SYMMETRIC CHERN-SIMONS-HIGGS VORTICES

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ABSTRACT. We prove the existence of radially symmetric vortices of the static nonself-dual Chern-Simons-Higgs equations with and without magnetic field in dimension 2. The vortex profiles are shown to be monotonically increasing and bounded. For a given vorticity n, when there is no magnetic field we prove that the n-vortices are stable for $n = 0, \pm 1$.

1. INTRODUCTION

The Chern-Simons-Higgs (CSH) theory generally refers to a wide category of field-theoretic models in $(2+1)$ dimensional Minkowski space that contain a Chern-Simons term in their action densities, see [2, 8, 9, 20]. These models have applications to several important problems in condensed matter physics such as high-temperature superconductivity and quantum and fractional Hall effect ([2, 20]). CSH theory is one of the simplest known anyonic models, i.e. a model that allows for quantized statistics of fractional values.

Define the Minkowski spacetime metric tensor $g = \mathrm{diag}(1, -1, -1)$, then in normalized units, the Lagrangian density of the CSH theory is written ([8], [9])

$$\mathcal{L}_{csh} = D_\alpha u \overline{D^\alpha u} + \frac{\mu}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - \lambda^2 |u|^2 \big(1 - |u|^2\big)^2 \tag{1.1}$$

where $\mathcal{A} = -iA_\alpha dx^\alpha$ with $A_\alpha : \mathbb{R}^{1,2} \to \mathbb{R}$ for $\alpha = 0, 1, 2$ is the gauge potential with covariant derivative $D_{\mathcal{A}} = d - i\mathcal{A}$. The corresponding curvature $F_{\mathcal{A}} = -\frac{1}{2} F_{\beta\gamma} dx^\beta \wedge dx^\gamma$ with $F_{\beta\gamma} = \partial_\beta A_\gamma - \partial_\gamma A_\beta$ defines the gauge field, and $u : \mathbb{R}^{1,2} \to \mathbb{C}$ is the Higgs scalar with $D_\alpha u = \partial_\alpha u - iA_\alpha u$, $\alpha = 0, 1, 2$. Furthermore, the antisymmetric Levi-Civita tensor $\epsilon^{\alpha\beta\gamma}$ is fixed by setting $\epsilon^{012} = 1$ and $\mu, \varepsilon > 0$ are the Chern-Simons coupling parameters. Here $\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma}$ is the Chern-Simons term. The Euler-Lagrange equations of (1.1) are

$$D_\alpha D^\alpha u + \lambda^2 u\big(|u|^2 - 1\big)\big(3|u|^2 - 1\big) = 0 \tag{1.2}$$

$$\frac{\mu}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} + \mathcal{J}^\alpha = 0 \tag{1.3}$$

where $\mathcal{J}^\alpha = (iu, D^\alpha u)$ is the matter current.

Since $\alpha = 0$ refers to time coordinates, we replace D_0 by $\partial_\Phi = \partial_t - i\Phi$ and replace D_α by $\nabla_A = \nabla - iA$ when $\alpha = 1, 2$, where $A = (A_1, A_2)$. Here (Φ, A) is the field potential. The curvature tensor is defined by

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & -h \\ E_2 & h & 0 \end{pmatrix}, \tag{1.4}$$

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where $h = \text{curl}A$ and $E_\alpha = \partial_t A_\alpha - \partial_\alpha \Phi$ are the induced magnetic and electric fields, respectively. We also use the standard current definition

$$\mathcal{J}^0 = (iu, \partial_\Phi u) = q, \quad \mathcal{J}^\alpha = (iu, \nabla_{A_\alpha} u) = j_{A_\alpha}$$

for $\alpha = 1, 2$ which are the charge and supercurrent, respectively. Hence we get the set of CSH equations as

$$\partial_\Phi^2 u = \nabla_A^2 u + \lambda^2 u\big(|u|^2 - 1\big)\big(3|u|^2 - 1\big) \tag{1.5}$$

$$q = -\frac{\mu}{2}\text{curl}A \tag{1.6}$$

$$j_A = \frac{\mu}{2}(E \times e_3). \tag{1.7}$$

Well-posedness for the initial value problem for equations (1.5)-(1.7) can be found in [3] and [4].

We look for static solutions. Setting $\partial_t u = 0$ then equations (1.5)-(1.7) becomes

$$-\Phi^2 u = \nabla_A^2 u + \lambda^2 u\big(|u|^2 - 1\big)\big(3|u|^2 - 1\big)$$

$$\Phi|u|^2 = \frac{\mu}{2}\text{curl}A$$

$$j_A(u) = \frac{\mu}{2}(\nabla\Phi \times e_3).$$

Removing the electic field potential Φ, we are left with a system of coupled elliptic PDE's

$$-\frac{\mu^2}{4}\frac{|\text{curl}A|^2}{|u|^4}u = \nabla_A^2 u + \lambda^2 u\big(|u|^2 - 1\big)\big(3|u|^2 - 1\big) \tag{1.8}$$

$$0 = -\frac{\mu^2}{4}\text{curl}\Big(\frac{\text{curl}A}{|u|^2}\Big) + j_A(u). \tag{1.9}$$

The above static equations can be viewed as the Euler-Lagrange equations of the following Chern-Simons-Higgs energy

$$G_{csh}(u, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A u|^2 + \frac{\mu^2}{4}\frac{|\text{curl}A|^2}{|u|^2} + \lambda^2|u|^2\big(1 - |u|^2\big)^2\, dx. \tag{1.10}$$

When there is no magnetic field the Chern-Simons-Higgs energy becomes

$$E_{csh}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \lambda^2|u|^2(1 - |u|^2)^2\, dx, \tag{1.11}$$

with the associated Euler-Lagrange equation

$$-\Delta u + \lambda^2 u(1 - |u|^2)(1 - 3|u|^2) = 0. \tag{1.12}$$

Due to the form of the potential in (1.10) and (1.11), locally minimizing configurations should satisfy

$$\begin{aligned} |u| \to 1, &\qquad \text{as } |x| \to \infty \\ \text{or } |u| \to 0, &\qquad \text{as } |x| \to \infty. \end{aligned}$$

We will only consider the first case which leads to the definition of the *topological degree*, $\deg(u)$, of such a configuration:

$$\deg(u) = \deg\Big(\frac{u}{|u|}\Big|_{|x|=R} : \mathbb{S}^1 \to \mathbb{S}^1\Big)$$

for R sufficiently large. The degree is related to the phenomenon of *flux quantization*. Indeed, an application of Stokes' theorem to (1.7) and using (1.6) shows that a locally minimizing energy configuration satisfies

$$\deg(u) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{curl} A \, dx = -\frac{1}{\mu\pi}\int_{\mathbb{R}^2} q \, dx$$

so long as there is good decay at infinity. Therefore, topological vortices in CSH theory carry both a quantized magnetic field and electrostatic charge.

1.1. **Prior results.** When $\mu = \frac{1}{\lambda}$, minimizers of the CSH energy satisfy a simpler system of first order PDE's. This self-dual mechanism was discovered by Hong-Kim-Pak [8] and Jackiw-Weinberg [9] and has been the subject of rich mathematical development. The resulting equations can allow for multivortex configurations and are similar to the Jaffee-Taubes self-dual Ginzburg-Landau theory. We point to Caffarelli-Yang [2] and Tarantello [18] for important results on the existence of such multivortex configurations in the self-dual regime. However, once the self-dual regime is left, the theory is underdeveloped. Immediately there are difficulties in understanding the term $\frac{\mu^2}{4}\frac{|\operatorname{curl} A|^2}{|u|^2}$ in the energy since u vanishes at least once whenever $\deg(u) \neq 0$.

In this paper we initiate a study of the CSH energies (1.10) and (1.11) on the plane outside of the self-dual regime. We study, in particular, *radially symmetric* fields of the form

$$u^{(n)} = f^{(n)}(r)e^{in\theta}, \tag{1.13}$$

$$A^{(n)} = n\frac{a_n(r)}{r}\vec{x}^{\perp}, \tag{1.14}$$

where (r, θ) are polar coordinates, $\vec{x}^{\perp} = (-x_2, x_1)^T$, n is an integer which corresponds to the degree of u, and $f^{(n)}, a_n : [0, \infty) \to \mathbb{R}$. We note Han [7] studied radial symmetric one-vortex solution in the self-dual regime $\mu = \frac{1}{\lambda}$.

There are some similarities to the rigorous study of planar vortex minimizers of the Ginzburg-Landau energy

$$G_{gl}(u, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A u|^2 + |\operatorname{curl} A|^2 + \lambda^2\left(1 - |u|^2\right)^2, \tag{1.15}$$

which originated with the work of Plohr [13] and Berger-Chen [1]. The stability of (1.15) was initiated by Guo [5] and completely characterized by Gustafson-Sigal [6]. When the magnetic field is not present, the Ginzburg-Landau energy simplifies to

$$E_{gl}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + \lambda^2\left(1 - |u|^2\right)^2. \tag{1.16}$$

Ovchinnikov-Sigal [12] established the existence and examined the stability of symmetric, planar minimizers of (1.16). Our CSH existence proofs rely on the existence results of [1, 12], as we look for minimizers in a constraint class of functions with finite Ginzburg-Landau energy.

1.2. **Main Results.** We consider the existence problem of CSH n-vortex solutions, in the cases when $A \equiv 0$ and $A \not\equiv 0$, as well as the stability problem of the n-vortex in the absence of the magnetic field potential A. One major difficulty with the existence problem is that there are trivial global minimizers. Therefore, we need to employ an unusual constraint to force a minimizing sequence $f^{(n)} \to 1$ as $r \to \infty$. The main results of this paper are the following:

Theorem 1.1. *For any n, there is a radially symmetric vortex solution $u^{(n)} = f^{(n)}e^{in\theta}$ of degree n to equation (1.12). Moreover, $f^{(n)}$ minimizes the renormalized energy functional E_{csh}^{ren} given in (3.1) over a certain admissible set defined in* (5.2). *For $n = 0, \pm 1$, $u^{(n)}$ are local minima of E_{csh}^{ren}.*

The proof of the existence part of Theorem 1.1 is similar in spirit to the methods developed in [12] for Ginzburg-Landau energy. The primary difficulty is the existence of a trivial global minimizer. In order to establish a nontrivial local minimizer, we examine a minimizing sequence of a renormalized CSH energy in a constraint class of functions with finite renormalized Ginzburg-Landau energy. The renormalized CSH energy functional yields the same Euler-Lagrange equation (1.12). To show coercivity of our minimizing sequence, we need to control the size of the set in which $|u| \leq \frac{1}{4}$. This is done via a covering argument, similar to methods developed for Allen-Cahn by Modica-Mortola [11], Ginzburg-Landau by Sandier [15], and Chern-Simons-Higgs by Kurzke-Spirn [10].

We want to point out that the potential term $\lambda^2|u|^2(1-|u|^2)^2$ in the energy functional (1.11) prevents us from getting partial convexity of the renormalized energy functional unlike the partial convexity found by Ovchinnikov-Sigal [12] for the reduced Ginzburg-Landau energy. Therefore, we are unable to prove uniqueness of the n-vortex solutions.

