ_*} solves the PDE in the domain R^{λ_*}, this violates the Serrin edge point lemma (see Theorem B.1). Therefore we must have $v^{\lambda_*} \equiv 0$ in R^{λ_*}, and hence h (and w) are symmetric about the axis $q_* := \lambda_*$. This proves the first part of the theorem.

Next, consider the strict monotonicity of h. For $\lambda < \lambda_*$ we have from the definition of λ_* that $v^\lambda > 0$ in R^λ. As v^λ vanishes on the right boundary of R^λ, it attains its minimum on $\overline{R^\lambda}$ there. The Hopf maximum principle then implies

$$h_q(\lambda, p) = -\frac{1}{2} v_q^\lambda(\lambda, p) > 0 \quad \text{for } \lambda < \lambda_*,\ -1 < p < 0. \tag{4.61}$$

Naturally, on the top boundary T^{λ_*}, $h_q \geq 0$ by continuity. Seeking a contradiction, suppose that h_q vanishes at some point $(\lambda, 0) \in T^{\lambda_*}$. Then, from differentiating the boundary condition (4.60) with respect to q, evaluating the result at $(\lambda, 0)$, and using the identities

$$h_q(\lambda,0) = -h_q^\lambda(\lambda,0) = 0, \qquad v^\lambda(\lambda,0) = v_q^\lambda(\lambda,0) = 0,$$
$$h_p(\lambda,0) = h_p^\lambda(\lambda,0), \qquad h_{qp}(\lambda,0) = -h_{qp}^\lambda(\lambda,0),$$

we conclude

$$2h_p(\lambda,0) v_{qp}^\lambda(\lambda,0) = 0,$$

and hence

$$v_{qp}^\lambda(\lambda,0) = 0.$$

Note that, as before, we have

$$v_p^\lambda(\lambda,0) = v_{pp}^\lambda(\lambda,0) = 0.$$

Thus, using the PDE to solve for v_{qq}^λ, we get $v_{qq}^\lambda(\lambda,0) = 0$. But v^λ satisfies (4.37) and so the Serrin edge point lemma is violated. We infer, therefore, that $h_q > 0$ on T^{λ_*}. Combining this with (4.61) confirms that $h_q > 0$ in $R^{\lambda_*} \cup T^{\lambda_*}$ as desired. The same holds for w_q. As $\{q = \lambda_*\}$ is an axis of even symmetry, the proof (4.35) is complete.

Case 2. If $\lambda_* = +\infty$, then $v^\lambda \geq 0$ in R^λ for all λ. Since v^λ solves (4.37) in R^λ, an application of the strong maximum principle ensures that $v^\lambda > 0$ in R^λ for all λ. Then, arguing as in the preceding paragraph, we see that $h_q > 0$ in $R^\lambda \cup T^\lambda$ for any λ, and hence $h_q > 0$ in R, indicating that h is a strictly monotone bore solution. In particular, the pointwise limits $h(q,p) \to H_\pm(p)$ as $q \to \pm\infty$ in (4.21) hold, with $H_- = H$ and $H_+ > H$ for $p > -1$, violating Theorem 4.8. Therefore we must have $\lambda_* < +\infty$, and thus the previous argument completes the proof of the theorem. □

4.5. **Asymptotic monotonicity and nodal properties.** In Section 5, we will establish the existence of a curve of small-amplitude supercritical waves of elevation bifurcating from the critical laminar flow. By the results in Section 4.4, these waves are symmetric and monotone. The global bifurcation theory will require us to show that these properties persist away from the point of initial bifurcation. In particular, this is a critical component of the proof that the solution set $\mathscr{F}^{-1}(0)$ is locally compact (cf. Lemma 6.7).

Notice, however, that monotonicity is neither an open nor closed property in the natural topology of our function spaces. It will therefore be necessary to exploit some structure of the equation itself, which will mostly come in the form of maximum principle arguments. We will actually consider a collection of sign conditions (4.65) that together imply monotonicity but also define a subset of $\mathscr{F}^{-1}(0)$ that is both open and closed in an appropriate sense.

An analysis of this kind is a standard part of many global bifurcation arguments, where conditions of the type (4.65) are referred to as "nodal properties." Typically, they are used to rule out undesirable topological alternatives such as the existence of closed loops of solutions. In our theory, however, they play a much different role as we will see in Section 6.

Before discussing the nodal properties, we first divide R into two regions: a finite rectangle and two semi-infinite tails where w is small. The finite rectangle can be dealt with in the same way as periodic solutions were in [94, Section 5]. For the tail, we will follow the strategy of [98, Section 2.2] by considering small solutions of (2.31) in the half strip

$$R^+ := \{(q,p) \in R : q > 0\}$$

with boundary components

$$L^+ := \{(0,p) : p \in [-1,0]\}, \qquad T^+ := \mathbb{R}_+ \times \{0\}, \qquad B^+ := \mathbb{R}_+ \times \{-1\}.$$

Proposition 4.19 (Asymptotic monotonicity)**.** *There exists $\delta > 0$ such that, if $w \in C_\mathrm{b}^3(R^+) \cap C_0^1(\overline{R^+})$ is a solution of equation* (2.31) *in R^+, $F > F_\mathrm{cr}$, and*

$$\|w\|_{C^2(R^+)} < \delta,$$

then w_q exhibits the following monotonicity property:

$$\text{If } \pm w_q \le 0 \text{ on } L^+, \text{ then } \pm w_q < 0 \text{ in } R^+ \cup T^+. \tag{4.62}$$

Proof of Proposition 4.19. We will use a maximum principle argument applied to $h_q = w_q$. Due to the translation invariance, we can "quasilinearize" the height equation by differentiating it with respect to q. This leads to a linear elliptic PDE for $v := h_q$ with coefficients depending on h:

$$\begin{cases} \left(-\dfrac{h_q v_q}{h_p^2} + \dfrac{(1+h_q^2)v_p}{h_p^3}\right)_p + \left(\dfrac{v_q}{h_p} - \dfrac{h_q v_p}{h_p^2}\right)_q - \dfrac{1}{F^2}\rho_p v = 0 & \text{in } R^+ \\ \dfrac{h_q v_q}{h_p^2} - \dfrac{(1+h_q^2)v_p}{h_p^3} + \dfrac{1}{F^2}\rho v = 0 & \text{on } T^+ \\ v = 0 & \text{on } B^+. \end{cases} \tag{4.63}$$

Let $v =: \Psi u$, where Ψ is chosen according Lemma 4.17. We can then rewrite (4.63) in terms of u. For instance, the equation in the interior becomes

$$\begin{aligned} 0 = {} & \left(\left(\frac{\Psi}{h_p^3} + \frac{\Psi h_q^2}{h_p^3}\right)u_p - \frac{\Psi h_q}{h_p^2}u_q\right)_p + \left(\frac{\Psi}{H_p}u_q - \frac{\Psi h_q}{h_p^2}u_p\right)_q \\ & + \frac{(1+h_q^2)\Psi}{h_p^3}u_p - \frac{h_q \Psi_p}{h_p^2}u_q + \left(\left(\frac{(1+h_q^2)\Psi_p}{h_p^3}\right)_p - \left(\frac{h_q\Psi_p}{h_p^2}\right)_q - \frac{1}{F^2}\rho_p \Psi\right)u. \end{aligned} \tag{4.64}$$

Taking $\|h - H\|_{C^2(R^+)} = \|w\|_{C^2(R^+)}$ sufficiently small ensures that this represents a uniformly elliptic operator acting on u on the set R^+. The key point is the zeroth order coefficient, which can be rewritten as

$$-\left(\frac{h_q \Psi_p}{h_p^2}\right)_q + \left(\left(\frac{1}{h_p^3} - \frac{1}{H_p^3}\right)\Psi_p + \frac{h_q^2}{h_p^3}\Psi_p\right)_p - \frac{1}{F^2}\varepsilon\Psi,$$

has the correct sign to apply the maximum principle since the first two terms on the right-hand side can be controlled by the final term by taking $\|h - H\|_{C^2}$ small. Thus (4.64) has the form

$$\sum_{i,j} \partial_i(a_{ij}\partial_j u) + \sum_i b_i \partial_i u + cu = 0,$$

where a_{ij} is a symmetric positive definite matrix and $c < 0$, both with uniform bounds for $\|h - H\|_{C^2}$ sufficiently small.

Likewise, the boundary conditions on T^+ in (4.63) can be written

$$-\frac{1+h_q^2}{h_p^3}\Psi u_p + \frac{h_q}{h_p^2}\Psi u_q + \left(-\frac{\Psi_p}{H_p^3} + \frac{1}{F^2}\rho\Psi - \left(\frac{1}{h_p^3} - \frac{1}{H_p^3}\right)\Psi_p\right)u = 0.$$

By (4.41), the coefficient of u is negative for $\|h - H\|_{C^1}$ small enough. Thus this represents a uniformly oblique boundary condition with the correct sign, and we can apply the maximum principle to conclude that (4.62) holds. $\square$

With Proposition 4.19 in hand, we now consider the "nodal properties"

$$w_q < 0 \qquad \text{in } R^+ \cup T^+, \tag{4.65a}$$
$$w_{qq} < 0 \qquad \text{on } L^+, \tag{4.65b}$$
$$w_{qp} < 0 \qquad \text{on } B^+, \tag{4.65c}$$
$$w_{qqp} < 0 \qquad \text{at } (0,-1), \tag{4.65d}$$
$$w_{qq} < 0 \qquad \text{at } (0,0). \tag{4.65e}$$

Lemma 4.20. *Let (w, F) be a solution of equation* (2.31) *where $w \in C^3_{\mathrm{b,e}}(\overline{R}) \cap C^1_0(\overline{R})$ is monotone in the sense that $w_q < 0$ in $R^+ \cup T^+$. Then w exhibits the nodal properties* (4.65)*.*

Proof. As in the proof of Proposition 4.19, $v := w_q = h_q$ satisfies the uniformly elliptic PDE (4.63). The strategy is now to use various maximum principle arguments to derive (4.65). Again, the zeroth order term in (4.63) comes with an adverse sign. Unlike in Proposition 4.19, we are not assuming that w is small. Instead, we are saved by the following observation. First, w is even in q and vanishes identically on the bed, hence

$$v = 0 \qquad \text{on } L^+ \cup B^+.$$

Second, by assumption we have that

$$v < 0 \qquad \text{in } R^+ \cup T^+.$$

Together, these two statements mean that we *can* apply the Hopf lemma, and the Serrin edge point lemma (Theorem B.1(iii)) at the points on $\overline{L^+ \cup B^+}$ where v achieves its maximum value.

With that in mind, let us consider in order the nodal properties (4.65). First, (4.65a) is satisfied by hypothesis. The Hopf lemma applied to L^+ and B^+ shows that (4.65b) and (4.65c) hold, respectively.

Now consider the corners. Since v vanishes identically on $L^+ \cup B^+$, we have that

$$v = v_p = v_q = v_{pp} = v_{qq} = 0 \qquad \text{at } (0,-1).$$

Thus, by the Serrin edge point lemma, (4.65d) holds.

Likewise, at the upper left corner point $(0, 0)$ we have

$$v(0,0) = v_p(0,0) = 0.$$

Using these facts, differentiating the top boundary condition in (2.28) twice in q, and evaluating at $(0, 0)$, one arrives at

$$\frac{1}{2h_p^2} v_q^2 + \frac{1}{h_p^3} v_{qp} + \frac{1}{F^2} \rho v_q = 0 \qquad \text{at } (0,0).$$

Seeking a contradiction, suppose that $v_q(0, 0) = w_{qq}(0, 0) = 0$. Then from the evenness of w and the line above we have

$$v,\ v_p,\ v_q,\ v_{pp},\ v_{qp} = 0 \qquad \text{at } (0,0).$$

Finally, we can use (4.63) to write v_{qq} in terms of the quantities above, which reveals that $v_{qq}(0, 0) = 0$ as well. This is a clear contradiction of the Serrin edge point lemma. We infer, therefore, that (4.65e) holds and the proof is complete. ☐

Notice that the hypotheses of Lemma 4.20 do not require that the solution be supercritical.

Lemma 4.21 (Open property)**.** *Let $(w, F), (\tilde{w}, \tilde{F})$ be two supercritical solutions of* (2.31) *with $w, \tilde{w} \in C^3_{\mathrm{b,e}}(\overline{R}) \cap C^1_0(\overline{R})$. If w satisfies the nodal properties* (4.65)*, then there exists $\varepsilon = \varepsilon(w) > 0$ such that*

$$\|w - \tilde{w}\|_{C^3(R)} + |F - \tilde{F}| < \varepsilon$$

implies that $\tilde{w}$ also satisfies (4.65)*.*

Proof. According to Lemma 4.20, to prove that $\tilde{w}$ satisfies (4.65) we need only confirm that $\tilde{w}_q < 0$ in $R^+ \cup T^+$. With that in mind, we begin by dividing R^+ into two overlapping regions, a finite extent rectangle and a "tail":

$$R_1^+ := \{(q,p) \in R^+ : q < 2K\}, \qquad R_2^+ := \{(q,p) \in R^+ : q > K\},$$

where $K > 0$ is to be determined. The top, bottom, and left boundary components we likewise write as $T_{1,2}^+$, $B_{1,2}^+$, and $L_{1,2}^+$.

First consider the finite rectangle R_1^+. Arguing as in [94, Section 5], it is possible to show that for each $K > 0$, there exists $\varepsilon_K > 0$ such that, if $\|w - \tilde{w}\|_{C^3(R^+)} + |F - \tilde{F}| < \varepsilon_K$, then $\tilde{w}_q < 0$ in $R_1^+ \cup T_1^+$. This is simply because the finite rectangle behaves exactly the same as in the periodic case.

On the other hand, because $w \in C_0^2(R)$, we can choose K to be large enough so that

$$\|w\|_{C^2(R_2^+)} < \frac{\delta}{2},$$

where δ is given as in the hypotheses of Proposition 4.19. By letting $\varepsilon := \min\{\delta/2, \varepsilon_K\}$, we have $\tilde{w}_q < 0$ in $R_1^+ \cup T_1^+$, which in particular means that $\tilde{w}_q \le 0$ on L_2^+. Applying Proposition 4.19 allows us to conclude that $\tilde{w}_q < 0$ in $R_2^+ \cup T_2^+$. Since $R_1^+ \cup R_2^+ = R^+$, the proof is complete. ☐

Lemma 4.22 (Closed property)**.** *Let $\{(w_n, F_n)\} \subset U$ be a sequence of solutions to* (2.31) *and suppose that there exists a solution $(w, F) \in U$ of* (2.31) *with*

$$(w_n, F_n) \to (w, F) \qquad \text{in } C_{\mathrm{b}}^3(\overline{R}) \times \mathbb{R}. \tag{4.66}$$

If each w_n satisfies the nodal properties (4.65)*, then w also satisfies* (4.65) *unless $w \equiv 0$.*

Proof. Let $\{(w_n, F_n)\}$ and (w, F) be given as above. Again, Lemma 4.20 shows that it is enough to confirm that $w_q < 0$ on $R^+ \cup T^+$. Let $v := w_q$. Simply by continuity, we know that $v \le 0$ on $\overline{R^+}$, and we have already seen that v satisfies the uniformly elliptic PDE (4.63) in R^+. As v vanishes identically on $L^+ \cup B^+$, the maximum principle implies that either (i) $v < 0$ in $R^+ \cup T^+$, or else (ii) there exists some point $(q_0, 0) \in T^+$ such that $v(q_0, 0) = 0$. Here we are once more using the facts that $\sup_{R^+} v = 0$ and $v \to 0$ as $q \to \infty$.

