MARTINGALE SOLUTIONS FOR THE THREE-DIMENSIONAL STOCHASTIC NONHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES EQUATIONS DRIVEN BY LÉVY PROCESSES

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Abstract. In this paper, the three-dimensional stochastic nonhomogeneous incompressible Navier-Stokes equations driven by Lévy processes consisting of the Brownian motion, the compensated Poisson random measure and the Poisson random measure are considered in a bounded domain. We obtain the existence of martingale solutions. The construction of the solution is based on the classical Galerkin approximation method, the stopping times, the stochastic compactness method and the Jakubowski-Skorokhod theorem.

1. Introduction

Lévy processes were introduced by Lévy in 1937. They are often applied to the term structure and credit risk areas. They also have many important applications including option pricing and the Black-Scholes formula. For example, in financial mathematics, the classical model for a stock price is a geometric Brownian motion. However, wars, decisions of the federal reserve and other central banks, and other news can cause the stock price to make a sudden shift. To model this, one would like to represent the stock price by a Lévy process which allows for jumps.

Another interesting application of the Lévy processes can be found in the study of the stochastic Navier-Stokes equations. The stochastic Navier-Stokes equations have a long history as a model to understand turbulence in fluid mechanics, structural vibrations in aeronautical applications, and unknown random external forces such as sun heating and industrial pollution in atmospheric dynamics. In real physical situations, the random external forces may exhibit jumps and hence a purely continuous process is not enough to capture the full dynamics. This again motivates the needs for introducing jump processes in the system.

In this work we are concerned with the study of Navier-Stokes equations driven by Lévy processes. Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂D and Ω be a sample space. We consider the following system of stochastic PDEs:

$$\begin{cases} \rho du + [\rho(u\cdot\nabla)u - \nu\Delta u + \nabla p]dt = \rho f(t,u)dt + \rho g(t,u)dW + \int_Z \rho L d\lambda, \\ d\rho + \mathrm{div}(\rho u)dt = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

in $\Omega \times [0,T] \times D$, with the initial data

$$\rho|_{t=0} = \rho_0,\ u|_{t=0} = u_0, \tag{1.2}$$

and the homogeneous boundary condition

$$u|_{\partial D} = 0. \tag{1.3}$$

Here $\rho \geq 0$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $p \in \mathbb{R}$ denote the density, the velocity and the pressure, respectively; the viscosity coefficient ν satisfies $\nu > 0$; $\rho f(t,u)$ is the deterministic external force;

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Z is a measurable metric space; the random external force is characterized by the Lévy processes $\rho g(t,u)dW + \int_Z \rho L d\lambda$, where W is an $\mathbb{R}^d$-valued standard Brownian motion, and

$$L d\lambda = \begin{cases} F\left(u(x,t-),z\right)\tilde{\pi}(dt,dz), & \text{if } |z|_Z < 1, \\ G\left(u(x,t-),z\right)\pi(dt,dz), & \text{if } |z|_Z \geq 1, \end{cases}$$

where F and G are two functions, $\pi(dt,dz)$ is a time homogeneous Poisson measure, $\tilde{\pi}(dt,dz)$ is the compensated Poisson measure associated to π which is defined as $\tilde{\pi}(dt,dz) = \pi(dt,dz) - dt\mu(dz)$, where $\mu(\cdot) = \mathbb{E}\pi(1,\cdot)$ is the intensity measure. Here $\mathbb{E}X = \int_\Omega X dP$ denotes the expectation of the random variable $X(\omega,t), \omega \in \Omega$ for a fixed t.

1.1. **History of the problem.** There have been extensive studies on the nonhomogeneous Navier-Stokes equations. In the deterministic case ($g = L = 0$), Kazhikhov [43] obtained weak solutions for initial density bounded away from zero. Simon [67] proved the global existence of strong solutions in two dimensions. For three-dimensional case, Ladyhzenskaya-Solonnikov [48], Padula [57, 58] and Salvi [63] established the local existence of strong solutions. The uniqueness of strong solutions in $\mathbb{R}^3$ was later proved by Choe-Kim [17].

When g or L does not vanish, the first and second equations in (1.1) are stochastic. For viscous compressible flows, Tornatore [70] obtained the existence and uniqueness of global solutions for the two-dimensional periodic barotropic fluids with an additive noise, i.e., the random external forcing is independent of the fluid velocity u. Feireisl-Maslowski-Novotný in [29] later considered the three-dimensional problem in Sobolev spaces where the noise, under suitable weak formulation, can be regarded as an additive one. They managed to show the existence of strong solutions by using an abstract measurability theorem [3] and proved that the weak solution generates a random variable. When the noise is *multiplicative*, that is, the random external forcing depends on u, the problem becomes more involved. Some recent development on the existence of martingale solutions can be found in [8, 68, 71].

For incompressible nonhomogeneous fluids, the existence of martingale solutions to the equations (1.1)-(1.3) driven by an additive noise was established by Yashima [72] with positive initial density. For general multiplicative noise, Cutland-Enright [20] constructed strong solutions by using the Loeb space techniques in two and three-dimensional bounded domains. For the well-posedness of the homogeneous stochastic incompressible Navier-Stokes equations, see [2, 3, 7, 11, 12, 13, 14, 25, 30, 32, 35, 45, 55, 56, 51, 53, 54, 64, 62, 69] and the references therein. See [5, 15, 18, 19, 22, 23, 26, 27, 31, 33, 34, 36, 37, 38, 39, 44, 50, 53] and the references therein for the studies and results on the incompressible stochastic Euler equations, ergodicity of stochastic partial differential equations, stochastic equations for turbulent flows, stochastic conservation laws, and so on.

1.2. **Main results.** In this paper we consider the existence of martingale solutions to the three-dimensional stochastic nonhomogeneous incompressible Navier-Stokes equations with Lévy processes. Our approach is based on the Galerkin approximation scheme and the stochastic compactness method. We will outline the main idea in the later part of the section.

First, we define the concept of solutions for the problems (1.1)-(1.3) as follows.

Definition 1.1. A martingale solution of (1.1)-(1.3) is a system $((\Omega,\mathcal{F},\mathcal{F}_t,P),W,\pi,\rho,u)$, which satisfies

(1) $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ is a filtered probability space with a filtration $\mathcal{F}_t$, i.e., a set of sub σ-fields of $\mathcal{F}$ with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$,

(2) W is a d-dimensional $\mathcal{F}_t$-standard Brownian motion,

(3) π is a time homogeneous Poisson random measure over $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ with the intensity measure μ,

(4) for almost every t, $\rho(t)$ and $u(t)$ are progressively measurable,

(5) $\rho \in L^\infty(\Omega; L^\infty(0,T;L^\infty(D)))$, $u \in L^2(\Omega;L^\infty(0,T;H)) \cap L^2(\Omega;L^2(0,T;V))$. For all $t \in [0,T]$, any $\varphi \in H^1(D)$ and $\psi \in V$ (see (2.1), (2.2) for definition of H and V), the following holds

P-a.s.

$$\int_D \rho(t)\varphi dx - \int_D \rho_0\varphi dx = \int_0^t \int_D \rho u \cdot \nabla\varphi dxds, \tag{1.4}$$

and

$$\begin{aligned}
&\int_D (\rho u)(t)\psi dx - \int_0^t \int_D (\rho uu\nabla\psi - \nu\nabla u \cdot \nabla\psi)\, dxds - \int_D \rho_0 u_0\psi dx \\
&= \int_0^t \int_D \rho f(s,u)\psi dxds + \int_0^t \int_D \rho g(s,u)\psi dxdW \\
&\quad + \int_0^t \int_{|z|_Z<1} \int_D \rho F(u(x,s-),z)\, \psi dx\tilde{\pi}(ds,dz) \\
&\quad + \int_0^t \int_{|z|_Z\geq 1} \int_D \rho G(u(x,s-),z)\, \psi dx\pi(ds,dz),
\end{aligned} \tag{1.5}$$

and

$$\rho|_{t=0} = \rho_0, \int_D \rho(0)u(0)\varphi dx = \int_D \rho_0 u_0\varphi dx. \tag{1.6}$$

In the above, all stochastic integrals are defined in the sense of Itô, see [1, 21, 28, 46, 42, 60, 61].

Throughout this paper, we assume that the Brownian motion W is independent of the compensated Poisson measure $\tilde{\pi}(dt,dz)$. The intensity measure μ on Z satisfies the conditions $\mu(\{0\}) = 0$, $\int_Z (1 \wedge |z|^2)\mu(dz) < \infty$ and $\int_{|z|_Z\geq 1} |z|^p\mu(dz) < \infty, \forall p \geq 1$. We also assume that $\{\mathcal{F}_t\}$ is a right continuous filtration over the probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{F}_0$ contains all P-negligible subsets of Ω.

Before we state our main theorem, we make the following assumptions on the external forces.
Assumption (A). Assume that $f : (0,T) \times H \to H$ and $g : (0,T) \times H \to H^{\times d}$ are continuous and nonlinear mappings, which satisfy the following condition: there exists a positive constant C such that

$$\|f(t,u) - f(t,v)\|_{L^2(D)} \leq C\|u - v\|_{L^2(D)}, \ \|g(t,u) - g(t,v)\|_{L^2(D)} \leq C\|u - v\|_{L^2(D)},$$

$$\|f(t,u)\|_{L^2(D)} \leq C\left(1 + \|u\|_{L^2(D)}\right), \ \|g(t,u)\|_{L^2(D)} \leq C\left(1 + \|u\|_{L^2(D)}\right),$$

where $H^{\times d}$ is the product of d copies of the space H which is defined in (2.1).
Assumption (B). For all $t \in [0,T]$, there exists a positive constant C such that

$$\begin{aligned}
&\int_{|z|_Z<1} \|F(u,z) - F(v,z)\|^2_{L^2(D)}\, \mu(dz) + \int_{|z|_Z\geq 1} \|G(u,z) - G(v,z)\|^2_{L^2(D)}\, \mu(dz) \\
&\quad \leq C\|u - v\|^2_{L^2(D)}.
\end{aligned} \tag{1.7}$$

For each $p \geq 2$ and all $t \in [0,T]$, there exists a positive constant C such that

$$\int_{|z|_Z<1} \|F(u,z)\|^p_{L^2(D)}\, \mu(dz) + \int_{|z|_Z\geq 1} \|G(u,z)\|^p_{L^2(D)}\, \mu(dz) \leq C\left(1 + \|u\|^p_{L^2(D)}\right). \tag{1.8}$$

Our main results are the following.

Theorem 1.1. *Let the assumptions (A) and (B) be satisfied and assume that $u_0 \in H$, $\rho_0 \in L^\infty(D)$ satisfying $0 < m \leq \rho_0 \leq M$. Then there exists a martingale solution of problems* (1.1)-(1.3) *in the sense of Definition 1.1.*

1.3. **Outline of ideas.** Theorem 1.1 will be proved through the following steps. First we use the Faedo-Galerkin method to construct the approximate solutions to the problem (1.1)-(1.3). More precisely, on the probability space $(\Omega, \mathcal{F}, P)$ with a given d-dimensional Brownian motion W and Poisson random measure π, for the finite-dimensional approximate system we use the Picard iteration to obtain a local solution $(W_n, \pi_n, \rho^n, u^n)$ in a short time interval $[0, T_n]$. Here, different from the deterministic situation, the velocity u in general exhibits jump discontinuity (in time), and hence one cannot apply the standard method of characteristics to solve the transport equation for ρ. To overcome this difficulty, we adapt the result of DiPerna-Lions [24] on transport theory for less regular vector fields u to obtain a solution $\rho \in L^\infty$. To obtain a uniform time interval $[0, T]$ of existence for all n, we need to derive the energy estimates. This can be done by applying the stopping times and the Burkholder-Davis-Gundy inequality.

The second step is to take a limit as $n \to \infty$ and prove the existence of martingale solutions. From energy estimates, the approximate solutions $(W_n, \pi_n, \rho^n, u^n)$ may converge on $[0, T]$. However the convergence is too weak to guarantee that the limit is a solution on $[0, T]$. In the two-dimensional case, it can be shown by using certain monotonicity principle that the nonlinear terms converge to the right limit and hence a global strong solution can be obtained [51]. But when the space dimension is three the monotonicity does not hold and to the best of our knowledge there is no result on the global strong solutions. This is why we pursue instead the martingale solutions. As is explained, the main issue is the convergence of the nonlinear terms.

