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The Hölder continuity of the solution map to the *b*-family equation in weak topology

Robin Ming Chen · Yue Liu · Pingzheng Zhang

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Abstract We prove that the solution map of the b-family equation is Hölder continuous as a map from a bounded set of $H^s(\mathbb{R})$, $s > \frac{3}{2}$ with $H^r(\mathbb{R})$ $(0 \le r < s)$ topology, to $C([0, T], H^r(\mathbb{R}))$ for some $T > 0$. Moreover, we show that the obtained exponent of the Hölder continuity is *optimal* when $s - 1 < r < s$.

1 Introduction

Russell's observation of solitary water waves [63], which are not predicted by purely linear models, motivated the development of nonlinear partial differential equations for modeling wave phenomena in fluids, plasmas, elastic bodies, etc. In the context of water waves, Boussinesq [6,7] developed the fundamental perturbation expansion method (see [66] for a modern presentation). At the asymptotic expansion to the first order in the small parameter representing the ratio of wave amplitude to undisturbed fluid depth and the square of the ratio of fluid depth to wave length (the smallness of such a parameter leads to the so-called "shallow water waves"), the well-known Boussinesq equation [5]

R. M. Chen
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
e-mail: mingchen@pitt.edu

Y. Liu (✉)
Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019-0408, USA
e-mail: yliu@uta.edu

P. Zhang
Department of Mathematics, University of Jiangsu, Jiangsu 212013, Peoples Republic of China
e-mail: pzzhang@ujs.edu.cn

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$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0 \tag{1.1}$$

and Korteweg-de Vries (KdV) equation [54]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.2}$$

arise.

Beyond Boussinesq and KdV, the asymptotic expansion at quadratic order produces a broad class of asymptotically equivalent quasilinear shallow water wave equations [28,29]. These equations are related by a continuous group of nonlinear, nonlocal transformations which was first introduced for determining normal forms of shallow water equations by Kodama [51–53]. Remarkably, three-dimensional incompressible versions of these equations also arose in the study of the turbulence closure problem, obtained by averaging the exact fluid equations at constant Lagrangian coordinate, then making the Taylor hypothesis for frozen-in turbulence (see [11,32,33,41,58,59] and references therein).

In this paper we will focus on a family of equations from the equivalent class derived in [28,29]

$$\begin{cases} u_t + uu_x + P_x = 0, \quad t > 0, \ x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{1.3}$$

where the source term P is a convolution in the space variable x for fixed t

$$P = P(t,x) = \frac{1}{2} e^{-|x|/\alpha_0} * \left(\frac{b}{2} u^2 + \frac{\alpha_0^2}{2} (3-b) u_x^2 \right), \tag{1.4}$$

with $u(t,x)$ the fluid velocity, $\alpha_0^2 \geq 0$ a constant and $b \neq -1$ a bifurcation parameter.

Equation (1.3) is also referred to as the b-family equation [26]. Note that Eqs. (1.3) generalizes the following types of equations

(1) When $\alpha_0^2 = b = 0$, (1.3) becomes the inviscid Burgers equation [8,43].
(2) When $b = 0$ and $\alpha_0^2 > 0$, (1.3) becomes the inviscid Burgers-Alpha equation [42].

The cases $b = 2$ and 3 are special values for the b-family equation.

(3) When $\alpha_0^2 = 1$ and $b = 2$, (1.3) becomes the Camassa-Holm (CH) equation [9,34].
(4) When $\alpha_0^2 = 1$ and $b = 3$, (1.3) becomes the Degasperis-Procesi (DP) equation [27].

The CH and DP equations are both integrable. These two cases exhaust the integrable candidates for the b-family [9,26], and most of the analytical properties of the b-family equation can be extracted from the CH and DP equations [31,35]. In particular, in a periodic setting, the CH equation, corresponding to $b = 2$, is the only case that can be realized as an Euler equation for the geodesic flow on the Lie group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ of all smooth and orientation preserving diffeomorphisms on the circle [30].

Note that in the momentum formulation, Eqs. (1.3)–(1.4) write

$$\begin{cases} m_t + um_x = -bmu_x, \\ m = (1 - \alpha_0^2 \partial_x^2)u, \end{cases} \tag{1.5}$$

which is similar to the vorticity formulation of the three-dimensional Euler equation for incompressible perfect fluids (U is the speed and ω its vorticity)

$$\begin{cases} \omega_t + (U \cdot \nabla)\omega = (\omega \cdot \nabla)U, \\ \text{div } U = 0, \\ \text{curl } U = \omega. \end{cases} \tag{1.6}$$

In both (1.5) and (1.6) there is a coupling between a transport equation and a stationary elliptic one. The stretching term $(\omega \cdot \nabla)U$ in (1.6) is also similar to the term $-bmu_x$ in (1.5).

Starting from Burgers [8] and Hopf [43], the Burgers equation (especially, the viscous Burgers-Hopf equation) has always been used as a simple model to study shocks, turbulence and other nonlinear phenomena in fluids. Equation (1.3) is a nonlinear nonlocal deformation of the Burgers equation. However, the qualitative nature of the solutions to (1.3) is very different from that of the Burgers equation. First, unlike the Burgers equation, the CH and DP equations admit special solutions, namely the peakons [9,26] of the form $u(t, x) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The peakons capture a feature that is characteristic for the waves of great height—waves of the largest amplitude that are exact solutions of the governing equations for water waves [16,65]. Both the CH and DP peakons are shown to be orbitally stable [22,23,56].

Second, the $H^1(\mathbb{R})$ norm of the solutions to the Burgers equation blows up in finite time as long as the initial data has a negative slope due to the strong nonlinear effect. On the other hand, when the nonlocal terms are at present, the $H^1(\mathbb{R})$ norm of the CH solution is conserved, and the $H^1(\mathbb{R})$ norm of the DP solution grows at most quadratic in time [57]. In fact the appropriate blow-up mechanism for the CH and DP equations is wave-breaking, i.e. the solution remains bounded while its slope becomes unbounded in finite time (see [9,13–15,17–19,60,66] for the CH equation and [12,57,62,67,68] for the DP equation).

What we will investigate here is another issue related to the well-posedness of (1.3)–(1.4), namely the continuous dependence on the initial data. It is known that if the local well-posedness of solutions to a certain evolutionary PDE can be established by a solely fixed point theorem for contraction mappings, then the data-to-solution map will be Lipschitz on the space where solutions live. For instance the KdV equation is shown to be well-posed in $H^s(\mathbb{R})$ with $s > -3/4$ and the solution map is Lipschitz on the same $H^s(\mathbb{R})$ [48]. One of the key ingredients is the Strichartz estimates achieved from the strong dispersive effect.

When there is no dispersion, c.f. the Burgers equation, however, Kato [46] proved the local well-posedness in $H^k(\mathbb{R})$ for any integer $k \geq 2$ and showed that the dependence on initial data in $H^k(\mathbb{R})$ is continuous but not Hölder continuous with any prescribed exponent. The phenomenon of not uniformly continuous for some dispersive equations can be found, for example in Kenig et al. [49].

In the case of nonlocal dispersive equations, there are also a number of papers dealing with nonuniform continuity, see for example, Koch and Tzvetkov [50] for the Benjamin-Ono equation, Himonas and Kenig [37], Himonas and Misiołek [38], and Himonas et al. [40] for the CH equation, Himonas and Misiołek [39] for the Euler equation.

On the other hand, if one relaxes the topology, the solution map may be uniformly continuous or even Lipschitz continuous with respect to the weaker topology. In fact in Theorem IV of [46], Kato obtained uniform continuity of solution maps to general quasi-linear symmetric hyperbolic systems on the Sobolev spaces with integer index. As stated in [46], the result is also valid for some equations with nonlocal term. It is not hard to see that applying the theorem in [46] to the b-family equation one could infer that the solution map is uniformly continuous from a bounded set of $H^s(\mathbb{R})$ ($s > \frac{3}{2}$ an integer), with the weaker $H^{s-2}(\mathbb{R})$ topology, to the space $C([0, T];\ H^{s-2}(\mathbb{R}))$ for some $T > 0$. In [64], by constructing a suitable gauge transformation of the Benjamin-Ono equation, Tao proved the Lipschitz continuity of the associated solution map from a bounded set of $H^1(\mathbb{R})$—the solution space, equipped with $L^2(\mathbb{R})$ norm—a weaker topology, to $C([0, T], L^2(\mathbb{R}))$. Such a method concerning the Lipschitz continuity in a weaker topology is also applied in [36] recently.

The goal of the present paper is to investigate whether or not the solution map in $H^s(\mathbb{R})$ of (1.3)–(1.4) is Hölder continuous or even Lipschtz continuous in terms in the $H^r(\mathbb{R})$ topology, $0 \leq r < s$, and to determine a possible *optimal* Hölder's index.

The rest of the paper is organized as follows. In Sect. 2 we state our main theorems on the Hölder continuity of the b-family data-to-solution map, and discuss the methods used in our proofs, especially in proving the optimal Hölder index. In Sect. 3 we give a refined estimates on the Burgers term and the nonlocal term and use that to establish the Hölder continuity of the solution map. In Sects. 4 and 5 we transform the b-family equation to a semilinear ODE system and establish the correspondence between the solution to the original PDE and the one to the transformed ODE. In the last section we construct special initial data to show that the Hölder exponent in Theorem 2.1 is optimal.

Notation As above and henceforth, we denote by c, c_1, c_2, and C the various positive constants depending only on b, s, r and h and by $A = O(B)$ the estimate $A \leq cB$. For any constant p with $1 \leq p < \infty$, let $L^p(\mathbb{R})$ be the space of all the Lebesgue measurable functions f such that $|f|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty$. We also denote by $L^\infty(\mathbb{R})$ the space of all essentially bounded functions f with the standard norm

$$|f|_\infty = \inf_{m(E)=0} \sup_{x \in \mathbb{R} \setminus E} |f(x)|.$$

For any number $s \in \mathbb{R}$, let $H^s(\mathbb{R})$ be the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_s = \left(\int\limits_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty$$

with $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$, $\hat{f}(\xi) = \int_{\mathbb{R}} e^{ix\xi} f(x)dx$. For any function $u = u(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ with $T > 0$, we denote its Fourier transform, L^p-norm, Sobolev norm and L^2 inner product with respect to the variable x by $\hat{u} = \hat{u}(t,\xi)$, $|u|_p = |u(t,\cdot)|_p$, $\|u\|_s = \|u(t,\cdot)\|_s$ and $(\cdot,\cdot)$. For $s \in \mathbb{R}$, define an integral operator $\Lambda^s = (I - \partial_x^2)^{\frac{s}{2}}$ on tempered distributions by

$$\Lambda^s f = \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{f}).$$

2 Main results

For simplicity, we will consider $\alpha_0^2 = 1$ in (1.4). The general $\alpha_0^2 > 0$ case can be treated the same way. In this way, the b-family equation is

$$\begin{cases} u_t + uu_x + P_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ P = \frac{1}{2} e^{-|x|} * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 \right), \end{cases} \tag{2.1}$$

with initial data

$$u(0,x) = u_0(x). \tag{2.2}$$

As seen in the expression of P, one sees that $\frac{1}{2}e^{-|x|}$ is the Fourier transformation of the Poisson kernel on $\mathbb{R}$, that is, $\Lambda^{-2} f = \frac{1}{2} e^{-|x|} * f$.

Let $s > \frac{3}{2}$ and write $B(0,h)$ as a ball in $H^s(\mathbb{R})$ centered on 0 with radius $h > 0$, i.e.

$$B(0,h) = \{u;\ \|u\|_s < h\}.$$

Decompose the set $\Omega = \{(s,r);\ s > \frac{3}{2},\ 0 \le r < s\}$ into (see Fig. 1)

$$\Omega_1 = \{(s,r);\ s > \frac{3}{2},\ 0 \le r \le s-1, r \ge 2-s\}, \tag{2.3}$$

$$\Omega_2 = \{(s,r);\ \frac{3}{2} < s < 2,\ 0 \le r < 2-s\},\ \Omega_3 = \{(s,r);\ s > \frac{3}{2},\ s-1 < r < s\}, \tag{2.4}$$

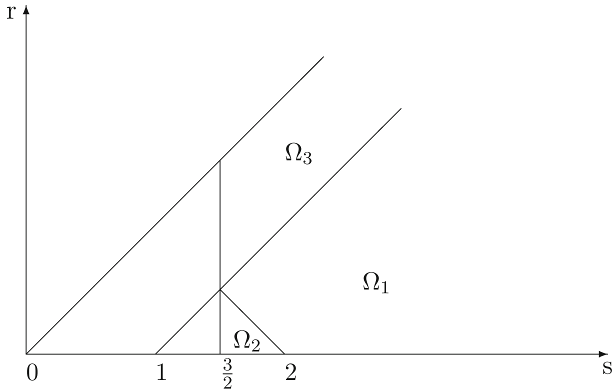

Fig. 1 A schematic depiction of the Hölder continuity with exponent $\alpha = \alpha(b, s, r)$ of the solution map of the b-family equation for variables s and r. The solution map is on a bounded set of $H^s(\mathbb{R})$ with topology of $H^r(\mathbb{R})$ and takes values in $C([0, T]; H^r(\mathbb{R}))$ for some $T > 0$. We have Lipschitz continuity when $(s, r) \in \Omega_1$, and Hölder continuity when $(s, r) \in \Omega_3$. When $b = 3$, we have Lipschitz continuity in Ω_2. When $b \neq 3$, we have Hölder continuity in Ω_2

and define the function $\alpha = \alpha(b, s, r)$ on Ω as

$$\alpha = \begin{cases} 1, & \text{if } (s, r) \in \Omega_1, \\ 1, & \text{if } b = 3 \text{ and } (s, r) \in \Omega_2, \\ \frac{2(s-1)}{s-r}, & \text{if } b \neq 3 \text{ and } (s, r) \in \Omega_2, \\ s - r, & \text{if } (s, r) \in \Omega_3. \end{cases} \tag{2.5}$$

With these notations we now give our two principal results of the paper on the Hölder (Lipschitz when $\alpha = 1$) continuity of the solution map of the b-family equation in a weaker topology.

Theorem 2.1 *Assume $s > \frac{3}{2}$ and $0 \leq r < s$. Then the solution map to (2.1)–(2.2) is Hölder continuous with exponent $\alpha = \alpha(b, s, r)$ defined in (2.5) as a map from $B(0, h)$, with $H^r(\mathbb{R})$ norm to $C([0, T], H^r(\mathbb{R}))$, that is, there exist positive constants T, c depending on b, s, r and h such that*

$$\|u(t) - \tilde{u}(t)\|_{C([0,T];\ H^r(\mathbb{R}))} \leq c\|u(0) - \tilde{u}(0)\|_r^{\alpha} \tag{2.6}$$

holds for all $u(0), \tilde{u}(0) \in B(0, h)$, where $u(t)$ and $\tilde{u}(t)$ are solutions of (2.1) with respectively initial data $u(0)$ and $\tilde{u}(0)$.

Moreover, we show that if $(s, r) \in \Omega_3$ the Hölder exponent $s - r$ of the solution map can not be improved to any larger number. More precisely, we have

Theorem 2.2 *Assume $s > \frac{3}{2}$ and $s - 1 < r < s$. Then the exponent $s - r$ of the solution map to (2.1)–(2.2) is optimal in the following sense: for any $\delta > 0$ there are constants $\lambda_0, t_0 > 0$, a family of constants $c^{\lambda} \to \infty$ as $\lambda \to 0$ and a family of functions $u^{\lambda}(0) \in B(0, h)$, $0 \leq \lambda \leq \lambda_0$ such that*

$$\|u^\lambda(t_0) - u^0(t_0)\|_r \geq c^\lambda \|u^\lambda(0) - u^0(0)\|_r^{s-r+\delta} \tag{2.7}$$

where $u^\lambda(t)$, $0 \leq \lambda \leq \lambda_0$ *are solutions of* (2.1) *with initial data* $u^\lambda(0)$ *and* h *is a positive constant depending only on* b *and* s.

Methodology To prove Theorem 2.1 about the Hölder continuity of the solution map, we need to estimate the Burgers nonlinear term uu_x and the nonlocal term P_x in the $H^r(\mathbb{R})$ topology, which is done in Sect. 3 using a commutator type estimate similar to the one in [55]. Applying the estimates to the difference of any two solutions with initial data in a bounded set $B(0, h)$ of $H^s(\mathbb{R})$ infers the $H^r(\mathbb{R})$-Lipschitz continuity of the solution map when the parameters (s, r) are in Ω_1. Then interpolation method implies the Hölder continuity of the solution map when those parameters are in Ω_2 and Ω_3, and Theorem 2.1 is obtained.