The second part of Theorem 1.1 concerns the stability property of the n-vortices for $n = 0, \pm 1$. When $n = 0$ it follows from the definition that a strict absolute minimum is given by $u^{(0)} \equiv z$ for any $z \in \mathbb{C}$ with $|z| = 1$. The proof for $n = \pm 1$ uses a block decomposition of the linearized operator for the energy functional which is similar to the argument in [12]. However, because the potential term in the energy functional does not imply partial convexity, the Hessian of the energy might induce some zero modes other than the ones due to the symmetry breaking. We are able to show that the possible extra zero mode is at most one-dimensional when $n = \pm 1$ and the vortices $u^{(\pm 1)}$ are still minimizing the energy along that direction. This, in turn, implies stability.

We now turn to the full CSH energy (1.10). Our primary result is:

Theorem 1.2. *For any n, there are radially symmetric field solutions of the form (1.13), (1.14) to equations (1.8) and (1.9). In particular, the radial functions $(f^{(n)}, a_n)$ minimize the radial energy functional (1.10) and $1 - f^{(n)}(r), 1 - a_n(r) \to 0$ as $r \to \infty$.*

The proof of Theorem 1.2 also relies on the results for the Ginzburg-Landau energy. We choose to minimize the energy functional over a constraint set suggested by Ginzburg-Landau vortices, see Berger-Chen [1]. The difficulty now comes from the term $\frac{\mu^2}{4}\frac{|\mathrm{curl} A|^2}{|u|^2}$ in the energy functional (1.10). We show the pointwise convergence of that term by recovering A (in particular $a_n(r)$) from the induced magnetic field term $\frac{\mu^2}{4}\frac{|\mathrm{curl} A|^2}{|u|^2}$. Then a combination of weak lower semi-continuity and Fatou's lemma gives the existence of minimizers, which is also a solution to equations (1.8) and (1.9).

We further investigate the basic properties of the minimizers of the energy functional (1.10). It is straightforward to establish regularity of the vortex profile; on the other hand establishing monotonicity and/or maximum principles turns out to be tricky. In the end, though, we are able to prove the following

Theorem 1.3. *For any n, the radial functions $(f^{(n)}, a_n)$ obtained in Theorem 1.2 are C^∞ on $(0, \infty)$ and have the following properties (for $n \neq 0$):*

(1) $0 < f^{(n)} < 1$ *on* $(0, \infty)$,
(2) $0 < a_n \leq 1$ *on* $(0, \infty)$,
(3) $a_n' \geq 0$, $f^{(n)\prime} > 0$.

The maximum principle and monotonicity for $a_n(r)$ can be established by a truncation argument, similar to the method used by Berger-Chen [1] to establish monotonicity of the planar, symmetric Ginzburg-Landau equations. The proof of the monotonicity for $f^{(n)}$ cannot be attacked in the same way due to the nonconventional structure of the CSH energy. Truncation of f does not work effectively, due to the offsetting behavior of the terms $n^2 \frac{f^2(1-a)^2}{r}$ and $\frac{\mu^2}{4}\frac{(a')^2}{r^2 f^2}$ in the energy. Furthermore, the elliptic equation for f:

$$\left[\frac{1}{2}\Delta_r - \lambda^2 f^2(3f^2-1)\right](f^2-1) = (f')^2 + \frac{n^2}{r^2}(1-a)^2 f^2 - \frac{\mu^2}{4}\frac{n^2}{r^2}\frac{(a')^2}{f^2}$$

does not have a definite sign on the right-hand-side, hence no simple application of the maximum principle. On the other hand, we use the first and second variations of the energy, along with the Euler-Lagrange equations to prove that $f^{(n)\prime}(r) > 0$. The bounds on $f^{(n)}$ follow.

1.3. **Discussion.** One quantity that we have difficulty describing is the induced magnetic field, $h = \text{curl}A = (n/r)a_n'(r)$ for $r \neq 0$. From Theorem 1.3 we know that $h(r) \geq 0$ and that $h \to 0$ as $r \to \infty$. Furthermore, the Euler-Lagrange equation for $(f^{(n)}, a_n)$ and the regularity result one has that $h \to 0$ as $r \to 0$. Since there is a quantized amount of magnetic field, we can conclude that h is roughly of annular shape in the plane. Although we are unable to determine much explicit behavior of h, we nonetheless assert

Conjecture 1.4. *The magnetic field profile $h(r)$ has exactly one local maximum for any positive μ and λ.*

Another issue which turns out to be difficult to analyze at this moment is the instability of vortices with large degree when $A \equiv 0$. The potential term in the energy indicates that the 0 state may also be preferable. Numerically this in turn gives rise to the existence of a sharp transition layer of $O(1)$ thickness, at a distance of $O(n^2)$ from the origin. So far no sharp analytical results can be obtained on the behavior of the transition layer, which seems necessary to excite an unstable mode. We offer

Conjecture 1.5. *When $|n| \geq 2$, the n-vortices $u^{(n)}$ obtained in Theorem 1.1 are saddle points of the renormalized energy, hence unstable.*

It is natural to study the stability of the full CSH energy (1.10) as was done by Gustafson-Sigal [6] for the Ginzburg-Landau energy (1.15). We note that the hessian of the CSH energy (1.10) is significantly more complicated than the hessian of the Ginzburg-Landau energy (1.15).

The rest of this paper is organized as follows. Sections 2-6 treat the case when $A \equiv 0$. In Section 2 we compute the linearized operator of equation (1.12) and identify the zero-modes of that operator due to symmetry-breaking. We renormalize the energy functional (1.11) in Section 3 and then consider minimizing the renormalized energy. In Section 4 we establish a certain covering property of the Ginzburg-Landau energy which controls the set on which the amplitude of solution is small, which enables us to choose a constraint set of the minimization problem. We provide an existence result of the n-vortex in Section 5. In Section 6 we make a block-decomposition for the linearized operator and give a spectral characterization of the operator, which provides the stability of the n-vortex for $n = 0, \pm 1$. In Section 7 we prove the existence of the n-vortex of the full equations (1.8) and (1.9), when $A \not\equiv 0$. In Section 8 we give some basic properties of those solutions.

2. SYMMETRY BREAKING

A central feature of the the static Chern-Simons-Higgs energy functional G_{csh} (and the CSH equations) is its infinite-dimensional symmetry group. Specifically, G_{csh} is invariant under $U(1)$ gauge transformations

$$u \mapsto e^{i\gamma} u \tag{2.1}$$

$$A \mapsto A + \nabla\gamma \tag{2.2}$$

for any smooth $\gamma : \mathbb{R}^2 \to \mathbb{R}$. In addition, G_{csh} is invariant under coordinate translations and rotation transformations. The same thing holds for E_{csh}.

The following theorem from [12] is crucial in our analysis

Theorem 2.1 (Ovchinnikov-Sigal [12])**.** *Let u_0 be a solution to the abstract equation $F(u) = 0$ breaking a one parameter subgroup $g(s)$ of the symmetry group of this equation. Let T be the generator of $g(s)$. Then $DF(u_0)Tu_0 = 0$, where $DF(u_0)$ is the linearized operator around u_0.*

When the magnetic field $A \equiv 0$ the Chern-Simons-Higgs energy reduces to (1.11). We let L_u be the linearized operator around u, i.e.

$$\lim_{\varepsilon,\delta\to 0} \partial_\varepsilon\partial_\delta E_{csh}(u + \varepsilon\xi + \delta\eta) = \langle L_u(\xi), \eta\rangle = \mathrm{Re}\int_{\mathbb{R}^2} \bar{\eta} L_u(\xi)\, dx.$$

A simple computation gives

$$L_u(\xi) = \Big[-\Delta + \lambda^2(9|u|^4 - 8|u|^2 + 1)\Big]\xi + \Big[\lambda^2(6|u|^2 - 4)\Big]u^2\bar{\xi}. \tag{2.3}$$

In the radially symmetric case, we are looking for solutions of the form (1.13). An immediate consequence of Theorem 2.1 is:

Corollary 2.2. *The functions $u^{(n)}_{x_1}, u^{(n)}_{x_2}$ and $iu^{(n)}$ solve the linearized equation*

$$L_{u^{(n)}}(\xi) = 0,$$

where $L_u(\xi)$ is given in (2.3).

We will also need the following lemma later

Lemma 2.3. *We have*

$$u^{(n)}_{x_1} = \frac{1}{2}\Big(f^{(n)\prime} - \frac{n}{r}f^{(n)}\Big)e^{i(n+1)\theta} + \frac{1}{2}\Big(f^{(n)\prime} + \frac{n}{r}f^{(n)}\Big)e^{i(n-1)\theta} \tag{2.4}$$

$$u^{(n)}_{x_2} = -\frac{i}{2}\Big(f^{(n)\prime} - \frac{n}{r}f^{(n)}\Big)e^{i(n+1)\theta} + \frac{i}{2}\Big(f^{(n)\prime} + \frac{n}{r}f^{(n)}\Big)e^{i(n-1)\theta}. \tag{2.5}$$

A proof of this lemma can be found in [12].

3. RENORMALIZED ENERGY FUNCTIONAL

Lemma 3.1. *If $u \in C^1(\mathbb{R}^2)$ such that $|u| \to 1$ as $|x| \to \infty$ and $\deg(u) \neq 0$, then $E_{csh} = \infty$, where E_{csh} is given in (1.11).*

Proof. Take $u = fe^{i\varphi}$ with $f = |u|$. Then

$$|\nabla u|^2 = f^2|\nabla\varphi|^2 + |\nabla f|^2 \geq f^2|\nabla\varphi|^2.$$

Because $f \to 1$ at ∞, $\exists R$ sufficiently large such that $|f|^2 > 1/2$ for all $|x| \geq R$. Hence

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx \;\geq \int_{|x|\geq R} \frac{1}{2}|\nabla\varphi|^2\, dx \;= \frac{1}{2}\int_R^\infty \int_0^{2\pi} r|\nabla\varphi|^2 d\theta dr.$$

We also have that for $r \geq R$

$$
\begin{aligned}
2\pi \mathrm{deg}(u) &= \int_{|x|=r} d(\mathrm{arg} u) \; = \int_{|x|=r} d\varphi \\
&\leq \int_{|x|=r} |d\varphi| \; = \int_0^{2\pi} r|\nabla\varphi| d\theta \\
&\leq r\Big(2\pi \int_0^{2\pi} |\nabla\varphi|^2 d\theta\Big).
\end{aligned}
$$

Therefore

$$
\int_0^{2\pi} |\nabla\varphi|^2 d\theta \; \geq \frac{2\pi(\mathrm{deg}(u))^2}{r^2},
$$

and then

$$
2E_{csh} \geq \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \; \geq \pi(\mathrm{deg}(u))^2 \int_R^\infty \frac{1}{r^2} r dr \; = \infty.
$$

□

We renormalize the energy as follows. Let $\chi(x) \in C^\infty(\mathbb{R}^2)$ such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ for for $|x| \geq 2$, and $\chi(x) = 0$ for $|x| \leq 1$. Define the *renormalized CSH* energy functional to be

$$
E_{csh}^{ren}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \Big(|\nabla u|^2 - \frac{(\mathrm{deg}(u))^2}{r^2}\chi + \lambda^2|u|^2(1-|u|^2)^2\Big) \, dx, \tag{3.1}
$$

where $r = |x|$. Then the renormalized energy functional has the same Euler-Lagrange equation (1.12).