To this end, we relax the restriction on the probability space and aim to prove a tightness result of the random variables $(W_n, \pi_n, \rho^n, u^n)$. This can be obtained by applying the Arzsela-Ascoli's Theorem combined with the Aubin-Simon Lemma [66]. Moreover in order to analyze the nonlinear terms, we prove the tightness of $\rho^n u^n$ as well. Then from the Jakubowski-Skorokhod Theorem [40] there exist a probability space $(\mathring{\Omega}, \mathring{\mathcal{F}}, \mathring{P})$ and random variables $(\mathring{W}_{n_j}, \mathring{\pi}_{n_j}, \mathring{\rho}^{n_j}, \mathring{u}^{n_j}, \mathring{\rho}^{n_j}\mathring{u}^{n_j}) \to (W, \pi, \rho, u, h)$, $\mathring{P}-a.s.$, with the property that the probability distribution of $(\mathring{W}_{n_j}, \mathring{\pi}_{n_j}, \mathring{\rho}^{n_j}, \mathring{u}^{n_j}, \mathring{\rho}^{n_j}\mathring{u}^{n_j})$ is the same as that of $(W_n, \pi_n, \rho^n, u^n, \rho^n u^n)$. By using a cut-off function we can also show that the random variables $(\mathring{W}_{n_j}, \mathring{\pi}_{n_j}, \mathring{\rho}^{n_j}, \mathring{u}^{n_j})$ satisfy the approximate equations in $(\mathring{\Omega}, \mathring{\mathcal{F}}, \mathring{P})$. When passing to a limit as $n \to \infty$, the usual method is to show that the limit process of the stochastic integral is a martingale, and to identify its quadratic variation. Then apply the representation theorem for martingales or the revised representation theorem (see [40]) to prove that it solves the equations. But here instead, we can prove that W is a Brownian motion and π is a time homogeneous Poisson random measure. Then in view of the uniform integrability criterion, Vitali's convergence theorem and mollification techniques, together with the almost sure convergence on $(\mathring{\Omega}, \mathring{\mathcal{F}}, \mathring{P})$, we can obtain that (W, π, ρ, u) satisfies the equations (1.1)-(1.3) by passing to the limit directly. Therefore it is a martingale solution of (1.1)-(1.3) in the sense of Definition 1.1.

The rest of the paper is organized as follows. We recall some analytic tools in Sobolev spaces and some basic theory of stochastic analysis in Section 2. In Section 3, we construct the solutions to an approximate scheme by the Faedo-Galerkin method. In Section 4, we prove the tightness property of the approximate solutions $(W_n, \pi_n, \rho^n, u^n)$ and then pass to the limit as $n \to \infty$.

Notation. Throughout the paper we drop the parameter $\omega \in \Omega$. Moreover, we use C to denote a generic constant which may vary in different estimates.

2. Preliminaries

Let $H^1(D)$ denote the Sobolev space of all $u \in L^2(D)$ for which there exist weak derivatives $\frac{\partial u}{\partial x_i} \in L^2(D), i = 1, 2, 3$. Let $C_c^\infty(D)$ denote the space of all $\mathbb{R}^3$-valued functions of class C^∞ with compact supports contained in D and define

$$\mathcal{V} := \{u \in C_c^\infty(D) : \text{div} u = 0\},$$

$$H := \text{the closure of } \ \mathcal{V} \ \text{ in } \ L^2(D), \tag{2.1}$$

$$V := \text{the closure of } \ \mathcal{V} \ \text{ in } \ H^1(D). \tag{2.2}$$

In the space H, we consider the scalar product and the norm inherited from $L^2(D)$ and denote them by $\langle\cdot,\cdot\rangle_H$ and $|\cdot|_H$ respectively, i.e.

$$\langle u,v\rangle_H = \langle u,v\rangle_{L^2(D)}, \qquad |u|_H = \|u\|_{L^2(D)}, \qquad u,v\in H.$$

In the space V we consider the scalar product inherited from $H^1(D)$, that is

$$\langle u,v\rangle_V = \langle u,v\rangle_H + \langle \nabla u,\nabla v\rangle_{L^2(D)}.$$

Let p_* denote the Sobolev conjugate in $\mathbb{R}^3$ which is defined as

$$p_* := \begin{cases} \dfrac{3p}{3-p}, & \text{if } 1\le p<3,\\ \text{any finite non-negative real number}, & \text{if } p=3,\\ \infty, & \text{if } p>3. \end{cases}$$

We first recall some properties of products in Sobolev spaces $W^{1,p}(D)$ with $p\ge 1$.

Lemma 2.1 ([67])**.** *For $1\le p\le q\le\infty$, $f\in W^{1,p}(D)$ and $g\in W^{1,q}(D)$, if $r\ge 1$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q_*}$, then $fg\in W^{1,r}(D)$ and*

$$\|fg\|_{W^{1,r}(D)} \le \|f\|_{W^{1,p}(D)}\|g\|_{W^{1,q}(D)}.$$

For $h\in W^{-1,q}(D)$, if $\frac{1}{p}+\frac{1}{q}\le 1$ and $\frac{1}{r}=\frac{1}{p_*}+\frac{1}{q}$, then $fh\in W^{-1,r}(D)$ and

$$\|fh\|_{W^{-1,r}(D)} \le \|f\|_{W^{1,p}(D)}\|h\|_{W^{-1,q}(D)}.$$

Lemma 2.2 ([49])**.** *Let $(g_k)_{k=1,2,\ldots}$ and g be functions in $L^q(0,T;L^q(D))$ for $q\in(1,\infty)$ such that $\|g_k\|_{L^q(0,T;L^q(D))}\le C$ for any k and $g_k\to g$ almost everywhere in $Q_T := D\times[0,T]$ as $k\to\infty$. Then g_k converges weakly to g in $L^q(0,T;L^q(D))$.*

For a probability space $(\Omega,\mathcal{F},P)$ and a Banach space X, denote by $L^p(\Omega;L^q(0,T;X))$ $(1\le p<\infty, 1\le q\le\infty)$ the space of random functions defined on Ω with value in $L^q(0,T;X)$, endowed with the norm:

$$\|u\|_{L^p(\Omega;L^q(0,T;X))} = \left(\mathbb{E}\,\|u\|^p_{L^q(0,T;X)}\right)^{\frac{1}{p}}.$$

If $p=\infty$, we write

$$\|u\|_{L^\infty(\Omega;L^q(0,T;X))} := \inf\left\{C; P[\|u\|_{L^q(0,T;X)}>C]=0\right\}.$$

Remark 2.1. Note that the result of Lemma 2.2 also holds for the space $L^q(\Omega;L^q(0,T;D))$ in $\Omega\times Q_T$.

We now list a few preliminary results of stochastic analysis and useful tools for the sake of convenience and completeness. For details, we refer the readers to [1, 21, 28, 46, 42, 60, 61] and the references therein. In particular, we will introduce the definitions of time homogenous Poisson random measure, Lévy process, stopping time, Itô's formula and the BDG inequality and so on.

Definition 2.1. A *filtration* on the parameter set $\mathbb{T}$ is an increasing family $\{\mathcal{F}_t : t\in\mathbb{T}\}$ of σ-algebra. A stochastic process X_t, $t\in\mathbb{T}$ is said to be adapted to $\{\mathcal{F}_t : t\in\mathbb{T}\}$ if for each t, the random variable X_t is $\mathcal{F}_t$-measurable.

Denote $\bar{\mathbb{N}} := \mathbb{N}\cup\{\infty\}$, $\mathbb{R}_+ := [0,\infty)$. Let $(Z,\mathcal{Z})$ be a measurable space. Then by $M(Z)$ we denote the set of all real valued measures on $(Z,\mathcal{Z})$, and $\mathcal{M}(Z)$ denotes the σ-field on $M(Z)$ generated by functions $i_B:\mu\mapsto\mu(B)\in\mathbb{R}$ for $\mu\in M(Z), B\in\mathcal{Z}$. Next, we denote the set of all non-negative measures on Z by $M_+(Z)$, and $\mathcal{M}_+(Z)$ denotes the σ-field on $M_+(Z)$ generated by functions $i_B : M_+(Z)\ni\mu\mapsto\mu(B)\in\mathbb{R}_+$, $B\in\mathcal{Z}$. Finally, by $M_{\bar{\mathbb{N}}}(Z)$ we denote the family of all $\bar{\mathbb{N}}$-valued measures on $(Z,\mathcal{Z})$, and $\mathcal{M}_{\bar{\mathbb{N}}}(Z)$ denotes the σ-field on $M_{\bar{\mathbb{N}}}(Z)$ generated by functions $i_B : M_{\bar{\mathbb{N}}}\ni\mu\mapsto\mu(B)\in\bar{\mathbb{N}}, B\in\mathcal{Z}$.

Definition 2.2. Let $(Z,\mathcal{Z})$ be a measurable space and $\mu\in\mathcal{M}_+(Z)$. A measurable function $\pi:(\Omega,\mathcal{F})\to(M_{\bar{\mathbb{N}}}(Z\times\mathbb{R}_+),\mathcal{M}_{\bar{\mathbb{N}}}(Z\times\mathbb{R}_+))$ is called a *time homogenous Poisson random measure* on $(Z,\mathcal{Z})$ over $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ if and only if the following conditions are satisfied

(1) for each $B\in\mathcal{Z}\otimes\mathcal{B}(\mathbb{R}_+)$, $\pi(B):=i_B\circ\pi:\Omega\to\bar{\mathbb{N}}$ is a Poisson random variable with parameter $\mathbb{E}\pi(B)$ (If $\mathbb{E}\pi(B)=\infty$, then $\pi(B)=\infty$);

(2) π is independently scattered, that is, if the sets $B_j\in\mathcal{Z}\otimes\mathcal{B}(\mathbb{R}_+), j=1,2,\dots,n$ are pair-wise disjoint, then the random variables $\pi(B_j), j=1,2,\dots,n$ are pair-wise independent;

(3) for all $B\in\mathcal{Z}$ and $I\in\mathcal{B}(\mathbb{R}_+)$, $\mathbb{E}[\pi(B\times I)]=\lambda(I)\mu(B)$, where λ is Lebesgue measure;

(4) for each $U\in\mathcal{Z}$, the $\bar{\mathbb{N}}$-valued process $(N(t,U))_{t\geq0}$ defined by $N(t,U):=\pi(U\times(0,t]), t\geq 0$ is $\mathcal{F}_t$-adapted and its increments are independent of the past, i.e. if $t>s\geq 0$, then $N(t,U)-N(s,U)=\pi(U\times(s,t])$ is independent of $\mathcal{F}_s$.

Now we turn to the definition of a Lévy process.

Definition 2.3. Let $\mathbb{B}$ be a Banach space. A stochastic process $L=\{L(t):t\geq 0\}$ over $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ is called an $\mathbb{B}$-valued *Lévy process* if the following conditions are satisfied.

(1) $L(t)$ is $\mathcal{F}_t$-measurable for any $t\geq 0$;

(2) the random variable $L(t)-L(s)$ is independent of $\mathcal{F}_s$ for any $0\leq s<t$;

(3) $L(0)=0$ a.s.;

(4) For all $0\leq s<t$, the law of $L(t+s)-L(s)$ does not depend on s;

(5) L is stochastically continuous;

(6) the trajectories of L are càdlàg in $\mathbb{B}$ P-a.s., i.e. which are right-continuous with left limits.

Note that we can construct a corresponding Poisson random measure from a Lévy process. For example, given a $\mathbb{B}$-valued Lévy process over $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, one can construct an integer-valued random measure in the following way: for each $(B,I)\in\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}_+)$, define

$$\pi_L(B,I):=\sharp\{t\in I\mid \Delta_t L\in B\}\in\bar{\mathbb{N}}.$$

where $\Delta_tL(t):=L(t)-L(t-)=L(t)-\lim_{s\uparrow t}L(s), t>0$ and $\Delta_0L:=0$. If $\mathbb{B}=\mathbb{R}^d$, then π_L is a time homogeneous Poisson random measure, for details see [65, Chapter 4, Theorem 19.2]. Conversely, given a Poisson random measure, we can also construct a corresponding Lévy process.

Definition 2.4. A random variable $\tau(\omega)$ with values in the parameter set $\mathbb{T}$ is a *stopping time* of the filtration $\mathcal{F}_t$ if $\{\omega:\tau(\omega)\leq t\}\in\mathcal{F}_t$ for each $t\in\mathbb{T}$.