As for the optimality in Ω_3, we need to construct initial data $u_0^\lambda(x) \to u_0^0$ as $\lambda \to 0$ so that the time evolution $u^\lambda(t)$ still converges to $u^0(t)$ in the $H^r(\mathbb{R})$ topology, but for any $\delta > 0$, $u^\lambda(t)$ does not converge to $u^0(t)$ uniformly in $H^r(\mathbb{R})$ with Hölder exponent $s - r + \delta$, that is,

$$\frac{\|u^\lambda(t_0) - u^0(t_0)\|_r}{\|u_0^\lambda - u_0^0\|_r^{s-r+\delta}} \to \infty \quad \text{as } \lambda \to 0 \tag{2.8}$$

for some $t_0 > 0$.

The proof of Theorem 2.2 is motivated from Example 5.2 in [46]. It is known that the solution to the Burgers equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R}$$

with initial value $u(0) = u_0$ can be written implicitly as

$$u = u_0(x - tu). \tag{2.9}$$

Let $s \geq 2$ be an integer and γ be a number such that $0 < \gamma - s + \frac{1}{2} < 1$. One may consider the following family of initial data

$$u_0^\lambda(x) = (\lambda + x_+^\gamma)\phi(x), \quad -1 \leq \lambda \leq 1$$

where $x_+ = \max\{x, 0\}$ and ϕ a smooth cutoff in $|x| \leq 2$. Then if $u^\lambda(t, x)$ is the solution corresponding to initial data u_0^λ, then at any later time the constant $\lambda \neq 0$ induces a spatial shift to the higher derivative of the difference $u^\lambda(t, x) - u^0(t, x)$. More explicitly,

$$\partial_x^s(u^\lambda - u^0)(t, x) = [1 + o(1)]\gamma \cdots (\gamma - s - 1)[(x - \lambda t)_+^{\gamma-s} - x_+^{\gamma-s}], \tag{2.10}$$

where $o(1) \to 0$ as x and λt tend to 0. Thus

$$\|u^\lambda(t) - u^0(t)\|_s \geq \|\partial_x^s(u^\lambda(t) - u^0(t))\|_0 \geq c|\lambda t|^{\gamma-s+\frac{1}{2}}.$$

Since $\|u_0^\lambda(0) - u_0^0(0)\|_s = \lambda\|\phi\|_s$ and $\gamma - s + \frac{1}{2}$ can be chosen arbitrarily small, it follows that the solution map of the Burgers equation cannot be Hölder continuous in the H^s-norm with any Hölder exponent.

In treating the b-family equation, there are two major difficulties. One difficulty is that now r can be any real number, one needs to find an explicit way for the calculation of the $H^r(\mathbb{R})$ norm. The other one comes from the nonlocal term. Due to the nonlocal term in the equation, the solution to the b-family equation does not have an implicit formula like (2.9) and therefore it is difficult to get an explicit estimate like (2.10). Thus it is not clear how to get a lower bound of the H^r-difference of solutions.

The first difficulty can be resolved using a formula for non-integer order Sobolev norm (see, for example Theorem 7.48 of [1]), while the other issue needs a more delicate argument. To understand the evolution of solutions of the b-family equation, we will first transform Eq. (2.1) to a semilinear ODE system using a new set of dependent variables. This idea is motivated by Constantin [14], Bressan and Constantin [3]. Furthermore we show that solutions $w(t, x)$ to the transformed ODE system lead back to the unique solution $u(t, x)$ of the b-family equation through an invertible transformation $\eta(t, x)$ with $\eta(0, \cdot) = \mathrm{Id}$. More explicitly, we have $u = w \circ \eta^{-1}$. We also prove that the transformations η and η^{-1} preserve the $H^r(\mathbb{R})$ norm in the sense that

$$c\|f\|_r \le \|f \circ \eta\|_r \le C\|f\|_r. \tag{2.11}$$

Therefore for two solutions $u, \bar{u}$ of (2.1), with the corresponding transformed ODE solutions $w, \bar{w}$ and the transformations $\eta, \bar{\eta}$, we have

$$\|u - \bar{u}\|_r \ge \|w \circ \eta^{-1} - w \circ \bar{\eta}^{-1}\|_r - \|w \circ \bar{\eta}^{-1} - \bar{w} \circ \bar{\eta}^{-1}\|_r.$$

From (2.11) and the Lipschitz continuity of solution maps of the ODE system, we see that

$$\|w \circ \bar{\eta}^{-1} - \bar{w} \circ \bar{\eta}^{-1}\|_r \le c\|w_0 - \bar{w}_0\|_r = c\|u_0 - \bar{u}_0\|_r.$$

From (2.11) we also see that

$$\|w \circ \eta^{-1} - w \circ \bar{\eta}^{-1}\|_r \ge c\left\|w\left(\eta^{-1} \circ \bar{\eta}\right) - w\right\|_r.$$

In this way, a lower bound of the $H^r(\mathbb{R})$ difference of solutions to the b-family equation (2.1) can be obtained by working with the ODE solutions. Using a similar initial data as in the Burgers case, the composition $\eta^{-1} \circ \bar{\eta}$ generates a "phase-shifting" and hence determines the optimal Hölder exponent.

3 The Hölder continuity

For $r \geq 0$ and any $\xi, \zeta \in \mathbb{R}$, an elementary calculation and mean value theorem yield

$$(1+\xi^2)^r \leq c\left[(1+(\xi-\zeta)^2\right]^r + (1+\zeta^2)^r, \tag{3.1}$$

and

$$\begin{aligned} &|(1+\xi^2)^{\frac{r}{2}} - (1+\zeta^2)^{\frac{r}{2}}| \\ &\leq \begin{cases} c|\xi-\zeta|\left[(1+(\xi-\zeta)^2)^{\frac{r-1}{2}} + (1+\zeta^2)^{\frac{r-1}{2}}\right], & \text{if } r>1, \\ c|\xi-\zeta|(1+\zeta^2)^{\frac{r-1}{2}}, & \text{if } 0\leq r\leq 1, \end{cases} \end{aligned} \tag{3.2}$$

see, for example, (2.1) and (2.2) in [55].

First we provide the following bas ic inequality which will be used in the proving of Theorm 2.1.

Lemma 3.1 *Given $s > \frac{3}{2}$ and $0 \leq r \leq s-1$, let w and f be any two functions such that $w \in H^r(\mathbb{R})$, and $f \in H^s(\mathbb{R})$. Then there is constant c depending only on s, r such that the following inequality holds*

$$\left|\int_{\mathbb{R}} \Lambda^r w \Lambda^r (wf)_x dx\right| \leq c\|f\|_s \|w\|_r^2. \tag{3.3}$$

Proof Using (3.1) and the Young inequality, we deduce that

$$\begin{aligned} &|\Lambda^r(wf_x)|_2 \\ &= \left(\int_{\mathbb{R}} (1+\xi^2)^r \left(\int_{\mathbb{R}} \hat{w}(\xi-\zeta)\widehat{f_x}(\zeta)d\zeta\right)^2 d\xi\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left[(1+(\xi-\zeta)^2)^{\frac{r}{2}} + (1+\zeta^2)^{\frac{r}{2}}\right]|\hat{w}(\xi-\zeta)\widehat{f_x}(\zeta)|d\zeta\right)^2 d\xi\right)^{\frac{1}{2}} \\ &\leq |\Lambda^r w|_2 |\widehat{f_x}|_1 + |\hat{w}|_p |(1+\zeta^2)^{\frac{r}{2}}\widehat{f_x}(\zeta)|_{\bar{p}}, \quad \text{with } \frac{1}{p}+\frac{1}{\bar{p}} = 1+\frac{1}{2}. \end{aligned}$$

Since $s > \frac{3}{2}$ one finds

$$|\widehat{f_x}|_1 = \int_{\mathbb{R}} |\widehat{f_x}(\xi)|d\xi \leq \left(\int_{\mathbb{R}} (1+\xi^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{(1+\xi^2)^{s-1}} d\xi\right)^{\frac{1}{2}} \leq c\|f\|_s.$$

If $r = 0$ we take $p = 2$, which implies that $\bar{p} = 1$. It then follows that

$$|\hat{w}|_p |(1+\zeta^2)^{\frac{r}{2}} \widehat{f_x}|_{\bar{p}} = |w|_2 |\widehat{f_x}|_1 \leq c\|w\|_0 \|f\|_s \leq c\|w\|_r \|f\|_s.$$

If $\frac{1}{2} < r \leq s-1$ we take $p = 1$. Hence $\bar{p} = 2$. It is also adduced that

$$|\hat{w}|_p |(1+\zeta^2)^{\frac{r}{2}} \widehat{f_x}|_{\bar{p}} = |\hat{w}|_1 |(1+\zeta^2)^{\frac{r}{2}} \widehat{f_x}|_2 \leq c\|w\|_r \|f\|_{r+1} \leq c\|w\|_r \|f\|_s.$$

The above estimates imply that for some fixed $r_0 \in (\frac{1}{2}, s-1]$

$$\|wf_x\|_0 = |\Lambda^0(wf_x)|_2 \leq c\|f\|_s \|w\|_0 \quad \text{for } w \in H^0(\mathbb{R}),$$

and

$$\|wf_x\|_{r_0} = |\Lambda^{r_0}(wf_x)|_2 \leq c\|f\|_s \|w\|_{r_0} \quad \text{for } w \in H^{r_0}(\mathbb{R}).$$

Using interpolation, the linear map $T : w \to wf_x$ is continuous from $H^r(\mathbb{R})$ to itself for all $0 < r \leq \frac{1}{2}$, that is, for those r one has

$$\|wf_x\|_r \leq c\|f\|_s \|w\|_r \quad \text{for } w \in H^r(\mathbb{R}),$$

and so it holds in the whole considered range $0 \leq r \leq s-1$. Therefore it follows from the Schwarz inequality that the following estimate

$$\left| \int_{\mathbb{R}} \Lambda^r w \Lambda^r (wf_x) dx \right| \leq |\Lambda^r w|_2 |\Lambda^r (wf_x)|_2 \leq c\|f\|_s \|w\|_r^2 \tag{3.4}$$

holds for $0 \leq r \leq s-1$. Secondly, integrating by parts leads to the relation

$$\left| \int_{\mathbb{R}} f \Lambda^r w \Lambda^r w_x dx \right| \leq \frac{1}{2} \left| \int_{\mathbb{R}} f_x (\Lambda^r w)^2 dx \right| \leq \frac{1}{2} |f_x|_\infty \|w\|_r^2 \leq c\|f\|_s \|w\|_r^2. \tag{3.5}$$

Next one has

$$\begin{aligned} &\left|\Lambda^r(w_x f) - f\Lambda^r w_x\right|_2 \\ &\quad = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} ((1+\xi^2)^{\frac{r}{2}} - (1+\zeta^2)^{\frac{r}{2}}) \hat{f}(\xi - \zeta) \widehat{w_x}(\zeta) d\zeta \right)^2 d\xi \right)^{\frac{1}{2}}. \end{aligned}$$

In the case $0 \le r \le 1$, the second estimate in (3.2) and the Young inequality imply

$$
\begin{aligned}
&\left|\Lambda^r(w_x f) - f\Lambda^r w_x\right|_2 \\
&\le c\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\xi-\zeta||\hat{f}(\xi-\zeta)|(1+\zeta^2)^{\frac{r-1}{2}}|\widehat{w_x}(\zeta)|d\zeta\right)^2 d\xi\right)^{\frac{1}{2}} \\
&\le c|\widehat{f_x}|_1|\Lambda^r w|_2 \le c\|f\|_s\|w\|_r.
\end{aligned} \tag{3.6}
$$

In the case $r > 1$, the first estimate in (3.2) and the Young inequality lead to

$$
\begin{aligned}
&\left|\Lambda^r(w_x f) - f\Lambda^r w_x\right|_2 \\
&\le \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \left[(1+(\xi-\zeta)^2)^{\frac{r-1}{2}} + (1+\zeta^2)^{\frac{r-1}{2}}\right]|\xi-\zeta||\hat{f}(\xi-\zeta)\widehat{w_x}(\zeta)|d\zeta\right)^2 d\xi\right)^{\frac{1}{2}} \\
&\le |\widehat{w_x}|_p|(1+\xi^2)^{\frac{r}{2}}\hat{f}(\xi)|_{\bar{p}} + |\Lambda^r w|_2|\widehat{f_x}|_1, \quad \text{with } \frac{1}{p}+\frac{1}{\bar{p}} = 1+\frac{1}{2}.
\end{aligned}
$$

Taking p such that $\max\{1, \frac{2}{2r-1}\} < p < 2$, one finds

$$
\begin{aligned}
|\widehat{w_x}|_p &= \left(\int_{\mathbb{R}} |\widehat{w_x}(\xi)|^p d\xi\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}} (1+\xi^2)^r|\hat{w}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{(1+\xi^2)^{\frac{(r-1)p}{2-p}}} d\xi\right)^{\frac{2-p}{2p}} \le c\|w\|_r.
\end{aligned}
$$

Notice $\bar{p} = \frac{2p}{3p-2}$. The assumption $r \le s-1$ gives

$$
\begin{aligned}
|(1+\xi^2)^{\frac{r}{2}}\hat{f}(\xi)|_{\bar{p}} &= \left(\int_{\mathbb{R}} (1+\xi^2)^{\frac{r\bar{p}}{2}}|\hat{f}(\xi)|^{\bar{p}} d\xi\right)^{\frac{1}{\bar{p}}} \le \left(\int_{\mathbb{R}} (1+\xi^2)^s|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}} \frac{1}{(1+\xi^2)^{\frac{(s-r)p}{2(p-1)}}} d\xi\right)^{\frac{p-1}{p}} \le c\|f\|_s.
\end{aligned}
$$

The above three estimates imply

$$
\left|\Lambda^r(w_x f) - f\Lambda^r w_x\right|_2 \le c\|f\|_s\|w\|_r,
$$

if $1 < r \leq s - 1$. Such estimate is confirmed in (3.6) under the condition $0 \leq r \leq 1$. Therefore it is true for all r with $0 \leq r \leq s-1$. By the Schwarz inequality, we deduce that

$$\left| \int_{\mathbb{R}} \Lambda^r w(\Lambda^r(w_x f) - f\Lambda^r w_x) dx \right| \leq |\Lambda^r w|_2 |\Lambda^r(w_x f) - f\Lambda^r w_x|_2^2$$

$$\leq c\|f\|_s \|w\|_r. \tag{3.7}$$

Using the decomposition gives

$$\Lambda^r w \Lambda^r(wf)_x = \Lambda^r w\big(\Lambda^r(wf_x) + f\Lambda^r w_x + (\Lambda^r(w_x f) - f\Lambda^r w_x)\big),$$

In view of (3.4), (3.5) and (3.7), we obtain the desired results (3.3). □

Next we give the following estimates related to the nonlocal terms.

Lemma 3.2 *Given $s > \frac{3}{2}$, let w and f be any two functions such that $w \in H^r(\mathbb{R})$, and $f \in H^s(\mathbb{R})$. Then there is a constant c depending only on s, r such that*

$$\left| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(wf)_x dx \right| \leq c\|f\|_s \|w\|_r^2 \tag{3.8}$$

holds for $0 \leq r \leq s$, and

$$\left| \int_{\mathbb{R}} \Lambda^r w \Lambda^{r-2}(w_x f_x)_x dx \right| \leq c\|f\|_s \|w\|_r^2 \tag{3.9}$$

holds for $0 \leq r \leq s$ and $s + r \geq 2$.