4. Further properties about Ginzburg-Landau energy

Consider the renormalized Ginzburg-Landau energy functional as in [12]

$$
E_{gl}^{ren}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \Big(|\nabla u|^2 - \frac{(\mathrm{deg} u)^2}{r^2}\chi + \lambda^2(1-|u|^2)^2\Big) \, dx, \tag{4.1}
$$

From [12] we know that for any n there exists a unique radially symmetric vortex $u^{(n)} = f^{(n)}e^{in\theta}$ such that $f^{(n)}$ minimizes

$$
E_{gl}^{ren}(f) = \frac{1}{2} \int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{n^2}{r^2}(f^2 - \chi) + \lambda^2(1-f^2)^2\Big) \, dx \tag{4.2}
$$

among all real f such that $E_{gl}^{ren}(f) < \infty$. Denote

$$
K^{(n)} = E_{gl}^{ren}(f^{(n)}).
$$

We also have the following estimate, via the Cauchy -Schwarz inequality

$$
\begin{aligned}
E_{gl}^{ren}(f) &= \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{n^2}{r^2}(f^2-\chi) + \lambda^2(1-f^2)^2\Big)\,dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{\lambda^2}{2}(1-f^2)^2\Big)\,dx - \frac{1}{2}\int_{1\leq r\leq 2}\frac{n^2}{r^2}\,dx \\
&\quad + \frac{1}{2}\int_{r\geq 2}\Big(\frac{n^2}{r^2}(f^2-1) + \frac{\lambda^2}{2}(1-f^2)^2\Big)\,dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{\lambda^2}{2}(1-f^2)^2\Big)\,dx - \frac{1}{2}\int_{1\leq r\leq 2}\frac{n^2}{r^2}\,dx - \int_{2\leq r}\frac{n^4}{\lambda^2 r^4}\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{\lambda^2}{2}(1-f^2)^2\Big)\,dx - \Big(n^2\pi\ln 2 + \frac{n^4\pi}{4\lambda^2}\Big) \\
&\geq \int_{\mathbb{R}^2} \frac{\lambda}{\sqrt{2}}|\nabla f||1-f^2|\,dx - \Big(n^2\pi\ln 2 + \frac{n^4\pi}{4\lambda^2}\Big).
\end{aligned}
$$

Let $N^{(n)} = \Big(n^2\pi\ln 2 + \frac{n^4\pi}{4\lambda^2}\Big)$, then

$$
\int_{\mathbb{R}^2} \frac{\lambda}{\sqrt{2}}|\nabla f||1-f^2|\,dx \leq \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{\lambda^2}{2}(1-f^2)^2\Big)\,dx \leq E_{gl}^{ren}(f) + N^{(n)}. \tag{4.3}
$$

For any open set Ω we define

$$
\mathcal{H}^1_\infty(\Omega) = \inf\big\{\Sigma 2r_j :\ \Omega\subset\bigcup_j B_{r_j}(x_j)\big\}, \tag{4.4}
$$

then $\mathcal{H}^1_\infty(\Omega)\leq\mathcal{H}^1(\partial\Omega)$, as noted in [15]. We can see that

$$
t\mapsto\mathcal{H}^1_\infty(\{x:\ f(x)\leq t\})
$$

is an increasing function.

Suppose there exists some large R such that $f\geq 1/2$ on ∂B_R. Hence

$$
\begin{aligned}
E_{gl}^{ren}(f) + N^{(n)} &\geq \frac{\lambda}{\sqrt{2}}\int_{\mathbb{R}^2} |\nabla f||1-f^2|\,dx \\
&= \frac{\lambda}{\sqrt{2}}\int_0^\infty |1-t^2|\mathcal{H}^1\big(f^{-1}(t)\big)dt \\
&\geq \frac{\lambda}{\sqrt{2}}\int_{1/4}^{1/2} |1-t^2|\mathcal{H}^1\big(f^{-1}(t)\big)dt \\
&\geq \frac{\lambda}{\sqrt{2}}\int_{1/4}^{1/2} |1-t^2|\mathcal{H}^1_\infty(\{x\in B_R:\ f(x)\leq t\})dt \\
&\geq \frac{\lambda}{\sqrt{2}}\mathcal{H}^1_\infty\big(\{x\in B_R:\ f(x)\leq\frac{1}{4}\}\big)\int_{1/4}^{1/2}(1-t^2)dt \\
&= \frac{41}{192}\frac{\lambda}{\sqrt{2}}\mathcal{H}^1_\infty\big(\{x\in B_R:\ f(x)\leq\frac{1}{4}\}\big) \\
&\geq \frac{\lambda}{10}\mathcal{H}^1_\infty\big(\{x\in B_R:\ f(x)\leq\frac{1}{4}\}\big).
\end{aligned}
$$

where we need the assumption that $f \geq 1/2$ on ∂B_R between the third and fourth lines. Therefore $\{x \in B_R : \ f(x) \leq 1/4\} \subset \bigcup_j B_{r_j}(x_j)$ with

$$\sum_j r_j \leq \frac{5}{\lambda}(E_{gl}^{ren}(f) + N^{(n)}). \tag{4.5}$$

Note that estimate (4.5) is **independent** of R.

5. Existence of radially symmetric vortices when $A \equiv 0$

Let $u = fe^{in\theta}$ with f real. Then $|\nabla u|^2 = n^2 f^2 |\nabla\theta|^2 + |\nabla f|^2$. Hence

$$E_{csh}^{ren}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \Big(|\nabla f|^2 + \frac{n^2}{r^2}(f^2 - \chi) + \lambda^2 f^2 (1-f^2)^2\Big)\, dx \equiv E(f). \tag{5.1}$$

Take a positive number M and consider the set

$$\mathcal{A}_M^{(n)} = \{f \text{ real} \mid \ E_{gl}^{ren}(f) < K^{(n)} + M, \ \ E(f) < \infty\}. \tag{5.2}$$

Recall that $E_{gl}^{ren}(f)$ is defined in (4.2).

Therefore $\mathcal{A}_M^{(n)}$ is not empty and from the continuity of the functionals $E_{gl}^{ren}(f)$ and $E(f)$ we know that $\mathcal{A}_M^{(n)}$ is open. Now we consider minimizing E over the constraint set $\mathcal{A}_M^{(n)}$.

The main result of this section is the following

Theorem 5.1. *For any given n, there is an M such that the functional $E(f)$ has a minimizer $f^{(n)}$ on $\mathcal{A}_M^{(n)}$. Such minimizer $f^{(n)}$ is radially symmetric, $0 \leq f^{(n)} \leq 1$, and $u^{(n)} = f^{(n)} e^{in\theta}$ is an n-vortex, i.e. solution to equation (1.12) of degree n. If $|n| \geq 1$, then $f^{(n)}$ is monotonically increasing.*

Proof. We first show that $E(f) > -\infty$ on $\mathcal{A}_M^{(n)}$. We provide two ways.

Method 1: Let $f \in \mathcal{A}_M^{(n)}$. Taking $\bar{f} = |f| \wedge 1$ we have

$$E(\bar{f}) \leq E(f), \ \ E_{gl}^{ren}(\bar{f}) \leq E_{gl}^{ren}(f),$$

which shows $\bar{f} \in \mathcal{A}_M^{(n)}$. Therefore it suffices to consider $0 \leq f \leq 1$. Let $v = \sqrt{\overline{f^2}}$, where $\bar{p}(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r,\theta)\, d\theta$. Then

$$v^2 = \overline{f^2}, \ \ v^4 \leq \overline{f^4}, \ \ |\nabla_r u|^2 \leq \overline{|\nabla_r f|^2}.$$

Hence

$$E_{gl}^{ren}(v) \leq E_{gl}^{ren}(f), \ \ E(v) \leq E(f),$$

which shows that it suffices to consider f being radially symmetric.

From (4.3) we know that $1 - f \in H^1(\mathbb{R}^2)$. For any $s > 0$,

$$\begin{aligned}\int_{1/s}^{s} |f'| dr &\leq \Big(\int_{1/s}^{s} \frac{1}{r} dr\Big)^{1/2} \Big(\int_{1/s}^{s} (f')^2 r dr\Big)^{1/2} \\ &\leq \Big(\int_{1/s}^{s} \frac{1}{r} dr\Big)^{1/2} \Big(\int_{\mathbb{R}^2} (f')^2 \, dx\Big)^{1/2} < \infty,\end{aligned}$$

so $f' \in L^1[1/s, s]$, which implies f is absolutely continuous on $[1/s, s]$. Therefore we get $f(r) \in C(0,\infty)$.

Due to the definition of $\mathcal{A}_M^{(n)}$ and (4.3) we know that $|f| \to 1$, we can take R large enough so that $f \geq 1/2$ for $|x| \geq R$. For $f \geq 1/4$ we have

$$\frac{n^2}{r^2}(f^2-1) + \lambda^2 f^2 (1-f^2)^2 \ \geq -\frac{n^4}{4\lambda^2 f^2 r^4} \ \geq -\frac{4n^4}{\lambda^2 r^4}.$$

Therefore we have

$$\begin{aligned}
E(f) &\geq \frac{1}{2}\int_{r\geq 1} \Big(\frac{n^2}{r^2}(f^2-1) + \lambda^2 f^2(1-f^2)^2\Big)\, dx \\
&\geq \frac{1}{2}\int_{r\geq 1, f\geq 1/4} \Big(\frac{n^2}{r^2}(f^2-1) + \lambda^2 f^2(1-f^2)^2\Big)\, dx + \frac{1}{2}\int_{r\geq 1, f<1/4} \frac{n^2}{r^2}(f^2-1)\, dx \\
&\geq \int_{r\geq 1} -\frac{4n^4}{\lambda^2 r^4}\, dx \ - \ n^2 \big|\{x:\ f(x)<1/4\}\big| \\
&\geq -\frac{2n^4}{\lambda^2} - \frac{25n^2}{\lambda^2}\big(K^{(n)} + M + N^{(n)}\big)^2 > -\infty, \qquad (5.3)
\end{aligned}$$

where we've used (4.5) in getting the fourth inequality.