Let us now recall the Itô formula for general Lévy-type stochastic integrals, see [1, 60, 61]. We define $\mathcal{P}_2(T,\mathbb{B})$ to be the set of all equivalence classes of mappings $f:[0,T]\times\mathbb{B}\times\Omega\to\mathbb{R}$ which coincide almost everywhere with respect to $\varrho\times P$ and satisfy the following conditions:

(1) f is predictable;

(2) $P\left(\int_0^T\int_{\mathbb{B}}|f(t,x)|^2\varrho(dt,dx)<\infty\right)=1$.

Here $\varrho(t,A)=m((0,t]\times A)$, where m is standard Lebesgue measure. With this we are ready to give the Itô formula [1, Page 251, Theorem 4.4.7] for general Lévy-type stochastic integrals. Let X be the following process

$$dX(t)=G(t)dt+F(t)dW(t)+\int_{|x|<1}H(t,x)\tilde{\pi}(dt,dx)+\int_{|x|\geq1}K(t,x)\pi(dt,dx), \tag{2.3}$$

where for each $t\geq 0$, $|G|^{\frac12},F\in\mathcal{P}_2(T):=\mathcal{P}_2(T,\{0\})$ and $H\in\mathcal{P}_2(T,\mathbb{B})$. Furthermore, we take $\mathbb{B}=\{x\in\mathbb{R}^d:0<|x|<1\}$ and K to be predictable (see [61, Page 7]). Denote

$$dX_c(t)=G(t)dt+F(t)dW(t),$$

and

$$dX_d(t)=\int_{|x|<1}H(t,x)\tilde{\pi}(dt,dx)+\int_{|x|\geq1}K(t,x)\pi(dt,dx),$$

so that for each $t \geq 0$, we have

$$X(t) = X(0) + X_c(t) + X_d(t).$$

Assume that for all $t > 0$, $\sup_{0\leq s\leq t,\ 0<|x|<1} H(s,x) < \infty$ a.s., then one has

Lemma 2.3 (Itô's formula,[1]). *If X is a Lévy-type stochastic integral of the form* (2.3)*, then for each $\Phi \in C^2(\mathbb{R}^n)$, $t \geq 0$, with probability 1 we have*

$$\begin{aligned}\Phi(X(t)) - \Phi(X(0)) &= \int_0^t \partial_i \Phi(X(s-)) dX_c^i(s) + \frac{1}{2}\int_0^t \partial_i\partial_j \Phi(X(s-)) d[X_c^i, X_c^j](s) \\ &+ \int_0^t \int_{|x|\geq 1} [\Phi(X(s-) + K(s,x)) - \Phi(X(s-))]\, \pi(ds,dx) \\ &+ \int_0^t \int_{|x|<1} [\Phi(X(s-) + H(s,x)) - \Phi(X(s-))]\, \tilde{\pi}(ds,dx) \\ &+ \int_0^t \int_{|x|<1} \left[\Phi(X(s-) + H(s,x)) - \Phi(X(s-)) - H^i(s,x)\partial_i \Phi(X(s-))\right] \mu(dx)ds\end{aligned} \tag{2.4}$$

We now recall the following so-called BDG inequality in stochastic analysis, see [60, Page 37, Theorem 3.50].

Lemma 2.4 (Burkholder-Davis-Gundy inequality). *Let $T > 0$, for every fixed $p \geq 1$, there is a constant $C_p \in (0,\infty)$ such that for every real-valued square integrable càadlàg martingale M_t with $M_0 = 0$, and for every $T \geq 0$,*

$$C_p^{-1}\mathbb{E}\left(\langle M\rangle_T^{\frac{p}{2}}\right) \leq \mathbb{E}\left(\max_{0\leq t\leq T} |M_t|^p\right) \leq C_p \mathbb{E}\left(\langle M\rangle_T^{\frac{p}{2}}\right),$$

where $\langle M\rangle_t$ is the quadratic variation of M_t and the constant C_p does not depend on the choice of M_t.

Definition 2.5. Let $\mathbb{B}$ be a separable Banach space and let $\mathcal{B}(\mathbb{B})$ be its Borel sets. A family of probability measures $\mathbb{P}$ on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ is *tight* if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathbb{B}$ such that $\Pi(K_\varepsilon) \geq 1 - \varepsilon$ for all $\Pi \in \mathbb{P}$. A sequence of measures $\{\Pi_n\}$ on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ is weakly convergent to a measure Π if for all continuous and bounded functions h on $\mathbb{B}$

$$\lim_{n\to\infty} \int_{\mathbb{B}} h(x)\Pi_n(dx) = \int_{\mathbb{B}} h(x)\Pi(dx).$$

Lemma 2.5 (Jakubowski-Skorokhod Theorem [40]). *Let $\mathcal{X}$ be a topological space such that there exists a sequence $\{h_m\}$ of continuous functions $h_m : \mathcal{X} \to \mathbb{R}$ that separate points of $\mathcal{X}$. Denote by $\mathcal{S}$ the σ-algebra generated by the maps $\{h_m\}$. Then*

(1) every compact subset of $\mathcal{X}$ is metrizable.

(2) every Borel subset of a σ-compact set in $\mathcal{X}$ belongs to $\mathcal{S}$.

(3) every probability measure supported by a σ-compact set in $\mathcal{X}$ has a unique Radon extension to the Borel σ-algebra on $\mathcal{X}$.

(4) if $\{\Pi_m\}$ is a tight sequence of probability measures on $(\mathcal{X}, \mathcal{S})$, there exist a subsequence $\{m_k\}$, and a probability space $(\Omega, \mathcal{F}, P)$ with $\mathcal{X}$-valued Borel measurable random variables X_k and X such that, Π_{m_k} is the distribution of X_k, and $X_k \to X$ a.s. on Ω. Moreover, the law of X is a Radon measure.

3. The Galerkin approximation and a priori estimates

3.1. The Galerkin approximation.

We will obtain the weak solution of the equations (1.1)-(1.3) via the Galerkin approximation: first we construct solutions of certain finite-dimensional approximations to (1.1)-(1.3) and then pass to the limits.

On the probability space $(\Omega, \mathcal{F}, P)$ with a given d-dimensional Brownian motion W and Poisson random measure π. In order to solve (1.1)-(1.3), we first consider a suitable orthogonal system formed by a family of smooth functions $\{w_n\}$ vanishing on ∂D. One can take the eigenfunctions of the Dirichlet problem for the Laplacian operator:

$$-\Delta w_n = \lambda_n w_n \quad \text{on} \quad D, \quad w_n|_{\partial D} = 0.$$

Now, we consider a sequence of finite-dimensional spaces

$$X_n = \operatorname{span}\{w_j\}_{j=1}^n, \ n = 1, 2, \ldots.$$

For each $n \in \mathbb{N}$, we will look for the sequences (ρ^n, u^n) satisfying the integral equation:

$$\begin{aligned}
&\int_D \rho^n u^n(t)\psi dx - \int_D \rho_0^n u_0^n \psi dx - \int_0^t \int_D [\nu \Delta u^n - \rho^n u^n \nabla u^n)] \psi dx ds \\
&= \int_0^t \int_D \rho^n f(s, u^n)\psi dx ds + \int_0^t \int_D \rho^n g(s, u^n)\psi dx dW \\
&\quad + \int_0^t \int_{|z|_Z<1} \int_D \rho^n F\left(u^n(x, s-), z\right) \psi dx \tilde{\pi}(ds, dz) \\
&\quad + \int_0^t \int_{|z|_Z\geq 1} \int_D \rho^n G\left(u^n(x, s-), z\right) \psi dx \pi(ds, dz),
\end{aligned} \tag{3.1}$$

for all $t \in [0, T]$ and any function $\psi \in X_n$, together with

$$(\rho^n)_t + (u^n \cdot \nabla)\rho^n = 0 \quad \text{in} \quad Q_T = D \times [0, T], \tag{3.2}$$

$$u^n|_{t=0} = u_0^n, \ \rho^n|_{t=0} = \rho_0^n \quad \text{in} \quad D. \tag{3.3}$$

Here we assume that

$$u_0^n \in X^n, \ u_0^n \to u_0 \quad \text{in} \quad L^2(D), \tag{3.4}$$

$$0 < m \leq \rho_0 \leq M, \ \rho_0^n \to \rho_0 \quad \text{in} \quad L^\infty(D) \quad \text{weakly star}. \tag{3.5}$$

We look for a function u^n in the following form

$$u^n = \sum_{k=1}^n \varphi_k^n(t) w_k(x). \tag{3.6}$$

It follows from [24, Proposition II.1] and the regularity of u^n that equation (3.2) with (3.3) admits a solution $\rho^n \in L^\infty(0, T; L^\infty(D))$ for any given u^n of the form (3.6). Specifically, there exists a solution map S such that $\rho^n = S(u^n)$. Similarly, from

$$\left(1/\rho^n\right)_t + u^n \cdot \nabla\left(1/\rho^n\right) = 0,$$

we know that $1/\rho^n \in L^\infty(0, T; L^\infty(D))$. Then ρ^n has lower and upper bound, that is

$$0 < \frac{1}{C} \leq \rho^n \leq C. \tag{3.7}$$

Here C is independent of n (only depends on ρ_0).

Next, we show the existence of a solution $u^n \in \mathbb{D}([0, T]; X_n)$ to (3.1). Here $\mathbb{D}([0, T]; X_n)$ is the space of all càdlàg functions $f : [0, T] \to X_n$. We equip $\mathbb{D}([0, T]; X_n)$ with the Skorokhod topology (see [52]).

Note that here $\rho^n = S(u^n)$. Choosing $\psi = w_1, w_2, \dots, w_n$ in (3.1), the coefficients φ_k^n satisfy the following stochastic ordinary differential equations:

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\int_D \rho^n w_k w_\ell\right) d\varphi_k^n(t) + \sum_{j,k=1}^{n}\int_D \rho^n (w_j\varphi_j^n\cdot\nabla)w_k\varphi_k^n w_\ell dxdt \\
&\quad - \int_D \rho^n f(t, \sum_{j=1}^{n} w_j\varphi_j^n) w_\ell dxdt + \nu\int_D \sum_{j=1}^{n}\varphi_j^n(t)\nabla w_j\nabla w_\ell dxdt \\
&= \int_D \rho^n g(t, \sum_{j=1}^{n} w_j\varphi_j^n(t)) w_\ell dxdW + \int_D\int_{|z|_Z<1}\rho^n F\left(\sum_{j=1}^{n} w_j\varphi_j^n(t), z\right) w_\ell\tilde{\pi}(dt,dz)dx \\
&\quad + \int_D\int_{|z|_Z\geq 1}\rho^n G\left(\sum_{j=1}^{n} w_j\varphi_j^n(t), z\right) w_\ell\pi(dt,dz)dx.
\end{aligned} \tag{3.8}$$

From (3.7) we know that the matrix $\left(\int_D \rho^n w_k w_\ell dx\right)$ is nondegenerate, and hence (3.8) can be reformulated as

$$\begin{aligned}
d\varphi_\ell^n + \tilde{F}_\ell(t,\varphi_1^n,\dots,\varphi_n^n)dt &= \tilde{G}_\ell^n(t,\varphi_1^n,\dots,\varphi_n^n)dW + \int_{|z|_Z\geq 1}\tilde{\Psi}_\ell(\varphi_1^n,\dots,\varphi_n^n,z)\pi(dt,dz) \\
&\quad + \int_{|z|_Z<1}\tilde{\Phi}_\ell(\varphi_1^n,\dots,\varphi_n^n,z)\tilde{\pi}(dt,dz),
\end{aligned} \tag{3.9}$$

with the initial data $\varphi_\ell^n(0) = \varphi_{\ell,0}^n$, where $\varphi_{\ell,0}^n$ are the coefficients of $u_0^n = \sum_{k=1}^n \varphi_{k,0}^n w_k$. In view of the assumptions (A) and (B), $\tilde{F}, \tilde{G}, \int_{|z|_Z\geq 1}\tilde{\Psi}\pi(dt,dz)$ and $\int_{|z|_Z<1}\tilde{\Phi}\tilde{\pi}(dt,dz)$ satisfy the Lipschitz and growth conditions. According to the existence theory [1, pp.367, Theorem 6.2.3] for the stochastic ordinary differential equations with jumps, we can apply a standard fixed point argument to show that there exist a time $T_n > 0$ and a function $\varphi_n = (\varphi_1^n, \dots, \varphi_n^n)$ satisfying equation (3.9) and the initial data for a.e. $t \in [0, T_n]$. This way u^n defined in (3.6) solves (3.8) for a.e. $t \in [0, T_n]$. Therefore we obtain a local solution (ρ^n, u^n) of the system (3.1)-(3.3).