Proof It is known that

$$|\Lambda^{r-2}(wf)_x|_2 = \left(\int_{\mathbb{R}} (1+\xi^2)^{r-2}\xi^2 \left(\int_{\mathbb{R}} \hat{w}(\xi - \zeta)\hat{f}(\zeta) d\zeta \right)^2 d\xi \right)^{\frac{1}{2}}.$$

If $0 \leq r \leq 1$ one finds

$$|\Lambda^{r-2}(wf)_x|_2 \leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\hat{w}(\xi - \zeta)\hat{f}(\zeta)| d\zeta \right)^2 d\xi \right)^{\frac{1}{2}} \leq |\hat{w}|_2 |\hat{f}|_1 \leq c\|f\|_s \|w\|_r.$$

If $1 < r \leq s$, it is inferred from (3.1) and the Young inequality that $|\Lambda^{r-2}(wf)_x|_2$ is less than

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left[(1+(\xi-\zeta)^2)^{\frac{r-1}{2}}+(1+\zeta^2)^{\frac{r-1}{2}}\right]|\hat{w}(\xi-\zeta)\hat{f}(\zeta)|d\zeta\right)^2 d\xi\right)^{\frac{1}{2}}$$
$$\leq \|w\|_{r-1}|\hat{f}|_1+|\hat{w}|_2|(1+\zeta^2)^{\frac{r-1}{2}}\hat{f}(\zeta)|_1 \leq c\|f\|_s\|w\|_r.$$

The Cauchy–Schwarz inequality is then applied to obtain (3.8).

We now turn to the estimate (3.9). Applying the Hölder inequality, it is found that

$$\begin{aligned}
|\Lambda^{r-2}(w_x f_x)_x|_2^2 &= \int_{\mathbb{R}}(1+\xi^2)^{r-2}\xi^2 d\xi\left(\int_{\mathbb{R}}(\xi-\zeta)\hat{w}(\xi-\zeta)\zeta\hat{f}(\zeta)d\zeta\right)^2\\
&\leq \int_{\mathbb{R}}(1+\xi^2)^{r-1}d\xi\int_{\mathbb{R}}\frac{|\xi-\zeta|^2|\hat{w}(\xi-\zeta)|^2}{(1+\zeta^2)^{s-1}}d\zeta\\
&\leq \int_{\mathbb{R}}(1+\zeta^2)^{s}|\hat{f}(\zeta)|^2 d\zeta\\
&= \|f\|_s^2\int_{\mathbb{R}}(1+\xi^2)^{r-1}d\xi\int_{\mathbb{R}}\frac{|\zeta|^2|\hat{w}(\zeta)|^2}{(1+(\xi-\zeta)^2)^{s-1}}d\zeta\\
&= \|f\|_s^2\int_{\mathbb{R}}|\zeta|^2|\hat{w}(\zeta)|^2 g(\zeta)d\zeta
\end{aligned}\tag{3.10}$$

with

$$g(\zeta)=\int_{\mathbb{R}}h(\xi,\zeta)d\xi \quad \text{and} \quad h(\xi,\zeta)=\frac{\left(1+(\xi-\zeta)^2\right)^{r-1}}{(1+\xi^2)^{s-1}}.$$

If $r=1$, one has $g(\zeta)=\int_{\mathbb{R}}\frac{c}{(1+\xi^2)^{s-1}}d\xi=c$. It thus transpires from (3.10) that

$$|\Lambda^{-1}(w_x f_x)_x|_2 \leq c\|f\|_s\|w\|_1.$$

If $r=s$, Kato-Ponce's estimates [47] and the assumption $s>\frac{3}{2}$ yield

$$\begin{aligned}
|\Lambda^{s-2}(w_x f_x)_x|_2 \leq |\Lambda^{s-1}(w_x f_x)|_2 &\leq |w_x|_\infty\|f_x\|_{s-1}+|f_x|_\infty\|w_x\|_{s-1}\\
&\leq c\|f\|_s\|w\|_s.
\end{aligned}$$

By interpolation, the linear map $T:\ w\to(w_x f_x)_x$ is continuous from $H^r(\mathbb{R})$ to $H^{r-2}(\mathbb{R})$ for all $1\leq r\leq s$, that is, for each such r one has

$$|\Lambda^{r-2}(w_x f_x)_x|_2 \le c\|f\|_s\|w\|_r \qquad \text{for } w \in H^r(\mathbb{R}). \tag{3.11}$$

Now assume $0 \le r < 1$ and $s + r \ge 2$. Without loss of generality, suppose $\zeta > 2$. Let $\Gamma \cup (\frac{\zeta}{2}, \zeta - 1] \cup (\zeta - 1, \zeta + 1] \cup (\zeta + 1, 2\zeta)$ be the decomposition of $\mathbb{R}$ with $\Gamma := \{\xi \in \mathbb{R};\ \xi \le \frac{\zeta}{2},\ \text{or } \xi > 2\zeta\}$. One has $|\xi - \zeta| \ge \frac{\zeta}{2}$ for $\xi \in \Gamma$, so

$$\int_\Gamma h(\xi,\zeta)d\xi \le \frac{c}{(1+\zeta^2)^{1-r}} \int_\Gamma \frac{1}{(1+\xi^2)^{s-1}} d\xi \le \frac{c}{(1+\zeta^2)^{1-r}}$$

with the help $s > \frac{3}{2}$. By change of variable $\xi = \zeta x$, one has

$$\begin{aligned}\int_{\frac{\zeta}{2}}^{\zeta-1} h(\xi,\zeta)d\xi &\le c\int_{\frac{\zeta}{2}}^{\zeta-1} \frac{1}{\xi^{2(s-1)}(\xi-\zeta)^{2(1-r)}}d\xi \\ &= \frac{c}{\zeta^{2(s-r)-1}} \int_{\frac{1}{2}}^{1-\frac{1}{\zeta}} \frac{1}{x^{2(s-1)}(x-1)^{2(1-r)}}dx \\ &\le \frac{c}{\zeta^{2(s-r)-1}} \int_{\frac{1}{2}}^{1-\frac{1}{\zeta}} \frac{1}{(x-1)^{2(1-r)}}dx.\end{aligned}$$

An explicit calculation shows

$$\int_{\frac{1}{2}}^{1-\frac{1}{\zeta}} \frac{1}{(x-1)^{2(1-r)}}dx = \begin{cases} \ln\frac{\zeta}{2}, & \text{if } r = \frac{1}{2}, \\ \frac{2^{1-2r}-\zeta^{1-2r}}{2r-1}, & \text{if } r \ne \frac{1}{2}, \end{cases} \le \begin{cases} \frac{2^{1-2r}}{2r-1}, & \text{if } \frac{1}{2} < r < 1, \\ \ln\frac{\zeta}{2}, & \text{if } r = \frac{1}{2}, \\ \frac{\zeta^{1-2r}}{1-2r}, & \text{if } 0 \le r < \frac{1}{2}. \end{cases}$$

Then the conditions $s + r \ge 2$ and $s > \frac{3}{2}$ imply

$$\int_{\frac{\zeta}{2}}^{\zeta-1} h(\xi,\zeta)d\xi \le \frac{c}{\zeta^{2(s-r)-1}} \cdot \begin{cases} \frac{2^{1-2r}}{2r-1}, & \text{if } \frac{1}{2} < r < 1, \\ \ln\frac{\zeta}{2}, & \text{if } r = \frac{1}{2}, \\ \frac{\zeta^{1-2r}}{1-2r}, & \text{if } 0 \le r < \frac{1}{2}. \end{cases} \le \frac{c}{(1+\zeta^2)^{1-r}}.$$

A similar argument yields

$$\int_{\zeta+1}^{2\zeta} h(\xi,\zeta)d\xi \le \frac{c}{(1+\zeta^2)^{1-r}}.$$

It is also found that

$$\int_{\zeta-1}^{\zeta+1} h(\xi,\zeta)d\xi \le c\int_{\zeta-1}^{\zeta+1} \frac{1}{(1+\zeta^2)^{s-1}}d\xi \le \frac{c}{(1+\zeta^2)^{1-r}},$$

by applying $s+r\ge 2$ again. The above estimates find that $g(\zeta)\le c(1+\zeta^2)^{r-1}$ when $0\le r<1$ and $s+r\ge 2$, which combines with (3.10) yield (3.11) for such parameters. So (3.11) is valid in the whole considered range $0\le r\le s$ and $s+r\ge 2$. Using the Cauchy-Schwarz inequality and (3.11) yields (3.9). □

It is known from [31,35] that the b-family equation is locally well-posed for $s>3/2$. It is also found that the time of existence of the solution has a lower bound T_m and the $H^s(\mathbb{R})$ norm of the solution $u(t)$ is controlled by the $H^s(\mathbb{R})$ norm of u_0 for any $t\in[0,T_m]$. This result was obtained by Himonas and Kenig in [37] for the CH equation, and it can be extended using the same argument to our case of the general b-family.

Proposition 3.3 ([37]) *Let $s>3/2$ and $u(t,x)\in C([0,T_0];H^s(\mathbb{R}))$ be the unique solution of the Cauchy problem (2.1)–(2.2). Then its lifespan (the maximal existence time) is greater than*

$$T_m := \frac{1}{2c_{b,s}}\frac{1}{\|u_0\|_s}, \tag{3.12}$$

where $c_{b,s}$ is a constant depending only on b and s. Also we have

$$\|u(t)\|_s \le 2\|u_0\|_s, \quad 0\le t\le T_m. \tag{3.13}$$

Proof of Theorem 2.1 Let $s>\frac{3}{2}$. By [31,35] for any initial data $u_0\in B(0,h)$ there exists a unique solution $u\in C([0,T_0];\ H^s(\mathbb{R}))\cap C^1([0,T_0];\ H^{s-1}(\mathbb{R}))$ to (2.1)–(2.2) for some $T_0>0$. By Proposition 3.3, the lifespan T_0 satisfies

$$T_0\ge T_m\ge \frac{1}{2c_{b,s}h} := T$$

for some constant $c_{b,s}$ depending only on b and s. Let $\tilde{u}$ be the solution to (2.1) corresponding to $\tilde{u}(0)\in B(0,h)$. Write $w=u-\tilde{u}$ and $f=\frac{1}{2}(u+\tilde{u})$. Then w satisfies the equation

$$w_t+(wf)_x+\Lambda^{-2}\left(bwf+(3-b)w_x f_x\right)_x=0, \quad t\in[0,T]. \tag{3.14}$$

We also have that initial data $w(0) = u(0) - \tilde{u}(0)$. By (3.13) we know $\|f(t)\|_s \le c$ for $t \in [0, T]$.

(i) The Lipschitz continuity in domain Ω_1. Let $0 \le r \le s-1$ and $s+r \ge 2$. Applying the operator Λ^r to the both sides of (3.14) and then multiplying the resulting expression by $\Lambda^r w$ and integrating over $\mathbb{R}$ with respect to x, it follows from (3.3), (3.8) and (3.9) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_r^2 &= -\Big(\Lambda^r w, \Lambda^r(wf)_x + b\Lambda^{r-2}(wf)_x + (3-b)\Lambda^{r-2}(w_x f_x)_x\Big)\\ &\le c\|w(t)\|_r^2 \end{aligned} \tag{3.15}$$

for $t \in [0, T]$. Then $\|w(t)\|_r \le e^{cT}\|w(0)\|_r \le c\|w(0)\|_r$, that is,

$$\|u(t) - \tilde{u}(t)\|_r \le c\|u(0) - \tilde{u}(0)\|_r$$

holds for $t \in [0, T]$ and the Lipschitz continuity in domain Ω_1 is obtained.

(ii) The Lipschitz continuity in domain Ω_2 when $b = 3$. When $b = 3$, applying (3.3) and (3.8), (3.15) becomes

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_r^2 = -\Big(\Lambda^r w, \Lambda^r(wf)_x + b\Lambda^{r-2}(wf)_x\Big) \le c\|w(t)\|_r^2$$

Thus the same argument as in (i) implies the Lipschitz continuity.

(iii) The Hölder continuity in domain Ω_2 when $b \ne 3$. Let $0 \le r < 2-s$ with $\frac{3}{2} < s < 2$. By interpolation, the boundedness of $\|w(t,\cdot)\|_s$ on $[0, T]$ and result in (i), it follows that

$$\begin{aligned}\|w(t)\|_r &\le \|w(t)\|_{2-s} \le c\|w(0)\|_{2-s} \le c\|w(0)\|_r^{\frac{2(s-1)}{s-r}}\|w(0)\|_s^{\frac{2-s-r}{s-r}}\\ &\le c\|w(0)\|_r^{\frac{2(s-1)}{s-r}}.\end{aligned}$$

This in turn implies the Hölder continuity of the solution map in $H^r(\mathbb{R})$ norm with exponent $\frac{2-s-r}{s-r}$.

(iv) The Hölder continuity in domain Ω_3. Let $s-1 < r < s$ with $s > \frac{3}{2}$. Applying the interpolation inequality, the boundedness of $\|w(t,\cdot)\|_s$ on $[0, T]$ and result in (i) yield

$$\|w(t)\|_r \le c\|w(t)\|_{s-1}^{s-r}\|w(t)\|_s^{1-s+r} \le c\|w(t)\|_{s-1}^{s-r} \le c\|w(0)\|_{s-1}^{s-r} \le c\|w(0)\|_r^{s-r}.$$

This completes the proof of Theorem 2.1. □

4 An equivalent semilinear system to the *b*-family equation

As mentioned in Sect. 2, we follow the ideas of [3] and [14] to transform formally the Cauchy problem (2.1)–(2.2) into an equivalent semilinear system (see 4.10–4.11).

Suppose that $u(t,\cdot) \in C([0,T_1]; H^s(\mathbb{R}))$ with $s > \frac{3}{2}$ and $T_1 > 0$ is a solution to problem (2.1)–(2.2). The associated Lagrangian scale of (2.1) is established by the initial value problem

$$\frac{d}{dt}\eta(t,x) = u(t,\eta(t,x)), \quad \eta(0,x) = x \tag{4.1}$$

for each fixed $x \in \mathbb{R}$. By functions u and η we set

$$w(t,x) := u(t,\eta(t,x)), \quad v(t,x) := u_x(t,\eta(t,x)), \quad q(t,x) := \eta_x(t,x). \tag{4.2}$$

Here $u_x(t,\eta(t,x)) = \frac{\partial}{\partial z}u(t,z)\Big|_{z=\eta(t,x)}$ and the convention is used in the following of this paper. The definition of q yields

$$\eta(t,x_1) - \eta(t,x) = \int_x^{x_1} q(t,x_2)dx_2. \tag{4.3}$$

Recalling the definition of P in (2.1), a change of variables $z = \eta(t,x_1)$ and (4.3) lead to

$$\begin{aligned}
P(t,\eta(t,x)) &= \frac{1}{4}\int_{\mathbb{R}} e^{-|\eta(t,x)-z|}\left[bu^2 + (3-b)u_x^2\right](t,z)dz\\
&= \frac{1}{4}\int_{\mathbb{R}} e^{-|\eta(t,x)-\eta(t,x_1)|}\left[bu^2 + (3-b)u_x^2\right](t,\eta(t,x_1))\eta_x(t,x_1)dx_1\\
&= \frac{1}{4}\int_{\mathbb{R}} e^{-\left|\int_x^{x_1} q(t,x_2)dx_2\right|}\left[bw^2q + (3-b)v^2q\right](t,x_1)dx_1,\\
P_x(t,\eta(t,x)) &= \frac{1}{4}\left(\int_{\eta(t,x)}^{\infty} - \int_{-\infty}^{\eta(t,x)}\right) e^{-|\eta(t,x)-z|}\left[bu^2 + (3-b)u_x^2\right](t,z)dz\\
&= \frac{1}{4}\left(\int_x^{\infty} - \int_{-\infty}^{x}\right) e^{-|\eta(t,x)-\eta(t,x_1)|}\left[bu^2 + (3-b)u_x^2\right](t,\eta(t,x_1))\\
&\quad \times \eta_x(t,x_1)dx_1\\
&= \frac{1}{4}\left(\int_x^{\infty} - \int_{-\infty}^{x}\right) e^{-\left|\int_x^{x_1} q(t,x_2)dx_2\right|}\left[bw^2q + (3-b)v^2q\right](t,x_1)dx_1
\end{aligned}$$

where the new variables w, v and q are given in (4.2). We further define $Q = Q(w,v,q)(t,x)$ and $R = R(w,v,q)(t,x)$ as

$$Q := \frac{1}{4}\int_{\mathbb{R}} e^{-\left|\int_x^{x_1} q(x_2)dx_2\right|}\left[bw^2q+(3-b)v^2q\right](x_1)dx_1, \tag{4.4}$$

$$R := \frac{1}{4}\left(\int_x^{\infty} - \int_{-\infty}^{x}\right) e^{-\left|\int_x^{x_1} q(x_2)dx_2\right|}\left[bw^2q+(3-b)v^2q\right](x_1)dx_1. \tag{4.5}$$