Thanks to Lemma 4.20, possibility (i) implies that w satisfies the nodal properties, so consider possibility (ii). The boundary condition on the top in (4.63) written in terms of v is

$$\frac{v v_q}{2h_p^2} - \frac{(1+v^2)v_p}{h_p^3} + \frac{1}{F^2}\rho v = 0 \qquad \text{on } T^+.$$

Were (ii) to hold, then evaluating the above line at $(q_0, 0)$ would give that $v_p(q_0, 0) = 0$. This contradicts the Hopf lemma unless v vanishes identically in $\overline{R^+}$, but this is equivalent to saying $w \equiv 0$. ☐

5. Small-amplitude existence theory

In this section we construct small-amplitude waves with nearly critical Froude numbers $F \approx F_{\mathrm{cr}}$. It will be convenient to introduce a small parameter ϵ defined by

$$\epsilon := \frac{1}{F_{\mathrm{cr}}^2} - \frac{1}{F^2} = \mu_{\mathrm{cr}} - \frac{1}{F^2}. \tag{5.1}$$

Note that with this convention, $\epsilon > 0$ corresponds to a supercritical Froude number. Solving (5.1) for F, we also introduce the notation

$$F = F^\epsilon := \left(\frac{1}{F_{\mathrm{cr}}^2} - \epsilon\right)^{-1/2}. \tag{5.2}$$

The main results of this section are summarized in the following theorem. In addition, we prove the existence of a family of subcritical periodic (cnoidal) waves, see Remark 5.8.

Theorem 5.1 (Small-amplitude solitary waves)**.** *There exists $\epsilon_* > 0$ and a curve*

$$\mathscr{C}_{\mathrm{loc}} = \{(w^\epsilon, F^\epsilon) : \epsilon \in (0, \epsilon_*)\} \subset U$$

of solutions to $\mathscr{F}(w, F) = 0$ with the following properties:

(i) (Continuity) *The map $\epsilon \mapsto w^\epsilon$ is continuous from $(0, \epsilon_*)$ to X, with $\|w^\epsilon\|_X \to 0$ as $\epsilon \to 0$.*
(ii) (Invertibility) *The linearized operator $\mathscr{F}_w(w^\epsilon, F^\epsilon)$ is invertible for each $\epsilon \in (0, \epsilon_*)$.*
(iii) (Uniqueness) *If $w \in X$ satisfies $w > 0$ on T and if $\|w\|_X$ is sufficiently small, then, for any $\epsilon \in (0, \epsilon_*)$, $\mathscr{F}(w, F^\epsilon) = 0$ implies $w = w^\epsilon$.*
(iv) (Elevation) *The waves (w^ϵ, F^ϵ) are waves of elevation in that $w^\epsilon > 0$ on $R \cup T$.*
(v) (Analyticity) *The curve $\mathscr{C}_{\mathrm{loc}}$ is real analytic in the sense that $\epsilon \mapsto w^\epsilon$ is real analytic.*

We will prove Theorem 5.1 incrementally. First, in Lemma 5.9, we use the center manifold reduction method to construct the family (w^ϵ, F^ϵ) (cf. Theorem B.2). Doing this requires proving a number of preparatory lemmas that show the linearized problem has the requisite spectral behavior and the nonlinearity is quadratic near the origin. Continuity and invertibility then follow from a straightforward adaptation of the arguments in [96, Theorem 4.1]. Elevation and uniqueness are proved in Lemma 5.9 and Lemma 5.10, respectively. These results are stitched together, along with an argument for the existence of an analytic reparameterization, in Section 5.5.

5.1. **Hamiltonian formulation.** Following [40], we will relate our nonlinear operator equation to a Hamiltonian system $(\mathcal{M}, \omega, \mathcal{H}^\epsilon)$ in which the horizontal variable q plays the role of time. The Hamiltonian function $\mathcal{H}^\epsilon$ will turn out to be the flow force $\mathscr{S}$ defined in Section 2.4 (cf. [11, 13]). For the position and momentum variables, we will use w and $r := w_q/(w_p + H_p)$, respectively.

With that in mind, in this section we think of $w = w(q, p)$ as a C^1 mapping $q \mapsto w(q, \cdot)$ taking values in a Hilbert space of p-dependent functions, and similarly for r. In particular, we will work with

$$\mathcal{X} := \{(w, r) \in H^1(-1, 0) \times L^2(-1, 0) : w(-1) = 0\},$$
$$\mathcal{Y} := \{(w, r) \in H^2(-1, 0) \times H^1(-1, 0) : w(-1) = 0\}.$$

Here we are abusing notation somewhat by suppressing the dependence of (w, r) on q; this will be a common practice throughout the section. We also define

$$\mathcal{M} := \{(w, r) \in \mathcal{Y} : w_p + H_p > 0 \text{ for } -1 \le p \le 0\}.$$

Clearly $\mathcal{M}$ is an open subset of $\mathcal{Y}$ containing the origin. Since $\mathcal{Y}$ is dense in $\mathcal{X}$ and the inclusion $\mathcal{Y} \hookrightarrow \mathcal{X}$ is smooth, $\mathcal{M}$ is therefore a so-called *manifold domain* of $\mathcal{X}$. The symplectic form $\omega \colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is defined by

$$\omega\big((w_1, r_1), (w_2, r_2)\big) := \int_{-1}^0 (r_2 w_1 - r_1 w_2)\, dp,$$

and finally the Hamiltonian $\mathcal{H}^\epsilon \in C^\infty(\mathcal{M}, \mathbb{R})$ is given by

$$\begin{aligned} \mathcal{H}^\epsilon(w, r) := \int_{-1}^0 \left[\int_0^p \frac{1}{(F^\epsilon)^2} \rho H_p\, dp' - \frac{1}{2H_p^2} + \frac{1}{2} r^2 - \frac{1}{2(w_p + H_p)^2} + \frac{1}{(F^\epsilon)^2} \rho w \right] (w_p + H_p)\, dp \\ + \int_{-1}^0 \left[H_p \int_0^p \frac{1}{(F^\epsilon)^2} \rho H_p\, dp' - \frac{1}{H_p} \right] dp. \end{aligned} \tag{5.3}$$

The second integral in (5.3) does not depend on w or r, and has been added to ensure $\mathcal{H}^\epsilon(0, 0) = 0$. Comparing with (2.30), we see that $\mathcal{H}^\epsilon$ is essentially $\mathscr{S}$ viewed as a functional acting on w and $r = w_q/(w_p + H_p)$.

A calculation shows that the domain $\mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$ of the Hamiltonian vector field $\mathcal{V}_{\mathcal{H}^\epsilon}$ corresponding to $(\mathcal{M}, \omega, \mathcal{H}^\epsilon)$ is

$$\begin{aligned}\mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon}) = \bigg\{ &(w,r) \in \mathcal{M} : r(-1) = 0, \\ &\qquad \left(\frac{r^2}{2} + \frac{1}{2(w_p+H_p)^2} - \frac{1}{2H_p^2} + \frac{1}{(F^\epsilon)^2}\rho w\right)\bigg|_{p=0} = 0 \bigg\},\end{aligned} \tag{5.4}$$

while Hamilton's equations are

$$\begin{cases} w_q = (w_p + H_p)r, \\ r_q = \left(\dfrac{r^2}{2} + \dfrac{1}{2(w_p+H_p)^2} - \dfrac{1}{2H_p^2}\right)_p + \dfrac{1}{(F^\epsilon)^2}\rho_p w. \end{cases} \tag{5.5}$$

It is easy to see that (5.5) together with $(w,r) \in \mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$ and $r = w_q/(w_p + H_p)$ is formally equivalent to the height equation (2.31). We note that these equations are *reversible* in that, if $(w,r)(q)$ is a solution, then so is $\mathcal{S}(w,r)(-q)$, where

$$\mathcal{S}(w,r) := (w, -r)$$

is called the *reverser*.

Linearizing the Hamiltonian system $(\mathcal{M}, \omega, \mathcal{H}^0)$ with $\epsilon = 0$ about the equilibrium $(w,r) = (0,0)$ yields the linear problem

$$(\dot{w}, \dot{r})_q = L(\dot{w}, \dot{r})$$

where $L \colon \mathcal{D}(L) \subset \mathcal{X} \to \mathcal{X}$ is the closed operator

$$L(\dot{w}, \dot{r}) := \left(H_p \dot{r},\ -\left(\frac{\dot{w}_p}{H_p^3}\right)_p + \frac{1}{F_{\mathrm{cr}}^2}\rho_p \dot{w} \right)$$

with domain

$$\mathcal{D}(L) := \left\{ (\dot{w}, \dot{r}) \in \mathcal{Y} : \dot{r}(-1) = 0,\ \left(-\frac{\dot{w}_p}{H_p^3} + \frac{1}{F_{\mathrm{cr}}^2}\rho \dot{w} \right)\bigg|_{p=0} = 0 \right\}.$$

The operator L is related to the Sturm–Liouville problem studied in Section 3.1. Using the results from that section we obtain the following.

Lemma 5.2 (Spectral properties of L)**.**

(i) *The spectrum of L consists of an eigenvalue at 0 with algebraic multiplicity 2, together with simple eigenvalues $\pm\sqrt{\nu_j}$, where ν_j are the nonzero eigenvalues of the corresponding Sturm–Liouville problem* (3.1)*. The eigenvector and generalized eigenvector associated with the 0 eigenvalue are $(\Phi_{\mathrm{cr}}, 0)$ and $(0, \Phi_{\mathrm{cr}}/H_p)$.*

(ii) *There exists $\Xi > 0$ and $C > 0$ such that*

$$\|u\|_{\mathcal{Y}} \le C\|(L - i\xi I)u\|_{\mathcal{X}}, \qquad \|u\|_{\mathcal{X}} \le \frac{C}{|\xi|}\|(L - i\xi I)u\|_{\mathcal{X}},$$

for all $u \in \mathcal{D}(L)$ and $\xi \in \mathbb{R}$ with $|\xi| > \Xi$.

Proof. We begin with part (i). Notice that λ is an eigenvalue of L provided that there exists a $(w,r) \in \mathcal{D}(L) \setminus \{0\}$ satisfying

$$H_p r = \lambda w, \qquad -\left(\frac{w_p}{H_p^3}\right)_p + \frac{1}{F_{\mathrm{cr}}^2}\rho_p w = \lambda r.$$

The boundary conditions encoded in the definition of $\mathcal{D}(L)$ along with the above equation imply that w is a weak solution of the Sturm–Liouville problem (3.1) with $\nu = \lambda^2$. By Lemma 3.3, then, the eigenvalues of L are precisely of the form $\pm\sqrt{\nu}$, for $\nu \in \Sigma$. In particular, 0 is an eigenvalue for L with multiplicity 2 while the rest of the spectrum consists of simple nonzero real eigenvalues.

We defer the proof of (ii) to Appendix A.1. □

5.2. **Further change of variables.** As in [40], before applying a center manifold reduction to $(\mathcal{M}, \omega, \mathcal{H}^\epsilon)$, we will perform a change of dependent variables (in a neighborhood of the origin in $\mathcal{Y}$) which flattens $\mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$. Set

$$\begin{aligned} f(w,r) &:= \left(\frac{r^2}{2} + \frac{1}{2(w_p+H_p)^2} - \frac{1}{2H_p^2} + \frac{1}{(F^\epsilon)^2}\rho w\right) - \left(-\frac{w_p}{H_p^3} + \frac{1}{(F^\epsilon)^2}\rho w\right) \\ &= \frac{r^2}{2} + \frac{1}{2(w_p+H_p)^2} - \frac{1}{2H_p^2} + \frac{w_p}{H_p^3}, \end{aligned}$$

so that the nonlinear boundary condition in the definition (5.4) of $\mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$ can be written as

$$\left(-\frac{w_p}{H_p^3} + \frac{1}{(F^\epsilon)^2}\rho w\right)\bigg|_{p=0} = f(w,r)\Big|_{p=0}.$$

We replace w with the new unknown ξ defined by

$$\xi := \zeta + \epsilon\rho(0)H_p^3(0)(1+p)\int_p^0 \zeta(s)\,ds,$$

where

$$\zeta := w + H_p^3(0)(1+p)\int_p^0 f(w,r)(s)\,ds.$$

A calculation then shows that, at $p=0$,

$$-\frac{\xi_p}{H_p^3} + \frac{1}{F_{\mathrm{cr}}^2}\rho\xi = -\frac{\zeta_p}{H_p^3} + \frac{1}{(F^\epsilon)^2}\rho\zeta = -\frac{w_p}{H_p^3} + \frac{1}{(F^\epsilon)^2}\rho w - f(w,r),$$

and hence that ξ satisfies the linearized boundary condition

$$-\frac{\xi_p(0)}{H_p^3(0)} + \frac{1}{F_{\mathrm{cr}}^2}\rho(0)\xi(0) = 0$$

if and only if $(w,r) \in \mathcal{D}(\mathcal{V}_{\mathcal{H}^\epsilon})$.

Denoting the change of variables mapping by $G^\epsilon(w,r) := (\xi, r)$, we have the following lemma.