Next we want to show that we can find a uniform time interval of existence for all n. This will follow from the a priori estimates established in the next subsection.

3.2. **A priori estimates.**

Now, we want to get the needed a priori estimates. To this end, taking $\psi = w_k$ in (3.1), multiplying the result by $\varphi_k^n(t)$ and then summing over $k = 1, 2, \dots, n$, we have

$$\begin{aligned}
\int_D (u^n\rho^n du^n(t))dx + &\int_D \rho^n u^n(u^n\cdot\nabla)u^n dxdt + \nu\int_D \nabla u^n\cdot\nabla u^n dxdt \\
&= \int_D \rho^n f(t,u^n)u^n dxdt + \int_D \rho^n g(t,u^n)u^n dxdW \\
&\quad + \int_D\int_{|z|_Z<1}\rho^n F\left(u^n(x,t-),z\right)u^n\tilde{\pi}(dt,dz)dx \\
&\quad + \int_D\int_{|z|_Z>1}\rho^n G\left(u^n(x,t-),z\right)u^n\pi(dt,dz)dx.
\end{aligned} \tag{3.10}$$

First, we introduce the following stopping times:

$$\tau_N = \begin{cases} \inf\{t>0 : \|\sqrt{\rho^n}u^n(t)\|_{L^2(D)} \geq N\}, & \text{if } \{\omega\in\Omega : \|\sqrt{\rho^n}u^n(t)\|_{L^2(D)} \geq N\} \neq \emptyset, \\ T, & \text{if } \{\omega\in\Omega : \|\sqrt{\rho^n}u^n(t)\|_{L^2(D)} \geq N\} = \emptyset. \end{cases} \tag{3.11}$$

Define the function $\Phi(\rho,q)=\int_D \frac{|q|^2}{\rho}dx$. Note that

$$\nabla_q\Phi(\rho,q)=\int_D \frac{2q}{\rho}dx,\quad \nabla_q^2\Phi(\rho,q)=\int_D \frac{2}{\rho}\mathbb{I}\,dx,\quad \partial_\rho\Phi(\rho,q)=-\int_D \frac{|q|^2}{\rho^2}dx,$$

where $\mathbb{I}$ is the identity matrix. Applying Itô's formula in Lemma 2.3 to the above function Φ with $(\rho,q)=(\rho^n,\rho^n u^n)$, from the first equation in (1.1), one deduces that

$$\begin{aligned}
d\int_D |\sqrt{\rho^n}u^n|^2dx=&\int_D u^nu^n\frac{\partial\rho^n(s)}{\partial s}dxds+2\nu\int_D u^n\Delta u^ndxds-2\int_D\rho^nu^n(u^n\cdot\nabla)u^ndxds\\
&+2\int_D\rho^nu^n\left[f(s,u^n)ds+g(s,u^n)dW\right]dx+\int_D|\sqrt{\rho^n}g(s,u^n)|^2dxds\\
&+\int_D\int_{|z|_Z<1}\left[2\rho^nu^nF\left(u^n(x,s-),z\right)+\rho^nF^2\left(u^n(x,s-),z\right)\right]\tilde{\pi}(ds,dz)dx\\
&+\int_D\int_{|z|_Z\geq1}\left[2\rho^nu^nG\left(u^n(x,s-),z\right)+\rho^nG^2\left(u^n(x,s-),z\right)\right]\tilde{\pi}(ds,dz)dx\\
&+\int_D\int_{|z|_Z<1}|\sqrt{\rho^n}F\left(u^n(x,s-),z\right)|^2\mu(dz)dx\\
&+\int_D\int_{|z|_Z\geq1}|\sqrt{\rho^n}G\left(u^n(x,s-),z\right)|^2\mu(dz)dx\\
&+2\int_D\int_{|z|_Z\geq1}\rho^nu^nG\left(u^n(x,s-),z\right)\mu(dz)dx,
\end{aligned}\tag{3.12}$$

where $s\in[0,t\wedge\tau_N]$, $t\in[0,T_n]$, $t\wedge\tau_N:=\min\{t,\tau_N\}$. From the second and third equations in (1.1), we can infer that

$$\begin{aligned}
0=\int_D\operatorname{div}(u^nu^n\rho^nu^n)dx&=\int_D[u^nu^n\operatorname{div}(\rho^nu^n)+\rho^nu^n\nabla(u^nu^n)]dx\\
&=\int_D[u^nu^n(u^n\cdot\nabla)\rho^n+2\rho^nu^n(u^n\cdot\nabla)u^n]dx,
\end{aligned}\tag{3.13}$$

where the first equality is due to the condition that u^n vanishes on $(0,T)\times\partial D$. It follows from the second equation in (1.1) that

$$\int_D u^nu^n\frac{\partial\rho^n(s)}{\partial s}dx=-\int_D u^nu^n(u^n\cdot\nabla)\rho^ndx=2\int_D\rho^nu^n(u^n\cdot\nabla)u^ndx. \tag{3.14}$$

Substituting (3.14) into (3.12), for all $s\in[0,t\wedge\tau_N]$, it holds that

$$\begin{aligned}
&\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+2\nu\int_0^s\|\nabla u^n(r)\|_{L^2(D)}^2dr\\
&\leq\left\|\sqrt{\rho_0^n}u_0^n\right\|_{L^2(D)}^2+\int_0^s2|\langle u^n,\rho^nf(r,u^n)\rangle|dr+\int_0^s\left\|\sqrt{\rho^n}g(r,u^n)\right\|_{L^2(D)}^2dr\\
&\quad+2\left|\int_0^s\langle u^n,\rho^ng(r,u^n)\rangle dW\right|+2\left|\int_0^s\int_{|z|_Z\geq1}\langle u^n,\rho^nG\left(u^n(x,r-),z\right)\rangle\mu(dz)dr\right|\\
&\quad+\int_0^s\int_{|z|_Z<1}\left(2\langle u^n,\rho^nF\left(u^n(x,r-),z\right)\rangle+\left\|\sqrt{\rho^n}F\right\|_{L^2(D)}^2\right)\tilde{\pi}(dr,dz)\\
&\quad+\int_0^s\int_{|z|_Z\geq1}\left(2\langle u^n,\rho^nG\left(u^n(x,r-),z\right)\rangle+\left\|\sqrt{\rho^n}G\right\|_{L^2(D)}^2\right)\tilde{\pi}(dr,dz)\\
&\quad+\int_0^s\int_D\int_{|z|_Z<1}|\sqrt{\rho^n}F\left(u^n(x,r-),z\right)|^2\mu(dz)dxdr
\end{aligned}\tag{3.15}$$

$$+\int_0^s\int_D\int_{|z|_Z\geq 1}|\sqrt{\rho^n}G\left(u^n(x,r-),z\right)|^2\mu(dz)dxdr.$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product. Taking supremum on both sides of (3.15) over the interval $[0,t\wedge\tau_N]$, and then taking the mathematical expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+2\nu\mathbb{E}\int_0^{t\wedge\tau_N}\|\nabla u^n(r)\|_{L^2(D)}^2\,dr\\
&\leq\mathbb{E}\left\|\sqrt{\rho_0^n}u_0^n\right\|_{L^2(D)}^2+\mathbb{E}\int_0^{t\wedge\tau_N}2|\langle u^n,\rho^n f(r,u^n)\rangle|dr+\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}g(r,u^n)\right\|_{L^2(D)}^2dr\\
&\quad+2\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left|\int_0^s\langle u^n,\rho^n g(r,u^n)\rangle dW\right|+2\mathbb{E}\left|\int_0^{t\wedge\tau_N}\int_{|z|_Z\geq 1}\langle u^n,G\left(u^n(x,r-),z\right)\rangle\mu(dz)dr\right|\\
&\quad+\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\int_0^s\int_{|z|_Z<1}\left(2\langle u^n,\rho^n F\left(u^n(x,r-),z\right)\rangle+\left\|\sqrt{\rho^n}F\right\|_{L^2(D)}^2\right)\tilde{\pi}(dr,dz) \qquad (3.16)\\
&\quad+\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\int_0^s\int_{|z|_Z\geq 1}\left(2\langle u^n,\rho^n G\left(u^n(x,r-),z\right)\rangle+\left\|\sqrt{\rho^n}G\right\|_{L^2(D)}^2\right)\tilde{\pi}(dr,dz)\\
&\quad+\mathbb{E}\int_0^{t\wedge\tau_N}\int_D\int_{|z|_Z<1}|\sqrt{\rho^n}F\left(u^n(x,r-),z\right)|^2\mu(dz)dxdr\\
&\quad+\mathbb{E}\int_0^{t\wedge\tau_N}\int_D\int_{|z|_Z\geq 1}|\sqrt{\rho^n}G\left(u^n(x,r-),z\right)|^2\mu(dz)dxdr\\
&:=I_0+I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.
\end{aligned}$$

Now, we shall estimate each term in the right-hand side of (3.16). First, for the term I_1, by Young's inequality and the hypothesis on f, it holds that

$$\begin{aligned}
I_1&\leq\varepsilon\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2ds+C_\varepsilon\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n(s)}f(s,u^n)\right\|_{L^2(D)}^2ds\\
&\leq C\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2ds+C.
\end{aligned} \qquad (3.17)$$

For the term I_2, by the assumption on g, the Hölder's inequality yields

$$I_2\leq C\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n(s)}u^n(s)\right\|_{L^2(D)}^2ds+C. \qquad (3.18)$$

Next, we shall estimate the term I_3, the hypothesis on g and Burkholder-Davis-Gundy inequality imply

$$\begin{aligned}
I_3&\leq C\mathbb{E}\left[\int_0^{t\wedge\tau_N}\langle\rho^n g(s,u^n),u^n\rangle^2ds\right]^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left[\int_0^{t\wedge\tau_N}\|\rho^n(s)\|_{L^\infty(D)}\left(1+\|u^n(s)\|_{L^2(D)}^2\right)\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2ds\right]^{\frac{1}{2}}\\
&\leq C\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}\left(\int_0^{t\wedge\tau_N}\|\rho^n(s)\|_{L^\infty(D)}\left(1+\|u^n(s)\|_{L^2(D)}^2\right)ds\right)^{\frac{1}{2}}\\
&\leq\varepsilon\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds.
\end{aligned} \qquad (3.19)$$

For the term I_4, it follows from the assumption on G and Hölder's inequality that

$$I_4\leq\mathbb{E}\int_0^{t\wedge\tau_N}\left|\int_{|z|_Z\geq 1}\langle\sqrt{\rho^n}u^n,\sqrt{\rho^n}G\left(u^n(x,r-),z\right)\rangle\mu(dz)dr\right|$$

$$
\begin{aligned}
&\leq \mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n\right\|_{L^2(D)}^2 ds+\mathbb{E}\left\|\rho^n\right\|_{L^\infty(D)}\int_0^{t\wedge\tau_N}\left|\int_{|z|_Z\geq 1}\left\|G\left(u^n(x,s-),z\right)\right\|_{L^2(D)}\mu(dz)\right|^2 ds\\
&\leq \mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n\right\|_{L^2(D)}^2 ds+C_\mu\mathbb{E}\|\rho^n\|_{L^\infty(D)}\int_0^{t\wedge\tau_N}\int_{|z|_Z\geq 1}\left\|G\left(u^n(x,s-),z\right)\right\|_{L^2(D)}^2\mu(dz)ds\\
&\leq \mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 ds+C_\mu\mathbb{E}\left\|\rho^n\right\|_{L^\infty(D)}\int_0^{t\wedge\tau_N}\left(1+\|u^n(s)\|_{L^2(D)}^2\right)ds\\
&\leq (C+1)\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 ds+C. \qquad (3.20)
\end{aligned}
$$