Method 2: We use an averaging method by Struwe [17]. From (4.3), (5.2) and $f \geq 0$ we know that there exists some C sufficiently large such that

$$\begin{aligned}
C &\geq \int_{R\leq |x|\leq 2R} |\nabla f|^2 + (1-f^2)^2\, dx \geq \int_{R\leq |x|\leq 2R} |\nabla f|^2 + (1-f)^2\, dx \\
&\geq R \inf_{r\in[R,2R]} \int_{\partial B_r} |\nabla f|^2 + (1-f)^2\, d\omega.
\end{aligned}$$

Hence there exists $r_* \in [R, 2R]$ such that

$$\int_{\partial B_{r_*}} |\nabla f|^2 + (1-f)^2\, d\omega \leq \frac{2C}{R},$$

which shows

$$\|1-f\|_{H^1(\partial B_{r_*})} \leq \sqrt{\frac{2C}{R}}.$$

Since we have the H^1 bound of $1-f$ on the circle, we may apply the Morrey's inequality to get

$$\|1-f\|_{C^{0,1/2}(\partial B_{r_*})} \leq \sqrt{\frac{2C}{R}}.$$

Suppose there is some point $r_* e^{i\theta_*} \in \partial B_{r_*}$ such that $f(r_* e^{i\theta_*}) \leq 1/2$ then

$$\begin{aligned}
|1-f(r_* e^{i\theta})| &\geq |1-f(r_* e^{i\theta_*})| - |f(r_* e^{i\theta_*}) - f(r_* e^{i\theta})| \\
&\geq \frac{1}{2} - |r_* e^{i\theta_*} - r_* e^{i\theta}|^{1/2}\sqrt{\frac{2C}{R}} \\
&\geq \frac{1}{2} - |\theta_* - \theta|^{1/2}\sqrt{\frac{2Cr_*}{R}} \\
&\geq \frac{1}{2} - 2\sqrt{C}|\theta_* - \theta| > \frac{1}{4}
\end{aligned}$$

if we consider $|\theta_* - \theta| < 1/(8\sqrt{C})$. Therefore

$$\begin{aligned}\frac{2C}{R} &\geq \int_{\partial B_{r_*} \bigcap \{|\theta_* - \theta| < 1/(8\sqrt{C})\}} |1-f|^2 r_* \, d\theta \\ &> \frac{1}{16} r_* \frac{1}{8\sqrt{C}} > \frac{R}{128\sqrt{C}},\end{aligned}$$

which is a contradiction if we choose R to be sufficiently large. Therefore, for any large R, there is some $r_* \in [R, 2R]$ such that $f \geq 1/2$ on ∂B_{r_*}, and (4.5) also holds for such r_*. We also notice that the estimate (4.5) is independent of R. The rest follows from the similar argument for (5.3).

Hence we've shown that $E(f)$ is bounded from below on $\mathcal{A}_M^{(n)}$. We take a minimizing sequence $f_m \in \mathcal{A}_M^{(n)}$ such that

$$\lim_{m\to\infty} E(f_m) = \inf_{u \in \mathcal{A}_M^{(n)}} E(u).$$

Without loss of generality we may assume $0 \leq f_m \leq 1$. Otherwise we consider $\bar{f}_m = |f_m| \wedge 1$. Since

$$E(\bar{f}_m) \leq E(f_m), \;\; E_{gl}^{ren}(\bar{f}_m) \leq E_{gl}^{ren}(f_m),$$

$\{\bar{f}_m\}$ would also be a minimizing sequence in $\mathcal{A}_M^{(n)}$.

Let $g_m = 1 - f_m$, then $0 \leq g_m \leq 1$. We have

$$\begin{aligned}K^{(n)} + M > E_{gl}^{ren}(f_m) &= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla g_m|^2 + \frac{n^2}{r^2}(f_m^2 - \chi) + \lambda^2 g_m^2 (1+f_m)^2\right) dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla g_m|^2 - \frac{2n^2}{r^2}\chi_{|x|\geq 1} g_m + \lambda^2 g_m^2\right) dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla g_m|^2 + \frac{\lambda^2}{2} g_m^2\right) dx - \int_{\mathbb{R}^2} \frac{1}{\lambda^2}\frac{n^4}{r^4}\chi_{|x|\geq 1}\, dx.\end{aligned}$$

Hence

$$\int_{\mathbb{R}^2} \left(|\nabla g_m|^2 + g_m^2\right) dx \leq C,$$

for some fixed $C < \infty$. Therefore we have up to a subsequence that

$$\begin{aligned} g_m &\to g_0 \quad \text{weakly in } \; H^1 \\ g_m &\to g_0 \quad \text{a.e. in } \; \mathbb{R}^2. \end{aligned}$$

Let $f_0 = 1 - g_0$, then

$$E(f_0) \leq \liminf_{m\to\infty} E(f_m).$$

On the other hand since $0 \leq f_m \leq 1$ we have that

$$E(f_m) \leq E_{gl}^{ren}(f_m) < K^{(n)} + M.$$

Therefore $f_0 \in \overline{\mathcal{A}_M^{(n)}}$, and

$$E(f_0) = \inf_{u \in \mathcal{A}_M^{(n)}} E(u).$$

Next we show that f_0 is radially symmetric, using the same method in [12].

Let $v = \sqrt{\overline{f_0^2}}$, where $\bar{p}(r) \equiv \frac{1}{2\pi}\int_0^{2\pi} p(r,\theta)\, d\theta$. Then

$$v^2 = \overline{f_0^2}, \;\; v^4 \le \overline{f_0^4}, \;\; |\nabla_r u|^2 \le \overline{|\nabla_r f_0|^2}.$$

Hence

$$E(v) \le E(f_0), \quad \text{and} \;\; E_{gl}^{ren}(v) \le E_{gl}^{ren}(f_0),$$

and the equality holds only if f_0 is radially symmetric. Thus f_0 must be radially symmetric.

Now we show that there exists an M such that f_0 is an interior minimizer. Let f_0^k be the minimizer of $E(f)$ over $\overline{\mathcal{A}^{(n)}_{M+1/k}}$ for $k = 1, 2, \ldots$. If f_0^k is not an interior minimizer then by previous argument we know $0 \le f_0^k \le 1$ and

$$E(f_0^k) = \inf\{E(u) : \; u \in \mathcal{A}^{(n)}_{M+1/k}\} \le E(f_0^{k-1}), \quad E_{gl}^{ren}(f_0^k) = K^{(n)} + M + 1/k. \tag{5.4}$$

Therefore all $E_{gl}^{ren}(f_0^k)$ are uniformly bounded by $K^{(n)} + M + 1$. From (4.3), $0 \le f_0^k \le 1$, and the fact that $\int_{\mathbb{R}^2} (1 - f_0^k)^2 \, dx \le \int_{\mathbb{R}^2} (1 - (f_0^k)^2)^2 \, dx$ we have that $\|1 - f_0^k\|_{H^1}$ is uniformly bounded. Hence $1 - f_0^k(r) \rightharpoonup 1 - f_0(r)$ in H^1. Since f_0^k are radial, from [19], radial H^1 functions on $\mathbb{R}^2$ decays like $r^{-1/2}$. We know further that $f_0^k(r) \to f_0(r)$ a.e.. Thus applying the weak-lower semicontinuity of norms, Fatou's lemma and (5.4) we get

$$E(f_0) \le \liminf_{k\to\infty} E(f_0^k) = E(f_0^1), \quad E_{gl}^{ren}(f_0) \le \liminf_{k\to\infty} E_{gl}^{ren}(f_0^k) = K^{(n)} + M.$$

In this way we see that f_0 is an interior minimizer of $\mathcal{A}^{(n)}_{M+1/k}$ for any k.

Now we prove that $f_0' > 0$. Since f_0 minimizes $E(f)$, $E'(f_0) = 0$, $E''(f_0) \ge 0$. We have

$$E'(f) = -\Delta_r f + \frac{n^2}{r^2} f + \lambda^2 f(1 - f^2)(1 - 3f^2) \tag{5.5}$$

$$E''(f) = -\Delta_r + \frac{n^2}{r^2} + \lambda^2(15f^4 - 12f^2 + 1). \tag{5.6}$$

Differentiating $E'(f_0)$ with respect to r to obtain

$$(E''(f_0) + \frac{1}{r^2}) f_0' = \frac{2n^2}{r^3} f_0.$$

Since $f_0 \ge 0$ and $f_0 \not\equiv 0$, applying the maximum principle (see, [16], Theorem B.4) we obtain that $f_0' > 0$ when $|n| \ge 1$. Hence we obtain further that $f_0(r) > 0$ for $r > 0$.

Lastly, since f_0 is radially symmetric, we have $\nabla f_0 \cdot \nabla\theta = 0$. Hence

$$\Delta(f_0 e^{in\theta}) = \Big(\Delta f_0 - \frac{n^2}{r^2} f_0\Big) e^{in\theta}$$

and therefore together with (5.5) we obtain that $f_0 e^{in\theta}$ satisfies the equation

$$-\Delta u + \lambda^2 u(1 - |u|^2)(1 - 3|u|^2) = 0.$$

Since $\deg(f_0 e^{in\theta}) = n$, we have completed the proof of Theorem 5.1. $\square$

Remark 5.2. When $n = 0$, it follows from the definition that a strict absolute minimum is given by $u^{(0)} \equiv z$ for any $z \in \mathbb{C}$ with $|z| = 1$.

6. Stability

We follow the argument in [12] to consider the stability of the critical points of the renormalized Chern-Simons-Higgs energy functional, $E_{csh}^{ren}(u)$. It is discussed in [12] that this question is related to the spectral property of the Hessian of the energy functional, $\mathrm{Hess}E_{csh}^{ren}(u)$.

We compute $\mathrm{Hess}E_{csh}^{ren}(u)$ to be

$$\mathrm{Hess}E_{csh}^{ren}(u) = \begin{pmatrix} -\Delta + \lambda^2(9|u|^4 - 8|u|^2 + 1) & 2\lambda^2(3|u|^2 - 2)u^2 \\ 2\lambda^2(3|u|^2 - 2)\bar{u}^2 & -\Delta + \lambda^2(9|u|^4 - 8|u|^2 + 1) \end{pmatrix}. \tag{6.1}$$

Denote

$$\vec{\xi} = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}, \tag{6.2}$$

then we also have

$$\mathrm{Hess}E_{csh}^{ren}(u)\vec{\xi} = \overrightarrow{L_u(\xi)}, \tag{6.3}$$

where $L_u(\xi)$ is the linearized operator given in (2.3). Using the same notation as in [12], we denote Sym Null $\mathrm{Hess}E_{csh}^{ren}(u)$ to be the maximal null space of $\mathrm{Hess}E_{csh}^{ren}(u)$ due to symmetry breaking. Then it is known that

(i) $\mathrm{Hess}E_{csh}^{ren}(u) \geq 0$ and Null $\mathrm{Hess}E_{csh}^{ren}(u) =$ Sym Null $\mathrm{Hess}E_{csh}^{ren}(u) \;\Rightarrow\; u$ is a local minimum of $E_{csh}^{ren}(u)$,

(ii) $\mathrm{Hess}E_{csh}^{ren}(u)$ has a negative eigenvalue $\;\Rightarrow\; u$ is a saddle point of $E_{csh}^{ren}(u)$.