Lemma 5.3. *Restricted to a sufficiently small neighborhood of the origin in $\mathbb{R}\times\mathcal{Y}$,*

(i) *G^ϵ is a diffeomorphism onto its image, with G^ϵ and $(G^\epsilon)^{-1}$ depending smoothly on ϵ;*
(ii) *the derivative $DG^\epsilon(w,r)\colon \mathcal{Y}\to\mathcal{Y}$ extends to an isomorphism $\widehat{DG}^\epsilon(w,r)\colon \mathcal{X}\to\mathcal{X}$. This isomorphism and its inverse depend smoothly on (w,r,ϵ); and*
(iii) *G^ϵ is a near-identity transformation in that $G^\epsilon(0)=0$ for all ϵ and $DG^0(0)=\mathrm{id}$.*

This follows from a standard argument that we omit; see, for example, [38, Lemma 4.1]. Applying the change of variables $u = G^\epsilon(w,r)$ transforms our Hamiltonian system $(\mathcal{M}, \omega, \mathcal{H}^\epsilon)$ into $(\mathcal{M}, \omega_*, \mathcal{H}_*^\epsilon)$, where now ω_* is the (position-dependent) symplectic form

$$\omega_*|_u(v_1,v_2) := \omega\left(\widehat{DG}^\epsilon\big((G^\epsilon)^{-1}(u)\big)v_1,\ \widehat{DG}^\epsilon\big((G^\epsilon)^{-1}(u)\big)v_2\right)$$

while

$$\mathcal{H}_*^\epsilon(u) := \mathcal{H}\big((G^\epsilon)^{-1}(u)\big).$$

Hamilton's equations can be written abstractly as

$$u_q = Lu + N^\epsilon(u). \tag{5.6}$$

Note that the reverser $\mathcal{S}$ is unchanged in these coordinates,

$$G^\epsilon \circ \mathcal{S} \circ (G^\epsilon)^{-1} = \mathcal{S}.$$

The following technical lemma details how the spatial dynamics formulation of the problem in (5.6) relates to the original operator equation $\mathscr{F}(w, F) = 0$.

Lemma 5.4.

(i) *Let (w, F^ϵ) be a solution of $\mathscr{F}(w, F^\epsilon) = 0$ with $\|w\|_X$ and $|\epsilon|$ sufficiently small, and set*

$$u = G^\epsilon \left(w, \frac{w_q}{H_p + w_p} \right). \tag{5.7}$$

Then $u \in C_0^2(\mathbb{R}, \mathcal{X}) \cap C_0^1(\mathbb{R}, \mathcal{U})$ solves Hamilton's equations $u_q = Lu + N^\epsilon(u)$ and is reversible in that $u(-q) = \mathcal{S}u(q)$.

(ii) *Conversely, suppose that $u \in C_0^4(\mathbb{R}, \mathcal{X}) \cap C_0^3(\mathbb{R}, \mathcal{U})$ satisfies $u_q = Lu + N^\epsilon(u)$ and $u(-q) = \mathcal{S}u(q)$. Then u is related by (5.7) to some $w \in X$ solving $\mathscr{F}(w, F^\epsilon) = 0$. Moreover, the correspondence $(\epsilon, u) \mapsto w$ is continuous $\mathbb{R} \times C_0^4(\mathbb{R}, \mathcal{X}) \cap C_0^3(\mathbb{R}, \mathcal{U}) \to X$.*

(iii) *With u and w as above, suppose that $\dot{w} \in X$ is a nontrivial solution of the linearized problem $\mathscr{F}_w(w, F^\epsilon)\dot{w} = 0$. Then there is an associated nontrivial solution $\dot{u} \in C_{\mathrm{b}}^2(\mathbb{R}, \mathcal{X}) \cap C_{\mathrm{b}}^1(\mathbb{R}, \mathcal{D}(L))$ of the linearized problem $\dot{u}_q = L\dot{u} + DN^\epsilon(u)\dot{u}$ which is reversible in that $\dot{u}(-q) = \mathcal{S}\dot{u}(q)$.*

Proof. Given Lemma 5.3, the proof of (i) is straightforward and hence omitted. Likewise the proofs of (iii) and (ii) proceed as in [96, Lemmas 4.3 and 4.4]. ☐

5.3. **Center manifold reduction.** Up to this point, we have succeeded in transforming the original Hamiltonian system into one that is suitable for analysis via the center manifold reduction method. In particular, by transitioning to $(\mathcal{M}, \omega_*, \mathcal{H}_*^\epsilon)$, we have obtained a reversible Hamiltonian system with linear boundary conditions. Moreover, because G^ϵ is a near-identity mapping, the spectral properties of the linearized operator established in Lemma 5.2 translate to the new system as well.

Let $\mathcal{X}^{\mathrm{c}} \subset \mathcal{X}$ be the two-dimensional center subspace associated with the eigenvalue 0 of L. We denote by P^{c} the associated spectral projection, and write $P^{\mathrm{su}} := I - P^{\mathrm{c}}$, $\mathcal{X}^{\mathrm{su}} := P^{\mathrm{su}}\mathcal{X}$. We will write $u^{\mathrm{c}} \in P^{\mathrm{c}}\mathcal{D}(L)$ as $u^{\mathrm{c}} = z_1 e_1 + z_2 e_2$, where

$$e_1 := c_0^{-1/2}(\Phi_{\mathrm{cr}}, 0), \qquad e_2 := c_0^{-1/2}(0, \Phi_{\mathrm{cr}}/H_p), \tag{5.8}$$

are the eigenvector and generalized eigenvector from Lemma 5.2, and the normalization constant $c_0 > 0$ is given by

$$c_0 := \omega\big((\Phi_{\mathrm{cr}}, 0), (0, \Phi_{\mathrm{cr}}/H_p)\big) = \int_{-1}^0 \frac{\Phi_{\mathrm{cr}}^2}{H_p}\, dp > 0. \tag{5.9}$$

Lemma 5.5 (Center manifold reduction)**.** *For any integer $k \geq 2$, there exists a neighborhood $\Lambda \times \mathcal{U}$ of the origin in $\mathbb{R} \times \mathcal{D}(L)$ such that, for each $\epsilon \in \Lambda$, there exists a two-dimensional manifold $\mathcal{W}^\epsilon \subset \mathcal{U}$ together with an invertible coordinate map*

$$\chi^\epsilon := P^{\mathrm{c}}|_{\mathcal{W}^\epsilon} \colon \mathcal{W}^\epsilon \to \mathcal{U}^{\mathrm{c}} := P^{\mathrm{c}}\mathcal{U}$$

with the following properties:

(i) *Defining $\Psi^\epsilon \colon \mathcal{U}^{\mathrm{c}} \to \mathcal{U}^{\mathrm{su}} := P^{\mathrm{su}}\mathcal{U}$ by*

$$u^{\mathrm{c}} + \Psi^\epsilon(u^{\mathrm{c}}) := (\chi^\epsilon)^{-1}(u^{\mathrm{c}}),$$

the map $(\epsilon, u) \mapsto \Psi^\epsilon(u)$ is $C^k(\Lambda \times \mathcal{U}^{\mathrm{c}}, \mathcal{U}^{\mathrm{su}})$. Moreover $\Psi^\epsilon(0) = 0$ for all $\epsilon \in \Lambda$ and $D\Psi^0(0) = 0$.

(ii) *Every initial condition $u_0 \in \mathcal{W}^\epsilon$ determines a unique solution u of $u_q = Lu + N^\epsilon(u)$ which remains in $\mathcal{W}^\epsilon$ as long as it remains in $\mathcal{U}$.*

(iii) *If u solves $u_q = Lu + N^\epsilon(u)$ and lies in $\mathcal{U}$ for all q, then u lies entirely in $\mathcal{W}^\epsilon$.*

(iv) *If $u^{\mathrm{c}} \in C^1((a,b),\mathcal{U}^{\mathrm{c}})$ solves the reduced system*

$$u^{\mathrm{c}}_q = f^\epsilon(u^{\mathrm{c}}) := Lu^{\mathrm{c}} + P^{\mathrm{c}} N^\epsilon(u^{\mathrm{c}} + \Psi^\epsilon(u^{\mathrm{c}})), \tag{5.10}$$

then $u = (\chi^\epsilon)^{-1}(u^{\mathrm{c}})$ solves the full system $u_q = Lu + N^\epsilon(u)$.

(v) *With u^{c} and u as above, if $\dot{u}^{\mathrm{c}} \in C^1(\mathbb{R},\mathcal{U}^{\mathrm{c}})$ solves the linearized reduced equation $\dot{u}^{\mathrm{c}}_q = Df^\epsilon(u^{\mathrm{c}})\dot{u}^{\mathrm{c}}$, then $\dot{u} = \dot{u}^{\mathrm{c}} + D_u\Psi^\epsilon(u^{\mathrm{c}})\dot{u}^{\mathrm{c}}$ solves the full linearized system $\dot{u}_q = L\dot{u} + D_u N^\epsilon(u)\dot{u}$.*

(vi) *The reduced system (5.10) can be transformed via a C^{k-1} change of variables into a Hamiltonian system $(U^{\mathrm{c}},\gamma,K^\epsilon)$, where U^{c} is a neighborhood of the origin in $\mathbb{R}^2$, γ is the canonical symplectic form*

$$\gamma((z_1,z_2),(z_1',z_2')) := z_1 z_2' - z_1' z_2, \qquad (z_1,z_2),\,(z_1',z_2') \in \mathbb{R}^2,$$

and the reduced Hamiltonian is

$$K^\epsilon(z_1,z_2) := \mathcal{H}^\epsilon(z_1 e_1 + z_2 e_2 + \Theta^\epsilon(z_1 e_1 + z_2 e_2)). \tag{5.11}$$

Here $(\epsilon,u^{\mathrm{c}}) \mapsto \Theta^\epsilon(u^{\mathrm{c}})$ is of class $C^{k-1}(\Lambda \times \mathcal{U}^{\mathrm{c}},\mathcal{U})$ and satisfies $\Theta^\epsilon(0)=0$ for all $\epsilon \in \Lambda$, and $D_{u^{\mathrm{c}}}\Theta^0(0) = 0$. The system is reversible with reverser $S(z_1,z_2) = (z_1,-z_2)$.

Remark 5.6. For a solution (z_1,z_2) to the reduced Hamiltonian system from part (vi), the corresponding solution (w,r) of the original Hamiltonian system is given by $(w,r) = z_1 e_1 + z_2 e_2 + \Theta^\epsilon(z_1 e_1 + z_2 e_2)$.

Proof. Observe that L satisfies (H1) and (H2) of Theorem B.2 in light of Lemma 5.2, and the only part of its spectrum lying on the imaginary axis is the eigenvalue 0, which has algebraic multiplicity 2. The nonlinearity N^ϵ is, in fact, C^∞ in its dependence on ϵ and u in any small neighborhood of the origin. Inspecting (5.5), and keeping in mind that G^ϵ is a near-identity transformation, it is easy to confirm that $N^0(0) = 0$ and $D_u N^0(0) = 0$. We may therefore apply Theorem B.2 to the system $(\mathcal{M},\omega^\epsilon_*,\mathcal{H}^\epsilon_*)$, obtaining a two-dimensional center manifold $\mathcal{W}^\epsilon$ satisfying (i)–(iv). Part (v) requires a small additional argument that is discussed, for example, in [96, Lemma 4.4].

Now consider statement (vi). We begin by undoing the transformation G^ϵ, since it is considerably simpler to work in the original variables. Define $\Upsilon^\epsilon \in C^k(\Lambda \times \mathcal{U}^{\mathrm{c}},\mathcal{U})$ by the relation

$$u^{\mathrm{c}} + \Upsilon^\epsilon(u^{\mathrm{c}}) := (G^\epsilon)^{-1}\left(u^{\mathrm{c}} + \Psi^\epsilon(u^{\mathrm{c}})\right).$$

Then $\Upsilon^\epsilon(0) = 0$ for all $\epsilon \in \Lambda$, and $D_{u^{\mathrm{c}}}\Upsilon^0(0) = 0$. Here we are simply relying on the fact that G^ϵ is near-identity.

From Theorem B.2(v), we know that the center manifold $(\mathcal{W}^\epsilon, \omega^\epsilon_*|_{\mathcal{W}^\epsilon}, \mathcal{H}^\epsilon_*|_{\mathcal{W}^\epsilon})$ is a symplectic submanifold of $(\mathcal{M},\omega^\epsilon_*,\mathcal{H}^\epsilon_*)$. Changing variables using G^ϵ we obtain a new manifold $\mathcal{V}^\epsilon$ that we equip with the chart $(\mathrm{id} + \Upsilon^\epsilon)^{-1} \colon \mathcal{V}^\epsilon \to \mathcal{U}^{\mathrm{c}}$. Working in these coordinates, the symplectic form $\omega^\epsilon_*|_{\mathcal{W}^\epsilon}$ becomes ϖ^ϵ, which, for $u^{\mathrm{c}} \in \mathcal{U}^{\mathrm{c}}$ and $v^{\mathrm{c}}_1, v^{\mathrm{c}}_2 \in \mathcal{X}^{\mathrm{c}}$, is given by

$$\begin{aligned} \varpi^\epsilon|_{u^{\mathrm{c}}}(v^{\mathrm{c}}_1,v^{\mathrm{c}}_2) &= \omega^\epsilon_*|_{u^{\mathrm{c}}+\Psi^\epsilon(u^{\mathrm{c}})}(v^{\mathrm{c}}_1 + D_{u^{\mathrm{c}}}\Psi^\epsilon(u^{\mathrm{c}})v^{\mathrm{c}}_1, v^{\mathrm{c}}_2 + D_{u^{\mathrm{c}}}\Psi^\epsilon(u^{\mathrm{c}})v^{\mathrm{c}}_2) \\ &= \omega(v^{\mathrm{c}}_1,v^{\mathrm{c}}_2) + \mathcal{O}(|\epsilon|\|u^{\mathrm{c}}\|_{\mathcal{X}}\|v^{\mathrm{c}}_1\|_{\mathcal{X}}\|v^{\mathrm{c}}_2\|_{\mathcal{X}}). \end{aligned} \tag{5.12}$$

Since χ^ϵ and $\mathcal{S}$ are linear maps which commute, it is easy to check that, in these coordinates, the reverser is simply $\mathcal{S}|_{\mathcal{U}^{\mathrm{c}}}$.

Thanks to (5.12), we can now employ a parameter-dependent Darboux transformation

$$u^{\mathrm{c}} \mapsto u^{\mathrm{c}} + \Xi^\epsilon(u^{\mathrm{c}}) \tag{5.13}$$

which is C^{k-1} and transforms ϖ^ϵ into ω in a neighborhood of the origin (see [18, Theorem 4]). This map is near-identity in that $\Xi^\epsilon(0) = 0$ and $D_{u^{\mathrm{c}}}\Xi^0(0) = 0$. We now equip $\mathcal{V}^\epsilon$ with the new chart $(\mathrm{id}+\Theta^\epsilon)^{-1} \colon \mathcal{V}^\epsilon \to \mathcal{U}^{\mathrm{c}}$ where Θ^ϵ is defined by

$$\mathrm{id}+\Theta^\epsilon := (\mathrm{id}+\Upsilon^\epsilon) \circ (\mathrm{id}+\Xi^\epsilon)^{-1}, \tag{5.14}$$

and $\mathrm{id}+\Theta^\epsilon$ is also near-identity. In these coordinates, our Hamiltonian system becomes $(\mathcal{U}^{\mathrm{c}}, \omega, \mathcal{K}^\epsilon)$, where the Hamiltonian $\mathcal{K}^\epsilon$ is

$$\mathcal{K}^\epsilon(u^{\mathrm{c}}) = \mathcal{H}^\epsilon(u^{\mathrm{c}} + \Theta^\epsilon(u^{\mathrm{c}})).$$

The Darboux transformation $\mathrm{id}+\Xi^\epsilon$ can be chosen so that the action of the reverser in these coordinates is still given by $\mathcal{S}|_{\mathcal{U}^{\mathrm{c}}}$; see the arguments leading to [71, Theorem 5.17].