In view of Eqs. (4.1) and (2.1) it is found that the variable $w(t,x)$ evolves with time in the form

$$\partial_t w(t,x) = (u_t+uu_x)(t,\eta(t,x)) = -P_x(u(t,\eta(t,x))) = -R \tag{4.6}$$

with R given in (4.5). Differentiating formally (2.1) with respect to x one finds

$$u_{xt}+uu_{xx}+u_x^2-\left(\frac{b}{2}u^2+\frac{3-b}{2}u_x^2\right)+P=0$$

by the identity $(1-\partial_x^2)P=\frac{b}{2}u^2+\frac{3-b}{2}u_x^2$. Combining this relation with (4.1) gives

$$\begin{aligned}\partial_t v(t,x) &= (u_{xt}+uu_{xx})(t,\eta(t,x)) = \left(\frac{b}{2}u^2-\frac{b-1}{2}u_x^2-P\right)(t,\eta(t,x))\\ &= \frac{b}{2}w^2-\frac{b-1}{2}v^2-Q\end{aligned} \tag{4.7}$$

with Q given in (4.4). From the definitions of η, q and v, we have

$$\begin{aligned}\partial_t q(t,x) &= \partial_t\partial_x\eta(t,x) = \partial_x\partial_t\eta(t,x) = \partial_x u(t,\eta(t,x))\\ &= u_x(t,\eta(t,x))\cdot\eta_x(t,x) = vq.\end{aligned} \tag{4.8}$$

From (4.1) and (4.2) it follows that for $x\in\mathbb{R}$

$$\begin{aligned}&w(0,x)=u(0,\eta(0,x))=u(0,x),\;\; v(0,x)=u_x(0,\eta(0,x))=u_x(0,x),\\ &q(0,x)=\eta_x(0,x)=1.\end{aligned} \tag{4.9}$$

For $s\in\mathbb{R}$, let $X := H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})\times H^{s-1}(\mathbb{R})$ be a Banach space with norm

$$\|(u_1,u_2,u_3)\|_X = (\|u_1\|_s^2+\|u_2\|_{s-1}^2+\|u_3\|_{s-1}^2)^{\frac{1}{2}},$$

and

$$Y := X+\{(0,0,1)\} = \{(w,v,q);\;(w,v,q-1)\in X\}$$

be a shifting of X. In connection with Eqs. (4.6)–(4.9) we consider the following abstract ordinary differential equations on a subset K (see (4.15) for its definition) of Y in terms of dependent variables (w,v,q)

$$\begin{cases} \frac{d}{dt} w = -R, \\ \frac{d}{dt} v = \frac{b}{2} w^2 - \frac{b-1}{2} v^2 - Q, \\ \frac{d}{dt} q = vq, \end{cases} \tag{4.10}$$

and initial data

$$\begin{cases} w(0,x) = u^0(x), \\ v(0,x) = u_x^0(x), \\ q(0,x) = 1, \end{cases} \tag{4.11}$$

where $Q = Q(w, v, q)$ and $R = R(w, v, q)$ are given respectively in (4.4) and (4.5), and $u^0 \in H^s(\mathbb{R})$. Solutions to the Cauchy problem (4.10)–(4.11) can be obtained as a fixed point of the following integral transformation

$$\mathcal{T} : (w, v, q) \longrightarrow (\mathrm{w}, \mathrm{v}, \mathrm{q}) \tag{4.12}$$

from $C([0, T_1]; K)$ to itself for sufficiently small $T_1 > 0$, where

$$\begin{cases} \mathrm{w}(t,x) = u^0(x) - \int_0^t R(\tau, x) d\tau, \\ \mathrm{v}(t,x) = u_x^0(x) + \frac{1}{2} \int_0^t \left[bw^2 - (b-1)v^2 - 2Q \right](\tau, x) d\tau, \\ \mathrm{q}(t,x) = 1 + \int_0^t (vq)(\tau, x) d\tau. \end{cases}$$

More precisely, we have

Theorem 4.1 *Assume $u^0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then the Cauchy problem (4.10) –(4.11) has a unique solution $(w^0, v^0, q^0) \in C^1([0, T_1]; Y)$ for some $T_1 > 0$.*

Remark 4.2 From the classical result of fixed point theorem applied to the integral equations, we have

(i) There exist positive numbers μ and $T_2 \le T_1$, depending on u^0, such that if $\|u(0) - u^0\|_s \le \mu$, then the solution (w, v, q) of (4.10) with initial value $(u(0), u_x(0), 1)$ exists on a common interval $[0, T_2]$.

(ii) The solution map of (4.10)–(4.11) is Lipschitz continuous in a neighborhood of u^0 in the following sense: the esitimate

$$\begin{aligned} \|(\tilde{w}, \tilde{v}, \tilde{q})(t) - (w, v, q)(t)\|_{C([0,T_2];\ X)} &\le c\|(\tilde{w}, \tilde{v}, \tilde{q})(0) - (w, v, q)(0)\|_X \\ &\le c\|\tilde{u}(0) - u(0)\|_s \end{aligned} \tag{4.13}$$

holds for all $\tilde{u}(0)$, $u(0)$ satisfying $\|\tilde{u}(0) - u^0\| \le \mu$ and $\|u(0) - u^0\| \le \mu$, where $(\tilde{w}, \tilde{v}, \tilde{q})$ and (w, v, q) are solutions of (4.10) with initial data $(\tilde{u}(0), \tilde{u}_x(0), 1)$ and $(u(0), u_x(0), 1)$, respectively.

To prove Theorem 4.1, we need the following lemma for $\rho > \frac{1}{2}$ (see Theorem 8.3.1 of [44] in one space dimension).

Lemma 4.3 (*Hörmander*) *If $\rho_1 + \rho_2 \geq 0$, then there is a constant $c > 0$ such that*

$$\|fg\|_\rho \leq c\|f\|_{\rho_1}\|g\|_{\rho_2} \quad \text{for } f \in H^{\rho_1}(\mathbb{R}),\ g \in H^{\rho_2}(\mathbb{R}) \tag{4.14}$$

provided $\rho \leq \rho_j,\ j = 1, 2$ and $\rho \leq \rho_1 + \rho_2 - \frac{1}{2}$, with the second inequality strict if ρ_1 or ρ_2 or $-\rho$ equals $\frac{1}{2}$.

Proof of Theorem 4.1 Let $K \subset Y$ be a bounded domain of the form

$$K = \{(w, v, q) \in Y;\ \|(w - u^0, v - u^0_x, q - 1)\|_X \leq q_0\} \tag{4.15}$$

with $0 < q_0 \ll 1$ and $u^0 \in H^s(\mathbb{R})$ by assumption. Notice that K is a convex subset of Y and closed under the topology of Y. Denote by $\mathcal{F}$ the map

$$(w, v, q) \longmapsto \Big(-R,\ bw^2 - (b-1)v^2 - 2Q,\ vq\Big). \tag{4.16}$$

We aim to prove that $\mathcal{F}$ maps K into X and it is Lipschitz continuous on K.

Applying the Sobolev embedding $H^{\frac{1}{2}+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (4.14), and using $q = (q-1) + 1$, the assumption $(w, v, q-1) \in X$ infers that

$$bw^2 - (b-1)v^2,\ vq,\ \Big[bw^2 + (3-b)v^2\Big]q \in H^{s-1}(\mathbb{R}), \tag{4.17}$$

and the following map

$$(w, v, q) \longmapsto \Big(bw^2 - (b-1)v^2,\ vq,\ \Big[bw^2 + (3-b)v^2\Big]q\Big) \tag{4.18}$$

is Lipschitz continuous from K into $H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R})$. The Sobolev embedding infers

$$|w|_\infty, \quad |v|_\infty \leq c, \quad 0 < c_1 < q \leq c_2 \tag{4.19}$$

for $(w, v, q) \in K$. In view of (4.19), the definition of Q in (4.4) yields

$$\begin{aligned} |Q|_2 &\leq \frac{1}{4}\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}} e^{-|\int_x^{x_1} q(x_2)dx_2|}\Big[bw^2q + (3-b)v^2q\Big](x_1)dx_1\right|^2 dx\right)^{\frac{1}{2}} \\ &\leq c|q|_\infty|e^{-c_1|\cdot|} * (w^2 + v^2)|_2 \leq c(|w|_2^2 + |v|_2^2)(\|q-1\|_{s-1} + 1), \end{aligned} \tag{4.20}$$

and the same estimate holds for $|R|_2$. Differentiating R and Q we obtain (in the sense of distribution in $\mathcal{D}'(\mathbb{R})$)

$$R_x = -\frac{1}{2}\Big[bw^2 + (3-b)v^2 - 2Q\Big]q,$$

$$R_{xx} = Rq^2 - [bww_x + (3-b)vv_x]q - \frac{1}{2}\left[bw^2 + (3-b)v^2 - 2Q\right]q_x, \ \cdots \tag{4.21}$$

and

$$\begin{aligned} Q_x &= Rq, \\ Q_{xx} &= -\frac{1}{2}\left[bw^2 + (3-b)v^2 - 2Q\right]q^2 + Rq_x, \\ Q_{xxx} &= Rq^3 - [bww_x + (3-b)vv_x]q^2 \\ &\quad -\frac{3}{2}\left[bw^2 + (3-b)v^2 - 2Q\right]qq_x + Rq_{xx}, \ \cdots \end{aligned} \tag{4.22}$$

The estimate (4.20) and the expression for R_x infer

$$\begin{aligned} |R_x|_2 &\le \left[\frac{b}{2}\|w\|_s^2 + \frac{3-b}{2}\|v\|_{s-1}^2 + |Q|_2\right]|q|_\infty \\ &\le c(\|w\|_s^2 + \|v\|_{s-1}^2)(\|q-1\|_{s-1}+1)^2, \end{aligned}$$

so $R_x \in L^2(\mathbb{R})$ and $R \in H^1(\mathbb{R})$. Similarly, $Q \in H^1(\mathbb{R})$ with the same estimate. Notice that for $k \ge 2$, $\partial_x^k R$ (and $\partial_x^k Q$) is a rational combination of w, v, q, R, Q and derivatives of w, v and q up to $k-1$ order. Moreover, it is linear in $\partial_x^{k-1}w$, $\partial_x^{k-1}v$ and $\partial_x^{k-1}q$. Let $s = m + \beta$ with m an integer and $0 \le \beta < 1$. Without loss of generality, one can assume $m = 2$. Then $(w, v, q-1) \in H^{2+\beta}(\mathbb{R}) \times H^{1+\beta}(\mathbb{R}) \times H^{1+\beta}(\mathbb{R})$ implies $w_x \in H^{1+\beta}(\mathbb{R})$ and $v_x, q_x \in H^\beta(\mathbb{R})$. Decomposing q as $(q-1)+1$ and applying (4.14) infer that every term of the right hand side of R_{xx} belongs to $H^\beta(\mathbb{R})$, for example, using (4.14) in the case $\rho_1 = 1$, $\rho_2 = \beta = \rho$ leads to $Qq_x \in H^\beta(\mathbb{R})$. Thus $R_{xx} \in H^\beta(\mathbb{R})$, therefore $R \in H^s(\mathbb{R})$. Similarly, (4.22) implies $Q \in H^s(\mathbb{R})$, and

$$\|R\|_s, \ \|Q\|_s \le c(\|w\|_s^2 + \|v\|_{s-1}^2)(\|q-1\|_{s-1}+1)^{[s]+1} \tag{4.23}$$

with $[s]$ being the largest integer less than or equal to s. It then follows from(4.17) and (4.23) that the transformation $\mathcal{F}$ defined in (4.16) maps K into X.

Now we show that the transformation $\mathcal{F}$ is Lipschitz continuous from the bounded domain K into X. In view of (4.18), we need only to show that the maps

$$(w, v, q) \longmapsto R, \quad (w, v, q) \longmapsto Q$$

from K into $H^s(\mathbb{R})$ and $H^{s-1}(\mathbb{R})$ respectively are Lipschitz continuous. We only consider the first map, the other one can be considered similarly. For (w_1, v_1, q_1) and $(w_2, v_2, q_2) \in K$, denote by

$$(\tilde{w}, \tilde{v}, \tilde{q}) = (w_2 - w_1, v_2 - v_1, q_2 - q_1), \quad (\bar{w}, \bar{v}, \bar{q}) = (w_1 + w_2, v_1 + v_2, q_1 + q_2),$$

and

$$\tilde{Q} = Q(w_2, v_2, q_2) - Q(w_1, v_1, q_1), \quad \tilde{R} = R(w_2, v_2, q_2) - R(w_1, v_1, q_1).$$

By (4.19), we have

$$|\bar{w}|_\infty,\ |\bar{v}|_\infty,\ |\bar{q}|_\infty \le c. \tag{4.24}$$

Adding and subtracting $e^{-|\int_x^{x_1} q_1(x_2)dx_2|}\left[bw_2^2 q_2 + (3-b)v_2^2 q_2\right](x_1)dx_1$ to the integrand of $4\tilde{R}$, we find

$$\begin{aligned} &4\tilde{R} \\ &\quad = \left(\int_x^\infty - \int_{-\infty}^x\right) e^{-|\int_x^{x_1} q_1(x_2)dx_2|}\Big[bw_1^2\tilde{q} + b\bar{w}\tilde{w}q_2 + (3-b)v_1^2\tilde{q} \\ &\qquad + (3-b)\bar{v}\tilde{v}q_2\Big](x_1)dx_1 \\ &\qquad + \left(\int_x^\infty - \int_{-\infty}^x\right) f\left[bw_2^2 q_2 + (3-b)v_2^2 q_2\right](x_1)dx_1 := I + II. \end{aligned} \tag{4.25}$$

with $f = e^{-|\int_x^{x_1} q_2(x_2)dx_2|} - e^{-|\int_x^{x_1} q_1(x_2)dx_2|}$. From (4.19) and (4.24) , it thus transpires that

$$\begin{aligned} \int_{\mathbb{R}} I^2 dx &\le \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{-c_1|x-x_1|}\left|bw_1^2\tilde{q}+b\bar{w}\tilde{w}q_2+(3-b)v_1^2\tilde{q}+(3-b)\bar{v}\tilde{v}q_2\right|dx_1\right|^2 dx \\ &= \left|e^{-c_1|x|} * |bw_1^2\tilde{q} + b\bar{w}\tilde{w}q_2 + (3-b)v_1^2\tilde{q} + (3-b)\bar{v}\tilde{v}q_2|\right|_2^2 \\ &\le c\left|e^{-c_1|x|} * (|\tilde{w}| + |\tilde{q}| + |\tilde{v}|)\right|_2^2 \le c|(\tilde{w}, \tilde{v}, \tilde{q})|_2^2. \end{aligned} \tag{4.26}$$

By the estimate $e^z - 1 = z\left(\sum_{n=1}^\infty \frac{z^{n-1}}{n!}\right)$, one has (without loss of generality, we assume $x_1 \ge x$)

$$f = e^{-|\int_x^{x_1} q_1(x_2)dx_2|}\left(e^{\int_x^{x_1}\tilde{q}(x_2)dx_2} - 1\right) = e^{-|\int_x^{x_1} q_1(x_2)dx_2|}\, z\left(\sum_{n=1}^\infty \frac{z^{n-1}}{n!}\right)$$

where $z = \int_x^{x_1} |\tilde{q}(x_2)|dx_2$. Applying the Hölder inequality and (4.19), there appears the relation

$$|f| \le ce^{-c_1|x-x_1|}|\tilde{q}|_2|x-x_1|^{\frac{1}{2}}\sum_{n=1}^\infty \frac{(2q_0)^{n-1}|x-x_1|^{n-1}}{n!}$$

$$\leq c|\tilde{q}|_2 e^{-c_1|x-x_1|+\frac{c_1}{2}|x-x_1|+2q_0|x-x_1|} \leq c|\tilde{q}|_2 e^{-\frac{c_1}{4}|x-x_1|}$$

where $q_0 \ll 1$ is given in (4.15). It then follows from (4.19) and the Young inequality that

$$\int_{\mathbb{R}} II^2 dx \leq c|\tilde{q}|_2^2 \left| e^{-\frac{c_1}{4}|x|} * \left[b w_2^2 + (3-b) v_2^2 \right] \right|_2^2 \leq c|\tilde{q}|_2^2,$$

which and (4.25), (4.26) lead to $|\tilde{R}|_2 \leq c|(\tilde{w}, \tilde{v}, \tilde{q})|_2$. Using similar arguments to obtaining (4.23), one gets $\|I\|_s, \|II\|_s \leq c\|(\tilde{w}, \tilde{v}, \tilde{q})\|_X$, and

$$\|\tilde{R}\|_s \leq c\|(\tilde{w}, \tilde{v}, \tilde{q})\|_X \tag{4.27}$$

Thus the Lipschitz continuity of $\mathcal{F}$ from K into X is established, and for $T_1 > 0$ sufficiently small, the transformation $\mathcal{T}$ of (4.12) is a contraction map on $C([0, T_1]; K)$. In consequence, the Cauchy problem (4.10)–(4.11) has a unique solution on $[0, T_1]$. This completes the proof of Theorem 4.1. $\square$

5 Solutions to the *b*-family equation

We show that any solution to the Cauchy problem (2.1)–(2.2) can be written as a composition of two functions related to the geodesic flows. More precisely, we have

Proposition 5.1 *Assume $u^0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let $(w, v, q) \in C^1([0, T_2]; Y)$ be the solution to (4.10) with initial value $(u(0), u_x(0), 1)$ such that $\|u(0) - u^0\|_s \leq \mu$ with μ and T_2 given in Remark 4.2. Then, for each fixed $t \in [0, T_2]$ the function*

$$\eta(t, x) := x + \int_0^t w(\tau, x) d\tau \tag{5.1}$$

has an inverse $x = x(t, \eta)$ on $\mathbb{R}$, and the function $u : [0, T_2] \times \mathbb{R} \to \mathbb{R}$ defined by

$$u(t, \eta(t, x)) := w(t, x), \tag{5.2}$$

is the unique solution to the b-family Eq. (2.1) with initial data $u(0)$ on $[0, T_2]$.