The main stability result of this section is

Theorem 6.1. *$u^{(n)}$ are local minima of $E_{csh}^{ren}(u)$ for $n = 0, \pm 1$.*

Theorem 6.1 says that the n-vortices are stable for $|n| = 0, 1$. From Remark 5.2 we know that when $n = 0$, $f^{(0)} \equiv 1$ and the corresponding 0-vortex $u^{(0)}$ is an absolute minimum. Hence we only argue the case when $n = \pm 1$

From the previous argument we know that we need to understand the spectrum of $\mathrm{Hess}E_{csh}^{ren}(u)$. Without loss of generality we assume $n = 1$. The case $n = -1$ can be treated the same way by observing that $\overline{u^{(1)}} = u^{(-1)}$.

We begin with an elementary harmonic analysis of the linearized operator $L_{u^{(n)}}$ which is closely related to $\mathrm{Hess}E_{csh}^{ren}(u)$ (see (6.3)). Consider a function $\xi(r, \theta)$ in polar coordinates and expand it in the Fourier series in θ

$$\xi(r, \theta) = \sum_{k=-\infty}^{\infty} \xi_k(r) e^{ik\theta},$$

where the Fourier coefficients are given by

$$\xi_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \xi(r, \theta) e^{-ik\theta} d\theta.$$

Consider a map Π of measurable functions $\xi : \mathbb{R}^2 \to \mathbb{C}$ into measurable functions $\hat{\xi} = \bigoplus_{k \geq n} \begin{pmatrix} \xi_k \\ \bar{\xi}_{2n-k} \end{pmatrix}$. If ξ's are endowed with inner product

$$\langle \vec{\xi}, \vec{\eta} \rangle = 2\mathrm{Re} \int_{\mathbb{R}^2} \bar{\xi} \eta \, dx,$$

where $\vec{\xi}$ is given in (6.2), then Π is unitary, provided $\hat{\xi}$'s are endowed with the inner product

$$\langle \hat{\xi}, \hat{\eta} \rangle = \mathrm{Re}\langle \xi_n, \eta_n \rangle + \sum_{k>n} \mathrm{Re}\Big\langle \begin{pmatrix} \xi_k \\ \bar{\xi}_{2n-k} \end{pmatrix}, \begin{pmatrix} \eta_k \\ \bar{\eta}_{2n-k} \end{pmatrix} \Big\rangle.$$

Define the real linear operator $\hat{L}_{u^{(n)}}$ on functions $\hat{\xi}$ by

$$\hat{L}_{u^{(n)}}(\Pi\xi) = \Pi L_{u^{(n)}}(\xi). \tag{6.4}$$

We first give the characterization of $\hat{L}_{u^{(n)}}$.

Lemma 6.2. *The operator $\hat{L}_{u^{(n)}}$ is block diagonal of the form*

$$\hat{L}_{u^{(n)}}(\xi) = \bigoplus_{k \geq n} L^k_{u^{(n)}} \begin{pmatrix} \xi_k \\ \bar{\xi}_{2n-k} \end{pmatrix}, \tag{6.5}$$

where $L^k_{u^{(n)}}$ are given by

$$L^k_{u^{(n)}} = \begin{pmatrix} -\Delta_r + \frac{k^2}{r^2} + \lambda^2(9|u^{(n)}|^4 - 8|u^{(n)}|^2 + 1) & 2\lambda^2|u^{(n)}|^2(3|u^{(n)}|^2 - 2) \\ 2\lambda^2|u^{(n)}|^2(3|u^{(n)}|^2 - 2) & -\Delta_r + \frac{(2n-k)^2}{r^2} + \lambda^2(9|u^{(n)}|^4 - 8|u^{(n)}|^2 + 1) \end{pmatrix}, \tag{6.6}$$

where $\Delta_r f = \frac{1}{r}\partial_r(r\partial_r f)$.

Proof. First it is easily checked that

$$\Big(L_{u^{(n)}}(\xi)\Big)_k = \Big[-\Delta_r + \frac{k^2}{r^2} + \lambda^2(9|u^{(n)}|^4 - 8|u^{(n)}|^2 + 1)\Big]\xi_k + \Big[\lambda^2(6|u^{(n)}|^2 - 4)\Big]|u^{(n)}|^2 \bar{\xi}_{2n-k}. \tag{6.7}$$

Since $\Delta = \Delta_r + r^{-2}\partial_\theta^2$, we have

$$(-\Delta\xi)_k = -\Delta_r \xi_k + \frac{k^2}{r^2}\xi_k.$$

Moreover we have

$$\begin{aligned} \Big((u^{(n)})^2\bar{\xi}\Big)_k &= |u^{(n)}|^2 (2\pi)^{-1/2} \int_0^{2\pi} e^{i2n\theta}\bar{\xi} e^{-ik\theta} d\theta \\ &= |u^{(n)}|^2 (2\pi)^{-1/2} \int_0^{2\pi} \xi e^{-i(2n-k)\theta} d\theta = |u^{(n)}|^2 \bar{\xi}_{2n-k}. \end{aligned}$$

Therefore (6.7) implies

$$\begin{pmatrix} (L_{u^{(n)}}\xi)_k \\ \overline{(L_{u^{(n)}}\xi)}_{2n-k} \end{pmatrix} = L^k_{u^{(n)}} \begin{pmatrix} \xi_k \\ \bar{\xi}_{2n-k} \end{pmatrix},$$

which, due to (6.4), yields (6.5). □

Lemma 6.3. *For $n \geq 1$, we have the following characterization of the linear operators:*

(1) $L^n_{u^{(n)}} \geq 0$ *and if 0 is an eigenvalue then it is non-degenerate.*
(2) $L^{n+1}_{u^{(n)}} \geq 0$ *and 0 is its non-degenerate eigenvalue.*
(3) $L^k_{u^{(n)}} \geq 0$ *for $k \geq 3n$ and 0 is not an eigenvalue.*
(4) *The continuous spectrum*

$$\textit{cont spec} L^k_{u^{(n)}} = [0, \infty),$$

for any k.

Proof.
(1) Due to the breaking of the gauge symmetry we have

$$L_{u^{(n)}}(iu^{(n)}) = 0.$$

After separating out the angular variable we obtain

$$\Big[-\Delta_r + \frac{n^2}{r^2} + \lambda^2(3f_n^4 - 4f_n^2 + 1)\Big] f_n = 0,$$

where $f_n = |u^{(n)}|$. We can rewrite the equation as

$$\Big[-\Delta_r + \frac{n^2}{r^2} + 4\lambda^2(1 - f_n^2) - 3\lambda^2(1 - f_n^4)\Big] f_n = 0.$$

Since $f_n > 0$, bounded and $\notin L^2(\mathbb{R}^+, rdr)$, we can apply Theorem B.1 in [12] to conclude that the operator $-\Delta_r + \frac{n^2}{r^2} + \lambda^2(3f_n^4 - 4f_n^2 + 1)$ is non-negative, 0 is not its eigenvalue, and any solution g to the equation

$$\Big[-\Delta_r + \frac{n^2}{r^2} + \lambda^2(3f_n^4 - 4f_n^2 + 1)\Big] g = 0$$

is of the form $g = cf$ where c is some constant.

Also since f_n minimizes $E(f)$ we have

$$E''(f_n) = -\Delta_r + \frac{n^2}{r^2} + \lambda^2(15f_n^4 - 12f_n^2 + 1) \geq 0.$$

Suppose $E''(f_n)$ has an eigenvalue 0, and the corresponding eigenfunction is $\psi_0(r)$. Then we can write

$$0 = E''(f_n)\psi_0 = \Big[-\Delta_r + \frac{n^2}{r^2} + 4\lambda^2(3f_n^4 - 3f_n^2 + 1) - 3\lambda^2(1 - f_n^4)\Big]\psi_0,$$

where we see that

$$4\lambda^2(3f_n^4 - 3f_n^2 + 1) \geq \lambda^2 > 0.$$

We now follow the idea from [12]. Let

$$L_0 = -\Delta_r + \frac{n^2}{r^2} + 4\lambda^2(3f_n^4 - 3f_n^2 + 1), \quad V = 3\lambda^2(1 - f_n^4) > 0.$$

Then $V = O(r^{-2})$ as $r \to \infty$. Let $R_0(\alpha) = (L_0 - \alpha)^{-1}$ with $\alpha \leq \lambda^2$ and consider the Birman-Schwinger-type operator function

$$K(\alpha) = \sqrt{V} R_0(\alpha) \sqrt{V}.$$

Then we have

- if $(E''(f_n) - \alpha)\psi = 0$, then $K(\alpha)\varphi = \varphi$ with $\varphi = \sqrt{V}\psi$. If $\psi \in L^2(\mathbb{R}^2)$ then $\varphi \in L^2(\mathbb{R}^2)$.
- if $K(\alpha)\varphi = \varphi$, then $(E''(f_n) - \alpha)\psi = 0$ with $\psi = R_0(\alpha)\sqrt{V}\varphi$. If $\varphi \in L^2(\mathbb{R}^2)$ then for $\alpha < \lambda^2$, $|\psi| \leq Ce^{-r\sqrt{\lambda^2-\alpha}}$.

We also have the following result

Lemma 6.4 (Ovchinnikov-Sigal [12])**.** *$K(\alpha)$ with $\alpha \leq \lambda^2$ is positivity improving, i.e. $K(\alpha)\varphi > 0$ (modulo a set of zero measure) whenever $\varphi \geq 0$, and*

(1) *α is an eigenvalue of $E''(f_n)$ iff 1 is an eigenvalue of $K(\alpha)$.*
(2) *α is the lowest eigenvalue of $E''(f_n)$ iff 1 is the largest eigenvalue of $K(\alpha)$.*

By assumption that $E''(f_n)$ has an eigenvalue 0 and the fact that $E''(f_n) \geq 0$ we know that 0 is its lowest eigenvalue. Hence from the above lemma, 1 is the largest eigenvalue of $K(0)$. Therefore we have (see [14], Theorem XIII.43) that the eigenfunction of $K(0)$, φ_0, is positive and the eigenvalue 1 is non-degenerate. Thus from Lemma 6.4, 0 is a non-degenerate eigenvalue of $E''(f_n)$ and ψ_0 can be taken to be $\psi_0 = L_0^{-1}\sqrt{V}\varphi_0 > 0$ and ψ_0 decays exponentially fast.

On the other hand when $k = n$, we have

$$RL^n_{u^{(n)}}R^T = L,$$

where

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} -\Delta_r + \frac{n^2}{r^2} + \lambda^2(3f_n^4 - 4f_n^2 + 1) & 0 \\ 0 & -\Delta_r + \frac{n^2}{r^2} + \lambda^2(15f_n^4 - 12f_n^2 + 1) \end{pmatrix}.$$

Therefore $L^n_{u^{(n)}} \geq 0$ and if 0 is an eigenvalue then it is non-degenerate, which proves (1).