Here $C_\mu = \int_{|z|_Z\geq 1}\mu(dz) < \infty$. For the term I_5, in view of the hypothesis on F, using the Burkholder-Davis-Gundy inequality, Hölder's and Young's inequality, we have

$$
\begin{aligned}
I_{5,1} &:= 2\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\int_0^s\int_{|z|_Z<1}\langle u^n,\rho^n F\left(u^n(x,r-),z\right)\rangle\tilde{\pi}(dr,dz)\\
&\leq 2\mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_{|z|<1}\langle u^n,\rho^n F(u^n(x,s-),z)\rangle^2\mu(dz)ds\right]^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left(\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\int_0^{t\wedge\tau_N}\int_{|z|_Z<1}\|\rho^n\|_{L^\infty(D)}\left\|F(u^n(x,s-),z)\right\|_{L^2(D)}^2\mu(dz)ds\right)^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}\left(\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds\right)^{\frac{1}{2}}\right] \qquad (3.21)\\
&\leq \varepsilon\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds,
\end{aligned}
$$

on the other hand, we can infer that

$$
\begin{aligned}
I_{5,2} &:= \mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\int_0^s\int_{|z|_Z<1}\left\|\sqrt{\rho^n}F\right\|_{L^2(D)}^2\tilde{\pi}(dr,dz)\\
&\leq C\mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_{|z|_Z<1}\left\|\sqrt{\rho^n}F\right\|_{L^2(D)}^4\mu(dz)ds\right]^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left[\int_0^{t\wedge\tau_N}\|\rho^n\|_{L^\infty(D)}^2\int_{|z|_Z<1}\|F\|_{L^2(D)}^4\mu(dz)ds\right]^{\frac{1}{2}} \qquad (3.22)\\
&\leq \varepsilon\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds,
\end{aligned}
$$

then

$$
I_5 \leq \varepsilon\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds. \qquad (3.23)
$$

For the term I_6, similarly to I_5, by the assumption on G, one has

$$
I_6 \leq \varepsilon\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2+C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds. \qquad (3.24)
$$

Finally, we shall estimate the last two terms. For the term I_7, in virtue of the assumption (B), using Hölder's inequality, we can infer that

$$\begin{aligned} I_7 &\le C\mathbb{E}\int_0^{t\wedge\tau_N} \|\rho^n(s)\|_{L^\infty(D)} \int_{|z|_Z<1} \|F(u^n(x,s-),z)\|_{L^2(D)}^2 \mu(dz)ds \\ &\le C\mathbb{E}\int_0^{t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 ds + C. \end{aligned} \tag{3.25}$$

Similarly to I_7, in view of the hypothesis on G, one deduces that

$$I_8 \le C\mathbb{E}\int_0^{t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 ds + C. \tag{3.26}$$

Substituting (3.17)-(3.26) into (3.16), for sufficiently small $\varepsilon > 0$, it holds that

$$\begin{aligned} &\mathbb{E}\sup_{0\le s\le t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 + 2\nu\mathbb{E}\int_0^{t\wedge\tau_N} \|\nabla u^n(s)\|_{L^2(D)}^2 ds \\ &\quad \le \mathbb{E}\left\|\sqrt{\rho_0^n}u_0^n\right\|_{L^2(D)}^2 + C\mathbb{E}\int_0^{t\wedge\tau_N} \left(1 + \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right) ds. \end{aligned} \tag{3.27}$$

By the Gronwall inequality, we have

$$\mathbb{E}\sup_{0\le s\le t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 + 2\nu\mathbb{E}\int_0^{t\wedge\tau_N} \|\nabla u^n(s)\|_{L^2(D)}^2 ds \le C. \tag{3.28}$$

Note that by the property of stopping time, we have $t\wedge\tau_N \to t$ as $N \to \infty$. Then letting $N \to \infty$ in (3.28), for any $t \in [0, T_n]$, we obtain

$$\mathbb{E}\sup_{0\le s\le t} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2 + 2\nu\mathbb{E}\int_0^{t} \|\nabla u^n(s)\|_{L^2(D)}^2 ds \le C. \tag{3.29}$$

Since the constant C is independent of n, then $T_n = T$.

Applying Itô's formula to (3.1) when $p \ge 2$, integrating over $[0, s]$, $s \in [0, t\wedge\tau_N]$, one has

$$\begin{aligned} &\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p + p\nu\int_0^s \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^{p-2} \|\nabla u^n(r)\|_{L^2(D)}^2 dr \\ &= p\int_0^s \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^{p-2} \langle u^n, \rho^n g(r,u^n)\rangle dW \\ &\quad + \frac{p}{2}\int_0^s \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^{p-2} \left[2\langle u^n, \rho^n f(r,u^n)\rangle + \left\|\sqrt{\rho^n}g(r,u^n)\right\|_{L^2(D)}^2\right] dr \\ &\quad + \frac{p}{2}\left(\frac{p}{2}-1\right)\int_0^s \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^{p-4} \langle u^n, \rho^n g(r,u^n)\rangle^2 dr + \sum_{i=1}^3 J_{i,s}, \end{aligned} \tag{3.30}$$

where

$$J_{1,s} := \int_0^s \int_{|z|_Z\ge 1} \left\{\left\|\sqrt{\rho^n}u^n(r) + \sqrt{\rho^n}G(u^n(r-),z)\right\|_{L^2(D)}^p - \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^p\right\} \pi(dz,dr), \tag{3.31}$$

$$J_{2,s} := \int_0^s \int_{|z|_Z<1} \left\{\left\|\sqrt{\rho^n}u^n(r) + \sqrt{\rho^n}F(u^n(r-),z)\right\|_{L^2(D)}^p - \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^p\right\} \tilde{\pi}(dz,dr), \tag{3.32}$$

and

$$\begin{aligned} J_{3,s} := \int_0^s \int_{|z|_Z<1} \Big\{ &\left\|\sqrt{\rho^n}u^n(r) + \sqrt{\rho^n}F(u^n(r-),z)\right\|_{L^2(D)}^p - \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^p \\ &- p\left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^{p-2} \langle\sqrt{\rho^n}u^n, F(u^n(r-),z)\rangle\Big\}\mu(dz)dr. \end{aligned} \tag{3.33}$$

Now, taking supremum up to time $t\wedge\tau_N$ and taking mathematical expectation in both sides of (3.30), and then we shall estimate each term of the resulting equation. For the first term, by the Burkholder-Davis-Gundy inequality and Hölder's inequality, in virtue of assumption (A) and (3.7), we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}\left|\int_0^s \left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^{p-2}\langle u^n,\rho^n g(r,u^n)\rangle dW\right|\\
&\le C\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2}\left(\int_0^{t\wedge\tau_N}\langle u^n,\rho^n g(s,u^n)\rangle^2 dt\right)^{\frac12}\right]\\
&\le C\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-1}\left(\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}g(s,u^n)\right\|_{L^2(D)}^2 dt\right)^{\frac12}\right]\\
&\le \varepsilon\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p + C\mathbb{E}\left(\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p\right)ds\right).
\end{aligned}\tag{3.34}$$

For the second, third and fourth terms, similar to the first term, Hölder's inequality, (3.7) and assumption (A) yield

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2}\langle u^n,\rho^n f(s,u^n)\rangle ds\\
&\le C\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds\\
&\le C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p\right)ds,
\end{aligned}\tag{3.35}$$

and

$$\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2}\left\|\sqrt{\rho^n}g(s,u^n)\right\|_{L^2(D)}^2 ds\le C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p\right)ds,\tag{3.36}$$

and

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-4}\langle u^n,\rho^n g(s,u^n)\rangle^2 ds\\
&\le C\mathbb{E}\int_0^{t\wedge\tau_N}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^2\right)ds\\
&\le C\mathbb{E}\int_0^{t\wedge\tau_N}\left(1+\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p\right)ds.
\end{aligned}\tag{3.37}$$

For the term $J_{1,s}$, it follows from the inequality $(a+b)^p\le 2^{p-1}(a^p+b^p)$ for all $p\ge1$ and $a,b\ge0$, (3.7) and assumption (B) that

$$\begin{aligned}
\mathbb{E}\sup_{0\le s\le t\wedge\tau_N}|J_{1,s}| &\le \mathbb{E}\sup_{0\le s\le t\wedge\tau_N}\int_0^s\int_{|z|_Z\ge1}\left\{\left\|\sqrt{\rho^n}u^n(r)+\sqrt{\rho^n}G(u^n(r-),z)\right\|_{L^2(D)}^p\right\}\pi(dz,dr)\\
&\le 2^{p-1}\mathbb{E}\int_0^{t\wedge\tau_N}\int_{|z|_Z\ge1}\left\{\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p+\left\|\sqrt{\rho^n}G(u^n(s-),z)\right\|_{L^2(D)}^p\right\}\mu(dz)ds\\
&\le C(p)\left(\mathbb{E}(t\wedge\tau_N)+\int_0^{t\wedge\tau_N}\mathbb{E}\left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p ds\right).
\end{aligned}\tag{3.38}$$

On the other hand, the terms $J_{2,s}$ and $J_{3,s}$ can be rewrite as

$$J_{2,s}+J_{3,s}=\int_0^s\int_{|z|_Z<1}\left\{\left\|\sqrt{\rho^n}u^n(r)+\sqrt{\rho^n}F(u^n(r-),z)\right\|_{L^2(D)}^p-\left\|\sqrt{\rho^n}u^n(r)\right\|_{L^2(D)}^p\right.$$

$$
\begin{aligned}
&\quad - p \left\|\sqrt{\rho^n} u^n(r)\right\|_{L^2(D)}^{p-2} \langle \sqrt{\rho^n} u^n, F(u^n(r-), z)\rangle \Big\}\pi(dz, dr) \\
&\quad + \int_0^s \int_{|z|_Z<1} p \left\|\sqrt{\rho^n} u^n(r)\right\|_{L^2(D)}^{p-2} \langle \sqrt{\rho^n} u^n, F(u^n(r-), z)\rangle \tilde{\pi}(dz, dr) \\
&:= J_{4,s} + J_{5,s}.
\end{aligned}
$$

For all $a, b \in H$ and $p \geq 2$, from Taylor's formula, it holds that

$$
\left| |a+b|_H^p - |a|_H^p - p|a|_H^{p-2}\langle a, b\rangle \right| \leq C(p)\left(|a|_H^{p-2}|b|_H^2 + |b|_H^p\right).
$$

From this above inequality, one has

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} |J_{4,s}| &\leq C(p)\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} \int_0^s \int_{|z|_Z<1} \Big\{ \left\|\sqrt{\rho^n} u^n(r)\right\|_{L^2(D)}^{(p-2)} \left\|\sqrt{\rho^n} F(u^n(r-), z)\right\|_{L^2(D)}^2 \\
&\quad + \left\|\sqrt{\rho^n} F(u^n(r-), z)\right\|_{L^2(D)}^p \Big\}\pi(dz, dr) \\
&\leq C(p)\mathbb{E}\left(\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^{(p-2)} \int_0^{t\wedge\tau_N} \int_{|z|_Z<1} \left\|\sqrt{\rho^n} F(u^n(s-), z)\right\|_{L^2(D)}^2 \mu(dz)ds\right) \\
&\quad + C(p)\mathbb{E}\int_0^{t\wedge\tau_N} \int_{|z|_Z<1} \left\|\sqrt{\rho^n} F(u^n(s-), z)\right\|_{L^2(D)}^p \mu(dz)ds.
\end{aligned}
$$

It follows from the assumption (B) and Hölder's inequality that

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} |J_{4,s}| &\leq \frac{1}{8}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p + C(p)\mathbb{E}\left[\int_0^{t\wedge\tau_N} \left(1 + \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^2\right) ds\right]^{\frac{p}{2}} \\
&\quad + C(p)\mathbb{E}\int_0^{t\wedge\tau_N} \left(1 + \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p\right) ds \\
&\leq \frac{1}{8}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p + C(p, T)\mathbb{E}\int_0^{t\wedge\tau_N} \left(1 + \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p\right) ds.
\end{aligned} \tag{3.39}
$$

For the term $J_{5,s}$, by the Burkholder-Davis-Gundy inequality and Young's inequality, we have

$$
\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} |J_{5,s}| &\leq C(p)\mathbb{E}\Bigg\{ \left(\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^{2(p-1)}\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^{t\wedge\tau_N} \int_{|z|_Z<1} \left\|\sqrt{\rho^n} F(u^n(s-), z)\right\|_{L^2(D)}^2 \mu(dz)ds\right)^{\frac{1}{2}} \Bigg\} \\
&\leq \frac{1}{8}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p + C(p)\mathbb{E}\left[\int_0^{t\wedge\tau_N} \left(1 + \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^2\right) ds\right]^{\frac{p}{2}} \\
&\leq \frac{1}{8}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p + C(p, T)\mathbb{E}\int_0^{t\wedge\tau_N} \left(1 + \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p\right) ds.
\end{aligned} \tag{3.40}
$$