Remark 5.2 It is known that $\eta_t = w$ by the definition of $\eta(t, x)$. This in turn implies that we can define u by setting $u = \eta_t \circ \eta^{-1}$, the composition of η_t and the inverse function of η. This formula is well related to the geodesic flows in the group of diffeomorphisms on $\mathbb{R}$ (see, for example, [14,61]).

The proof of Proposition 5.1 is approached via a series of lemmas.

Lemma 5.3 *Under the assumption of Proposition 5.1, the function u defined in (5.2) satisfies the b-family Eq. (2.1) in $C([0, T_2] \times \mathbb{R})$.*

Proof Since X is a reflexive space, we deduce from the Rademacher Theorem [2] in infinite dimensional form that the functions on the left hand side of (4.10) are well defined in X, for a.e. $t \in [0, T_2]$, then (4.10) can be interpreted as equalities in X for a.e. $t \in [0, T_2]$. Let $C_b(\mathbb{R})$ be the space of bounded continuous functions in $\mathbb{R}$. By the Sobolev embedding of $H^\rho(\mathbb{R})$ with $\rho > \frac{1}{2}$ into $C_b(\mathbb{R})$ one infers for a.e. $t \in [0, T_2]$ the equations

$$\begin{cases} \frac{d}{dt} w = -R, \\ \frac{d}{dt} w_x = -R_x, \\ \frac{d}{dt} v = \frac{b}{2} w^2 - \frac{b-1}{2} v^2 - Q, \\ \frac{d}{dt} q = vq, \end{cases} \tag{5.3}$$

hold in $C_b(\mathbb{R})$. As w, v and q are Lipschitz continuous with respect to t for each $x \in \mathbb{R}$, (4.4) and (4.5) imply R, R_x and Q have the same property. The last three equations in (5.3) and the expression of R_x in (4.21) yield

$$\frac{d}{dt} w_x = \left(\frac{b}{2} w^2 + \frac{3-b}{2} v^2 - Q \right) q = \frac{d}{dt}(vq),$$

which, together with $w_x(0, x) = u_x(0, x) = (vq)(0, x)$, imply for each $x \in \mathbb{R}$

$$w_x(t, x) = (vq)(t, x) \quad \text{for } t \in [0, T_2]. \tag{5.4}$$

This in turn implies from the last equation of (5.3) that

$$\frac{d}{dt} q = w_x \quad \text{with} \quad q(0, x) = 1.$$

On the other hand, the definition of η infers that, for each fixed $x \in \mathbb{R}$, η_x satisfies the same differential equation with the same initial value, it then follows that

$$\eta_x = q \quad \text{for } t \in [0, T_2],\ x \in \mathbb{R}. \tag{5.5}$$

By (4.13) $\|v(t, \cdot)\|_{s-1} \le c$ for $t \in [0, T_2]$, and hence from Sobolev embedding $|v| \le c$ on $[0, T_2] \times \mathbb{R}$. We can conclude from the last equation in (5.3) that

$$0 < c_1 \le \eta_x(t, x) = q(t, x) = e^{\int_0^t v(\tau, x) d\tau} \le c_2 \tag{5.6}$$

for $t \in [0, T_2]$ and all $x \in \mathbb{R}$. Hence $\eta(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is invertible for every $t \in [0, T_2]$ and its inverse function $x(t, \eta)$ satisfies

$$0 < c_2^{-1} \le x_\eta(t, \eta) = \frac{1}{\eta_x(t, x)} < c_1^{-1}. \tag{5.7}$$

Differentiating (5.2) with respect to x and taking into account (5.4), one gets

$$u_x(t,\eta(t,x))\cdot\eta_x(t,x)=w_x(t,x)=(vq)(t,x),$$

which and $\eta_x=q>0$ in (5.6) yield

$$v(t,x)=u_x(t,\eta(t,x)). \tag{5.8}$$

On the other hand, the definition of R in (4.5) and the identity $\eta_x=q$ in (5.5) imply

$$R(t,x)=-\frac{1}{4}\left(\int_x^{\infty}-\int_{-\infty}^{x}\right)e^{-|\eta(t,x)-\eta(t,x_1)|}\left[bw^2+(3-b)v^2\right](x_1)d\eta(t,x_1).$$

Then we deduce from (5.8) and the first equation of (5.3) that

$$w_t(t,x(t,z))=-R(t,x(t,z))=-P_z(t,z).$$

Differentiating both sides of (5.2) with respect to t, we have

$$u_t(t,\eta(t,x))+u_x(t,\eta(t,x))\cdot\eta_t(t,x)=w_t(t,x).$$

Replacing x by $x(t,z)$ and using the definition $u=\eta_t\circ\eta^{-1}$ yield

$$u_t(t,z)+u_x(t,z)u(t,z)=w_t(t,x(t,z))=-P_z(t,z),\quad z\in\mathbb{R}.$$

Consequently, u satisfies (2.1) in $C([0,T_2]\times\mathbb{R})$. This completes the proof of Lemma 5.3.

Remark 5.4 Recall that $w_x=vq,\ \eta_x=q$ in (5.4) and (5.5). In the case when $b=2$, (2.1) refers to the CH equation. It then follows from (5.3) that

$$\frac{d}{dt}\|u(t,\cdot)\|_1^2=\frac{d}{dt}\int_{\mathbb{R}}(w^2q+v^2q)(t,x)dx=0$$

Hence $\|u(t)\|_1$ is conserved on $[0,T_2]$. The Sobolev embedding theorem (see e.g. [14] for best constant) may be applied to obtain

$$|w(t,\cdot)|_\infty=|u(t,\cdot)|_\infty\le\frac{1}{\sqrt{2}}\|u(t,\cdot)\|_1=\frac{1}{\sqrt{2}}\|u(0)\|_1 \tag{5.9}$$

for $t\in[0,T_2]$. Given some $x_0\in\mathbb{R}$ such that $u_x(0,x_0)<-\|u(0)\|_1$, then a wave-breaking phenomenon can be found from the third equation in (5.3) and $Q\ge 0$. So the solution of equations in (4.10) would blow up in finite time. This is different from the semilinear equations in Bressan and Constantin [3], where the solutions exist globally in time.

Lemma 5.5 *Under the assumption of Proposition 5.1, we have for each* $t \in [0, T_2]$

$$\eta(t,\cdot) - \mathrm{Id},\ x(t,\cdot) - \mathrm{Id} \in C^1([0,T_2]; H^s(\mathbb{R})), \tag{5.10}$$

where Id *is the identity function on* $\mathbb{R}$.

Proof The proof is divided into two steps.

Step 1 The definition of $\eta(t,x)$ in (5.1) yields $\eta(t,x) - x = \int_0^t w(\tau,x)d\tau$. For $t \in [0,T_2]$ fixed, the Minkowski inequality then gives rise to

$$\begin{aligned} |\eta(t,x) - x|_2 &= \left(\int_{\mathbb{R}} \left| \int_0^t w(\tau,x)d\tau \right|^2 dx \right)^{\frac{1}{2}} \leq \int_0^t \left(\int_{\mathbb{R}} |w(\tau,x)|^2 dx \right)^{\frac{1}{2}} d\tau \\ &\leq T_2 \max_{0\leq\tau\leq T_2} |w(\tau,\cdot)|_2. \end{aligned} \tag{5.11}$$

Let $\eta^{(k)}(t,x)$ denote the kth derivative of η with respect to the space variable x. Then a similar arguments as in (5.11) implies

$$|(\eta(t,x) - x)^{(k)}|_2 \leq T_2 \max_{0\leq\tau\leq T_2} |w^{(k)}(\tau,\cdot)|_2, \tag{5.12}$$

for integers $1 \leq k \leq m$ with m an integer such that $s = m + \beta$ and $0 \leq \beta < 1$.

If $\beta = 0$, then (5.11) and (5.12) lead to $\eta(t,\cdot) - \mathrm{Id} \in H^s(\mathbb{R})$.

If $0 < \beta < 1$, differentiating both sides of (5.1) m times with respect to x, one has

$$\frac{|\eta^{(m)}(t,x_1) - \eta^{(m)}(t,x_2)|^2}{|x_1 - x_2|^{1+2\beta}} = \left| \int_0^t \frac{w^{(m)}(\tau,x_1) - w^{(m)}(\tau,x_2)}{|x_1 - x_2|^{\frac{1}{2}+\beta}} d\tau \right|^2.$$

Integrating both sides on $\mathbb{R}^2$ and applying the Minkowski inequality yield

$$\begin{aligned} \|\eta^{(m)}(t,\cdot)\|_\beta &\leq \int_0^t \left(\int_{\mathbb{R}^2} \frac{|w^{(m)}(\tau,x_1) - w^{(m)}(\tau,x_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 \right)^{\frac{1}{2}} d\tau \\ &\leq T_2 \max_{0\leq\tau\leq T_2} \|w^{(m)}(\tau,\cdot)\|_\beta \end{aligned} \tag{5.13}$$

with the notation

$$\|f^{(m)}\|_\beta = \left(\int_{\mathbb{R}^2} \frac{|f^{(m)}(x_1) - f^{(m)}(x_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 \right)^{\frac{1}{2}}. \tag{5.14}$$

By Theorem 7.48 of [1], $\|f\|_s^2 = \|f\|_m^2 + \|f^{(m)}\|_\beta^2$ for $f \in H^s(\mathbb{R})$. Then (5.11)–(5.13) lead to $\eta(t,\cdot) - \mathrm{Id} \in H^s(\mathbb{R})$ for each $t \in [0, T_2]$. The above arguments and the equality $\eta(t_2, x) - \eta(t_1, x) = \int_{t_1}^{t_2} w(\tau, x)d\tau$ yield the C^1 continuity with respect to t. Hence, the first property of (5.10) holds.

Step 2 Replacing x in (5.1) by $x(t,\eta)$ yields $x(t,\eta) - \eta = -\int_0^t w(\tau, x(t,\eta))d\tau$ for $t \in [0, T_2]$ and $\eta \in \mathbb{R}$. Let $t \in [0, T_2]$ be fixed. The Minkowski inequality then implies

$$\begin{aligned} |x(t,\eta) - \eta|_2 &= \left(\int_{\mathbb{R}} \left|\int_0^t w(\tau, x(t,\eta))d\tau\right|^2 d\eta\right)^{\frac{1}{2}} \le \int_0^t \left(\int_{\mathbb{R}} |w(\tau, x(t,\eta))|^2 d\eta\right)^{\frac{1}{2}} d\tau \\ &\le c\int_0^t \left(\int_{\mathbb{R}} |w(\tau, x)|^2 dx\right)^{\frac{1}{2}} d\tau \le cT_2 \max_{0\le\tau\le T_2} |w(\tau,\cdot)|_2. \end{aligned} \tag{5.15}$$

Here the estimate $d\eta = \eta_x(t,x)dx \le cdx$ is applied in the third inequality. Denote $x^{(k)}(t,\eta)$ the kth derivative of $x(t,\eta)$ with respect to the space variable η. As $x(t,\eta)$ is a bi-Lipschitz function of η, and $w(\tau, x)$ has an L^2 derivative with respect to x, the composite function $w(\tau, x(t,\eta))$ has an L^2 derivative with respect to η by Theorem 2.2.2. of [69], and for $\tau, t \in [0, T_2]$

$$(w(\tau, x(t,\eta)))^{(1)} = w^{(1)}(\tau, x(t,\eta))x^{(1)}(t,\eta) \tag{5.16}$$

in the sense of $L^2(\mathbb{R})$. Thus

$$(x(t,\eta) - \eta)^{(1)} = -\int_0^t w^{(1)}(\tau, x(t,\eta))x^{(1)}(t,\eta)d\tau.$$

By (5.7), a similar argument as in (5.15) leads to

$$|(x(t,\eta) - \eta)^{(1)}|_2 \le c_2 T_2 \max_{0\le\tau\le T_2} |w^{(1)}(\tau,\cdot)|_2. \tag{5.17}$$

If $m \ge 2$, applying the bi-Lipschitz change of coordinates formula (5.16) for those functions again gives

$$\begin{aligned} x^{(2)}(\eta) &= -\frac{1}{(\eta^{(1)}(x))^3}\eta^{(2)}(x), \\ x^{(3)}(\eta) &= -\frac{1}{(\eta^{(1)}(x))^4}\eta^{(3)}(x) + \frac{3}{(\eta^{(1)}(x))^5}(\eta^{(2)}(x))^2, \\ &\cdots \\ x^{(m)}(\eta) &= -\frac{1}{(\eta^{(1)}(x))^{m+1}}\eta^{(m)}(x) + \cdots + \frac{1}{(\eta^{(1)}(x))^{i_n}}H_n, \end{aligned} \tag{5.18}$$

where H_i, $1 \le i \le n$, are polynomials of $\eta^{(2)}(t,x), \ldots, \eta^{(m-1)}(t,x)$ with no zero order term $\eta(t,x)$, where $n, i_1, \ldots, i_n$ are positive integers depending on m. From (5.6), (5.11) and (5.12) we deduce

$$|x^{(k)}(t,\cdot)|_2 \le c, \quad k = 2, \ldots, m. \tag{5.19}$$

If $\beta = 0$ then (5.15), (5.17) and (5.19) lead to $x(t,\cdot) - \mathrm{Id} \in H^s(\mathbb{R})$.

If $0 < \beta < 1$, we consider only the case $m = 3$ for simplicity of notation. It is inferred from (5.6) that

$$|\eta_1 - \eta_2| \ge c|x_1 - x_2| \tag{5.20}$$

for $\eta_i = \eta(t, x_i) \in \mathbb{R}$, $i = 1, 2$ and any $t \in [0, T_2]$ fixed. Applying (5.6), (5.18) and (5.20) gives

$$\begin{aligned} \|(x(t,\eta) - \eta)^{(3)}\|_\beta &= \int_{\mathbb{R}^2} \frac{|x^{(3)}(\eta_1) - x^{(3)}(\eta_2)|^2}{|\eta_1 - \eta_2|^{1+2\beta}} d\eta_1 d\eta_2 \\ &\le c \int_{\mathbb{R}^2} \frac{|x^{(3)}(\eta_1) - x^{(3)}(\eta_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 \le cI + cII \end{aligned} \tag{5.21}$$

with

$$\begin{aligned} I &= \int_{\mathbb{R}^2} \frac{|\eta^{(3)}(x_1)(\eta^{(1)}(x_2))^4 - \eta^{(3)}(x_2)(\eta^{(1)}(x_1))^4|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2, \\ II &= \int_{\mathbb{R}^2} \frac{|(\eta^{(2)}(x_2))^2(\eta^{(1)}(x_1))^5 - (\eta^{(2)}(x_1))^2(\eta^{(1)}(x_2))^5|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2. \end{aligned}$$

This in turn implies that

$$\begin{aligned} I &\le \int_{\mathbb{R}^2} \frac{|\eta^{(3)}(x_1) - \eta^{(3)}(x_2)|^2 \cdot |\eta^{(1)}(x_2)|^8}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 \\ &\quad + \int_{\mathbb{R}^2} \frac{|\eta^{(3)}(x_2)|^2 \cdot |(\eta^{(1)}(x_2))^4 - (\eta^{(1)}(x_1))^4|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 \\ &\le c \int_{\mathbb{R}^2} \frac{|\eta^{(3)}(x_1) - \eta^{(3)}(x_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 \\ &\quad + c \int_{\mathbb{R}^2} \frac{|\eta^{(3)}(x_2)|^2 \cdot |\eta^{(1)}(x_2) - \eta^{(1)}(x_1)|^2}{|x_1 - x_2|^{1+2\beta}} dx_1 dx_2 := I_1 + I_2. \end{aligned}$$

The fact that $\eta(t,x)-x\in H^s(\mathbb{R})$ in *Step 1* and the definition of the intrinsic $H^s(\mathbb{R})$ norm then imply

$$I_1 = c\|(\eta(t,x)-x)^{(3)}\|_\beta \le c\|\eta(t,\cdot)-\mathrm{Id}\|_s.$$

The calculation of I_2 is decomposed into two cases.