(2) We perform the similar argument as in (1), but instead of the zero mode due to breaking the gauge symmetry we use the zero mode due to breaking the translation symmetry. Such a zero mode is $\nabla u^{(n)}$. Due to Lemma 2.3 and since $n-1 = 2n-k$ for $k = n+1$, $\widehat{\partial_{x_j} u^{(n)}}$ contains only the $k = n+1$ block, $(n+1, n-1)$:

$$\widehat{\partial_{x_1} u^{(n)}} = \bigoplus_{k\geq n} g^{(n)}\delta_{k,n+1}, \qquad \widehat{\partial_{x_2} u^{(n)}} = \bigoplus_{k\geq n} -ig^{(n)}\delta_{k,n+1},$$

where $g^{(n)} = \frac{1}{2}\begin{pmatrix} f^{(n)\prime} - \frac{n}{r}f^{(n)} \\ f^{(n)\prime} + \frac{n}{r}f^{(n)} \end{pmatrix}$. Hence

$$0 = \hat{L}_{u^{(n)}}(\widehat{\partial_{x_1} u^{(n)}}) = \bigoplus_{k\geq n}\Big(L^{n+1}_{u^{(n)}}g^{(n)}\Big)\delta_{k,n+1}$$

and therefore

$$L^{n+1}_{u^{(n)}}g^{(n)} = 0. \tag{6.8}$$

The zero mode $\widehat{\partial_{x_1} u^{(n)}}$ leads to the same equation.

Notice that $Rg^{(n)} = \begin{pmatrix} \frac{n}{r}f^{(n)} \\ f^{(n)\prime} \end{pmatrix}$. Since $f^{(n)} > 0$ and $f^{(n)\prime} > 0$, $r > 0$, $Rg^{(n)}$ has positive entries. Hence (6.8) together with Appendix B in [12] shows that 0 is the lowest eigenvalue of $L^{n+1}_{u^{(n)}}$ and is non-degenerate. This and statement (4) imply statement (2).

(3) We have

$$L^k_{u^{(n)}} - L^{n+1}_{u^{(n)}} = \begin{pmatrix} \frac{k^2-(n+1)^2}{r^2} & 0 \\ 0 & \frac{(k-2n)^2-(n-1)^2}{r^2} \end{pmatrix} > 0,$$

for $k \geq 3n$. From (2) we obtain (3).

(4) As $|x| \to \infty$, we have

$$L^k_{u^{(n)}} \to \begin{pmatrix} -\Delta + 2\lambda^2 & 2\lambda^2 \\ 2\lambda^2 & -\Delta + 2\lambda^2 \end{pmatrix} =: L_0.$$

We know that

$$\text{cont spec} L^k_{u^{(n)}} = \text{spec} L_0. \tag{6.9}$$

Using the transformation matrix R to diagonalize L_0 as

$$RL_0R^T = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + 4\lambda^2 \end{pmatrix}.$$

Thus spec$L = [0,\infty) \bigcup [4\lambda^2, \infty)$, which together with (6.9) yields (4). $\square$

Proof of Theorem 6.1. When $n = 1$, due to Lemma 6.2 and Lemma 6.3, $\hat{L}_{u^{(1)}} \geq 0$, i.e. Hess$E^{ren}_{csh}(u^{(1)}) \geq 0$ with zero modes determined either completely by the symmetry breaking or by symmetry breaking and an extra mode $\psi_0 e^{i\theta}$. In the first case we obviously know that $u^{(1)}$ is a local minimum.

If $(\psi_0 e^{i\theta}, \psi_0 e^{-i\theta})^T \in$ Null Hess$E^{ren}_{csh}(u^{(1)})$. Then we know that except for the direction generated by $\psi_0 e^{i\theta}$, $u^{(1)}$ minimizes E^{ren}_{csh} locally. Along this direction we compute the second variation of the renormalized energy to be

$$\lim_{\varepsilon\to\infty} \partial^2_\varepsilon E^{ren}_{csh}(u^{(1)} + \varepsilon\psi) = \mathrm{Re}\int_{\mathbb{R}^2} \bar{\psi} L_{u^{(1)}}(\psi)\, dx, \tag{6.10}$$

where $\psi = c\psi_0 e^{i\theta}$ for some $c \in \mathbb{C}$ and L_u is given in (2.3). If $c = a + ib$ where $a, b \in \mathbb{R}$, further computation gives

$$\begin{aligned}
&\mathrm{Re}\int_{\mathbb{R}^2} \bar{\psi} L_{u^{(1)}}(\psi)\, dx \\
&= \mathrm{Re}\int_{\mathbb{R}^2} |c|^2\psi_0 \cdot \Big[-\Delta_r + \frac{n^2}{r^2} + \lambda^2(9f_1^4 - 8f_1^2 + 1)\Big]\psi_0 + \bar{c}^2\lambda^2(6f_1^2 - 4)f_1^2\psi_0^2\, dx \\
&= a^2\langle E''(f_1)\psi_0, \psi_0\rangle_{L^2} + b^2\Big\langle \Big[-\Delta_r + \frac{n^2}{r^2} + \lambda^2(3f_1^4 - 4f_1^2 + 1)\Big]\psi_0, \psi_0\Big\rangle_{L^2} \\
&= b^2\Big\langle \Big[-\Delta_r + \frac{n^2}{r^2} + \lambda^2(3f_1^4 - 4f_1^2 + 1)\Big]\psi_0, \psi_0\Big\rangle_{L^2} > 0
\end{aligned}$$

if $b \neq 0$ (from Lemma 6.3). Hence

$$E^{ren}_{csh}(u^{(1)} + c\psi_0 e^{i\theta}) > E^{ren}_{csh}(u^{(1)})$$

for $|c|$ sufficiently small. Therefore we only need to check the case when $c \in \mathbb{R}$. In this case, from (5.1) we know that

$$E^{ren}_{csh}(u^{(1)} + \psi) = E^{ren}_{csh}((f_1 + c\psi_0)e^{i\theta}) = E(f_1 + c\psi_0).$$

Hence we can apply Theorem 5.1 to obtain that $u^{(1)}$ is also a local minimizer along this direction $\psi_0 e^{i\theta}$. Thus we conclude to obtain Theorem 6.1.

$\square$

7. Existence of radially symmetric vortices when $A \not\equiv 0$

When the magnetic field $A \not\equiv 0$, the CSH energy functional and the Euler-Lagrange functions become more complicated. We look for minimizers of the CSH energy (1.10) among all symmetric vortices of the form (1.13), (1.14), with $f^{(n)}, a_n \to 1$ as $r \to \infty$ (This means we are looking for topological symmetric vortices).

Let

$$u = f(r)e^{in\theta}, \qquad A = n\frac{a(r)}{r}\vec{x}^{\perp}. \tag{7.1}$$

Then the CSH energy functional takes the following radial form

$$G^r_{csh}(f, a) = \frac{1}{2}\int_{\mathbb{R}^2} |f'|^2 + \frac{n^2}{r^2}(1-a)^2 f^2 + \frac{\mu^2}{4}\frac{n^2}{r^2}\Big(\frac{a'}{f}\Big)^2 + \lambda^2 f^2\big(1 - f^2\big)^2\, dx. \tag{7.2}$$

Our argument is also based on the results for Ginzburg-Landau vortices. Following [1], we define the spaces

$C_f =$ the set of real-valued radially symmetric functions $f(|x|)$ defined on $\mathbb{R}^2$ such that $f \geq 0$ a.e. and $1 - f \in H^1(\mathbb{R}^2)$.

$C_a =$ the set of real-valued radially symmetric functions $a(|x|)$ defined on $\mathbb{R}^2$ such that $a/r \in L^2_{\text{loc}}(\mathbb{R}^2)$ and $a'/r \in L^2(\mathbb{R}^2)$ where the derivative a' is in the distributional sense.

Recall the following results for Ginzburg-Landau equations.

Lemma 7.1 (Berger-Chen [1])**.** *The C_a and C_f spaces satisfy the following:*

(1) *C_a with the inner product which induces the norm $\|a\|_{C_a} = \|a'/r\|_{L^2(\mathbb{R}^2)}$ is a Hilbert space.*
(2) *For $f \in C_f$, $f(r) \in C(0, \infty)$.*
(3) *For $a \in C_a$, $a(r) \in C[0, \infty)$, $a(0) = 0, a(r) = \int_0^r a'(s)ds$, and*

$$\sup_{r \in (0,\infty)} |a/r| \leq \|a\|_{C_a}.$$

(4) *If $f \in C_f, a \in C_a$, and $G^r_{gl}(f, a) < \infty$, where $G^r_{gl}(f, a)$ is the radial Ginzburg-Landau energy given by*

$$G^r_{gl}(f, a) = \frac{1}{2} \int_{\mathbb{R}^2} |f'|^2 + \frac{n^2}{r^2}(1 - a)^2 f^2 + \frac{n^2(a')^2}{r^2} + \lambda^2 \big(1 - f^2\big)^2 \, dx, \tag{7.3}$$

then $f \in C[0, \infty)$ and $f(0) = 0$.

Theorem 7.2 (Berger-Chen [1])**.** *For any integer n and λ, there is a solution $(u^{(n)}, A^{(n)})$ to the Ginzburg-Landau equation which is of the form (1.13), (1.14). In particular, $(f^{(n)}, a_n) \in C_f \bigoplus C_a$ minimizes the radial Ginzburg-Landau energy $G^r_{gl}(f, a)$ defined in (7.3).*

First we note that $G^r_{csh}(f, a) \geq 0$ and $G^r_{csh}(1, 1) = 0$.Therefore $u \equiv 1$ and $A \equiv 1$ give a trivial solution to the CSH equations and minimize $G^r_{csh}(f, a)$ without restricting $G^r_{csh}(f, a)$ by any vortex number $n \neq 0$. On the other hand if we take (u, A) of the form (7.1) with $f, a \to 1$ at ∞ and if $m = \inf_{C_f \bigoplus C_a} G^r_{csh}(f, a)$ is attained at (f_0, a_0), then $m > 0$. Otherwise $\int_{\mathbb{R}^2} f_0^2(1 - f_0^2)^2 \, dx = \int_{\mathbb{R}^2} (1 - a_0)^2 f_0^2/r^2 \, dx = 0$. From Lemma 7.1we obtain the continuity of f_0 and a_0. Therefore we know from the integral identies that $f_0 \equiv 1$ and $a_0 \equiv 1$, which contradicts that $a_0/r \in L^2_{\text{loc}}(\mathbb{R}^2)$. Similarly we know that $\inf_{C_f \bigoplus C_a} G^r_{gl}(f, a) > 0$.

Let $m_0 = \inf_{C_f \bigoplus C_a} G^r_{gl}(f, a) > 0$. From Theorem 7.2 we know that m_0 is attained in $C_f \bigoplus C_a$. For any $M > 0$, let

$$\mathcal{B}_M = \{(f, a) \in C_f \bigoplus C_a : \; G^r_{gl}(f, a) < m_0 + M\}. \tag{7.4}$$

The main result of this section is the following

Theorem 7.3. *There is an M such that the infimum of G^r_{csh} over $\mathcal{B}_M$ is attained and is positive.*

Proof. Since $G^r_{csh} \geq 0$, we can take a minimizing sequence $(f_m, a_m) \in \mathcal{B}_M$. Therefore we have
(i) $G^r_{csh}(f_m, a_m) < K$ for some $K < \infty$, and
(ii) $G^r_{gl}(f_m, a_m) < m_0 + M$.