Plugging (3.34)-(3.40) into (3.30), one deduces that

$$
\begin{aligned}
&\frac{3}{8}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p + C\mathbb{E}\int_0^{t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^{p-2} \left\|\nabla u^n(s)\right\|_{L^2(D)}^2 ds \\
&\qquad \leq \mathbb{E}\left\|\sqrt{\rho_0^n} u_0^n\right\|_{L^2(D)}^p + C(p, T)\left(1 + \mathbb{E}\int_0^{t\wedge\tau_N} \left\|\sqrt{\rho^n} u^n(s)\right\|_{L^2(D)}^p ds\right).
\end{aligned} \tag{3.41}
$$

Applying the Gronwall inequality to (3.41), we can infer that

$$\mathbb{E} \sup_{0\le s\le t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p \le C\left(1+\mathbb{E}\left\|\sqrt{\rho_0^n}u_0^n\right\|_{L^2(D)}^p\right). \tag{3.42}$$

Therefore,

$$\begin{aligned} \mathbb{E} \sup_{0\le s\le t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p + C\mathbb{E}\int_0^{t\wedge\tau_N} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2} \left\|\nabla u^n(s)\right\|_{L^2(D)}^2 ds \\ \le C\left(1+\mathbb{E}\left\|\sqrt{\rho_0^n}u_0^n\right\|_{L^2(D)}^p\right). \end{aligned} \tag{3.43}$$

By the argument similar to the case of $p=2$, since $t\wedge\tau_N\to t$ as $N\to\infty$, letting $N\to\infty$ in (3.43), then

$$\begin{aligned} \mathbb{E} \sup_{0\le s\le t} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^p + C\mathbb{E}\int_0^{t} \left\|\sqrt{\rho^n}u^n(s)\right\|_{L^2(D)}^{p-2} \left\|\nabla u^n(s)\right\|_{L^2(D)}^2 ds \\ \le C\left(1+\mathbb{E}\left\|\sqrt{\rho_0^n}u_0^n\right\|_{L^2(D)}^p\right). \end{aligned} \tag{3.44}$$

Taking the power $p\ge 1$ to (3.15), it follows from (3.44) that

$$\mathbb{E}\left(\int_0^T \left\|\nabla u^n(s)\right\|_{L^2(D)}^2 ds\right)^p \le C. \tag{3.45}$$

Next, in order to get the tightness of $\rho^n u^n$, we need to estimate some increments in time of $\rho^n u^n$ in the space V'. For this, we need the following estimates. First, it follows from (3.44) and $\rho^n\in L^\infty(0,T;L^\infty(D))$ that

$$\rho^n u^n \in L^p(\Omega;L^\infty(0,T;L^2(D))). \tag{3.46}$$

Then we have

$$\nabla(\rho^n u^n) \in L^p(\Omega;L^\infty(0,T;H^{-1}(D))). \tag{3.47}$$

From this, the continuity equation implies that

$$\partial_t\rho^n \in L^p(\Omega;L^\infty(0,T;H^{-1}(D))). \tag{3.48}$$

For $p\ge 1$, from (3.45), it holds that

$$u^n \in L^p(\Omega;L^2(0,T;V)).$$

Since $V\hookrightarrow L^6$, then

$$u^n \in L^p(\Omega;L^2(0,T;L^6(D))), \tag{3.49}$$

and

$$\rho^n u^n \in L^p(\Omega;L^2(0,T;L^6(D))). \tag{3.50}$$

To get the another estimate of $\rho^n u^n$, we need the following lemma in [4, Theorem 1.1.1].

Lemma 3.1. *Let $\mathcal{T}$ be a linear operator from $L^{p_1}(0,T)$ into $L^{p_2}(D)$ and from $L^{q_1}(0,T)$ into $L^{q_2}(D)$ with $q_1\ge p_1$ and $q_2\le p_2$. Then for any $s\in(0,1)$, $\mathcal{T}$ maps $L^{r_1}(0,T)$ into $L^{r_2}(D)$, where $r_1=\frac{1}{s/p_1+(1-s)/q_1}, r_2=\frac{1}{s/p_2+(1-s)/q_2}$.*

When $p_1=2, p_2=6, q_1=\infty, q_2=2$ and $s=\frac{3}{4}$, it follows from Lemma 3.1, (3.46) and (3.50) that

$$\rho^n u^n,\ u^n \in L^p(\Omega;L^{8/3}(0,T;L^4(D))). \tag{3.51}$$

By Hölder's inequality and (3.51), it holds that

$$\begin{aligned}\int_0^T \left(\int_D (\rho^n u^n u^n)^2 dx\right)^{2/3} dt &\le \int_0^T \left(\int_D (\rho^n u^n)^4 dx\right)^{1/3} \left(\int_D (u^n)^4 dx\right)^{1/3} dt \\ &\le \left(\left(\int_D (\rho^n u^n)^4 dx\right)^{2/3} dt\right)^{1/2} \left(\int_0^T \left(\int_D (u^n)^4 dx\right)^{2/3} dt\right)^{1/2} \\ &\le C\int_0^T \left(\int_D (\rho^n u^n)^4 dx\right)^{2/3} dt + C\int_0^T \left(\int_D (u^n)^4 dx\right)^{2/3} dt \\ &\le C.\end{aligned} \tag{3.52}$$

Then, we have

$$\rho^n u^n u^n \in L^p(\Omega; L^{4/3}(0,T; L^2(D))), \tag{3.53}$$

and

$$\nabla(\rho^n u^n u^n) \in L^p(\Omega; L^{4/3}(0,T; H^{-1}(D))). \tag{3.54}$$

In virtue of the definition of the norm of V', one deduces that

$$\|\rho^n u^n(t+\theta) - \rho^n u^n(t)\|_{V'} = \sup_{\varphi\in V,\ \|\varphi\|_V=1} \int_D \left[\rho^n u^n(t+\theta) - \rho^n u^n(t)\right] \varphi dx.$$

Thus it follows from (3.1) that

$$\begin{aligned}\mathbb{E}\int_0^{T-\theta} \|\rho^n u^n(t+\theta) - \rho^n u^n(t)\|_{V'}^2 dt &= \mathbb{E}\int_0^{T-\theta} \left\|\int_t^{t+\theta} d(\rho^n u^n) ds\right\|_{V'}^2 dt \\ &\le \mathbb{E}\int_0^{T-\theta} (J_1 + J_2 + J_3 + J_4 + J_5 + J_6) dt,\end{aligned} \tag{3.55}$$

where

$$\begin{aligned} J_1(t) &:= \left\|\int_t^{t+\theta} \operatorname{div}(\rho^n u^n \otimes u^n) ds\right\|_{V'}^2,\ J_2(t) := \left\|\int_t^{t+\theta} \bar{\mu}\Delta u^n ds\right\|_{V'}^2, \\ J_3(t) &:= \left\|\int_t^{t+\theta} \rho^n f(s, u^n) ds\right\|_{V'}^2,\ J_4(t) := \left\|\int_t^{t+\theta} \rho^n g(s, u^n) dW\right\|_{V'}^2, \\ J_5(t) &:= \left\|\int_t^{t+\theta} \int_{|z|_Z<1} \rho^n F\left(u^n(x, s-), z\right) \tilde{\pi}(ds, dz)\right\|_{V'}^2, \\ J_6(t) &:= \left\|\int_t^{t+\theta} \int_{|z|_Z\ge 1} \rho^n G\left(u^n(x, s-), z\right) \pi(ds, dz)\right\|_{V'}^2. \end{aligned}$$

For the term $J_1(t)$, one has

$$\begin{aligned} J_1^{1/2} &= \sup_{\varphi\in V;\ \|\varphi\|_V=1} \left\{\int_D \left(\int_t^{t+\theta} \operatorname{div}(\rho^n u^n \otimes u^n) ds\right) \varphi(x) dx\right\} \\ &\le C\int_t^{t+\theta} \|\rho^n u^n u^n\|_{L^2(D)} ds. \end{aligned}$$

By Hölder's inequality and (3.53), then

$$\mathbb{E}\int_0^{T-\theta} J_1(t) dt \lesssim \theta^{1/2} \left[\mathbb{E}\left(\int_0^T \|\rho^n u^n u^n\|_{L^2(D)}^{4/3} dt\right)^2\right]^{3/4} \le C\theta^{1/2}. \tag{3.56}$$

For the term $J_2(t)$, similarly to (3.56), we have

$$\begin{aligned}\mathbb{E}\int_0^{T-\theta} J_2(t)dt &\le \mathbb{E}\int_0^{T-\theta}\left(\int_t^{t+\theta}\|\nabla u^n\|_{L^2(D)}\,ds\right)^2 dt\\ &\le \theta\mathbb{E}\int_0^{T-\theta}\int_t^{t+\theta}\|\nabla u^n\|^2_{L^2(D)}\,dsdt \le C\theta.\end{aligned}\tag{3.57}$$

For the term $J_3(t)$, by Hölder's inequality, it follows from the assumption (A) and (3.7) that

$$\begin{aligned}\mathbb{E}\int_0^{T-\theta} J_3(t)dt &\le C\mathbb{E}\int_0^{T-\theta}\left[\int_t^{t+\theta}\|\rho^n(s)\|_{L^\infty(D)}\left(1+\|u^n\|^2_{L^2(D)}\right)ds\right]^2 dt\\ &\le C\theta\mathbb{E}\left[\|\rho^n\|^2_{L^\infty(0,T;L^\infty(D))}\int_0^{T-\theta}\int_t^{t+\theta}\left(1+\|u^n(s)\|^2_{L^2(D)}\right)dsdt\right] \le C\theta.\end{aligned}\tag{3.58}$$

For the term J_4, in virtue of the Burkholder-Davis-Gundy inequality, Hölder's inequality and the condition on g, we obtain

$$\begin{aligned}J_4 &\le \int_0^T \mathbb{E}\left(\sup_{\varphi\in V,\|\varphi\|_V=1}\int_t^{t+\theta}\int_D \rho^n g(s,u^n)\varphi dxdW\right)^2 dt\\ &\le \int_0^T \mathbb{E}\left(\sup_{\varphi\in V,\|\varphi\|_V=1}\int_t^{t+\theta}\left(\int_D \rho^n g(s,u^n)\varphi dx\right)^2 ds\right)dt\\ &\le \int_0^T\left(\mathbb{E}\int_t^{t+\theta}\|\rho^n\|^2_{L^\infty(D)}\|g(s,u^n)\|^2_{L^2(D)}\,ds\right)dt\\ &\le \int_0^T\left(\mathbb{E}\int_t^{t+\theta}\|\rho^n\|^2_{L^\infty(D)}\left(1+\|u^n\|^2_{L^2(D)}\right)ds\right)dt\\ &\le C\theta\mathbb{E}\int_0^T\sup_{0\le t\le T}\|\rho^n\|^2_{L^\infty(D)}\left(1+\|u^n\|^2_{L^2(D)}\right)dt \le C\theta.\end{aligned}\tag{3.59}$$

For the term J_5, by the Burkholder-Davis-Gundy inequality and Hölder's inequality, the assumption (B) and (3.7) imply

$$\begin{aligned}J_5 &\le \int_0^T \mathbb{E}\left(\sup_{\varphi\in V,\|\varphi\|_V=1}\int_t^{t+\theta}\int_{|z|_Z<1}\int_D \rho^n G\left(u^n(x,s-),z\right)\varphi dx\tilde{\pi}(ds,dz)\right)^2 dt\\ &\le C\int_0^T \mathbb{E}\sup_{\varphi\in V,\|\varphi\|_V=1}\int_t^{t+\theta}\int_{|z|_Z<1}\left(\int_D \rho^n G\left(u^n(x,s-),z\right)\varphi dx\right)^2 \mu(dz)dsdt\\ &\le C\int_0^T \mathbb{E}\int_t^{t+\theta}\int_{|z|_Z<1}\|\rho^n\|^2_{L^\infty(D)}\|G\left(u^n(x,s-),z\right)\|^2_{L^2(D)}\,\mu(dz)dsdt\\ &\le C\int_0^T \mathbb{E}\int_t^{t+\theta}\|\rho^n\|^2_{L^\infty(D)}\left(1+\|u^n\|^2_{L^2(D)}\right)dsdt\\ &\le C\mathbb{E}\int_0^T\theta\sup_{0\le t\le T}\|\rho^n\|^2_{L^\infty(D)}\left(1+\|u^n\|^2_{L^2(D)}\right)dt \le C\theta.\end{aligned}\tag{3.60}$$