(a) $0<\beta\le\frac{1}{2}$. Write the set $E=\{(x_1,x_2)\in\mathbb{R}^2:\ |x_1-x_2|<1\}$. Applying the Lipshcitz continuity of $\eta^{(1)}(x)$ with respect to x in the domain E and the boundedness of $\eta^{(1)}(x)$ in the complement of E lead to

$$\begin{aligned}
I_2 &\le c\int\limits_E \frac{|\eta^{(3)}(x_2)|^2}{|x_1-x_2|^{2\beta-1}}dx_1dx_2 + c\int\limits_{\mathbb{R}^2\setminus E}\frac{|\eta^{(3)}(x_2)|^2}{|x_1-x_2|^{1+2\beta}}dx_1dx_2\\
&= c\int\limits_{|z|<1}\frac{1}{|z|^{2\beta-1}}dz\int\limits_{\mathbb{R}}|\eta^{(3)}(x_2)|^2dx_2 + c\int\limits_{|z|>1}\frac{1}{|z|^{1+2\beta}}dz\int\limits_{\mathbb{R}}|\eta^{(3)}(x_2)|^2dx_2\\
&\le c|\eta^{(3)}(t,\cdot)|_2 \le c\|\eta(t,\cdot)-\mathrm{Id}\|_s^2.
\end{aligned}$$

(b) $\frac{1}{2}<\beta<1$. In this case $\eta^{(3)}(t,\cdot)\in L^\infty$ from Sobolev embedding. Since $s-1\ge\beta$, we have $(\eta(t,\cdot)-\mathrm{Id})^{(1)}\in H^{s-1}(\mathbb{R})\subset H^\beta(\mathbb{R})$. Using intrinsic Sobolev we get

$$\begin{aligned}
I_2 &\le |\eta^{(3)}(t,\cdot)|_\infty^2\int\limits_{\mathbb{R}^2}\frac{|\eta^{(1)}(x_2)-\eta^{(1)}(x_1)|^2}{|x_1-x_2|^{1+2\beta}}dx_1dx_2\\
&= |\eta^{(3)}(t,\cdot)|_\infty^2\|(\eta(t,x)-x)^{(1)}\|_\beta^2 \le c\|\eta(t,\cdot)-\mathrm{Id}\|_s^2.
\end{aligned}$$

With a slightly more complicated calculation, the term II can be estimated by the same term as in (5.21). Combining (5.15), (5.17), (5.19) and (5.21) we find $x(t,\cdot)-\mathrm{Id}\in H^s(\mathbb{R})$. The identities $\eta(t,x(t,\eta))=\eta$, $\eta_t=w$ and $\eta_x=q$ infer that $x_t(t,\eta)=\frac{w(t,x(t,\eta))}{q(t,x(t,\eta))}$. Using similar arguments as before we get $\frac{w(t,x(t,\cdot))}{q(t,x(t,\cdot))}\in H^s(\mathbb{R})$, then the equation

$$x(t,\eta)-\eta=\int\limits_0^t\frac{w(\tau,x(\tau,\eta))}{q(\tau,x(\tau,\eta))}d\tau$$

implies the C^1 continuity of $x(t,\cdot)-\mathrm{Id}$ with respect to t in $H^s(\mathbb{R})$. □

Next we prove that the transformations η and η^{-1} "preserve" the $H^r(\mathbb{R})$ norm.

Lemma 5.6 *Under the assumptions of Proposition 5.1 one gets $f(x(t,\cdot))\in H^s(\mathbb{R})$ for any $f\in H^s(\mathbb{R})$ and any $t\in[0,T_2]$, and the estimates*

$$c_1\|f\|_r\le\|f(x(t,\cdot))\|_r\le c_2\|f\|_r \tag{5.22}$$

are valid for $0 \le r \le s$*, where the constants* c_1, c_2 *depend only on* u^0*,* μ *and* T_2 *(see Remark 4.2 for* μ *and* T_2*), and is independent of* r*.*

Proof Write $g(t,\eta) = f(x(t,\eta))$. Applying chain rule as in (5.16), the assumption that $f \in H^s(\mathbb{R})$ and the fact $x(t,\eta) - \eta \in H^s(\mathbb{R})$ with $s = m + \beta$ in (5.10) yield

$$\begin{aligned} g^{(1)}(\eta) &= f^{(1)}(x))x^{(1)}(\eta), \\ g^{(2)}(\eta) &= f^{(2)}(x))(x^{(1)}(\eta))^2 + f^{(1)}(x))x^{(2)}(\eta), \\ &\cdots \\ g^{(m)}(\eta) &= f^{(m)}(x))(x^{(1)}(\eta))^m + \cdots + f^{(1)}(x))x^{(m)}(\eta). \end{aligned} \tag{5.23}$$

Let $r = n + \gamma$ with n an integer and $0 \le \gamma < 1$. For simplicity of notation, we only consider the case $n = 2$ and $\gamma > 0$. We have

$$\begin{aligned} \|g^{(2)}(t,\cdot)\|_\gamma^2 &= \int_{\mathbb{R}^2} \frac{|g^{(2)}(\eta_1) - g^{(2)}(\eta_2)|^2}{|\eta_1 - \eta_2|^{1+2\gamma}} d\eta_1 d\eta_2 \\ &\le c \int_{\mathbb{R}^2} \frac{|f^{(2)}(x(\eta_1)))(x^{(1)}(\eta_1))^2 - f^{(2)}(x(\eta_2)))(x^{(1)}(\eta_2))^2|^2}{|\eta_1 - \eta_2|^{1+2\gamma}} d\eta_1 d\eta_2 \\ &\quad + \int_{\mathbb{R}^2} \frac{|f^{(1)}(x(\eta_1)))x^{(2)}(\eta_1) - f^{(1)}(x(\eta_2))x^{(2)}(\eta_2)|^2}{|\eta_1 - \eta_2|^{1+2\gamma}} d\eta_1 d\eta_2. \end{aligned}$$

The same arguments as in the proof of Lemma 5.5 yield the second inequality of (5.22). This also infers $\|\bar{f}(\eta(t,\cdot))\|_r \le c\|\bar{f}\|_r$ for any $\bar{f} \in H^s$ and any $t \in [0, T_2]$. Replacing $\bar{f}(\cdot)$ by $f(x(t,\cdot))$ and applying the identity $x(t, \eta(t,\cdot)) = \mathrm{Id}$ yield the first inequality of (5.22). □

Proof of Proposition 5.1 From (5.22) we have $u(t,\cdot) = w(t, x(t,\cdot)) \in H^s(\mathbb{R})$ for $t \in [0, T_2]$. Next we show

$$u(t,\cdot) \in C^1([0,T_2];\ H^{s-1}(\mathbb{R})) \quad \text{and} \quad u(t,\cdot) \in C([0,T_2];\ H^s(\mathbb{R})). \tag{5.24}$$

For $t_0, t \in [0, T_2]$, the definition of u gives

$$u(t,\eta) - u(t_0,\eta) = h_1(t,\eta) + h_2(t,\eta),$$

where

$$h_1(t,\eta) = w(t, x(t,\eta)) - w(t, x(t_0,\eta)),\ h_2(t,\eta) = w(t, x(t_0,\eta)) - w(t_0, x(t_0,\eta)).$$

From Theorem 4.1 we know $w(t,\cdot) \in C^1([0,T_2];\ H^s(\mathbb{R}))$. This, together with (5.22) imply that for $0 \le r \le s$

$$\|h_2(t,\cdot)\|_r \le c\|w(t,\cdot) - w(t_0,\cdot)\|_r \le c|t - t_0|. \tag{5.25}$$

Applying mean value theorem we have for $0 \le k \le m-1$

$$w^{(k)}(t, x(t,\cdot)) - w^{(k)}(t, x(t_0,\cdot)) = \int_0^1 w^{(k+1)}(t, \theta(t,\cdot))d\theta \cdot (x(t,\cdot) - x(t_0,\cdot)),$$

where $\theta(t,\cdot) = \theta x(t,\cdot) + (1-\theta)x(t_0,\cdot)$ is bi-Lipschitz about space variables and $\theta(t,\cdot) - \mathrm{Id} \in H^s(\mathbb{R})$. Then (5.10) implies

$$\begin{aligned} &|w^{(l)}(t, x(t,\cdot)) - w^{(l)}(t, x(t_0,\cdot))|_2 = O(|t-t_0|) \quad \text{for } 0 \le l \le m-2, \text{ and} \\ &\|w^{(m-1)}(t, x(t,\cdot)) - w^{(m-1)}(t, x(t_0,\cdot))\|_\beta = O(|t-t_0|). \end{aligned} \tag{5.26}$$

From (4.14), (5.23), (5.10), and (5.26) we obtain $\|h_1(t,\cdot)\|_{s-1} \le c|t-t_0|$, which together with (5.25) prove the first part of (5.24).

Let $\varphi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi(x)dx = 1$ and $\varphi_n(x) = n\varphi(nx),\ n \ge 1$. We have

$$\begin{aligned} &\lim_{t\to t_0} \left| w^{(m)}(t, x(t,\cdot)) - w^{(m)}(t, x(t_0,\cdot) \right|_2 \\ &\quad = \lim_{t\to t_0} \lim_{n\to\infty} \left| (w^{(m)}(t, x(t,\cdot)) - w^{(m)}(t, x(t_0,\cdot))) * \varphi_n \right|_2 \\ &\quad = \lim_{n\to\infty} \lim_{t\to t_0} \left| (\varphi_n(\cdot - x(t,\cdot)) - \varphi_n(\cdot - x(t_0,\cdot))) * w^{(m)}(t,\cdot) \right|_2 = 0 \end{aligned} \tag{5.27}$$

where we used the Lebesgue dominated convergence theorem to interchange the limits.

If $\beta = 0$, from (4.14) and (5.23), we deduce from (5.26), (5.27) and (5.10) that $\lim_{t\to t_0} \|h_1(t,\cdot)\|_m = 0$, which together with (5.25) lead to the second part of (5.24).

If $0 < \beta < 1$, we have

$$\begin{aligned} &\lim_{t\to t_0} \|w^{(m)}(t, x(t,\cdot)) - w^{(m)}(t, x(t_0,\cdot))\|_\beta^2 = \lim_{t\to t_0} \int_{\mathbb{R}^2} \frac{|\tilde{w}(t,\eta_1,\eta_2)|^2}{|\eta_1-\eta_2|^{1+2\beta}} d\eta_1 d\eta_2 \\ &= \lim_{t\to t_0} \lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{|(\tilde{w}(t,\eta_1,\eta_2)) * \varphi_n|^2}{|\eta_1-\eta_2|^{1+2\beta}} d\eta_1 d\eta_2 \\ &= \lim_{n\to\infty} \lim_{t\to t_0} \int_{\mathbb{R}^2} \frac{|\tilde{\varphi}_n(t,\eta_1,\eta_2,\cdot) * w^{(m)}(t,\cdot)|^2}{|\eta_1-\eta_2|^{1+2\beta}} d\eta_1 d\eta_2 = 0, \end{aligned}$$

where $\tilde{w}(t,\eta_1,\eta_2) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for fixed $t \in [0, T_2]$ is given as

$$w^{(m)}(t, x(t,\eta_1)) - w^{(m)}(t, x(t_0,\eta_1)) - w^{(m)}(t, x(t,\eta_2)) + w^{(m)}(t, x(t_0,\eta_2)),$$

and $\tilde{\varphi}_n(t,\eta_1,\eta_2,z)$ represents

$$\varphi_n(z - x(t,\eta_1)) - \varphi_n(z - x(t_0,\eta_1)) - \varphi_n(z - x(t,\eta_2)) + \varphi_n(z - x(t_0,\eta_2)).$$

Then the same arguments as in the case $\beta = 0$ implies $\lim_{t\to t_0} \|h_1(t,\cdot)\|_s = 0$ and hence proves the second part of (5.24).

Since $u(t,\cdot) \in C^1([0,T_2];\ H^{s-1}(\mathbb{R}))$ satisfies (2.1) by (5.24), the estimates of the Hölder continuity in Theorem 2.1 in turn imply the uniqueness of u in the space $C([0,T_2];\ H^s(\mathbb{R}))$. □

6 Proof of Theorem 2.2

Let $s = m + \beta > \frac{3}{2}$ with m an integer and $\beta \in [0,1)$. Let $r \in (s-1, s)$. Define

$$\beta_0 = \begin{cases} \frac{1}{2} - \beta, & \text{when } \beta \in [0, \frac{1}{2}) \\ \frac{3}{4} - \frac{\beta}{2}, & \text{when } \beta \in [\frac{1}{2}, 1). \end{cases} \tag{6.1}$$

This in turn implies that $\beta_0 \in (0, \frac{1}{2}]$. Set

$$s_1 = s - \frac{1}{2} - \varepsilon \quad \text{and} \quad \beta_1 = \beta - \frac{1}{2} + \varepsilon \tag{6.2}$$

with

$$\varepsilon = \frac{1}{2}\min\{\beta_0, 1 - s + r, \delta\} \tag{6.3}$$

where $\delta > 0$ is given in the assumption of Theorem 2.2. Then we know

$$\begin{aligned} &\beta_1 \in (-1/2, 0) \quad \text{when} \quad \beta \in [0, 1/2), \\ &\beta_1 \in (0, 1-\varepsilon) \quad \text{when} \quad \beta \in [1/2, 1), \\ &s_1 > m - 1, \quad m - 1 + \beta_1 > 0. \end{aligned} \tag{6.4}$$

Varying ε slightly, we can suppose that s_1 is not an integer.

Motivated by Example 5.2 in [46], we choose the following family of initial data

$$u^\lambda(0,x) = (\lambda + \varepsilon x_+^{m+\beta_1})\phi(x), \qquad 0 \le \lambda \le 1, \tag{6.5}$$

where $x_+ = \max\{0, x\}$ and $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(x) = 1$ for $|x| \le 2$ and $\phi(x) = 0$ for $|x| \ge 4$.