We identify $\mathcal{U}^{\mathrm{c}}$ with a neighborhood of the origin $U^{\mathrm{c}} \subset \mathbb{R}^2$ via the mapping $U^{\mathrm{c}} \ni (z_1, z_2) \mapsto z_1 e_1 + z_2 e_2 \in \mathcal{U}^{\mathrm{c}}$. Notice that

$$\begin{aligned} \omega(z_1 e_1 + z_2 e_2, z_1' e_1 + z_2' e_2) &= z_1 z_2' \omega(e_1, e_2) + z_1' z_2 \omega(e_2, e_1) \\ &= \gamma((z_1, z_2), (z_1', z_2')), \end{aligned}$$

for all $(z_1, z_2), (z_1', z_2') \in \mathbb{R}^2$, and that

$$\mathcal{S}(z_1 e_1 + z_2 e_2) = z_1 e_2 - z_2 e_2.$$

Thus, we obtain the reversible and canonical Hamiltonian system $(U^{\mathrm{c}}, \gamma, K^\epsilon)$, proving (vi). $\square$

Thanks to the clever choice of coordinates outlined in [40] and leading to the system described in Lemma 5.5(vi), we can Taylor expand the reduced system (5.10) by using (5.11) directly, and avoid dealing with the implicitly defined intermediate Hamiltonian $\mathcal{H}_*^\epsilon$ entirely. The result of this calculation, which is presented in Appendix A.2, is that the reduced system (5.10) is equivalent to the following ODE set in $\mathbb{R}^2$:

$$\begin{cases} z_{1q} = z_2 + \mathcal{R}_1(z_1, z_2, \epsilon) \\ z_{2q} = \epsilon c_0^{-1} c_1 z_1 - \dfrac{3}{2} c_0^{-3/2} c_2 z_1^2 + \mathcal{R}_2(z_1, z_2, \epsilon), \end{cases}$$

where c_0 was defined in (5.9),

$$c_1 := \rho(0)\Phi_{\mathrm{cr}}(0)^2 - \int_{-1}^0 \rho_p \Phi_{\mathrm{cr}}^2 \, dp, \qquad c_2 := \int_{-1}^0 \frac{(\partial_p \Phi_{\mathrm{cr}})^3}{H_p^4} \, dp, \tag{5.15}$$

and

$$\begin{aligned} \mathcal{R}_1(z_1, z_2, \epsilon) &= \mathcal{O}(|(z_1, z_2)|^2 + |z_2||(\epsilon, z_1, z_2)|^2 + |\epsilon||(z_1, z_2)|), \\ \mathcal{R}_2(z_1, z_2, \epsilon) &= \mathcal{O}(|z_1||(\epsilon, z_2)|^2 + |z_2||(\epsilon, z_1)|), \end{aligned}$$

are higher order remainder terms. The reversal symmetry implies that $\mathcal{R}_1$ is odd in z_2 and $\mathcal{R}_2$ is even in z_2. We can simplify things even further by introducing the scaled variables (Z_1, Z_2) and Q defined as

$$z_1 =: |\epsilon| c_0^{1/2} c_1 c_2^{-1} Z_1, \qquad z_2 =: |\epsilon|^{3/2} c_1^{3/2} c_2^{-1} Z_2, \qquad q =: |\epsilon|^{-1/2} c_0^{1/2} c_1^{-1/2} Q. \tag{5.16}$$

This transforms the system into

$$\begin{cases} Z_{1Q} = Z_2 + \mathcal{R}_3(Z_1, Z_2, \epsilon) \\ Z_{2Q} = Z_1 - \dfrac{3}{2} (\operatorname{sgn} \epsilon) Z_1^2 + \mathcal{R}_4(Z_1, Z_2, \epsilon), \end{cases} \tag{5.17}$$

where $\mathcal{R}_3$ and $\mathcal{R}_4$ are new remainder terms. From the estimates of $\mathcal{R}_1, \mathcal{R}_2$ and the change of variable formulas above, it is clear that

$$\mathcal{R}_3, \ \mathcal{R}_4, \ D_{(Z_1,Z_2)}\mathcal{R}_3, \ D_{(Z_1,Z_2)}\mathcal{R}_4 = \mathcal{O}(|\epsilon|^{1/2}).$$

These calculations lead directly to the following result.

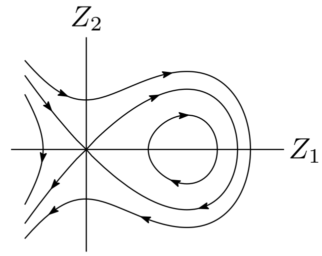


FIGURE 5. Phase portrait for the reduced system (5.17) when $\epsilon = 0$.

Lemma 5.7 (Existence of w^ϵ)**.** *There exists $\epsilon_* > 0$ such that, for each $\epsilon \in (0, \epsilon_*)$, there is a solution $(w^\epsilon, F^\epsilon) \in X \times \mathbb{R}$ to the height equation* (2.31)*. Moreover, the map $\epsilon \mapsto w^\epsilon$ is continuous $(0, \epsilon_*) \to X$, and $\|w^\epsilon\|_X \to 0$ as $\epsilon \to 0$.*

Proof. Observe that, when $\epsilon = 0$, the reduced and rescaled system (5.17) becomes

$$-Z_1 + \frac{3}{2} Z_1^2 + Z_{1QQ} = 0, \tag{5.18}$$

whose phase portrait is shown in Figure 5. This is exactly the equation satisfied by the Korteweg–de Vries soliton $Z_1 = \operatorname{sech}^2(Q/2)$ (with unit wave speed). We conclude, therefore, that at $\epsilon = 0$, (5.17) has an orbit (Z_1^0, Z_2^0) homoclinic to 0. The reversibility of the system guarantees that there is a nearby reversible homoclinic orbit $(Z_1^\epsilon, Z_2^\epsilon)$ for $\epsilon > 0$ sufficiently small (see, e.g., [53, Proposition 5.1] and the surrounding comments). By choosing the k in Lemma 5.5 appropriately, we can ensure that the map $\epsilon \mapsto (Z_1^\epsilon, Z_2^\epsilon)$ is continuous with values in $C_0^4(\mathbb{R}, \mathbb{R}^2)$.

By Lemma 5.5, for each $\epsilon > 0$ sufficiently small, there is a corresponding solitary wave solution

$$(w^\epsilon, r^\epsilon, F^\epsilon) \in \big(C_0^4(\mathbb{R}, \mathcal{X}) \cap C_0^3(\mathbb{R}, \mathcal{U})\big) \times \mathbb{R}$$

of the full system (5.5). By Lemma 5.4(ii), there exists a corresponding solution $(w^\epsilon, F^\epsilon) \in X \times \mathbb{R}$ of the original height equation (2.31). Thus, there is an $\epsilon_* > 0$ and a local curve of solutions $\mathscr{C}_{\mathrm{loc}} = \{(w^\epsilon, F^\epsilon) : \epsilon \in (0, \epsilon_*)\}$. Finally, undoing the scaling and the various changes of variable, we see that $\epsilon \mapsto w^\epsilon$ is continuous $(0, \epsilon_*) \to X$ with $\|w^\epsilon\|_X \to 0$ as $\epsilon \to 0$. ☐

Remark 5.8. The waves constructed in Lemma 5.7 are solitary waves. As in [40], for $\epsilon < 0$ with $|\epsilon|$ sufficiently small, a similar argument guarantees the existence of a family of symmetric periodic waves of cnoidal type with period $\mathcal{O}(|\epsilon|^{1/2})$.

5.4. **Uniqueness and elevation.** In this subsection we show that the waves (w^ϵ, F^ϵ) constructed in Lemma 5.7 are waves of elevation. Moreover, they are the unique waves of elevation with w and ϵ sufficiently small.

Lemma 5.9 (Elevation)**.** *After possibly shrinking the range of ϵ, all of the waves w^ϵ constructed above are waves of elevation in that $w^\epsilon > 0$ on $R \cup T$.*

Proof. For each $\epsilon \geq 0$, consider the rescaled solution $(Z_1^\epsilon, Z_2^\epsilon)$ of (5.17). Since $Z_1^0(Q) > 0$ for all Q, the continuous dependence of Z_1^ϵ as well as the stable and unstable manifolds of (5.17) at the origin on ϵ imply that $Z_1^\epsilon(Q) > 0$ for all Q when ϵ is sufficiently small. Since

$$\lim_{Q \to \pm\infty} \frac{Z_2^\epsilon(Q)}{Z_1^\epsilon(Q)} = \pm 1 + \mathcal{O}(\epsilon^{1/2}), \tag{5.19}$$

(see Figure 5) this continuous dependence also means that, shrinking ϵ further,

$$|Z_2^\epsilon(Q)| \leq C |Z_1^\epsilon(Q)| \qquad \text{for all } Q,$$

where C is independent of ϵ. Undoing the scaling in (5.16), we find that $z_1^\epsilon(q) > 0$ and

$$|z_2^\epsilon(q)| \le C\epsilon^{1/2}|z_1^\epsilon(q)|. \tag{5.20}$$

Going back through the many changes of variables (see Remark 5.6), we see that we can write

$$w^\epsilon(q,p) = c_0^{-1/2} z_1^\epsilon(q)\Phi_{\mathrm{cr}}(p) + \mathcal{R}(q,p), \tag{5.21}$$

where the remainder term satisfies

$$\|\mathcal{R}(q,\,\cdot\,)\|_{H^2} \le C\big(\epsilon + |z_1^\epsilon(q)| + |z_2^\epsilon(q)|\big)\big(|z_1^\epsilon(q)| + |z_2^\epsilon(q)|\big).$$

In particular, taking ϵ small enough, (5.20) ensures that

$$\|\mathcal{R}_p(q,\,\cdot\,)\|_{L^\infty} \le \frac{c_0^{-1/2}}{2}\min[(\Phi_{\mathrm{cr}})_p] z_1^\epsilon(q), \tag{5.22}$$

where we recall from Lemma 3.1(b) that $(\Phi_{\mathrm{cr}})_p > 0$. Differentiating (5.21) with respect to p then yields

$$w_p^\epsilon(q,p) \ge c_0^{-1/2} z_1^\epsilon(q)(\Phi_{\mathrm{cr}})_p(p) - \frac{c_0^{-1/2}}{2}\min[(\Phi_{\mathrm{cr}})_p] z_1^\epsilon(q) > 0$$

for all $(q,p) \in \overline{R}$. Integrating with respect to p using the fact that $w = 0$ on $B = \{p = -1\}$, we conclude that $w^\epsilon > 0$ on $R \cup T$ as desired. ☐

Lemma 5.10 (Uniqueness)**.** *Suppose that $(w, F^\epsilon) \in X \times \mathbb{R}$ is a solution of* (2.31) *with F^ϵ defined as in* (5.2)*. If $\epsilon > 0$ and $\|w\|_{C^2(R)}$ are sufficiently small and $w > 0$ on T, then $w = w^\epsilon$.*

Proof. Suppose that we have a solution (w, F^ϵ) of (2.31) with $\epsilon + \|w\|_{C^2(R)} < \delta$, where $\delta > 0$ is to be determined, and assume that $w \not\equiv w^\epsilon$. We will show that w is not a wave of elevation.

By the properties of the center manifold, w is determined by a homoclinic orbit (z_1, z_2) of (5.10). Since this equation already has a homoclinic orbit, namely $(z_1^\epsilon, z_2^\epsilon)$, and w cannot be a translate of w^ϵ, (z_1, z_2) is not a translate of $(z_1^\epsilon, z_2^\epsilon)$. Looking at the phase portrait at the origin (see Figure 5), we conclude that $z_1 < 0$ for $|q|$ sufficiently large, and also that (compare with the signs in (5.19))

$$\lim_{q\to\pm\infty} \frac{z_2(q)}{z_1(q)} = \mp\epsilon^{1/2} + \mathcal{O}(\epsilon).$$

Thus, as in the proof of Lemma 5.9 above, we can shrink δ so that

$$w(q,p) = c_0^{-1/2} z_1(q)\Phi_{\mathrm{cr}}(p) + \mathcal{R}(q,p). \tag{5.23}$$

where the remainder term has the bound

$$\|\mathcal{R}_p(q,\,\cdot\,)\|_{L^\infty} \le \frac{c_0^{-1/2}}{2}\min[(\Phi_{\mathrm{cr}})_p]|z_1(q)|. \tag{5.24}$$

Differentiating (5.23) with respect to p and plugging in (5.24), we conclude

$$w_p(q,p) \le c_0^{-1/2} z_1(q)(\Phi_{\mathrm{cr}})_p(p) + \frac{c_0^{-1/2}}{2}\min[(\Phi_{\mathrm{cr}})_p]|z_1(q)| < 0$$

as soon as q is large enough that $z_1(q) < 0$. Integrating with respect to p, we find, for instance, that $w(q,0) < 0$ for such q, and hence in particular that w cannot be a wave of elevation. ☐

5.5. **Proof of small-amplitude existence.** We now have all of the major components necessary to prove our main theorem.

Proof of Theorem 5.1. We already established the existence of the one-parameter family (w^ϵ, F^ϵ) as well as (i) in Lemma 5.7. Likewise, parts (iii) and (iv) of the theorem statement were already proved in Lemmas 5.9 and 5.10.

Next we prove (ii). Since $\epsilon > 0$ implies $F^\epsilon > F_{\mathrm{cr}}$, Lemma 3.6 guarantees that $\mathscr{F}_w(w^\epsilon, F^\epsilon)$ is Fredholm with index 0, and so it suffices to show that it has trivial kernel. If $\dot{w} \in X$ is an element of the kernel, then Lemma 5.4(iii) gives us a corresponding reversible solution $\dot{u}$ of the linearized Hamiltonian system

$$\dot{u}_q = L\dot{u} + DN^\epsilon(u^\epsilon)\dot{u}, \tag{5.25}$$

where u^ϵ is related to w^ϵ via (5.7). As in [96, Lemmas 4.14 and 4.15], a two-dimensional space of solutions to (5.25) can be lifted from solutions to the two-dimensional reduced equation $\dot{u}^{\mathrm{c}}_q = Df^\epsilon(P^{\mathrm{c}}u^\epsilon)\dot{u}^{\mathrm{c}}$ via Lemma 5.5(v). Solutions to (5.25) off this set grow exponentially as $q \to \infty$ or as $q \to -\infty$. But one of these solutions coming from the reduced equation is just u^ϵ_q, which is not reversible, and the other has mild exponential growth as $|q| \to \infty$. Thus the only possibility is $\dot{u} \equiv 0$, which forces $\dot{w} \equiv 0$ as desired.