From (ii) we know that

$$\begin{aligned}K &> \int_{\mathbb{R}^2} |f_m'|^2 + \lambda^2(1-f_m^2)^2\,dx = \int_{\mathbb{R}^2} |f_m'|^2 + \lambda^2(1-f_m)^2(1+f_m)^2\,dx \\ &\geq \min\{1,\lambda^2\}\int_{\mathbb{R}^2} |f_m'|^2 + (1-f_m)^2\,dx = \min\{1,\lambda^2\}\|1-f_m\|^2_{H^1(\mathbb{R}^2)}, \\ K &> \int_{\mathbb{R}^2} \frac{n^2(a_m')^2}{r^2}\,dx = \|a_m\|^2_{C_a}.\end{aligned}$$

Therefore $1-f_m$ is bounded in $H^1(\mathbb{R}^2)$ and a_m is bounded in C_a. Hence we may extract a subsequence, still denoted (f_m, a_m), such that

$$1-f_m \rightharpoonup 1-f_0 \text{ weakly in } H^1(\mathbb{R}^2), \quad a_m \rightharpoonup a_0 \text{ weakly in } C_a,$$

with $(f_0, a_0) \in C_f \bigoplus C_a$. Thus the Rellich-Kondrachov embedding theorem implies strong convergence in $L^p_{\text{loc}}(\mathbb{R}^2)$. From [19] we know that radial H^1 functions in $\mathbb{R}^2$ have good decay properties, like $r^{-1/2}$. Hence it is easy to see that

$$f_m(r) \to f_0(r), \;\; a_m(r) \to a_0(r) \quad \text{a.e.}$$

Using Fatou's Lemma and the weak lower semicontinuity of L^2-norm of f_m' we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^2} |f_0'|^2 + \frac{n^2}{r^2}(1-a_0)^2 f_0^2 + \lambda^2 f_0^2\big(1-f_0^2\big)^2\,dx \\ &\qquad \leq \liminf_{m\to\infty}\int_{\mathbb{R}^2} |f_m'|^2 + \frac{n^2}{r^2}(1-a_m)^2 f_m^2 + \lambda^2 f_m^2\big(1-f_m^2\big)^2\,dx. \qquad (7.5)\end{aligned}$$

It's also easy to see from the Ginzburg-Landau energy form (7.3) that

$$G^r_{gl}(f_0, a_0) \leq \liminf_{m\to\infty} G^r_{gl}(f_m, a_m) \leq m_0 + M.$$

Hence $(f_0, a_0) \in \overline{\mathcal{B}_M}$.

From (i) we know that

$$\frac{(a_m'/f_m)}{r} \in L^2(\mathbb{R}^2).$$

Let $g_m' = a_m'/f_m$. Then since $g_m'/r \in L^2(\mathbb{R}^2)$,

$$\begin{aligned}\int_0^r |g_m'|ds &\leq \Big(\int_0^r s ds\Big)^{1/2}\Big(\int_0^r \frac{1}{r}|g_m'|^2 ds\Big)^{1/2} \\ &\leq \frac{r}{\sqrt{2}}\Big(\int_{\mathbb{R}^2} \frac{1}{r^2}|g_m'|^2 dx\Big)^{1/2} < \frac{2\sqrt{2}K}{\mu^2 n^2} < \infty.\end{aligned}$$

Therefore $g_m' \in L^1_{\text{loc}}(\mathbb{R}^+)$, hence

$$g_m(r) - g_m(0) = \int_0^r g_m'(s)ds.$$

It's also easy to see that $\frac{1}{r}(g_m(r) - g_m(0)) \in L^2_{\text{loc}}(\mathbb{R}^2)$. Let $h_m = g_m(r) - g_m(0)$, then $h_m \in C_a$ and $\|h_m\|_{C_a} < K$. So there is a subsequence, still denoted h_m, such that $h_m \rightharpoonup h_0$ weakly in C_a and $h_m(r) \to h_0(r)$ a.e.

For any $r > 0$, since $a_m \in C_a$, we know that $a_m' \in L^1_{\text{loc}}(\mathbb{R}^+)$. Thus

$$a_m(r) = a_m(l) + \int_l^r a_m'(s)ds,$$

for $0 < l < r$. Plugging in $a_m' = f_m h_m'$ and perfoming integration by parts we obtain

$$a_m(r) = a_m(l) + f_m h_m\Big|_l^r - \int_l^r f_m' h_m \, ds.$$

Letting $m \to \infty$ we get

$$a_m(r) - a_m(l) \to a_0(r) - a_0(l), \qquad f_m h_m\Big|_l^r \to f_0 h_0\Big|_l^r.$$

$$\begin{aligned}
&\Big|\int_l^r f_m' h_m \, ds - \int_l^r f_0' h_0 \, ds\Big| = \Big|\int_l^r f_m'(h_m - h_0) \, ds + \int_l^r (f_m' - f_0') h_0 \, ds\Big| \\
&\le \|h_m - h_0\|_{L^\infty[l,r]} \Big(\int_l^r \frac{1}{s} ds\Big)^{1/2} \Big(\int_{\mathbb{R}^2} |f_m'|^2 \, dx\Big)^{1/2} + \int_l^r (f_m' - f_0') h_0 \, ds \\
&\to 0 \ \text{ as } m \to \infty.
\end{aligned}$$

Therefore

$$a_0(r) = a_0(l) + f_0 h_0\Big|_l^r - \int_l^r f_0' h_0 \, ds,$$

which in turn gives that $a_0' = f_0 h_0'$. Thus

$$\frac{a_m'/f_m}{r} \rightharpoonup \frac{a_0'/f_0}{r}, \text{ weakly in } L^2(\mathbb{R}^2).$$

Hence the weak lower semicontinuity of the L^2-norm implies that

$$\int_{\mathbb{R}^2} \frac{\mu^2}{4} \frac{n^2}{r^2} \Big(\frac{a_0'}{f_0}\Big)^2 \, dx \le \liminf_{m\to\infty} \int_{\mathbb{R}^2} \frac{\mu^2}{4} \frac{n^2}{r^2} \Big(\frac{a_m'}{f_m}\Big)^2 \, dx. \tag{7.6}$$

Combining (7.5) and (7.6) we obtain that

$$G_{csh}^r(f_0, a_0) \le \liminf_{m\to\infty} G_{csh}^r(f_m, a_m). \tag{7.7}$$

Thus we have $G_{csh}^r(f_0, a_0) = \inf_{\mathcal{B}_M} G_{csh}^r(f, a)$.

Next we show that there is some M such that (f_0, a_0) is an interior minimizer. The argument is similar to the one in Theorem 5.1. Let (f_0^k, a_0^k) be minimizers of G_{csh}^r over $\overline{\mathcal{B}_{M+1/k}}$ for $k = 1, 2, \ldots$. If they are not interior minimizers, then

$$G_{csh}^r(f_0^k, a_0^k) = \inf_{\mathcal{B}_{M+1/k}} G_{csh}^r \le G_{csh}^r(f_0^{k-1}, a_0^{k-1}), \ \ G_{gl}^r(f_0^k, a_0^k) = m_0 + M + 1/k. \tag{7.8}$$

Hence $G_{gl}^r(f_0^k, a_0^k)$ is uniformly bounded and then as is discussed before, we have

$$\begin{aligned}
&1 - f_0^k(r) \rightharpoonup 1 - f_0(r) \text{ in } H^1, && a_0^k(r) \rightharpoonup a_0(r) \text{ in } C_a. \\
&f_0^k(r) \to f_0(r), \ \ a_0^k(r) \to a_0(r) \text{ a.e.}, && \frac{(a_0^k)'/f_0^k}{r} \rightharpoonup \frac{a_0'/f_0}{r} \text{ in } L^2.
\end{aligned}$$

Therefore applying lower-semicontinuity of norms, Fatou's lemma and (7.8) we obtain

$$\begin{aligned}
G_{csh}^r(f_0, a_0) &\le \liminf_{k\to\infty} G_{csh}^r(f_0^k, a_0^k) = G_{csh}^r(f_0^1, a_0^1), \\
G_{gl}^r(f_0, a_0) &\le \liminf_{k\to\infty} G_{gl}^r(f_0^k, a_0^k) = m_0 + M.
\end{aligned}$$

Thus (f_0, a_0) is an interior minimizer of $\mathcal{B}_{M+1/k}$ for any k.

The positivity of the infimum is given in the argument before the theorem. $\square$

It is easy to check that the minimizing solution (f_0, a_0) obtained in Theorem 7.3 solves the following equations on $\mathbb{R}^2\backslash\{0\}$

$$\Delta_r f = \frac{n^2}{r^2}(1-a)^2 f - \frac{\mu^2}{4}\frac{n^2}{r^2}\frac{(a')^2}{f^3} + \lambda^2(1-f^2)(1-3f^2)f, \tag{7.9}$$

$$-\frac{\mu^2}{4}\Big(\frac{a'}{rf^2}\Big)' = \frac{f^2(1-a)}{r}, \tag{7.10}$$

where $\Delta_r = \frac{1}{r}\partial_r(r\partial_r)$ is the radial Laplacian.

8. Basic properties of symmetric vortices

In Section 5 we showed the existence of symmetric vortices in absence of the magnetic field, and discovered some properties of the vortices. In this section we will develop some basic properties of the symmetric vortices when $A \not\equiv 0$.

1. Regularity.

Theorem 8.1. *Suppose $(u^{(n)}, A^{(n)})$ is of the form (1.13), (1.14), where $(f^{(n)}(r), a_n(r))$ is obtained from Theorem 7.3. Then $f^{(n)}(r), a_n(r) \in C^\infty(0,\infty)$, hence $(u^{(n)}, A^{(n)})$ is C^2 on $\mathbb{R}^2\backslash\{0\}$.*

Proof. For simplicity, we omit the superscript and subscript n in $(f^{(n)}, a_n)$ in the proof of Theorem 8.1. We know from the previous section that (f, a) satisfy equations (7.9) and (7.10) on $\mathbb{R}^2\backslash\{0\}$.

From Lemma 7.1 we know that $f, a \in C[0,\infty)$, $f(0) = a(0) = 0$. Moreover, since $f, a \to 1$ as $r \to \infty$, we have $f, a \in L^\infty$.