Proposition 6.1 *There exists a positive constant h such that $\|u^\lambda(0,\cdot)\|_s^2 \le h$ for all $0 < \delta \le 1$ and $0 \le \lambda \le 1$.*

Proof If $\beta = 0$, the proof is trivial. If $0 < \beta < 1$, from the intrinsic Sobolev norm (5.14) and symmetry we obtain that

$$\|(x_+^{m+\beta_1}\phi)^{(m)}\|_\beta^2 = 2(I_1 + I_2 + II)$$

where

$$I_1 = \int_0^\infty dx_1 \int_0^{\frac{x_1}{2}} \frac{|x_1^{\beta_1}\phi(x_1) - x_2^{\beta_1}\phi(x_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_2 \le c \int_0^8 dx_1 \int_0^{\frac{x_1}{2}} \frac{x_1^{2\beta-1+2\varepsilon}}{x_1^{1+2\beta}} dx_2$$

$$+c \int_8^\infty dx_1 \int_0^4 \frac{x_2^{2\beta-1+2\varepsilon}}{x_1^{1+2\beta}} dx_2 = O(\varepsilon^{-1}) + O(1) = O(\varepsilon^{-1}),$$

$$I_2 = \int_0^\infty dx_1 \int_{\frac{x_1}{2}}^{x_1} \frac{|x_1^{\beta_1}\phi(x_1) - x_2^{\beta_1}\phi(x_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_2 \le c \int_0^8 dx_1 \int_{\frac{x_1}{2}}^{x_1} \frac{x_1^{2\beta_1-2}}{|x_1 - x_2|^{2\beta-1}} dx_2$$

$$\le c \int_0^8 x_1^{2\beta-3+2\varepsilon} dx_1 \int_0^{\frac{x_1}{2}} \eta^{1-2\beta} d\eta = O(\varepsilon^{-1}),$$

and

$$II = \int_{-\infty}^0 dx_1 \int_0^\infty \frac{|x_2^{\beta_1}\phi(x_2)|^2}{|x_1 - x_2|^{1+2\beta}} dx_2 \le c \int_{-1}^0 dx_1 \int_0^4 \frac{x_2^{2\beta-1+2\varepsilon}}{x_2^{2\beta+\varepsilon}|x_1|^{1-\varepsilon}} dx_2$$

$$+c \int_{-\infty}^{-1} dx_1 \int_0^4 \frac{1}{|x_1|^{1+2\beta}} dx_2 = O(\varepsilon^{-2}) + O(1) = O(\varepsilon^{-2}),$$

where use has been made of the mean-value theorem and change of variables in calculating I_2. Thus

$$\|\varepsilon x_+^{m+\beta_1}\phi\|_s^2 = \|\varepsilon x_+^{m+\beta_1}\phi\|_m^2 + \|(\varepsilon x_+^{m+\beta_1}\phi)^{(m)}\|_\beta^2 = O(1).$$

Observing $\|\lambda\phi\|_s = O(\lambda)$, the claim $\|u^\lambda(0,\cdot)\|_s^2 \le h$ for some $h > 0$ follows. □

Moreover, it is easy to see that for $r \ge 0$

$$\|u^\lambda(0) - u^0(0)\|_r = \|\lambda\phi(\cdot)\|_r = \lambda\|\phi\|_r, \tag{6.6}$$

and $u^\lambda(0, x)$ is smooth except at the origin.

For any nonnegative integer k, let $C_{\mathrm{loc}}^k(\mathbb{R}\backslash\{0\})$ be the space of continuous functions in $\mathbb{R}\backslash\{0\}$ with continuous derivatives of order up to k (not necessarily bounded). If $\rho = k + \gamma$ with $0 < \gamma < 1$, the space of *locally Hölder class* $C_{\mathrm{loc}}^\rho(\mathbb{R}\backslash\{0\})$ is the set of all of functions u such that for any compact interval $I \subset \mathbb{R}\backslash\{0\}$ there is a constant c_I depending on I such that

$$\max_{0\le i\le k} \sup_{x\in I} |u^{(i)}(x)| + \sup_{x,\eta\in I, x\neq\eta} |u^{(k)}(x) - u^{(k)}(\eta)||x-\eta|^{-\gamma} \le c_I.$$

Replacing $\mathbb{R}\backslash\{0\}$ by $\mathbb{R}$ and cancel the condition that the constant c_I depends on I, the above definition yields the space $C^\rho(\mathbb{R})$ of *Hölder class* on $\mathbb{R}$.

We need some multiplicative properties and the Sobolev embedding for the Hölder class. To this end, a weaker version of Proposition 8.6.8 and 8.6.10 of [44] will be contented:

(i) if $\rho > 0$ and it not an integer, then there is a constant c_ρ such that

$$\|u_1 u_2\|_{C^\rho(\mathbb{R})} \leq c_\rho \|u_1\|_{C^\rho(\mathbb{R})} \|u_2\|_{C^\rho(\mathbb{R})}; \tag{6.7}$$

(ii) if $\rho > \frac{1}{2}$, then it follows

$$\|u\|_{C^{\rho-\frac{1}{2}-\nu}(\mathbb{R})} \leq c_\rho \|u\|_\rho \tag{6.8}$$

for any $\nu > 0$ such that $\rho - \frac{1}{2} - \nu > 0$ is not an integer.

For any $\delta > 0$ we choose ε and $u^\lambda(0, \cdot)$ as at the beginning of this section. From (6.6) and Remark 4.2 there exist $\lambda_0, T_2 > 0$ such that for each initial data $u^\lambda(0, \cdot)$ with $0 \leq \lambda \leq \lambda_0$ system (4.10) has a unique solution $(w^\lambda(t, \cdot), v^\lambda(t, \cdot), q^\lambda(t, \cdot)) \in C([0, T_2]; Y)$. For simplicity of notation, in the rest of this paper we suppress the superscript of $(w^0(t, \cdot), v^0(t, \cdot), q^0(t, \cdot))$ and denote $Q = Q(w, v, q)$ and $R = R(w, v, q)$ as in (4.4) and (4.5). First we show some properties of $v^{(m-1)}$ in a neighborhood of $x = 0$.

Lemma 6.2 *Assume $s = m + \beta > \frac{3}{2}$ with m an integer and $0 \leq \beta < 1$. Then there exists a positive number $\bar{x}$ such that for $t \in [0, T_2]$*

(i) *for $\beta \in [0, 1)$ and $x < 0$*

$$v^{(m-1)}(x) = O(1); \tag{6.9}$$

(ii) *for $\beta \in [0, \frac{1}{2})$ and $0 < x < \bar{x}$*

$$c_1 x^{\beta_1} \leq v^{(m-1)}(x) \leq c_2 x^{\beta_1}; \tag{6.10}$$

(iii) *for $\beta \in [\frac{1}{2}, 1)$ and $x_1 \in (-4x_0, x_0)$, $x_2 \in (2x_0, 3x_0)$ with $0 < 4x_0 \leq \bar{x}$*

$$c_1 x_0^{\beta_1} \leq v^{(m-1)}(x_2) - v^{(m-1)}(x_1) \leq c_2 x_0^{\beta_1}, \tag{6.11}$$

$$v^{(m-1)}(x_2 - 4x_0) - v^{(m-1)}(x_1 - 4x_0) = O(x_0^{1-\varepsilon}). \tag{6.12}$$

Proof First, we prove

$$w(t, \cdot) \in C^m_{\text{loc}}(\mathbb{R}\backslash\{0\}) \quad \text{and} \quad v(t, \cdot),\ q(t, \cdot) \in C^{m-1}_{\text{loc}}(\mathbb{R}\backslash\{0\}). \tag{6.13}$$

By (6.8) one finds that $(w(t,\cdot), v(t,\cdot), q(t,\cdot))$ satisfies

$$\begin{cases} \frac{d}{dt} w = -R, \\ \frac{d}{dt} v = \frac{b}{2} w^2 - \frac{b-1}{2} v^2 - Q, \\ \frac{d}{dt} q = vq, \end{cases} \tag{6.14}$$

in the Hölder classes $C^{s_1}(\mathbb{R}) \times C^{s_1-1}(\mathbb{R}) \times C^{s_1-1}(\mathbb{R})$ with $R(t,\cdot), Q(t,\cdot) \in C^{s_1}(\mathbb{R})$ for $t \in [0, T_2]$, where s_1 is defined in (6.2). Observe that $w^2(t,\cdot) \in C^{s_1}(\mathbb{R})$ from (6.7), and we know that the initial data $v(0,\cdot) = u_x^0(0,\cdot) \in C^{s_1}_{\text{loc}}(\mathbb{R}\backslash\{0\})$. We can then apply the fixed point theorem to the second equation of (6.14) in the space $C([0, T_2], C^{s_1}(I))$ on any compact interval $I \subset \mathbb{R}\backslash\{0\}$, with terms w^2 and Q being regarded as independent of v. Then we have

$$v(t,\cdot) \in C^{s_1}_{\text{loc}}(\mathbb{R}\backslash\{0\}). \tag{6.15}$$

Using (6.15), $q(0,\cdot) = 1 \in C^{s_1}_{\text{loc}}(\mathbb{R}\backslash\{0\})$ and the third equation of (6.14), the above arguments lead to

$$q(t,\cdot) \in C^{s_1}_{\text{loc}}(\mathbb{R}\backslash\{0\}). \tag{6.16}$$

Applying (4.21) and (6.7), (6.15) and (6.16) improves the regularity of $R(t,\cdot)$ to $C^{s_1+1}_{\text{loc}}(\mathbb{R}\backslash\{0\})$. Now $w(0,\cdot) = u^0(0,\cdot) \in C^{s_1+1}_{\text{loc}}(\mathbb{R}\backslash\{0\})$ and the first equation of (6.14) yield $w(t,\cdot) \in C^{s_1+1}_{\text{loc}}(\mathbb{R}\backslash\{0\})$. Thus $s_1 > m-1$ in (6.4) leads to (6.13).

We are now in a position to consider the estimates for $v^{(m-1)}$. We claim that the boundedness of $(w^2)^{(m-1)}$, $Q^{(m-1)}$, $v, \ldots, v^{(m-2)}$ gives the following estimate

$$-c|v^{(m-1)}(t,x)| - c \le \frac{d}{dt} v^{(m-1)}(t,x) \le c|v^{(m-1)}(t,x)| + c$$

for $t \in [0, T_2]$ and a fixed $x \in \mathbb{R}\backslash\{0\}$. In fact, it is clearly true for $m = 1$. If $m \ge 2$, differentiating the second equation of (6.14) with respect to x $(m-1)$ times implies that the coefficient of $v^{(m-1)}$ is $-(b-1)v$. Moreover $v^{(m-1)}(0,x) = 0$ for $x < 0$ from (6.5), which leads to (6.9). On the other hand, since $v^{(m-1)}(0,x) = \varepsilon x^{\beta_1} > 0$ for $0 < x \le 1$, we see that for $t \in [0, T_2]$

$$c\varepsilon x^{\beta_1} - c \le v^{(m-1)}(t,x) \le c\varepsilon x^{\beta_1} + c. \tag{6.17}$$

If $\beta \in [0, \frac{1}{2})$ then $\beta_1 < 0$ by (6.4). Taking $\bar{x} > 0$ small enough, we obtain (6.10) from (6.17).

If $\beta \in [\frac{1}{2}, 1)$, then $(w^2)^{(m-1)}$, $Q^{(m-1)}$, $v \in C^{1-\varepsilon}(\mathbb{R})$, and hence

$$(w^2)^{(m-1)}(x_2) - (w^2)^{(m-1)}(x_1),\ Q^{(m-1)}(x_2) - Q^{(m-1)}(x_1) = O(x_0^{1-\varepsilon}). \tag{6.18}$$

Set $f(t) = v^{(m-1)}(t,x_2) - v^{(m-1)}(t,x_1)$. We claim for $t \in [0, T_2]$

$$(v^2)^{(m-1)}(x_2) - (v^2)^{(m-1)}(x_1) = O(f(t)) + O(x_0^{1-\varepsilon}). \tag{6.19}$$

As this is obvious for $m = 1$, we can assume $m \geq 2$. From Leibniz formula, $(v^2)^{(m-1)}$ is a rational combination of

$$vv^{(m-1)},\ v^{(1)}v^{(m-2)},\ \ldots,\ v^{([\frac{m-1}{2}])}v^{([\frac{m}{2}])}. \tag{6.20}$$

Notice that $0 < v^{(m-1)}(0, x) \leq c$ for $x > 0$, then relation (6.17) yields $v^{(m-1)}(t, x) = O(1)$, which together with $v \in C^{1-\varepsilon}(\mathbb{R})$ imply

$$\begin{aligned} v(x_2)v^{(m-1)}(x_2) - v(x_1)v^{(m-1)}(x_1) &= v(x_2)f(t) + (v(x_2) - v(x_1))v^{(m-1)}(x_1) \\ &= f(t)v(x_2) + O(x_0^{1-\varepsilon}). \end{aligned}$$

The estimates concerning the other terms in (6.20) are of the form $O(x_0^{1-\varepsilon})$, i.e.,

$$v^{(k)}(x_2) - v^{(k)}(x_1) = O(x_0^{1-\varepsilon}), \quad \text{and} \quad v^{(k)}(x) = O(1)$$

for $k = 0, \ldots, m-2$. Thus (6.19) holds. In view of (6.18) and (6.19), it follows from the second equation of (6.14) that

$$-c|f(t)| - cx_0^{1-\varepsilon} \leq \frac{d}{dt}f(t) \leq c|f(t)| + cx_0^{1-\varepsilon} \quad \text{for } t \in [0, T_2]. \tag{6.21}$$

Hence from $c_1x_0^{\beta_1} \leq f(0) \leq c_2x_0^{\beta_1}$ we have

$$c_1x_0^{\beta_1} - cx_0^{1-\varepsilon} \leq f(t) \leq c_2x_0^{\beta_1} + cx_0^{1-\varepsilon}.$$

Since $\beta_1 < 1 - \varepsilon$ by (6.4), taking $\bar{x} > 0$ sufficiently small we obtain (6.11). The differential inequality (6.21) is also valid if $f(t)$ is replaced by $v^{(m-1)}(t, x_2 - 4x_0) - v^{(m-1)}(t, x_1 - 4x_0)$ and then $f(0) = 0$. Thus we obtain (6.12) for $t \in [0, T_2]$. □

Now we give some estimates regarding the intrinsic Sobolev norms which will be needed in the proof of Theorem 2.2.

Lemma 6.3 *Let $s > \frac{3}{2}$. There exist positive numbers $\bar{x}$, $t_0 \in (0, T_2]$ and $a \in (0, 1)$ such that the estimates*

$$g(t_0),\ |h(t_0)| \geq cx_0^{\beta_1} \quad \text{and} \quad b(t_0) \geq cx_0^{\beta_1+1} \tag{6.22}$$

hold for $x_1 \in (ax_0, 2ax_0)$ and $x_2 \in (2x_0, 3x_0)$ with $0 < 4x_0 \leq \bar{x}$, where

$$\begin{aligned} &g(t) := (w^{(m)}(x_2) - w^{(m)}(x_2 - 4x_0))(t), \\ &h(t) := (w^{(m)}(x_2) - w^{(m)}(x_1) - w^{(m)}(x_2 - 4x_0) + w^{(m)}(x_1 - 4x_0))(t), \\ &b(t) := (w^{(m-1)}(x_2) - w^{(m-1)}(x_1) - w^{(m-1)}(x_2 - 4x_0) + w^{(m-1)}(x_1 - 4x_0))(t). \end{aligned}$$

Proof We divide the proof of Lemma 6.3 into three cases.

Case 1 $\beta \in [0, \frac{1}{2})$. Then $m \geq 2$ since $s > \frac{3}{2}$. We have $q(t,x) = e^{\int_0^t v(\tau,x)d\tau}$ from the third equation of (6.14) and the initial condition $q(0) = 1$. Applying Leibniz formula, we have

$$\begin{aligned} q^{(1)} &= e^{\int_0^t v(\tau)d\tau} \int_0^t v^{(1)}(\tau)d\tau, \\ &\cdots \\ q^{(m-1)} &= e^{\int_0^t v(\tau)d\tau} \left(\int_0^t v^{(1)}(\tau)d\tau \right)^{m-1} + \cdots + e^{\int_0^t v(\tau)d\tau} \int_0^t v^{(m-1)}(\tau)d\tau. \end{aligned} \tag{6.23}$$

From $w^{(1)} = vq$ and (6.13) one gets for $x \in \mathbb{R}\backslash\{0\}$

$$w^{(m)} = (vq)^{(m-1)} = v^{(m-1)}q + (m-1)v^{(m-2)}q^{(1)} + \cdots + vq^{(m-1)}. \tag{6.24}$$

Note that all terms in (6.24) except the first and the last are bounded because of Sobolev embedding. From (6.10) and (6.23) we know $|q^{(m-1)}(t,x)| \leq ctx^{\beta_1}$ for $0 < x < \bar{x}$, and then (6.24), $|v| \leq c$ and $\beta_1 < 0$ by (6.4) yield

$$\begin{aligned} &g(t_0) \geq cx_0^{\beta_1} - ct_0x_0^{\beta_1} - c \geq cx_0^{\beta_1}, \\ &h(t_0) \leq c(1+t_0)x_0^{\beta_1} - c(1-t_0)(ax_0)^{\beta_1} + c \leq -cx_0^{\beta_1} \end{aligned}$$

provided that $\bar{x}, t_0, a > 0$ are sufficiently small. Notice that

$$w^{(m-1)} = v^{(m-2)}q + (m-2)v^{(m-3)}q^{(1)} + \cdots + vq^{(m-2)}. \tag{6.25}$$

Using mean value theorem, (6.9), $q \geq c > 0$ and (6.23) we have

$$\begin{aligned} (v^{(m-2)}(x_2) - v^{(m-2)}(x_1))q(x_2) &= v^{(m-1)}(\bar{x}_1)q(x_2) \cdot (x_2 - x_1) \geq cx_0^{\beta_1+1}, \\ v^{(m-2)}(x_1)(q(x_2) - q(x_1)) &= v^{(m-2)}(x_1)q^{(1)}(\bar{x}_2) \cdot (x_2 - x_1) = O(tx_0^{\beta_1+1}) \end{aligned}$$

with $\bar{x}_1, \bar{x}_2 \in (x_1, x_2)$. Summing up the above two yields

$$I_1 := (v^{(m-2)}(x_2)q(x_2) - v^{(m-2)}(x_1)q(x_1))(t_0) \geq cx_0^{\beta_1+1}$$

if t_0 is chosen small. Using (6.10), similar arguments lead to

$$I_2 := (v^{(m-2)}(x_2 - 4x_0)q(x_2 - 4x_0) - v^{(m-2)}(x_1 - 4x_0)q(x_1 - 4x_0))(t_0) = O(x_0).$$

Hence $I_1 - I_2 \geq cx_0^{\beta_1+1}$. Such estimates of the intermediate and the last terms in (6.25) are of the order $O(x_0)$ and $O(tx_0^{\beta_1+1})$, respectively. So we proved

$$b(t_0) \geq cx_0^{\beta_1+1} - ct_0x_0^{\beta_1+1} - cx_0 \geq cx_0^{\beta_1+1}$$

provided that $\bar{x}, t_0 > 0$ are sufficiently small.