All that remains, therefore, is part (v). But, for any $\epsilon \in (0, \epsilon_*)$, $\mathscr{F}_w(w^\epsilon, F^\epsilon)$ is an isomorphism by the above reasoning, and moreover $\mathscr{F}$ is real analytic. Applying the real analytic implicit function theorem, we find that w^ϵ depends analytically on $\epsilon \in (0, \epsilon_*)$, which completes the theorem. □

6. Large-amplitude existence theory

The final step in the proof of Theorem 1.1 is to show that the local solution curve $\mathscr{C}_{\mathrm{loc}}$ can be continued to obtain a curve $\mathscr{C}$ of large-amplitude solutions. This will be accomplished using an argument based on analytic global bifurcation theory. Unfortunately, the existing literature does not immediately apply to our problem for two reasons: (i) the point of bifurcation $(0, F_{\mathrm{cr}})$ is singular in the sense that $\mathscr{F}_w(0, F_{\mathrm{cr}})$ fails to be a Fredholm operator with index 0; and (ii) it is far from obvious that $\mathscr{F}^{-1}(0)$ is locally compact.

With that in mind, in the next subsection we provide an abstract global-bifurcation-theoretic result that does hold in this more general context. The price we pay for this is the appearance of a new (and undesirable) possibility for the behavior of the global solution curve (see Figure 6(a) for an illustration). For a large class of elliptic systems set on infinite cylinders, we show that this undesirable alternative occurs precisely when there exists a sequence of translated solutions along the curve that converges to a front. At last, in Section 6.2, these results are applied to our problem and combined with the qualitative theory developed in Section 4 to complete the proof of Theorem 1.1.

6.1. **Global continuation.** Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $\mathcal{I}$ an open interval (possibly unbounded) with $0 \in \overline{\mathcal{I}}$, and $\mathcal{U} \subset \mathcal{X}$ an open set with $0 \in \partial\mathcal{U}$. Consider the abstract operator equation

$$\mathcal{F}(x, \lambda) = 0,$$

where $\mathcal{F} \colon \mathcal{U} \times \mathcal{I} \to \mathcal{Y}$ is an analytic mapping. Assume that for any $(x, \lambda) \in \mathcal{U} \times \mathcal{I}$ with $\mathcal{F}(x, \lambda) = 0$, the Fréchet derivative $\mathcal{F}_x(x, \lambda) \colon \mathcal{X} \to \mathcal{Y}$ is Fredholm with index 0.

Theorem 6.1 (Global continuation)**.** *Suppose that there exists a continuous curve $\mathscr{C}_{\mathrm{loc}}$ of solutions to $\mathcal{F}(x, \lambda) = 0$, parametrized as*

$$\mathscr{C}_{\mathrm{loc}} := \{(\tilde{x}(\lambda), \lambda) : 0 < \lambda < \lambda_*\} \subset \mathcal{F}^{-1}(0)$$

for some $\lambda_* > 0$ and continuous $\tilde{x} \colon (0, \lambda_*) \to \mathcal{U}$. If

$$\lim_{\lambda \searrow 0} \tilde{x}(\lambda) = 0 \in \partial\mathcal{U}, \qquad \mathcal{F}_x(\tilde{x}(\lambda), \lambda) \colon \mathcal{X} \to \mathcal{Y} \text{ is invertible for all } \lambda, \tag{6.1}$$

then $\mathscr{C}_{\mathrm{loc}}$ is contained in a curve of solutions $\mathscr{C}$, parametrized as

$$\mathscr{C} := \{(x(s), \lambda(s)) : 0 < s < \infty\} \subset \mathcal{F}^{-1}(0)$$

for some continuous $(0,\infty) \ni s \mapsto (x(s), \lambda(s)) \in \mathcal{U} \times \mathcal{I}$, with the following properties.

(a) *One of the following alternatives holds:*
 (i) (Blowup) *As $s \to \infty$,*

$$N(s) := \|x(s)\|_{\mathcal{X}} + \frac{1}{\operatorname{dist}(x(s), \partial\mathcal{U})} + \lambda(s) + \frac{1}{\operatorname{dist}(\lambda(s), \partial\mathcal{I})} \to \infty. \tag{6.2}$$

 (ii) (Loss of compactness) *There exists a sequence $s_n \to \infty$ such that* $\sup_n N(s_n) < \infty$ *but $\{x(s_n)\}$ has no subsequences converging in $\mathcal{X}$.*

(b) *Near each point $(x(s_0), \lambda(s_0)) \in \mathscr{C}$, we can reparametrize $\mathscr{C}$ so that $s \mapsto (x(s), \lambda(s))$ is real analytic.*

(c) *$(x(s), \lambda(s)) \notin \mathscr{C}_{\mathrm{loc}}$ for s sufficiently large.*

Proof. Set $(x_0, \lambda_0) := (0, 0) \in \partial(\mathcal{U} \times \mathcal{I})$. Since we are not guaranteed that $\mathcal{F}(x_0, \lambda_0)$ exists or that $\mathcal{F}_x(x_0, \lambda_0)$ is Fredholm, we cannot start our continuation argument there. Instead, we will start our argument at $(x_{1/2}, \lambda_{1/2}) := (\tilde{x}(\lambda_*/2), \lambda_*/2)$.

Since $\mathcal{F}$ is real analytic and $\mathcal{F}_x(x_{1/2}, \lambda_{1/2})$ is invertible by (6.1), $(x_{1/2}, \lambda_{1/2})$ lies in some connected component A_0 of

$$\mathcal{A} := \Big\{(x, \lambda) \in \mathcal{U} \times \mathcal{I} : \mathcal{F}(x, \lambda) = 0,\ \mathcal{F}_x(x, \lambda) \text{ is invertible}\Big\}.$$

Following [20], we will call such connected components *distinguished arcs*. The analytic implicit function theorem guarantees that, like all distinguished arcs, A_0 is a graph:

$$A_0 = \{(\widetilde{x}_0(\lambda), \lambda) : \lambda \in I_0\},$$

where $I_0 \subset \mathcal{I}$ is a (possibly unbounded) open interval and $\widetilde{x}_0 \colon I_0 \to \mathcal{X}$ is analytic. For notational convenience, we will also reparametrize A_0 as

$$A_0 = \{(x(s), \lambda(s)) : 0 < s < 1\}$$

where $\lambda(s)$ is increasing.

An easy argument using the implicit function theorem shows that $\mathscr{C}_{\mathrm{loc}}$ lies entirely in A_0. Thus $I_0 = (0, \lambda_1)$ for some $\lambda_1 \geq \lambda_*$, and A_0 has "starting endpoint"

$$\lim_{s \searrow 0} (x(s), \lambda(s)) = (x_0, \lambda_0) = (0, 0)$$

by (6.1). Next we turn our attention to the limit $s \nearrow 1$. We will show that either $\mathscr{C} := A_0$ satisfies (i) or (ii) (after reparametrization) or that A_0 connects to another distinguished arc A_1. If $N(s) \to \infty$ as $s \nearrow 1$, then, after a trivial reparametrization mapping $(0,1)$ to $(0,\infty)$, (i) occurs and we are done. So assume that (6.2) does not hold as $s \nearrow 1$, in which case we can find a sequence $\{s_n\} \subset (1/2, 1)$ with $s_n \nearrow 1$ so that $N(s_n) \leq M < \infty$ for all n. Without loss of generality we can assume that $\lambda(s_n) \to \lambda_1 \in \mathcal{I}$. If $\{x(s_n)\}$ has no convergent subsequences in $\mathcal{X}$, then (ii) occurs, and again we are done after a trivial reparametrization. Assume instead, passing to a subsequence, that we have $x(s_n) \to x_1 \in \mathcal{U}$.

By continuity, $\mathcal{F}(x_1, \lambda_1) = 0$, and hence $\mathcal{F}_x(x_1, \lambda_1)$ is Fredholm with index 0. Let $1 \leq k < \infty$ be the dimension of its null space. Using the real-analytic Lyapunov–Schmidt procedure as in [20, Theorem 8.2.1 and Remark 8.2.2], the problem of solving $\mathcal{F}(x, \lambda) = 0$ near (x_1, λ_1) can be reduced to a finite-dimensional system. That is, there exists a real-analytic bifurcation function $\mathcal{B} \colon \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$, and a real-analytic bijection mapping the zero-set of $\mathcal{B}$ in neighborhood of the origin to the zero-set of $\mathcal{F}$ near (x_1, λ_1). Complexifying $\mathcal{B}$ in the natural way, we see that its zero-set represents a complex-analytic variety, which implies a great deal about its structure; see [20, Theorem 7.4.7]. Arguing as in Steps 1 and 2 of the proof of [20, Theorem 9.1.1], we find that

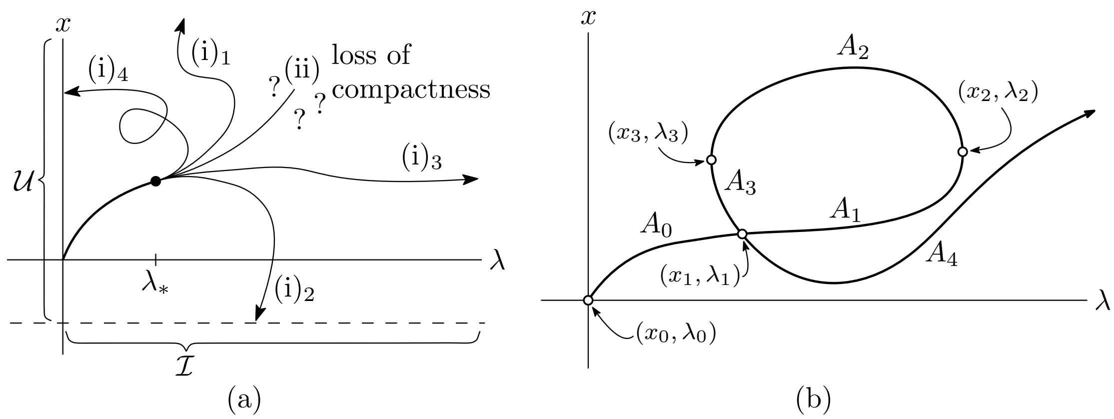


FIGURE 6. (a) Several ways for the alternatives (i) and (ii) in Theorem 6.1 to hold. In $(\mathrm{i})_1$, the first term on the right-hand side of 6.2 tends to infinity as $s \to \infty$. In $(\mathrm{i})_2$, it is the second term on the right-hand side of 6.2, and so on. Alternative (ii) holds if the curve loses compactness even while $N(s)$ remains bounded. (b) The distinguished arcs in the proof of Theorem 6.1.

$(x(s), \lambda(s)) \to (x_1, \lambda_1)$ as $s \nearrow 1$, and that, at (x_1, λ_1), the distinguished arc A_0 is uniquely paired to another (different) distinguished arc A_1, parametrized as

$$A_1 = \{(x(s), \lambda(s)) : 1 < s < 2\}$$

and with $(x(s), \lambda(s)) \to (x_1, \lambda_1)$ as $s \searrow 1$. Moreover, we can choose $s \mapsto (x(s), \lambda(s))$ to be analytic in a neighborhood of $s = 1$, perhaps at the cost of giving up analyticity for other values of s. Note that, by the implicit function theorem, the closures $\overline{A_0}$ and $\overline{A_1}$ can only meet at (x_1, λ_1) and perhaps also at the other endpoint (x_0, λ_0). In particular, $\overline{A_1} \cap \mathscr{C}_{\mathrm{loc}} = \varnothing$.

We can now argue for A_1 as we did for A_0. We find that either (after a trivial reparametrization) $\mathscr{C} := A_0 \cup A_1$ satisfies (i) or (ii), or that A_1 terminates in a second endpoint (x_2, λ_2) as $s \nearrow 2$, and that at this point A_1 is uniquely paired with another distinguished arc A_2; see Figure 6(b). The parametrization can be chosen to be analytic near (x_2, λ_2), and $\overline{A_1 \cup A_2} \cap \mathscr{C}_{\mathrm{loc}} = \varnothing$. Continuing in this way, we can assume that we have an infinite sequence of distinguished arcs

$$A_n = \{(x(s), \lambda(s)) : n < s < n + 1\}$$

with (two-sided) limits $(x(s), \lambda(s)) \to (x_n, \lambda_n)$ as $s \to n$. The unique way consecutive arcs are paired at the endpoints (x_n, λ_n), together with the fact that the only "unpaired" endpoint (x_0, λ_0) lies in $\partial(\mathcal{U} \times \mathcal{I})$ and not in $\mathcal{U} \times \mathcal{I}$, guarantees that all of the A_n are distinct. Setting $\mathscr{C} := \cup_n A_n$, the implicit function theorem guarantees as before that $\cup_{n \geq 1} \overline{A_n} \cap \mathscr{C}_{\mathrm{loc}} = \varnothing$, which implies (c).

It remains to show, in this case, that either (i) or (ii) holds. Seeking a contradiction, suppose that neither holds, so that we can find a sequence $s_n \to \infty$ with $N(s_n) \leq M < \infty$ and where $(x(s_n), \lambda(s_n)) \to (x_\infty, \lambda_\infty) \in \mathcal{U} \times \mathcal{I}$ and $\mathcal{F}(x_\infty, \lambda_\infty) = 0$. Therefore any neighborhood of the point $(x_\infty, \lambda_\infty)$ intersects infinitely many distinguished arcs. But, applying the analytic Lyapunov–Schmidt reduction near $(x_\infty, \lambda_\infty)$ along with the structure theorem for real-analytic varieties as in Step 4 of the proof of [20, Theorem 9.1.1], we see that sufficiently small neighborhoods of $(x_\infty, \lambda_\infty)$ only meet a finite number of distinguished arcs, a contradiction.

Finally, we point out that this construction also yields (b): in the interior of a distinguished arc, one can apply the analytic implicit function theorem to reparameterize, and at the end points, we get it from the analytic Lyapunov–Schmidt reduction instead (cf. [20, Theorem 9.1.1(d)]). $\square$

Remark 6.2. This theorem represents a somewhat atypical global bifurcation result. More commonly, one begins with a local curve that bifurcates from a trivial solution and perhaps lies inside some open set $\mathcal{O}$. Provided that the necessary compactness hypotheses are met, then one expects to obtain a statement of the form: there exists a global continuation of the local curve that either is unbounded, contains a sequence limiting to $\partial\mathcal{O}$, or is a closed loop.