From (7.10) we have for any $l > 0$,

$$\begin{aligned}
\int_{1/l}^{l} \Big|\Big(\frac{a'}{rf^2}\Big)'\Big| dr &= \frac{4}{\mu^2}\int_{1/l}^{l}\frac{f^2|1-a|}{r}dr \\
&\le \frac{4}{\mu^2}\Big(\int_{1/l}^{l}\frac{f^2}{r}dr\Big)^{1/2}\Big(\int_0^l \frac{f^2(1-a)^2}{r^2} r dr\Big)^{1/2} \\
&\le \frac{4}{\mu^2}\|f\|_{L^\infty}\Big(\int_{1/l}^{l}\frac{1}{r}dr\Big)^{1/2}\Big(2G^r_{csh}(f,a)\Big)^{1/2} < \infty.
\end{aligned}$$

Thus $\left(\frac{a'}{rf^2}\right)' \in L^1[1/l, l]$ for any $l > 0$, which implies that $\frac{a'}{rf^2} \in C(0,\infty)$. Hence $a \in C^1(0,\infty)$. Using standard elliptic theory on equation (7.9) we obtain $f \in C^2(0,\infty)$. A standard iterative bootstrap argument shows that $f, a \in C^\infty(0,\infty)$. Hence $f, a \in C^2(\mathbb{R}^2\backslash\{0\})$, which implies that $u^{(n)}, A^{(n)} \in C^2(\mathbb{R}^2\backslash\{0\})$. □

2. Maximum Principle.

Theorem 8.2. *For any $M > 0$, if (f, a) minimizes G^r_{csh} over $\mathcal{B}_M$, then*

(i) $0 < a(r) \le 1$, *for* $r \in (0,\infty)$;
(ii) $a'(r) \ge 0$;
(iii) $f'(r) > 0$;
(iv) $0 < f(r) < 1$, *for* $r \in (0,\infty)$.

Proof. Since (f, a) minimizes G^r_{csh} over $\mathcal{B}_M$, we know that (f, a) also solves equations (7.9) and (7.10) on $\mathbb{R}^2\backslash\{0\}$.

(i) First we use a truncation argument to show that $0 \le a(r) \le 1$. Suppose not. Then the set $D_a = \{r \in (0,\infty) : a(r) < 0 \text{ or } a(r) > 1\}$ is not empty. We define a truncated function $\bar{a}(r)$ by

$$\bar{a}(r) = \begin{cases} 0 & \text{if } a(r) < 0, \\ n & \text{if } a(r) > 1, \\ a(r) & \text{otherwise.} \end{cases} \tag{8.1}$$

Since $a(r) \in C^2(0,\infty)$ and D_a is not empty, we know

$$\int_{\mathbb{R}^2} \frac{n^2}{r^2}(1-\bar{a})^2 f^2 + \frac{\mu^2}{4}\frac{n^2}{r^2}\Big(\frac{\bar{a}'}{f}\Big)^2\,dx < \int_{\mathbb{R}^2} \frac{n^2}{r^2}(1-a)^2 f^2 + \frac{\mu^2}{4}\frac{n^2}{r^2}\Big(\frac{a'}{f}\Big)^2\,dx,$$
$$\int_{\mathbb{R}^2} \frac{n^2}{r^2}(1-\bar{a})^2 f^2 + \frac{n^2}{r^2}(\bar{a}')^2\,dx < \int_{\mathbb{R}^2} \frac{n^2}{r^2}(1-a)^2 f^2 + \frac{n^2}{r^2}(a')^2\,dx.$$

So $(f,\bar{a}) \in \mathcal{B}_M$ and $G^r_{csh}(f,\bar{a}) < G^r_{csh}(f,a) = \inf_{\mathcal{B}_M} G^r_{csh}$, a contradiction. Therefore $0 \le a(r) \le 1$.

We then make use of the second Euler-Lagrange equation (7.10), which can also be written as

$$\frac{a''}{rf^2} + a'\Big(\frac{1}{rf^2}\Big)' = -\frac{4}{\mu^2}\frac{f^2(1-a)}{r} \le 0. \tag{8.2}$$

Hence maximum principle implies that either $a(r) > 0$ on $(0,\infty)$ or $a(r) \equiv 0$.

If $a(r) \equiv 0$ then from the energy we know

$$\int_{\mathbb{R}^2} \frac{n^2}{r^2} f^2\,dx < \infty.$$

From the fact that $(f,a) \in \mathcal{B}_M$ we also have

$$\int_{\mathbb{R}^2} (1-f^2)^2\,dx \le \frac{1}{\lambda^2} G^r_{gl}(f,a) < \infty.$$

Therefore

$$\int_{r\ge 1} \frac{n^2(1-f^2)}{r^2}\,dx \le \Big(\int_{r\ge 1} \frac{n^4}{r^4}\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^2} (1-f^2)^2\Big)^{1/2} < \infty,$$

which implies that

$$\int_{r\ge 1} \frac{n^2}{r^2}\,dx < \infty,$$

a contradiction. Hence $a(r) > 0$ on $(0,\infty)$.

(ii) Suppose $a(r)$ is not nondecreasing. Then there are $r_1 < r_2 \in [0,\infty]$ such that $a(r_1) > a(r_2)$. Now let

$$\bar{a}(r) = \begin{cases} a(r) & \text{for } r \in [\,0, r_1], \\ \max\{a(r), a(r_1)\} & \text{for } r \in [\,r_1,\infty]. \end{cases} \tag{8.3}$$

Then the distributional derivative of $\bar{a}$ is equal to the classical derivative a.e. and $\bar{a} \in C_a$. Since $f = 0$ forces a' to be zero and hence a equals a constant, we know that $f \not\equiv 0$ in $[\,r_1, r_2]$. Therefore

$$\int_{\mathbb{R}^2} \frac{f^2(1-\bar{a})^2}{r^2}\,dx < \int_{\mathbb{R}^2} \frac{f^2(1-a)^2}{r^2}\,dx.$$

By the continuity of f, a and the fact that $|\bar{a}'| \le |a'|$ we know that $(f,\bar{a}) \in \mathcal{B}_M$ and

$$G^r_{csh}(f,\bar{a}) < G^r_{csh}(f,a),$$

which contradicts the minimality of $G^r_{csh}(f,a)$. Hence $a'(r) \ge 0$.

Moreover, when $a(r)$ is between 0 and 1, from (iv) $0 < f(r) < 1$ for $r > 0$, we know that the right-hand-side of (8.2) is negative. Therefore by a maximum principle we get that $a'(r) > 0$ when $0 < a(r) < 1$.

(iii) Since (f, a) minimizes G^r_{csh} over $\mathcal{B}_M$, we have that

$$(G^r_{csh})'(f,a) = 0, \text{ and } (G^r_{csh})''(f,a) \geq 0.$$

An explicit computation gives

$$(G^r_{csh})'(f,a) = \left(\begin{array}{c} -\Delta_r f + \frac{n^2}{r^2}(1-a)^2 f - \frac{\mu^2 n^2}{4r^2}\frac{(a')^2}{f^3} + \lambda^2 f(1-f^2)(1-3f^2) \\ -\frac{\mu^2 n^2}{4r}\left(\frac{a'}{rf^2}\right)' - \frac{n^2}{r^2}(1-a)f^2 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \tag{8.4}$$

$$(G^r_{csh})''(f,a) = \left(\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right) \geq 0, \tag{8.5}$$

where

$$\begin{aligned} G_{11} &= -\Delta_r + \frac{n^2}{r^2}(1-a)^2 + \frac{3\mu^2 n^2}{4r^2}\frac{(a')^2}{f^4} + \lambda^2(15f^4 - 12f^2 + 1), \\ G_{12} &= G_{21} = -\frac{2n^2}{r^2}(1-a)f + \frac{\mu^2 n^2}{2r}\Big(\frac{a'}{rf^3}\Big)' \\ G_{22} &= -\frac{\mu^2 n^2}{4r^2}\Big[\frac{1}{f^2}\partial_r^2 - (\frac{1}{rf^2} + \frac{2}{f^3})\partial_r\Big] + \frac{n^2}{r^2}f^2. \end{aligned}$$

Differentiating equation (8.4) with respect to r and using (8.5) we obtain

$$\left(\begin{array}{c} 0 \\ 0 \end{array} \right) = \frac{d}{dr}(G^r_{csh})'(f,a) = \Big[(G^r_{csh})''(f,a) + Q\Big]\left(\begin{array}{c} f' \\ a' \end{array} \right) + R, \tag{8.6}$$

where

$$Q = \left(\begin{array}{cc} \frac{1}{r^2} & -\frac{\mu^2 n^2}{r}\left(\frac{a'}{rf^3}\right)' \\ 0 & \frac{\mu^2 n^2}{2r^3 f}\partial_r - \frac{3\mu^2 n^2}{4r^4 f^2} - \frac{\mu^2 n^2}{r^3 f^3} \end{array} \right),$$

$$R = \left(\begin{array}{c} -\frac{2n^2}{r^3}(1-a)^2 f - \frac{3\mu^2 n^2}{2r^2}\frac{(a')^2}{f^4} \\ \frac{2n^2}{r^3}(1-a)f^2 \end{array} \right).$$

The first-row equation in (8.6) is

$$\Big(G_{11} + \frac{1}{r^2}\Big)f' + \Big[G_{12} - \frac{\mu^2 n^2}{r}\Big(\frac{a'}{rf^3}\Big)'\Big]a' = \frac{2n^2}{r^3}(1-a)^2 f + \frac{3\mu^2 n^2}{2r^2}\frac{(a')^2}{f^4}. \tag{8.7}$$

On the other hand we have

$$\Big[G_{12} - \frac{\mu^2 n^2}{r}\Big(\frac{a'}{rf^3}\Big)'\Big]a' = \Big[-\frac{2n^2}{r^2}(1-a)f - \frac{\mu^2 n^2}{2r}\Big(\frac{a'}{rf^3}\Big)'\Big]a'.$$

From the second-row equation in (8.4) we know that

$$-\frac{2n^2}{r^2}(1-a)f = \frac{\mu^2 n^2}{2rf}\Big(\frac{a'}{rf^2}\Big)'.$$

Hence the previous expression becomes

$$\Big[G_{12} - \frac{\mu^2 n^2}{r}\Big(\frac{a'}{rf^3}\Big)'\Big]a' = \Big[\frac{\mu^2 n^2}{2rf}\Big(\frac{a'}{rf^2}\Big)' - \frac{\mu^2 n^2}{2r}\Big(\frac{a'}{rf^3}\Big)'\Big]a' = \frac{\mu^2 n^2}{2r^2}\frac{a'}{f^4}f'.$$

Therefore (8.7) becomes

$$\Big(G_{11}+\frac{1}{r^2}+\frac{\mu^2n^2}{2r^2}\frac{a'}{f^4}\Big)f'=\frac{2n^2}{r^3}(1-a)^2f+\frac{3\mu^2n^2}{2r^2}\frac{(a')^2}{f^4}. \tag{8.8}$$

Since $(G^r_{csh})''(f,a)\geq 0$, we know $G_{11}\geq 0$. From part (i), (ii) and the fact that $f\geq 0$, $f\not\equiv 0$, the right hand side of (8.8) is nonnegative and $\frac{\mu^2n^2}{2r^2}\frac{a'}{f^4}\geq 0$. Therefore using the maximum principle (see [16], Theorem B.4) we know that $f'>0$.

(iv) Combining the regularity result and (iii) we conclude that $0<f<1$. $\square$

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