Finally, for the term J_6, similarly to (3.60), one has

$$J_6 \le C\mathbb{E}\int_0^{T-\theta}\theta\sup_{0\le t\le T}\|\rho^n\|^2_{L^\infty(D)}\left(1+\|u^n\|^2_{L^2(D)}\right)dt \le C\theta.\tag{3.61}$$

Plugging (3.56)-(3.61) into (3.55), we obtain

$$\mathbb{E}\int_0^{T-\theta} \|\rho^n(t+\theta)u^n(t+\theta)-\rho^n(t)u^n(t)\|_{V'}^2\,dt \le C\theta^{\frac{1}{2}}. \tag{3.62}$$

Now, we want to get the estimates of the increment of u^n for the tightness of u^n. Note that

$$\rho^n(t+\theta)\left[u^n(t+\theta)-u^n(t)\right] = \rho^n(t+\theta)u^n(t+\theta)-\rho^n(t)u^n(t)-u^n(t)\left[\rho^n(t+\theta)-\rho^n(t)\right], \tag{3.63}$$

then we only need to estimate the term $u^n(t)\left[\rho^n(t+\theta)-\rho^n(t)\right]$. To this end, it follows from Sobolev embedding theorem, Lemma 2.1, (3.45) and (3.48) that

$$\begin{aligned}
&\mathbb{E}\int_0^{T-\theta} \|u^n(t)\left[\rho^n(t+\theta)-\rho^n(t)\right]\|^2_{W^{-1,\frac{3}{2}}(D)}\,dt\\
&\quad\le \mathbb{E}\int_0^{T-\theta}\left(\|u^n\|_{H^1(D)}\int_t^{t+\theta}\|\rho_s^n\|_{W^{-1,\frac{3}{2}}(D)}\,ds\right)^2 dt\\
&\quad\le C\theta^2\mathbb{E}\left(\|\rho_s^n\|^2_{L^\infty(0,T;H^{-1}(D))}\int_0^T\|u^n(t)\|^2_{H^1(D)}\,dt\right)\le C\theta^2.
\end{aligned} \tag{3.64}$$

This combining with (3.62) and (3.63) yields

$$\mathbb{E}\int_0^{T-\theta} \|\rho^n(t+\theta)\left[u^n(t+\theta)-u^n(t)\right]\|^2_{W^{-1,\frac{3}{2}}(D)}\,dt \le C\theta^{\frac{1}{2}}. \tag{3.65}$$

Applying (3.7), then we have

$$\mathbb{E}\int_0^{T-\theta} \|u^n(t+\theta)-u^n(t)\|^2_{W^{-1,\frac{3}{2}}(D)}\,dt \le C\theta^{\frac{1}{2}}. \tag{3.66}$$

4. Tightness property for the approximation solutions and convergence

4.1. **Tightness property for the approximation solutions.**

In this subsection, we shall show the tightness property for the approximation solutions in the following lemma.

Lemma 4.1. *Define*

$$\begin{aligned}
S = {}& C(0,T;\mathbb{R}^d)\times M_{\mathbb{N}}(Z\times[0,T])\times L^\infty(0,T;W^{-1,\infty}(D))\\
&\times L^2(0,T;L^2(D))\times L^2(0,T;W^{-\alpha,2}(D)),\ 0<\alpha<\frac{6}{13}
\end{aligned}$$

equipped with its Borel σ-algebra. Let Π^n be the probability on S which is the image of P on Ω by the map: $\omega\mapsto(W_n(\omega,\cdot),\pi_n(\omega,\cdot),\rho^n(\omega,\cdot),u^n(\omega,\cdot),\rho^n u^n(\omega,\cdot))$, that is, for any $B\subseteq S$,

$$\Pi^n(B) = P\left\{\omega\in\Omega : (W_n(\omega,\cdot),\pi_n(\omega,\cdot),\rho^n(\omega,\cdot),u^n(\omega,\cdot),\rho^n u^n(\omega,\cdot))\in B\right\}.$$

Then the family Π^n is tight.

Remark 4.1. We can also prove that u^n is tight in $\mathbb{D}([0,T];H_w)$ as [52, 55], where $\mathbb{D}([0,T];H_w)$ denotes the space of H-valued weakly càdlàg functions.

In order to get the tightness of ρ^n, u^n and $\rho^n u^n$, we need the following Proposition in [66].

Proposition 4.1. Let X, B and Y be Banach spaces such that $X\subset\subset B\subset Y$. Assume $1\le p\le\infty$, $\mathcal{E}$ is a set bounded in $L^p(0,T;X)$ and $\|y(t+\theta)-y(t)\|_{L^p(0,T-\theta;Y)}\to 0$ as $\theta\to 0$ uniformly for $y\in\mathcal{E}$. Then $\mathcal{E}$ is relatively compact in $L^p(0,T;B)$.

Proof of Lemma 4.1. Denote $C(0,T;\mathbb{R}^d)\times L^\infty(0,T;W^{-1,\infty}(D))\times L^2((0,T)\times D)\times L^2(0,T;W^{-\alpha,2}(D))$ by S_1. Let Π_1^n be the probability on S_1. We shall find for any ε subsets: $\Sigma_\varepsilon \in C(0,T;\mathbb{R}^d), X_\varepsilon \in L^\infty(0,T;W^{-1,\infty}(D)), Y_\varepsilon \in L^2(0,T;L^2(D))$ and $Z_\varepsilon \in L^2(0,T;W^{-\alpha,2}(D))$ are compact, such that $P(W_n \notin \Sigma_\varepsilon) \le \frac{\varepsilon}{4}, P(\rho^n \notin X_\varepsilon) \le \frac{\varepsilon}{4}$, $P(u^n \notin Y_\varepsilon) \le \frac{\varepsilon}{4}$ and $P(\rho^n u^n \notin Z_\varepsilon) \le \frac{\varepsilon}{4}$. We will prove these results in the following four steps.

Step 1: Find a $\Sigma_\varepsilon \in C(0,T;\mathbb{R}^d)$ which is compact, such that $P(W_n \notin \Sigma_\varepsilon) \le \frac{\varepsilon}{4}$. To this end. For Σ_ε we rely on classical results concerning the Brownian motion. For a constant L_ε to be chosen later, we consider the set

$$\Sigma_\varepsilon = \left\{ W(\cdot) \in C(0,T;\mathbb{R}^d) : \sup_{t_1,t_2\in[0,T],|t_1-t_2|<\frac{1}{m^6}} m|W(t_2)-W(t_1)| \le L_\varepsilon,\ \forall m \in \mathbb{N} \right\}.$$

Σ_ε is relatively compact in $C(0,T;\mathbb{R}^d)$ by Arzsela-Ascoli's Theorem. Furthermore Σ_ε is closed in $C(0,T;\mathbb{R}^d)$. Therefore Σ_ε is a compact subset of $C(0,T;\mathbb{R}^d)$. We can show that $P(W_n \notin \Sigma_\varepsilon) \le \frac{\varepsilon}{4}$. In fact, by the Chebyshev inequality $P\{\omega : \xi(\omega) \ge r\} \le \frac{1}{r^k}\mathbb{E}\left[|\xi(\omega)|^k\right]$, we have

$$\begin{aligned}
P\{\omega : W_n(\omega,\cdot) \notin \Sigma_\varepsilon\} &\le P\left[\bigcup_{m=1}^{\infty}\left\{\omega : \sup_{t_1,t_2\in[0,T],|t_1-t_2|<m^{-6}} |W_n(t_1)-W_n(t_2)| > \frac{L_\varepsilon}{m}\right\}\right] \\
&\le \sum_{m=1}^{\infty}\sum_{i=0}^{m^6-1}\left(\frac{m}{L_\varepsilon}\right)^4 \mathbb{E}\left[\sup_{iTm^{-6}\le t\le(i+1)Tm^{-6}} \left|W_n(t)-W_n\left(iTm^{-6}\right)\right|^4\right] \\
&\le C\sum_{m=1}^{\infty}\left(\frac{m}{L_\varepsilon}\right)^4\left(Tm^{-6}\right)^2 m^6 = \frac{C}{L_\varepsilon^4}\sum_{m=1}^{\infty}\frac{1}{m^2}.
\end{aligned}$$

Therefore choosing $L_\varepsilon^4 = \frac{1}{4C\varepsilon}\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)^{-1}$ we obtain that $P\{\omega : W_n(\omega,\cdot) \in \Sigma_\varepsilon\} \ge 1 - \frac{\varepsilon}{4}$.

Step 2: To find an $X_\varepsilon \in L^\infty(0,T;W^{-1,\infty}(D))$ that is compact, such that $P(\rho^n \notin X_\varepsilon) \le \frac{\varepsilon}{4}$. For this, we introduce the space $\mathcal{Y}^1$ with the norm

$$\|y\|_{\mathcal{Y}^1} := \|y\|_{L^\infty(0,T;L^\infty(D))} + \left\|\frac{\partial y}{\partial t}\right\|_{L^\infty(0,T;H^{-1}(D))},$$

then $\mathcal{Y}^1$ is a Banach space. For $q > 0$, $\mathbb{E}\|y\|^q_{L^\infty(0,T;L^\infty(D))} \le C$ and $\mathbb{E}\left\|\frac{\partial y}{\partial t}\right\|^q_{L^\infty(0,T;H^{-1}(D))} \le C$, define $\|y\|_{\mathcal{Y}^1_{\mathbb{E}}}$ as the space of random variables y endowed with the norm

$$\|y\|_{\mathcal{Y}^1_{\mathbb{E}}} := \left(\mathbb{E}\|y\|^q_{L^\infty(0,T;L^\infty(D))}\right)^{\frac{1}{q}} + \left(\mathbb{E}\left\|\frac{\partial y}{\partial t}\right\|^q_{L^\infty(0,T;H^{-1}(D))}\right)^{\frac{1}{q}}.$$

We choose X_ε as a closed ball of radius r_ε centered at 0 in $L^\infty(0,T;W^{-1,\infty}(D))$ with the norm $\|\cdot\|_{\mathcal{Y}^1}$. By Proposition 4.1, then X_ε is compact.

It follows from (3.7), (3.48) and Chebyshev's inequality that

$$P(\rho^n \notin X_\varepsilon) = P\left(\|\rho^n\|_{\mathcal{Y}^1} > r_\varepsilon\right) \le \frac{1}{r_\varepsilon}\mathbb{E}\left(\|\rho^n\|_{\mathcal{Y}^1}\right) \le \frac{1}{r_\varepsilon}\|y\|_{\mathcal{Y}^1_{\mathbb{E}}} \le \frac{C}{r_\varepsilon}.$$

Choosing $r_\varepsilon = 4C\varepsilon^{-1}$, we have $P(\rho^n \notin X_\varepsilon) \le \frac{\varepsilon}{4}$. Then $P\{\omega : \rho^n(\omega,\cdot) \in X_\varepsilon\} \ge 1 - \frac{\varepsilon}{4}$.