The situation here is quite different as the trivial solution is not the initial point of bifurcation, and in fact it lies in $\partial(\mathcal{U} \times \mathcal{I})$. It is therefore impossible to have a closed loop in this case, since $\mathscr{C} = \bigcup_n A_n$, where each distinguished arc A_n is *distinct*.

Next let us look more closely at alternative (ii) of Theorem 6.1. In doing so, we will restrict our attention to the case of an elliptic PDE that generalizes the height equation (2.31). Fix an integer $k \geq 0$, and let $\Omega := \mathbb{R} \times \mathcal{B}$ be an infinite cylinder whose base $\mathcal{B} \subset \mathbb{R}^{n-1}$ is a bounded $C^{k+2+\alpha}$ domain. We will denote points in Ω as (x, y) where $x \in \mathbb{R}$ and $y \in \mathcal{B}$. Partition the components of $\partial\mathcal{B}$ as $\partial\mathcal{B} = \partial_1\mathcal{B} \cup \partial_2\mathcal{B}$ (either may be empty). Consider a nonlinear elliptic problem

$$\begin{cases} \mathcal{F}(y, u, Du, D^2u, \lambda) = 0 \text{ in } \Omega, \\ \mathcal{G}(y, u, Du, \lambda) = 0 \text{ on } \mathbb{R} \times \partial_1\mathcal{B}, \\ u = 0 \text{ on } \mathbb{R} \times \partial_2\mathcal{B}, \end{cases} \tag{6.3}$$

where the parameter $\lambda \in \mathbb{R}^m$ and where $\mathcal{F}$ and $\mathcal{G}$ have the regularity

$$\mathcal{F} \in C_{\mathrm{b}}^{k+1+\alpha}(\overline{\mathcal{B}} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n\times n} \times \mathbb{R}^m), \qquad \mathcal{G} \in C_{\mathrm{b}}^{k+2+\alpha}(\mathbb{R} \times \partial_1\mathcal{B} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m).$$

Moreover, assume that $\mathcal{F}$ is uniformly elliptic in that

$$\mathcal{F}_{r^{ij}}(y, z, \xi, r, \lambda)\eta_i\eta_j \geq c|\eta|^2 \tag{6.4}$$

for all $y, z, \xi, r, \eta, \lambda$ while $\mathcal{G}$ is uniformly oblique in that

$$\mathcal{G}_{\xi^i}(y, z, \xi, \lambda)\nu_i > c \tag{6.5}$$

for all z, ξ, λ and $y \in \partial_1\mathcal{B}$, where here ν is the outward pointing normal to $\partial\Omega$.

Lemma 6.3 (Compactness or front). *If $\{(u_n, \lambda_n)\}$ is a sequence of solutions to* (6.3) *that is uniformly bounded in $C_{\mathrm{b}}^{k+2+\alpha}(\overline{\Omega}) \times \mathbb{R}^m$, with the additional monotonicity property*

$$u_n(x, y) \text{ is even in } x \text{ and has } \partial_x u_n \leq 0 \text{ for } x \geq 0 \tag{6.6}$$

for each n as well as the asymptotic condition

$$\lim_{|x|\to\infty} u_n(x, y) = U(y) \quad \text{uniformly in } y \tag{6.7}$$

for some fixed function $U \in C_{\mathrm{b}}^{k+2+\alpha}(\overline{\mathcal{B}})$, then either

(i) *we can extract a subsequence so that $u_n \to u$ in $C_{\mathrm{b}}^{k+2+\alpha}(\overline{\Omega})$; or*
(ii) *we can extract a subsequence and find $x_n \to +\infty$ so that the translated sequence $\{\widetilde{u}_n\}$ defined by $\widetilde{u}_n = u_n(\,\cdot\, + x_n, \,\cdot\,)$ converges in $C_{\mathrm{loc}}^{k+2}(\overline{\Omega})$ to some $\widetilde{u} \in C_{\mathrm{b}}^{k+2+\alpha}(\overline{\Omega})$ which solves* (6.3) *and has $\widetilde{u} \not\equiv U$ and $\partial_x\widetilde{u} \leq 0$.*

Proof. Without loss of generality we can assume that $\lambda_n \to \lambda \in \mathbb{R}^m$. Suppose first that

$$\lim_{|x|\to\infty} \sup_n \sup_y |u_n(x, y) - U(y)| = 0. \tag{6.8}$$

We will show that alternative (i) occurs. Using Arzelà–Ascoli, (6.8), a diagonalization argument, and Remark 2.2, we can extract a subsequence so that $u_n \to u$ in $C_{\mathrm{loc}}^{k+2}(\overline{\Omega})$ and $C_{\mathrm{b}}^{0}(\overline{\Omega})$ for some $u \in C_{\mathrm{b}}^{k+2+\alpha}(\overline{\Omega})$ satisfying (6.7). It remains to show that $u_n \to u$ in $C_{\mathrm{b}}^{k+2+\alpha}(\overline{\Omega})$. For this we observe that $v_n := u_n - u$ satisfies a linear elliptic equation

$$\begin{cases} a_n^{ij}D_{ij}v_n + b_n^iD_iv_n + c_nv_n = 0 & \text{in } \Omega, \\ \beta^iD_iv_n + \mu_nv_n = 0 & \text{on } \mathbb{R} \times \partial_1\mathcal{B}, \\ v_n = 0 & \text{on } \mathbb{R} \times \partial_2\mathcal{B}, \end{cases} \tag{6.9}$$

where the coefficients $a_n^{ij}, b_n^i, c_n, \beta_n^i, \mu_n$ are defined in terms of the convex combinations $u_n^{(s)} := su_n + (1-s)u$ and $\lambda^{(s)} := s\lambda_n + (1-s)\lambda$ by

$$a_n^{ij} := \int_0^1 \mathcal{F}_{r^{ij}}(y, u_n^{(s)}, Du_n^{(s)}, D^2u_n^{(s)}, \lambda^{(s)})\, ds, \qquad b_n^i := \int_0^1 \mathcal{F}_{\xi^i}(y, u_n^{(s)}, Du_n^{(s)}, D^2u_n^{(s)}, \lambda^{(s)})\, ds,$$
$$c_n := \int_0^1 \mathcal{F}_z(y, u_n^{(s)}, Du_n^{(s)}, D^2u_n^{(s)}, \lambda^{(s)})\, ds, \qquad \beta_n^i := \int_0^1 \mathcal{G}_{\xi^i}(y, u_n^{(s)}, Du_n^{(s)}, \lambda^{(s)})\, ds,$$
$$\mu_n := \int_0^1 \mathcal{G}_z(y, u_n^{(s)}, Du_n^{(s)}, \lambda^{(s)})\, ds.$$

From the assumptions on $\mathcal{F}, \mathcal{G}$ and the uniform bounds on u_n and u in $C^{k+\alpha}_{\mathrm{b}}(\overline{\Omega})$, the $C^{k+\alpha}$ norms of a_n^{ij}, b_n^i, c_n as well as the $C^{k+1+\alpha}$ norms of β_n^i, μ_n are bounded uniformly in n. From (6.4) and (6.5) we also see that $a_n^{ij}\eta_i\eta_j \geq c|\eta|^2$ and $\beta^i\nu_i > c$ so that so that (6.9) is uniformly elliptic with uniformly oblique boundary condition. Thus we have a Schauder estimate

$$\|v_n\|_{C^{k+2+\alpha}(\Omega)} \leq C\|v_n\|_{C^0(\Omega)}$$

where the constant C is independent of n. Since $u_n \to u$ and hence $v_n \to 0$ in $C^0_{\mathrm{b}}(\overline{\Omega})$, this proves that $u_n \to u$ in $C^{k+2+\alpha}_{\mathrm{b}}(\overline{\Omega})$ as desired.

Now assume that (6.8) does not hold; we will show that (ii) occurs. We can find a sequence $\{(x_n, y_n)\} \subset \Omega$ with $x_n \to +\infty$ and $\varepsilon > 0$ so that

$$|u_n(x_n, y_n) - U(y_n)| \geq \varepsilon$$

for all n. Extracting a subsequence we can assume that $y_n \to y_\infty \in \overline{\mathcal{B}}$. Consider the translated sequence $\{\widetilde{u}_n\}$ defined by

$$\widetilde{u}_n(x, y) := u_n(x + x_n, y).$$

Thanks to the uniform bounds on u_n and hence $\widetilde{u}_n$, we can extract a further subsequence so that $\widetilde{u}_n \to \widetilde{u}$ in $C^{k+2}_{\mathrm{loc}}(\overline{\Omega})$ for some $\widetilde{u} \in C^{k+2+\alpha}_{\mathrm{b}}(\overline{\Omega})$. Since $\mathcal{F}$ and $\mathcal{G}$ have no explicit dependence on x, the $\widetilde{u}_n$ are also solutions to (6.3), and therefore the C^{k+2}_{loc} limit implies that $\widetilde{u}$ also solves (6.3).

By (6.6), we have $\partial_x \widetilde{u}_n \leq 0$ for $x \geq -x_n$, and hence $\partial_x \widetilde{u} \leq 0$ on $\overline{\Omega}$. Finally, to see that $\widetilde{u} \not\equiv U$, we simply note that

$$|\widetilde{u}(0, y_\infty) - U(y_\infty)| = \lim_{n\to\infty} |u_n(x_n, y_n) - U(y_n)| \geq \varepsilon > 0. \qquad \square$$

Remark 6.4. The above proof can easily be genearlized to the case where the u_n are not necessarily even and where (6.6) is replaced by the monotonicity of the u_n for $|x| > M$ for some fixed $M > 0$.

6.2. **Proof of the main result.** Recall from Section 2.5 that we can formulate the height equation (2.31) as a nonlinear operator equation $\mathscr{F}(w, F) = 0$ with $\mathscr{F} \colon U \to Y$ given in (2.33) and (2.34). Obviously $\mathscr{F}$ is real analytic on U, and from Lemma 3.6 it follows that $\mathscr{F}_w(w, F)$ is Fredholm with index 0 whenever $(w, F) \in U$. By Theorem 5.1, we know that there is a local curve

$$\mathscr{C}_{\mathrm{loc}} = \{(w^\epsilon, F^\epsilon) : 0 < \epsilon < \epsilon_*\} \subset U$$

of nontrivial symmetric and monotone waves of elevation with F slightly larger than F_{cr}. (Recall from (5.2) that $F^\epsilon = (1/F_{\mathrm{cr}}^2 - \epsilon)^{-1/2}$.) Moreover, $\mathscr{F}_w$ is invertible along $\mathscr{C}_{\mathrm{loc}}$.

Applying Theorem 6.1 to our nonlinear operator $\mathscr{F} \colon U \to Y$ with $\mathcal{U} \times \mathcal{I} = U$, we obtain the following.

Theorem 6.5 (Global continuation)**.** *The local curve $\mathscr{C}_{\mathrm{loc}}$ is contained in a continuous curve of solutions, parametrized as*

$$\mathscr{C} = \{(w(s), F(s)) : 0 < s < \infty\} \subset U$$

with the following properties.

(a) *One of two alternatives must hold: either*
   (i) (Blowup) *as* $s \to \infty$,
$$N(s) := \|w(s)\|_X + \frac{1}{\inf_R(w_p(s)+H_p)} + F(s) + \frac{1}{F(s)-F_{\mathrm{cr}}} \to \infty; \quad \textit{or} \tag{6.10}$$
   (ii) (Loss of compactness) *there exists a sequence* $s_n \to \infty$ *such that* $\sup_n N(s_n) < \infty$ *but* $\{w(s_n)\}$ *has no subsequences converging in* X.

(b) *Near each point* $(w(s_0), F(s_0)) \in \mathscr{C}$, *we can reparametrize* $\mathscr{C}$ *so that the mapping* $s \mapsto (w(s), F(s))$ *is real analytic.*

(c) $(w(s), F(s)) \notin \mathscr{C}_{\mathrm{loc}}$ *for* s *sufficiently large.*

We will now use the qualitative results from Section 4 to pare down the alternatives in Theorem 6.5 until we are left with only $\sup_R(w_p(s)+H_p) \to \infty$, proving Theorem 1.1.

First we consider alternative (ii). Here we would like to apply Lemma 6.3, but first we need to know that these solutions have the required monotonicity properties (6.7).

Lemma 6.6. *The nodal properties* (4.65) *hold along the global bifurcation curve* $\mathscr{C}$.

Proof. First we claim that (4.65) holds along the local bifurcation curve $\mathscr{C}_{\mathrm{loc}}$. By Theorem 5.1(iv), any $(w,F) \in \mathscr{C}_{\mathrm{loc}}$ is a wave of elevation in that $w > 0$ on $R \cup T$. Thus we can apply Theorem 4.13 to get $w_q < 0$ on $R \cup T$, and hence by Lemma 4.20 that (4.65) holds.

Let $V \subset \mathscr{C}$ denote the set of all $(w,F) \in \mathscr{C}$ satisfying (4.65). Since $\mathscr{C}$ is a continuous curve, it is connected in $X \times \mathbb{R}$. By Lemmas 4.21 and 4.22, $V \subset \mathscr{C}$ is both relatively open and relatively closed. Since $\mathscr{C}_{\mathrm{loc}} \subset V$, V is nonempty, and we conclude that $V = \mathscr{C}$ as desired. ☐

Now we are ready to eliminate alternative (ii) in Theorem 6.5. We state the lemma in a slightly more general form for later convenience.