Case 2 $\beta \in (\frac{1}{2}, 1)$ and $m = 1$. Since $e^z - 1 = z + o(z)$ for $|z| \ll 1$, from (6.11) it follows that

$$\begin{aligned}|q(t, x_2) - q(t, x_1)| &= e^{\int_0^t v(\tau,x_1)d\tau}|e^{\int_0^t (v(\tau,x_2)-v(\tau,x_1))d\tau} - 1|\\ &\leq c\int_0^t |v(\tau, x_2) - v(\tau, x_1)|d\tau + o(x_0^{\beta_1}) \leq ctx_0^{\beta_1}.\end{aligned}$$

Then $q \geq c > 0,\ |v| \leq c$ and (6.11) give

$$\begin{aligned}(w^{(1)}(x_2) - w^{(1)}(x_1))(t_0) &= ((v(x_2) - v(x_1))q(x_2) + v(x_1)(q(x_2) - q(x_1)))(t_0)\\ &\geq cx_0^{\beta_1} - ct_0x_0^{\beta_1} \geq cx_0^{\beta_1}\end{aligned} \tag{6.26}$$

provided t_0 is chosen small. Similarly applying (6.12) it follows

$$w^{(1)}(x_2 - 4x_0) - w^{(1)}(x_1 - 4x_0))(t_0) = O(x_0^{1-\varepsilon}),$$

which together (6.26) and $\beta_1 < 1 - \varepsilon$ by (6.4) lead to

$$h(t_0) \geq cx_0^{\beta_1} - cx_0^{1-\varepsilon} \geq cx_0^{\beta_1}$$

if $\bar{x} > 0$ is small. The estimate for $g(t_0)$ can be obtained directly from (6.26). By the mean value theorem and (6.26), one has

$$b(t_0) = (w^{(1)}(\bar{x}_1) - w^{(1)}(\bar{x}_2))(t_0) \cdot (x_2 - x_1) \geq cx_0^{\beta_1} \cdot x_0 \geq cx_0^{\beta_1+1}$$

provided that $\bar{x} > 0$ is small, where $\bar{x}_1 \in (x_1, x_2), \bar{x}_2 \in (x_1 - 4x_0, x_2 - 4x_0)$.

Case 3 $\beta \in [\frac{1}{2}, 1)$ and $m \geq 2$. Denote by

$$\begin{aligned}I :&= v^{(m-1)}(x_2)q(x_2) - v^{(m-1)}(x_2 - 4x_0)q(x_2 - 4x_0)\\ &= (v^{(m-1)}(x_2) - v^{(m-1)}(x_2 - 4x_0))q(x_2)\\ &\quad + (q(x_2) - q(x_2 - 4x_0))v^{(m-1)}(x_2 - 4x_0).\end{aligned}$$

Recall that $q \geq c > 0$ and $|v^{(m-1)}| \leq c$. Then (6.11) and the Hölder continuity of q yield

$$I \geq cx_0^{\beta_1} - cx_0^{1-\varepsilon} \geq cx_0^{\beta_1}. \tag{6.27}$$

From (6.23), estimates of the intermediate terms and the last term in (6.24) are of order $O(x_0^{1-\varepsilon})$ and $O(tx_0^{\beta_1})$, respectively. Thus

$$g(t_0) \geq cx_0^{\beta_1} - ct_0 x_0^{\beta_1} - cx_0^{1-\varepsilon} \geq cx_0^{\beta_1}$$

provided that $\bar{x}$, $t_0 > 0$ are small (since $\beta_1 < 1 - \varepsilon$ by (6.4)). Applying the similar arguments, we have

$$\begin{aligned} &v^{(m-1)}(x_2)q(x_2) - v^{(m-1)}(x_1)q(x_1) - v^{(m-1)}(x_2 - 4x_0)q(x_2 - 4x_0) \\ &\quad + v^{(m-1)}(x_1 - 4x_0)q(x_1 - 4x_0) \geq cx_0^{\beta_1}. \end{aligned}$$

Thus

$$h(t_0) \geq cx_0^{\beta_1} - ct_0 x_0^{\beta_1} - cx_0^{1-\varepsilon} \geq cx_0^{\beta_1}$$

if $\bar{x}$, $t_0 > 0$ are sufficiently small.

The estimate for $b(t_0)$ can be obtained by combining the arguments in *Case 1* for $b(t_0)$ and the above for $g(t_0)$. □

Denote $\eta^\lambda(t, x), 0 \leq \lambda \leq \lambda_0$, the function determined by (5.1) corresponding to the initial data $u^\lambda(0, \cdot)$ given in (6.5), and $x^\lambda(t, \eta)$ its inverse function. Next we establish the estimates concerning the space shift.

Lemma 6.4 *Assume $s > \frac{3}{2}$. Then for $0 < \lambda \leq 1$ the estimate*

$$x^\lambda(t, \eta^0(t, x)) = x - \lambda t[1 + O(\lambda^{m-1+\beta_1}) + O(t)] \tag{6.28}$$

holds for $|x| \leq 2\lambda$ and $t \in [0, T_2]$.

Proof Let $R^\lambda = R(w^\lambda, v^\lambda, q^\lambda)$. Since $s > \frac{3}{2}$, one infers from $\|R^\lambda(\sigma, \cdot)\|_s \leq c$ for $0 \leq \sigma \leq T_2$ and $0 < \lambda \leq 1$ that $|R_x^\lambda(\sigma, \cdot)|_\infty \leq c$. By mean value theorem there is a $\theta = \theta(\lambda, \sigma, x)$ such that

$$R^\lambda(\sigma, x) = R^\lambda(\sigma, 0) + R_x^\lambda(\sigma, \theta x) \cdot x$$

for $0 \leq \sigma \leq T_2$, $|x| \leq 2\lambda$ and $0 < \lambda \leq 1$. Integrating both sides of the first equation of (6.14) on $[0, \tau]$ gives

$$w^\lambda(\tau, x) = w^\lambda(0, x) - \int_0^\tau R^\lambda(\sigma, x) d\sigma.$$

Applying (5.1) and $w^\lambda(0, x) = u^\lambda(0, x)$ in (6.5), we deduce from the last two equations that for $|x| \leq 2\lambda$ and $t \in [0, T_2]$

$$
\begin{aligned}
\eta^\lambda(t,x) &= x + \int_0^t \left(w^\lambda(0,x) - \int_0^\tau R^\lambda(\sigma,x)d\sigma \right) d\tau \\
&= x + t(\lambda + \varepsilon x_+^{m+\beta_1})\phi(x) - \int_0^t \int_0^\tau (R^\lambda(\sigma,0) + R_x^\lambda(\sigma,\theta x)\cdot x) d\sigma d\tau \\
&= x + \lambda t \Big[1 + O(\lambda^{m-1+\beta_1}) + O(t) \Big] - \int_0^t \int_0^\tau R^\lambda(\sigma,0) d\sigma d\tau \qquad (6.29)
\end{aligned}
$$

where we have used $\phi(x) = 1$ for $|x| \le 2$ and $|R_x^\lambda(\sigma,\cdot)|_\infty \le c$. The same argument is valid for $\eta^0 = \eta^0(t,x)$ and it follows that

$$
\eta^0(t,x) = x + \lambda t \Big[O(\lambda^{m-1+\beta_1}) + O(t) \Big] - \int_0^t \int_0^\tau R(\sigma,0) d\sigma d\tau
$$

for $|x| \le 2\lambda$ and $t \in [0,T_2]$, where $R = R(w^0, v^0, q^0)$ with (w^0, v^0, q^0) the solution of (4.10) corresponding to initial data $u^0(0,\cdot)$. Note (6.29) gives

$$
x^\lambda(t,\eta) = \eta - \lambda t \Big[1 + O(\lambda^{m-1+\beta_1}) + O(t) \Big] + \int_0^t \int_0^\tau R^\lambda(\sigma,0) d\sigma d\tau.
$$

Using (4.27), (4.13) and (6.6) one has

$$
\begin{aligned}
|R^\lambda(\sigma,0) - R(\sigma,0)| \le |R^\lambda(\sigma,\cdot) - R(\sigma,\cdot)|_\infty &\le c\|R^\lambda(\sigma,\cdot) - R(\sigma,\cdot)\|_s \\
&\le c\|u^\lambda(0) - u^0(0)\|_s \le c\lambda.
\end{aligned}
$$

It then follows from the last three equations that

$$
\begin{aligned}
x^\lambda(t,\eta^0(t,x)) &= x - \lambda t \Big[1 + O(\lambda^{m-1+\beta_1}) + O(t) \Big] + O(\lambda t^2) \\
&= x - \lambda t \Big[1 + O(\lambda^{m-1+\beta_1}) + O(t) \Big]
\end{aligned}
$$

for $|x| \le 2\lambda$ and $t \in [0,T_2]$. Hence, the proof of (6.28) is complete. □

Proof of Theorem 2.2 Assume $s = m + \beta$ with m an integer, $0 \le \beta < 1$, and $r \in (s-1,s)$. From (5.22) we have

$$
\|w^\lambda(x^\lambda)(t) - w(x^\lambda)(t)\|_r \le c\|w^\lambda(t) - w(t)\|_r \le c\|w^\lambda(t) - w(t)\|_s
$$

for $t \in [0,T_2]$. It then follows from (4.13) and (6.6) that

$$
\|w^\lambda(x^\lambda)(t) - w(x^\lambda)(t)\|_r \le c\|u^\lambda(0) - u^0(0)\|_s \le c\lambda. \qquad (6.30)
$$

Using the identity $x^0(t, \eta^0(t,\cdot)) = \mathrm{Id}$, the first inequality of (5.22) yields

$$\|w(x^\lambda)(t) - w(x^0)(t)\|_r \ge c\|w(x^\lambda(\eta^0(\cdot)))(t) - w(\cdot)(t)\|_r. \tag{6.31}$$

Note $m - 1 + \beta_1 > 0$ by (6.4). We can take $t_0 \in (0, T_2]$ and $\lambda_0 \in (0, 1]$ sufficiently small such that $|O(\lambda_0^{m-1+\beta_1})| + |O(t_0)| \le \frac{1}{2}$, and t_0 and $\bar{x} = 5\lambda_0 t_0$ satisfy the assumptions of Lemma 6.2 and Lemma 6.3. We rewrite (6.28) as

$$x^\lambda(t_0, \eta^0(t_0, x)) = x - 4x_0 \quad \text{for } |x| \le 2\lambda \text{ and } 0 < \lambda \le \lambda_0,$$

with

$$x_0 := \frac{\lambda t_0}{4}[1 + O(\lambda_0^{m-1+\beta_1}) + O(t_0)].$$

Then $x_0 \ge \frac{1}{8}\lambda t_0$. Write

$$f^\lambda(x) := \left(w^{(m)}(x - 4x_0) - w^{(m)}(x)\right)(t_0).$$

Again, there are three cases to be considered.

Case 1 $r = m$. By the first estimate of (6.22), we have

$$|f^\lambda|_2^2 = \int_{\mathbb{R}} |f^\lambda(x)|^2 dx \ge c \int_{2x_0}^{3x_0} (\lambda t_0)^{2\beta-1+2\varepsilon} dx \ge c(\lambda t_0)^{2(\beta+\varepsilon)}.$$

This in turn implies that

$$\|w\left(x^\lambda(\eta^0(\cdot))\right)(t_0) - w(\cdot)(t_0)\|_r \ge |f^\lambda|_2 \ge c(\lambda t_0)^{\beta+\varepsilon} = c(\lambda t_0)^{s-r+\varepsilon},$$

which together with (6.31) lead to

$$\|w(x^\lambda)(t_0) - w(x^0)(t_0)\|_r \ge c(\lambda t_0)^{s-r+\varepsilon}. \tag{6.32}$$

Case 2 $r = m + \gamma$ with $0 < \gamma < \beta$. Denote by $\Delta := (ax_0, 2ax_0) \times (2x_0, 3x_0)$. The second estimate of (6.22) implies

$$|f^\lambda(x_1) - f^\lambda(x_2)| \ge c(\lambda t_0)^{\beta_1} \quad \text{for } (x_1, x_2) \in \Delta.$$

Thus

$$\begin{aligned}\|f^{\lambda}\|_{\gamma} &= \iint\limits_{\mathbb{R}\times\mathbb{R}} \frac{\left|f^{\lambda}(x_1)-f^{\lambda}(x_2)\right|^2}{|x_1-x_2|^{1+2\gamma}}dx_1dx_2 \geq \iint\limits_{\Delta} \frac{\left|f^{\lambda}(x_1)-f^{\lambda}(x_2)\right|^2}{|x_1-x_2|^{1+2\gamma}}dx_1dx_2 \\ &\geq c\int_{ax_0}^{2ax_0}\int_{2x_0}^{3x_0}(\lambda t_0)^{2\beta-1+2\varepsilon}(\lambda t_0)^{-1-2\gamma}dx_1dx_2 = c(\lambda t_0)^{2(\beta-\gamma+\varepsilon)}.\end{aligned}$$

Then using the intrinsic Sobolev norm we have

$$\begin{aligned}\|w(x^{\lambda}(\eta^0(\cdot)))(t_0)-w(\cdot)(t_0)\|_r &\geq \|(w(x^{\lambda}(\eta^0(\cdot))(t_0)-w(\cdot)(t_0))^{(m)}\|_{\gamma} \\ &\geq c(\lambda t_0)^{\beta-\gamma+\varepsilon} = c(\lambda t_0)^{s-r+\varepsilon}\end{aligned}$$

since $\beta-\gamma=s-r$. Hence (6.31) also implies (6.32) in this case.

Case 3 $r=(m-1)+\gamma$ with $0\leq\beta<\gamma<1$. Using the third estimate of (6.22), relation (6.32) can be verified in a similar way as in *Case 2* and the proof is therefore omitted.

Applying (6.30) and (6.32) we obtain

$$\begin{aligned}\|u^{\lambda}(t_0)-u^0(t_0)\|_r &= \|w^{\lambda}(x^{\lambda})(t_0)-w(x^0)(t_0)\|_r \\ &\geq \|w(x^{\lambda})(t_0)-w(x^0)(t_0)\|_r - \|w^{\lambda}(x^{\lambda})(t_0)-w(x^{\lambda})(t_0)\|_r \\ &\geq c(\lambda t_0)^{s-r+\varepsilon}-c\lambda \geq c(\lambda t_0)^{s-r+\varepsilon}\end{aligned}$$

for $\lambda>0$ small with the help $s-r+\varepsilon<1$ by (6.3). Now for any $\delta>0$ the last estimate and (6.6) infer

$$\|u^{\lambda}(t_0)-u^0(t_0)\|_r \geq c(\lambda t_0)^{s-r+\varepsilon} = ct_0^{s-r+\varepsilon}\lambda^{\varepsilon-\delta}\|u^{\lambda}(0)-u^0(0)\|_r^{s-r+\delta}$$

which gives (2.7) with $c^{\lambda}=ct_0^{s-r+\varepsilon}\lambda^{\varepsilon-\delta}$. Since $\varepsilon<\delta$ by (6.3), we have $c^{\lambda}\to\infty$ as $\lambda\to 0$. This completes the proof of Theorem 2.2. □

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