Lemma 6.7. *Given a sequence of solutions* $\{(w_n, F_n)\} \subset \mathscr{C}$ *to* $\mathscr{F}(w,F) = 0$ *with* $\|w_n\|_X$ *uniformly bounded, we can extract a subsequence so that* (w_n, F_n) *converges in* $X \times \mathbb{R}$ *to a solution* (w,F) *of* $\mathscr{F}(w,F) = 0$. *In particular, alternative* (ii) *in Theorem 6.5 cannot occur.*

Proof. We will apply a slight variant of Lemma 6.3 to (2.28) in non-divergence form, setting $k := 1$, $\lambda := F$, $u := h$, $U := H$,
$$\mathcal{B} := [-1,0], \quad \partial_1\mathcal{B} := \{0\}, \quad \partial_2\mathcal{B} := \{-1\},$$
and
$$\begin{aligned} \mathcal{F}(p,z,\xi,r,F) &:= (1+\xi_1^2)r_{22} - 2\xi_1\xi_2 r_{12} + \xi_2^2 r_{11} + \Big(\frac{1}{2H_p^2}\Big)_p \xi_2^3 + \frac{1}{F^2}\rho_p(H-z)\xi_2^3, \\ \mathcal{G}(z,\xi,F) &:= \frac{1+\xi_1^2}{2\xi_2^2} - \frac{1}{2H_p(0)^2} + \frac{1}{F^2}\rho(0)(z-1). \end{aligned} \tag{6.11}$$

Given a sequence $\{(w_n, F_n)\}$ as in the statement of the lemma, Lemma 6.8 at once furnishes a uniform bound
$$\|h_n\|_X + \frac{1}{\inf_R \partial_p h_n} + F_n \le M < \infty.$$
Thus, for all n and at any $(q,p) \in \overline{R}$, $(p, h_n, Dh_n, D^2h_n, F_n) \in \mathscr{D}_M$, where
$$\mathscr{D}_M := \big\{(p,z,\xi,r,F) \in [-1,0] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^{2\times 2} \times \mathbb{R} : \xi_2 \ge \tfrac{1}{M},\ F_{\mathrm{cr}} \le F \le M\big\}$$
Here $\mathbb{S}^{2\times 2}$ is the set of symmetric 2×2 real matrices. We easily check that $\mathcal{F}, \mathcal{G}$ satisfy the requirements of Lemma 6.3 when restricted to $\mathscr{D}_M$. Since $\mathscr{D}_M$ is convex, the proof of Lemma 6.3 is easily extended to this setting. We conclude that, after extracting a subsequence, we can arrange

for $F_n \to F \geq F_{\mathrm{cr}}$ and find $q_n \to +\infty$ so that $\tilde{h}_n(q_n + \;\cdot\;, \;\cdot\;)$ converges in $C^3_{\mathrm{loc}}(\overline{R})$ to a solution $\tilde{h} \in C^{3+\alpha}_{\mathrm{b}}(\overline{R})$ of (2.28) with $\tilde{h}_q \leq 0$,

$$\tilde{h} \not\equiv H, \tag{6.12}$$

and $\tilde{h}_p \geq 1/M$. Here we have used Remark 2.2 to conclude that $\tilde{h} \in C^{3+\alpha}_{\mathrm{b}}(\overline{R})$. Moreover, since each $h_n \geq H$, we have $\tilde{h} \geq H$, and since $\mathscr{S}(h_n) = \mathscr{S}(H)$, $\mathscr{S}(\tilde{h}) = \mathscr{S}(H)$.

Because $\tilde{h}_q \leq 0$ and $\tilde{h}$ is bounded, for all p we must have pointwise limits

$$H_\pm(p) := \lim_{q\to\pm\infty} \tilde{h}(q,p).$$

As $\tilde{h} \geq H$, we have

$$H \leq H_- \leq \tilde{h} \leq H_+. \tag{6.13}$$

Moreover, since $\mathscr{S}(\tilde{h}) = \mathscr{S}(H)$, $\mathscr{S}(H_-) = \mathscr{S}(H_+) = \mathscr{S}(H)$. Combining these facts with Corollary 4.11, we conclude that $H_+ \equiv H_- \equiv H$. But then (6.13) forces $\tilde{h} \equiv H$, contradicting (6.12). $\square$

Now we turn to alternative (i) in Theorem 6.5, and clarify the way in which the first and last terms on the left-hand side of (6.10) may be unbounded. First we apply Corollary 4.2 and Theorem 4.7 to bound the middle two terms in (6.10) by the first term.

Lemma 6.8 (Finite Froude number and velocity)**.** *Let $(w,F) \in U$ solve $\mathscr{F}(w,F) = 0$. If $\|w\|_{C^1(R)} \leq K$, there is a constant C depending only on K so that*

$$\frac{1}{\inf_R(w_p + H_p)} + F < C.$$

Proof. Corollary 4.2 immediately gives $\inf_R(w_p + H_p) > \delta_*$, while Theorem 4.7 gives

$$F^2 \leq C\|h_p\|_{L^\infty} \leq C\|H_p + w_p\|_{L^\infty} \leq C(1 + \|w\|_{C^1(R)}) \leq C. \qquad \square$$

Next we deal with the fourth term in (6.10) by showing that uniform bounds on $\|w(s)\|_X$ imply that $F(s)$ is bounded away from F_{cr}.

Lemma 6.9 (Asymptotic supercriticality)**.** *If $\|w(s)\|_X$ is uniformly bounded along $\mathscr{C}$, then*

$$\liminf_{s\to\infty} F(s) > F_{\mathrm{cr}}. \tag{6.14}$$

Proof. Arguing by contradiction, suppose that there exists a sequence $s_n \to \infty$ with

$$\limsup_{n\to\infty} \|w(s_n)\|_X < \infty \qquad \text{and} \qquad F(s_n) \to F_{\mathrm{cr}}. \tag{6.15}$$

Applying Lemma 6.7, we can extract a subsequence so that $\{(w(s_n), F(s_n))\}$ converges in $X \times \mathbb{R}$ to a solution (w^*, F^*) of $\mathscr{F}(w,F) = 0$ with critical Froude number $F^* = F_{\mathrm{cr}}$. By Theorem 4.4(ii), we know that this can only happen if $w^* = 0$. Consequently, $\|w(s_n)\|_X \to 0$. Since, by Lemma 6.6, each $w(s_n)$ is a wave of elevation, we have $(w(s_n), F(s_n)) \in \mathscr{C}_{\mathrm{loc}}$ for n sufficiently large by Lemma 5.10. But by Theorem 6.5(c), $(w(s), F(s)) \notin \mathscr{C}_{\mathrm{loc}}$ for s sufficiently large, so this is a contradiction. $\square$

Together, Lemmas 6.7, 6.8, 6.9, and Theorem 6.5 show that $\|w\|_X$ is necessarily unbounded along $\mathscr{C}$. The next result explains more precisely which derivatives of w are growing in the limit.

Theorem 6.10 (Uniform regularity)**.** *For each $K > 0$ there exists a constant $C = C(K) > 0$ such that, if $(w,F) \in \mathscr{C}$ with $\|w_p\|_{C^0(R)} < C$, then $\|w\|_{C^{3+\alpha}(R)} < K$.*

In other words, if $\|w(s)\|_X \to \infty$, then $\|w_p(s)\|_{C^0(R)} \to \infty$ as well. This is a consequence of the structure of the height equation and elliptic regularity theory. Statements of this type are well known in the context of steady water waves (see, e.g., [24, Section 6], [94, Section 6], [96, Section 5], and [95, Section 5]), so we only provide a sketch of the argument and relegate it to Appendix A.3. The crucial first step will be to apply Corollary 4.2.

At last, we are prepared to prove the main result. Most of the work has already been done, and so all that remains is to assemble the various pieces laid out above.

Proof of Theorem 1.1. Let $\mathscr{C}$ be given as in Theorem 6.5. Applying Lemma 6.7, we conclude that alternative (i) in Theorem 6.5 holds, i.e.,

$$\|w(s)\|_X + \frac{1}{\inf_R(w_p(s)+H_p)} + F(s) + \frac{1}{F(s)-F_{\mathrm{cr}}} \to \infty$$

as $s \to \infty$. Using Lemma 6.8, we can simplify this to

$$\|w(s)\|_X + \frac{1}{F(s)-F_{\mathrm{cr}}} \to \infty$$

and using Lemma 6.9 we conclude $\|w(s)\|_X \to \infty$. Theorem 6.10 now yields

$$\|w_p(s)\|_{C^0(\overline{R})} \to \infty.$$

Using (2.26), we can re-express w_p in Eulerian variables. At the same time, we also reintroduce the tildes distinguishing dimensional and dimensionless Eulerian quantities. We will not put tildes on the Dubreil-Jacotin variables h, ρ, and p even though they are dimensionless. For s sufficiently large, this gives

$$\inf_{\tilde{\Omega}(s)}(\tilde{c}-\tilde{u}(s)) = \frac{1}{\sup_R|\sqrt{\rho}h_p|} \le \frac{1}{\min\sqrt{\rho}}\frac{1}{\sup_R|w_p| - \max H_p} \to 0, \tag{6.16}$$

where $\tilde{\Omega}(s)$ is the dimensionless fluid domain corresponding to $(w(s), F(s))$. To prove a similar statement for the dimensional horizontal velocity

$$u - c = \frac{m}{\sqrt{\rho_0}d}(\tilde{u}-\tilde{c}) = F\sqrt{gd}(\tilde{u}-\tilde{c}),$$

we combine (6.16) with the bound the bound (4.20) from Theorem 4.7,

$$F^2(s) \le \frac{C}{\inf_{\Omega(s)}(\tilde{c}-\tilde{u}(s))},$$

to get

$$\inf_{\Omega(s)}(c-u(s)) = F\sqrt{gd}\inf_{\tilde{\Omega}(s)}(\tilde{c}-\tilde{u}) \le C\sqrt{\inf_{\tilde{\Omega}(s)}(\tilde{c}-\tilde{u})} \to 0.$$

The regularity statements about the parameterization are inherited from those of Theorem 6.5. ☐

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Appendix A. Proofs and calculations

This appendix collects some of the more technical proofs and straightforward calculations, organized by section.

A.1. **Proofs and calculations from Section 5.**

Proof of Lemma 5.2(ii). Let $u = (w, r) \in \mathcal{D}(L)$, $\xi \in \mathbb{R}$ be given, and denote $(f, g) := (L - i\xi)u$, that is,

$$f = H_p r - i\xi w, \qquad g = -\left(\frac{w_p}{H_p^3}\right)_p + \frac{1}{F_{\mathrm{cr}}^2}\rho_p w - i\xi r. \tag{A.1}$$

Recall also that the definition of $\mathcal{D}(L)$ implies that

$$w(-1) = r(-1) = 0, \qquad \left(-\frac{w_p}{H_p^3} + \frac{1}{F_{\mathrm{cr}}^2}\rho w\right)\bigg|_{p=0} = 0. \tag{A.2}$$

Following [39, Lemma 3.4] and [96, Lemma 4.9], we integrate by parts in (A.1) using (A.2),

$$\begin{aligned} C(\|f\|_{H^1}^2 + \|g\|_{L^2}^2) \geq \int_{-1}^0 \left(\frac{|f_p|^2}{H_p^3} + H_p|g|^2\right)\, dp \\ &\geq \|w_{pp}\|_{L^2}^2 + \|r_p\|_{L^2}^2 + |\xi|^2(\|w_p\|_{L^2}^2 + \|r\|_{L^2}^2) \\ &\quad - (|\xi| + C)\left(\|w\|_{H^1}^2 + \|r\|_{L^2}^2\right) - C|\xi||r(0)\bar{w}(0)|. \end{aligned}$$

Here we have used that $H \in C^{3+\alpha}$ and $H_p > 0$. From the bottom boundary conditions, Sobolev embedding theorem, Poincaré inequality, and an interpolation argument, we have

$$\|w\|_{L^\infty} \lesssim \|w\|_{H^1} \lesssim \|w_p\|_{L^2}, \quad \|r\|_{L^\infty} \lesssim \|r\|_{H^1} \lesssim \|r_p\|_{L^2}, \qquad \|w\|_{H^2} \lesssim \|w_{pp}\|_{L^2}.$$

So we can bound $w(0)$ by $|w(0)| \leq C\|w_p\|_{L^2}$. To control $|r(0)|$, we note from (A.2) that

$$|H_p(0)r(0)|^2 = 2\int_{-1}^0 \operatorname{Re}\left(H_p\bar{r}(H_p r)_p\right)\, dp.$$

Then using the first equation in (A.1) it follows that

$$|H_p(0)r(0)|^2 = 2\operatorname{Re}\int_{-1}^0 H_p\bar{r}(f_p + i\xi w_p)\, dp \leq \delta^2\left(\|f_p\|_{L^2}^2 + |\xi|^2\|w_p\|_{L^2}^2\right) + \frac{1}{\delta^2}\|r\|_{L^2}^2,$$

for some $\delta > 0$. Thus choosing $|\xi|$ sufficiently large and δ sufficiently small, we finally have

$$\begin{aligned} C(\|f\|_{H^1}^2 + \|g\|_{L^2}^2) &\geq \|w_{pp}\|_{L^2}^2 + \|r_p\|_{L^2}^2 + |\xi|^2(\|w_p\|_{L^2}^2 + \|r\|_{L^2}^2) \\ &\gtrsim \|w\|_{H^2}^2 + \|r\|_{H^1}^2 + |\xi|^2(\|w\|_{H^1}^2 + \|r\|_{L^2}^2), \end{aligned}$$

which completes the proof. $\square$

A.2. **Calculation of the reduced system.** In this subsection, we present the computation of the leading order part of the reduced system in Lemma 5.5(vi).

Recall that in Lemma 5.2(i) we found that the center space is spanned by the e_1, e_2 defined in (5.8). We record below the first through third variations of $\mathcal{H}$ evaluated at $u = 0$ and in taken in the direction $u^{\mathrm{c}} := z_1 e_1 + z_2 e_2$,

$$\begin{aligned} &\mathcal{H}_u^\epsilon(0)[u^{\mathrm{c}}] = \mathcal{H}_{u\epsilon}^\epsilon(0)[u^{\mathrm{c}}] = 0, \qquad \mathcal{H}_{uu}^\epsilon(0)[u^{\mathrm{c}}, u^{\mathrm{c}}] = z_2^2, \\ &\mathcal{H}_{uu\epsilon}^\epsilon(0)[u^{\mathrm{c}}, u^{\mathrm{c}}] = -c_0^{-1}c_1 z_1^2, \qquad \mathcal{H}_{uuu}^\epsilon(0)[u^{\mathrm{c}}, u^{\mathrm{c}}, u^{\mathrm{c}}] = 3c_0^{-3/2}c_2 z_1^3 + 3c_0^{-1/2}z_1 z_2^2, \end{aligned} \tag{A.3}$$

where c_0 is defined in (5.9) and c_1, c_2 are defined in (5.15). Also, an elementary computation reveals that $\mathcal{H}_{uu}^0(0)[e_1, \,\cdot\,] \equiv 0$, given the equation satisfied by Φ_{cr}.

Consider the quantity

$$\mathcal{K}^\epsilon(u^{\mathrm{c}}) := \mathcal{H}^\epsilon(u^{\mathrm{c}} + \Theta^\epsilon(u^{\mathrm{c}})),$$

where Θ^ϵ is the reduction function of Lemma 5.5(vi). The reduced Hamiltonian K^ϵ given in (5.11) corresponds to viewing $\mathcal{K}^\epsilon$ as a function of (z_1, z_2). The derivatives of $\mathcal{K}^\epsilon$ can be computed readily using (A.3). It is clear, for instance, that $\mathcal{K}^\epsilon_u(0)$ vanishes identically, while

$$\mathcal{K}^0_{\epsilon u^c u^c}(0)[\dot{u}^c, \dot{u}^c] = -c_0^{-1} c_1 z_1^2 + 2\mathcal{H}^0_{uu}(0)\left[e_2, \Theta^0_{\epsilon u^c}(0)e_1\right] z_1 z_2 + 2\mathcal{H}^0_{uu}(0)\left[e_2, \Theta^0_{\epsilon u^c}(0)e_2\right] z_2^2,$$

and thus

$$\mathcal{K}^\epsilon_{u^c u^c}(0)[u^c, u^c] = z_2^2 - \epsilon c_0^{-1} c_1 z_1^2 + \mathcal{O}(|\epsilon||(z_1,z_2)||z_2|) + \mathcal{O}(\epsilon^2|(z_1,z_2)|^2). \tag{A.4}$$

Likewise,

$$\begin{aligned}\mathcal{K}^\epsilon_{u^c u^c u^c}(0)[u^c, u^c, u^c] &= \mathcal{H}^\epsilon_{uuu}(0)[u^c, u^c, u^c] + 2\mathcal{H}^\epsilon_{uu}(0)[u^c,\, \Theta^\epsilon_{u^c u^c}(0)[u^c, u^c]] \\ &= 3c_0^{-3/2} c_2 z_1^3 + \mathcal{O}(|z_2||(z_1,z_2)|^2),\end{aligned} \tag{A.5}$$

in light of (A.3) and the fact that $\Theta^\epsilon_{u^c}(0) = 0$.

Combining (A.4), and (A.5), and expressing the result in terms of $(z_1, z_2) \in \mathbb{R}^2$, we arrive at the following expansion for the reduced Hamiltonian K^ϵ:

$$\begin{aligned}K^\epsilon(z_1,z_2) = \frac{1}{2}z_2^2 - \frac{1}{2}\epsilon c_0^{-1} c_1 z_1^2 + \frac{1}{2}c_0^{-3/2} c_2 z_1^3 \\ + \mathcal{O}(|z_2||(z_1,z_2)|^2) + \mathcal{O}(|\epsilon||z_2||(z_1,z_2)|) + \mathcal{O}(|(\epsilon,z_1,z_2)|^2|(z_1,z_2)|^2).\end{aligned} \tag{A.6}$$

A.3. **Proofs from Section 6.**

Proof of Theorem 6.10. Let $K > 0$ be given. Throughout the proof, we let $C > 0$ denote a generic constant that depends only on $\|w_p\|_{C^0(R)}$. In light of Corollary 4.2, we already know that $\|w\|_{C^1(R)}$ can be controlled by $\|w_p\|_{C^0(R)}$. It remains now to bound the higher order derivatives.

First, we establish uniform Hölder norm estimates for the gradient. Note that the height equation (2.28) can be written abstractly as

$$\mathcal{F}(p, h, Dh, D^2h, F) = 0 \text{ in } R, \qquad \mathcal{G}(h, Dh, F) = 0 \text{ on } T, \qquad h = 0 \text{ on } B.$$

where

$$\mathcal{F}\colon [-1,0] \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^{2\times 2} \times \mathbb{R}_+ \to \mathbb{R}, \qquad \mathcal{G}\colon \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$$

are defined by (6.11). We are interested in deriving bounds that are uniform for

$$\sup_R h_p > \delta_* > 0, \qquad \|h\|_{C^1(R)} + F + \frac{1}{F} < C.$$

Translating this to the notation above, this means that one should consider the restriction of $\mathcal{F}$ and $\mathcal{G}$ to sets of the form

$$V := \left\{ (z, \xi, F) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}_+ :\ \xi_2 > \delta_*,\ z > 0,\ z + |\xi_1| + |\xi_2| + F + \frac{1}{F} < C \right\}.$$

This can be achieved in the usual way by using cutoff functions. For $(z, \xi, F) \in V$, $p \in [-1, 0]$, and $r \in \mathbb{S}^{2\times 2}$, it is easy to confirm that

$$\begin{gathered} c_1 I \le \mathcal{F}_r(p, z, \xi, r, F) \le c_1 c_2 I, \qquad |\mathcal{G}_\xi(z, \xi, F)| > c_3, \\ |\mathcal{F}(p, z, \xi, 0, F)| < c_1 c_4 |p|^{\alpha - 1}, \\ (1 + |r|)|\mathcal{F}_\xi(p, z, \xi, r, F)| + |\mathcal{F}_z(p, z, \xi, r, F)| + |\mathcal{F}_p(p, z, \xi, r, F)| \le c_1 c_5 (|r|^2 + |p|^{\alpha-2}), \end{gathered}$$

where I is the 2×2 identity matrix and c_1, c_2, c_3, c_4, and c_5 are positive constants depending only on C and δ_*. Moreover, for any (z, ξ, F), $(z', \xi', F) \in V$ there exists a positive constant $c_6 > 0$, depending only on C and δ_*, such that

$$|\mathcal{G}(z, \xi, F) - \mathcal{G}(z', \xi', F)| \le c_3 c_6 \left(|z - z'|^\alpha + |\xi - \xi'|^\alpha \right).$$

These structural properties permit us to apply quasilinear elliptic estimates up to the boundary as in [64, Theorem 1] to conclude that $\|h\|_{C^{1+\alpha'}(R)} < C$, for some $\alpha' \in (0, \alpha]$. Here we have also used that the fact that $h \in X \subset W^{3,2}_{\mathrm{loc}}(R)$ and h is uniformly bounded in the local Lipshitz norm by C.

Next, we consider the higher-order derivatives. For this we yet again exploit the height equation's translation invariance in q to quasi-linearize it by applying ∂_q. That is, h_q is the solution of a uniformly elliptic second-order divergence form PDE with a uniformly oblique boundary condition (4.63). Our efforts thus far show that the coefficients of this PDE are uniformly bounded in $C^{\alpha'}(\overline{R})$, thus linear Schauder estimates are enough to get control of h_q in $C^{1+\alpha'}(\overline{R})$ (see, e.g., [25, Theorem 3]). Lastly, to bound h_{pp} in $C^{\alpha'}(\overline{R})$, we use the full height equation (2.28) to express it in terms of h, h_q, h_p, h_{qq}, and h_{qp}.

Thus, h is uniformly controlled in $C^{2+\alpha'}(\overline{R})$. But then it is in particular bounded uniformly in $C^{1+\alpha}(\overline{R})$. Repeating the same argument above, we see that the coefficients of the linear PDE for h_q are in $C^{\alpha}(\overline{R})$, hence h_q is controlled uniformly in $C^{3+\alpha}(\overline{R})$. As before, this is enough to conclude that $\|h\|_{C^{2+\alpha}(R)} < C$. It is straightforward to continue in this fashion and obtain uniform bounds of h in $C^{3+\alpha}(\overline{R})$, which finishes the proof. □

Appendix B. Quoted results

First, let us recall the maximum principle, Hopf boundary lemma, and Serrin edge point lemma [75]. In particular, note that we are using the version that allows for an adverse sign of the zeroth order term provided that the sign of the solution is known; see, for example, [33], [75, Lemma 1], and [37, Lemma S].

Theorem B.1. *Let $\Omega \subset \mathbb{R}^n$ be a connected, open set (possibly unbounded), and consider the second-order operator L given by*

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x) \tag{B.1}$$

where $\partial_i := \partial_{x_i}$ and the coefficients a_{ij}, b_i, c are of class $C^0(\overline{\Omega})$. We assume that L is uniformly elliptic in the sense that there exists $\lambda > 0$ with

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \qquad \text{for all } \xi \in \mathbb{R}^n,\ x \in \overline{\Omega}, \tag{B.2}$$

and that a_{ij} is symmetric. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a classical solution of $Lu = 0$ in Ω.

(i) (Strong maximum principle) *Suppose u attains its maximum value on $\overline{\Omega}$ at a point in the interior of Ω. If $c \leq 0$ in Ω, or if $\sup_\Omega u = 0$, then u is a constant function.*

(ii) (Hopf boundary lemma) *Suppose that u attains its maximum value on $\overline{\Omega}$ at a point $x_0 \in \partial\Omega$ for which there exists an open ball $B \subset \Omega$ with $\overline{B} \cap \partial\Omega = \{x_0\}$. Assume that either $c \leq 0$ in Ω, or else $\sup_B u = 0$. Then u is a constant function or*

$$\nu \cdot \nabla u(x_0) > 0,$$

where ν is the outward unit normal to Ω at x_0.

(iii) (Serrin edge point lemma) *Let $x_0 \in \partial\Omega$ be an "edge point" in the sense that near x_0 the boundary $\partial\Omega$ consists of two transversally intersecting C^2 hypersurfaces $\{\gamma(x) = 0\}$ and $\{\sigma(x) = 0\}$. Suppose that $\gamma, \sigma < 0$ in Ω. If $u \in C^2(\overline{\Omega})$, $u > 0$ in Ω and $u(x_0) = 0$. Assume further that $a_{ij} \in C^2$ in a neighborhood of x_0,*

$$B(x_0) = 0, \quad \text{and} \quad \partial_\tau B(x_0) = 0 \tag{B.3}$$

for every differential operator ∂_τ tangential to $\{\gamma = 0\} \cap \{\sigma = 0\}$ at x_0. Then for any unit vector s outward from Ω at x_0, either

$$\partial_s u(x_0) < 0 \ \text{or} \ \ \partial_s^2 u(x_0) < 0.$$

We quote below the center manifold reduction theorem that forms the basis of the small-amplitude existence theory in Section 5 (cf. [70] and [42] for a general discussion). The version that we use is specifically designed to take advantage of the Hamiltonian structure of the system.

Theorem B.2 (Buffoni, Groves, and Toland [19])**.** *Suppose that $(\mathcal{X}, \omega^\epsilon, \mathcal{H}^\epsilon)$ is a one-parameter family of reversible Hamiltonian systems, where $\mathcal{X}$ is a Hilbert space, ω^ϵ a symplectic form on $\mathcal{X}$, and $\mathcal{H}^\epsilon$ the Hamiltonian. Write the corresponding Hamilton equation in the form*

$$u_q = Lu + N^\epsilon(u), \tag{B.4}$$

where $u(q)$ is assumed to lie in $\mathcal{X}$ for each q. We assume that $L \colon \mathcal{D}(L) \subset \mathcal{X} \to \mathcal{X}$ is a densely defined, closed linear operator. Suppose that 0 is an equilibrium for (B.4) *at $\epsilon = 0$ and that the following conditions hold.*

(H1) *The spectrum $\sigma(L)$ of L contains at most finitely many eigenvalues on the imaginary axis, each of which has finite multiplicity. Moreover, $\sigma(L) \cap i\mathbb{R}$ is separated from $\sigma(L) \setminus i\mathbb{R}$ in the sense of Kato. Let P^{c} denote the spectral projection corresponding to $\sigma(L) \cap i\mathbb{R}$ and put $\mathcal{X}^{\mathrm{c}} := P^{\mathrm{c}}\mathcal{X}$, $\mathcal{X}^{\mathrm{su}} := (1 - P^{\mathrm{c}})\mathcal{X}$. We let n be the (finite) dimension of $\mathcal{X}^{\mathrm{c}}$.*

(H2) *There exists $C > 0$ such that the operator L satisfies the resolvent estimate*

$$\|u\|_{\mathcal{X}} \leq \frac{C}{1 + |\xi|} \|(L - i\xi I)u\|_{\mathcal{X}}, \tag{B.5}$$

for all $\xi \in \mathbb{R}$ and $u \in \mathcal{X}^{\mathrm{su}}$.

(H3) *There exists a natural number k, an interval $\Lambda \subset \mathbb{R}$ containing 0, and a neighborhood $\mathcal{U}$ of 0 in $\mathcal{D}(L)$ such that N is C^{k+1} in its dependence on (ϵ, u) on $\Lambda \times \mathcal{U}$. Moreover, $N^0(0) = 0$ and $D_u N^0(0) = 0$.*

Then, after possibly shrinking the interval Λ and neighborhood $\mathcal{U}$, we have that, for each $\epsilon \in \Lambda$, there exists an n-dimensional local center manifold $\mathcal{W}^\epsilon \subset \mathcal{U}$ together with an invertible coordinate map

$$\chi^\epsilon := P^{\mathrm{c}}|_{\mathcal{W}^\epsilon} \colon \mathcal{W}^\epsilon \to \mathcal{U}^{\mathrm{c}} := P^{\mathrm{c}}\mathcal{U}$$

with the following properties:

(i) *Defining $\Psi^\epsilon \colon \mathcal{U}^{\mathrm{c}} \to \mathcal{U}^{\mathrm{su}} := P^{\mathrm{su}}\mathcal{U}$ by $u^{\mathrm{c}} + \Psi^\epsilon(u^{\mathrm{c}}) = (\chi^\epsilon)^{-1}(u^{\mathrm{c}})$, the map $(\epsilon, u) \mapsto \Psi^\epsilon(u)$ is $C^k(\Lambda \times \mathcal{U}^{\mathrm{c}}, \mathcal{U}^{\mathrm{su}})$. Moreover $\Psi^\epsilon = 0$ for all $\epsilon \in \Lambda$ and $D_u \Psi^0(0) = 0$.*

(ii) *Every initial condition $u_0 \in \mathcal{W}^\epsilon$ determines a unique solution u of* (B.4) *which remains in $\mathcal{W}^\epsilon$ as long as it remains in $\mathcal{U}$.*

(iii) *If u solves* (B.4) *and lies in U for all q, then u lies entirely in $\mathcal{W}^\epsilon$.*

(iv) *If $u^{\mathrm{c}} \in C^1((a, b), \mathcal{U}^{\mathrm{c}})$ solves the reduced system*

$$u^{\mathrm{c}}_q = f^\epsilon(u^{\mathrm{c}}) := Lu^{\mathrm{c}} + P^{\mathrm{c}} N^\epsilon(u^{\mathrm{c}} + \Psi^\epsilon(u^{\mathrm{c}})), \tag{B.6}$$

then $u = (\chi^\epsilon)^{-1}(u^{\mathrm{c}})$ solves the full system (B.4).

(v) *$\mathcal{M}^\epsilon$ is a symplectic submanifold of $\mathcal{X}$ when equipped with the symplectic form $\omega^\epsilon|_{\mathcal{M}^\epsilon}$ and Hamiltonian $\mathcal{K}^\epsilon(u^{\mathrm{c}}) = \mathcal{H}^\epsilon(u^{\mathrm{c}} + \Psi^\epsilon(u^{\mathrm{c}}))$. The reduced system* (B.6) *corresponds to the Hamiltonian flow for $(\mathcal{M}^\epsilon, \omega^\epsilon|_{\mathcal{M}^\epsilon}, \mathcal{K}^\epsilon)$. In fact, it is reversible and coincides with the restriction of the full Hamiltonian to the center manifold.*

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