Step 3: Find a $Y_\varepsilon \in L^2(0,T;L^2(D))$ that is compact, such that $P(u^n \notin Y_\varepsilon) \le \frac{\varepsilon}{4}$. For this, we introduce $\mathcal{Y}^2$ with the norm:

$$\begin{aligned}
\|y\|_{\mathcal{Y}^2} := &\sup_{0\le t\le T}\|y(t)\|_{L^2(D)} + \left(\int_0^T \|y(t)\|_V^2 ds\right)^{\frac{1}{2}} \\
&+ \left(\int_0^{T-\theta} \|y(t+\theta)-y(t)\|^2_{W^{-1,\frac{3}{2}}(D)} dt\right)^{\frac{1}{2}},
\end{aligned}$$

then $\mathcal{Y}^2$ is a Banach space. For $1 \le p < \infty$, $\mathbb{E}\sup_{0\le t\le T}\|y(t)\|^p_{L^2(D)} \le C$, $\mathbb{E}\left(\int_0^T \|y(t)\|_V^2 ds\right)^{\frac{p}{2}} \le C$ and $\mathbb{E}\int_0^{T-\theta}\|y(t+\theta)-y(t)\|^2_{W^{-1,\frac{3}{2}}(D)}dt \le C$, define

$$\|y\|_{\mathcal{Y}^2_{\mathbb{E}}} := \left(\mathbb{E}\sup_{0\le t\le T}\|y(t)\|^p_{L^2(D)}\right)^{\frac{1}{p}} + \left[\mathbb{E}\left(\int_0^T \|y(t)\|_V^2 ds\right)^{\frac{p}{2}}\right]^{\frac{2}{p}} + \mathbb{E}\left(\int_0^{T-\theta}\|y(t+\theta)-y(t)\|^2_{W^{-1,\frac{3}{2}}(D)}dt\right)^{\frac{1}{2}}.$$

Choose Y_ε as a closed ball of radius r'_ε centered at 0 in $\mathcal{Y}^2$ with the norm $\|\cdot\|_{\mathcal{Y}^2}$. Then Proposition 4.1 yields that Y_ε is compact in $L^2(0,T;L^2(D))$. From (3.66) and Chebyshev's inequality, one deduces that

$$P(u^n \notin Y_\varepsilon) = P\left(\|u^n\|_{\mathcal{Y}^2} > r'_\varepsilon\right) \le \frac{1}{r'_\varepsilon}\mathbb{E}\left(\|u^n\|_{\mathcal{Y}^2}\right) \le \frac{1}{r'_\varepsilon}\|y\|_{\mathcal{Y}^2_{\mathbb{E}}} \le \frac{C}{r'_\varepsilon}.$$

Choosing $r'_\varepsilon = 4C\varepsilon^{-1}$, we have $P(u^n \notin Y_\varepsilon) \le \frac{\varepsilon}{4}$. Then $P\{\omega : u^n(\omega,\cdot) \in Y_\varepsilon\} \ge 1 - \frac{\varepsilon}{4}$.

Step 4: Find a $Z_\varepsilon \in L^2(0,T;W^{-\alpha,2}(D)), 0 < \alpha < \frac{6}{13}$ that is compact, such that $P(\rho^n u^n \notin Z_\varepsilon) \le \frac{\varepsilon}{4}$. To this end, define the space $\mathcal{Y}^3$ with the norm:

$$\|y\|_{\mathcal{Y}^3} := \|y(t)\|_{L^{\frac{8}{3}}(0,T;L^4(D))} + \left(\int_0^{T-\theta}\|y(t+\theta)-y(t)\|^2_{V'}dt\right)^{\frac{1}{2}},$$

then $\mathcal{Y}^3$ is a Banach space. For $1 \le p < \infty$ and y such that $\mathbb{E}\|y(t)\|^p_{L^{\frac{8}{3}}(0,T;L^4(D))} \le C$ and $\mathbb{E}\int_0^{T-\theta}\|y(t+\theta)-y(t)\|^2_{V'}dt \le C$, define

$$\|y\|_{\mathcal{Y}^3_{\mathbb{E}}} := \left(\mathbb{E}\|y(t)\|^p_{L^{\frac{8}{3}}(0,T;L^4(D))}\right)^{\frac{1}{p}} + \mathbb{E}\left(\int_0^{T-\theta}\|y(t+\theta)-y(t)\|^2_{V'}dt\right)^{\frac{1}{2}}.$$

Take Z_ε as a closed ball of radius $\tilde{r}_\varepsilon$ centered at 0 in $\mathcal{Y}^3$ with the norm $\|\cdot\|_{\mathcal{Y}^3}$. From Proposition 4.1, it holds that Z_ε is compact in $L^2(0,T;W^{-\alpha,2}(D))$. On the other hand, (3.55) and Chebyshev's inequality imply

$$P(\rho^n u^n \notin Z_\varepsilon) = P\left(\|\rho^n u^n\|_{\mathcal{Y}^3} > \tilde{r}_\varepsilon\right) \le \frac{1}{\tilde{r}_\varepsilon}\mathbb{E}\left(\|\rho^n u^n\|_{\mathcal{Y}^3}\right) \le \frac{1}{\tilde{r}_\varepsilon}\|y\|_{\mathcal{Y}^3_{\mathbb{E}}} \le \frac{C}{\tilde{r}_\varepsilon}.$$

Choosing $\tilde{r} = 4C\varepsilon^{-1}$, we have $P(\rho^n u^n \notin Z_\varepsilon) \le \frac{\varepsilon}{4}$. Then $P\{\omega : \rho^n u^n(\omega,\cdot) \in Z_\varepsilon\} \ge 1 - \frac{\varepsilon}{4}$.

To summarize, we can find suitable θ such that

$$P\left\{\omega : W_n \in \Sigma_\varepsilon, \rho^n \in X_\varepsilon, u^n \in Y_\varepsilon, \rho^n u^n \in Z_\varepsilon\right\} \ge 1 - \varepsilon.$$

Hence

$$\Pi_1^n(\Sigma_\varepsilon \times X_\varepsilon \times Y_\varepsilon \times Z_\varepsilon) \ge 1 - \varepsilon. \tag{4.1}$$

Since $M_{\mathbb{N}}(Z\times[0,T])$ endowed with the Prohorov's metric is a separable metric space, by Theorem 3.2 in [59, pp.29], then it holds that the distributions of the family $\{\pi_n, n \in \mathbb{N}\}$ are tight on $M_{\mathbb{N}}(Z\times[0,T])$. Therefore, it follows from Corollary 1.3 in [47, pp.16] that the distribution of the joint processes

$$\{(W_n(\omega,\cdot), \pi_n(\omega,\cdot), \rho^n(\omega,\cdot), u^n(\omega,\cdot), \rho^n u^n(\omega,\cdot)) : n \in \mathbb{N}\}$$

are tight on S. The tightness property of Π^n is thus proved. □

4.2. **Convergence.**

In this subsection, we shall pass to the limit directly to get the solution. The following proof differs from the approach of Da Parto-Zabczyk in [21]. It is based on the method used by Bensoussan in [2]. In the above Subsection, we know that Π^n is tight on the space $S = C(0,T;\mathbb{R}^d) \times M_{\mathbb{N}}(Z\times[0,T]) \times L^\infty(0,T;W^{-1,\infty}(D)) \times L^2(0,T;L^2(D)) \times L^2(0,T;W^{-\alpha,2}(D)),\ 0<\alpha<\frac{6}{13}$. According to Jakubowski-Skorohod's theorem, there exist a subsequence $\{n_j\}$, and a probability space $(\mathring{\Omega},\mathring{\mathcal{F}},\mathring{P})$ and random variables $(\mathring{W}_{n_j},\mathring{\pi}_{n_j},\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{\rho}^{n_j}\mathring{u}^{n_j})$; (W,π,ρ,u,h) with values in S, such that the probability distribution of $(\mathring{W}_{n_j},\mathring{\pi}_{n_j},\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{\rho}^{n_j}\mathring{u}^{n_j})$ is Π^{n_j} and the probability distribution of (W,π,ρ,u,h) is a Radon measure, and

$$(\mathring{W}_{n_j},\mathring{\pi}_{n_j},\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{\rho}^{n_j}\mathring{u}^{n_j}) \to (W,\pi,\rho,u,h) \ \text{ in } \ S,\ \mathring{P}-a.s. \tag{4.2}$$

Here W is $\mathring{\mathcal{F}}_t=\sigma\{\rho(s),u(s),W(s),\pi(s)\}_{0\le s\le t}$-standard Brownian motion. In fact, we need to prove that for $s\le t$ and $i^2=-1$

$$\mathring{\mathbb{E}}\left[\exp\{i\lambda(W(t)-W(s)\}\right] = \exp\left(-\frac{\lambda^2}{2}(t-s)\right). \tag{4.3}$$

It is sufficient to show that

$$\mathring{\mathbb{E}}\left[\exp\{i\lambda(W(t)-W(s)\}\,|\mathring{\mathcal{F}}_s\right] = \exp\left(-\frac{\lambda^2}{2}(t-s)\right). \tag{4.4}$$

Here $\mathring{\mathbb{E}}$ denotes the mathematical expectation with respect to the probability space $(\mathring{\Omega},\mathring{\mathcal{F}},\mathring{P})$. Note that if X is $\mathring{\mathcal{F}}$ measurable and $\mathring{\mathbb{E}}(|Y|),\mathring{\mathbb{E}}(|XY|)<\infty$, then

$$\mathring{\mathbb{E}}(XY|\mathring{\mathcal{F}}) = X\mathring{\mathbb{E}}(Y|\mathring{\mathcal{F}}),\ \mathring{\mathbb{E}}(\mathring{\mathbb{E}}(Y|\mathring{\mathcal{F}})) = \mathring{\mathbb{E}}(Y). \tag{4.5}$$

Thus

$$\mathring{\mathbb{E}}(XY) = \mathring{\mathbb{E}}\left(X\mathring{\mathbb{E}}(Y|\mathring{\mathcal{F}})\right). \tag{4.6}$$

Using (4.6), we can prove (4.4) if the following equality is satisfied.

$$\mathring{\mathbb{E}}\left[\exp\{i\lambda(W(t)-W(s))\}\Lambda(\rho,u,W,\pi)\right] = \exp\left(-\frac{\lambda^2(t-s)}{2}\right)\mathring{\mathbb{E}}\left(\Lambda(\rho,u,W,\pi)\right), \tag{4.7}$$

for any continuous bounded functional $\Lambda(\rho,u,W,\pi)$ on S depending only on the values of ρ,u,W,π on $(0,s)$. Since $\mathring{W}_{n_j}(t)-\mathring{W}_{n_j}(s)$ is independent of $\Lambda(\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{W}_{n_j},\mathring{\pi}_{n_j})$ and $\mathring{W}_{n_j}$ is a Brownian motion, then we have

$$\begin{aligned}
&\mathring{\mathbb{E}}\left[\exp\{i\lambda(\mathring{W}_{n_j}(t)-\mathring{W}_{n_j}(s))\}\Lambda(\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{W}_{n_j},\mathring{\pi}_{n_j})\right]\\
&=\mathring{\mathbb{E}}\left[\exp\{i\lambda(\mathring{W}_{n_j}(t)-\mathring{W}_{n_j}(s))\}\right]\mathring{\mathbb{E}}(\Lambda(\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{W}_{n_j},\mathring{\pi}_{n_j}))\\
&=\exp\left(-\frac{\lambda^2(t-s)}{2}\right)\mathring{\mathbb{E}}(\Lambda(\mathring{\rho}^{n_j},\mathring{u}^{n_j},\mathring{W}_{n_j},\mathring{\pi}_{n_j})).
\end{aligned} \tag{4.8}$$

Taking $j\to\infty$, (4.2) and the continuity of Λ imply (4.7). Then $W(t)$ is a $\mathring{\mathcal{F}}_t$-standard Brownian motion.

For a random measure η on $Z\times[0,T]$ and for any $A\in\mathcal{Z}$, where $\mathcal{Z}$ is the Borel sets on Z, define the measure valued process $N_\eta(t)$ by $N_\eta(t)=\{A\mapsto N_\eta(t,A):=\eta(A\times(0,t])\}$, $t\in[0,T]$. We have the following proposition:

Proposition 4.2. $\mathring{\pi}_{n_j}$ and π are time homogeneous Poisson random measures on $\mathcal{B}(Z)\times\mathcal{B}([0,T])$ over $(\mathring{\Omega},\mathring{\mathcal{F}},\mathring{P})$ with intensity measure μ.

Proof. We shall prove the Proposition by the definition. Since $\mathring{\pi}_n$ and π_n have the same distribution and π_n is a time homogeneous Poisson random measure, from [9, Proposition A.5, Remark A.6], it holds that $\mathring{\pi}_n$ satisfies (1)-(3) of Definition 2.2.

Therefore, we only need to prove that $\mathring{\pi}_n$ satisfies (4) of Definition 2.2 with the filtration $\mathring{\mathcal{F}}_t = \sigma\left\{W(s), \pi(s), \rho^n(s), u^n(s), \rho, u; 0 \le s \le t\right\}, t \in [0,T]$. To this end, fix $n \in \mathbb{N}, t_0 \in [0,T]$ and $t_0 \le s \le t$. The definition of $\mathring{\mathcal{F}}_t$ implies that $\mathring{\pi}_n$ is $\mathring{\mathcal{F}}_t$-adapted. Then it remains to prove that $\mathring{X}_n = N_{\mathring{\pi}_n}(t) - N_{\mathring{\pi}_n}(s)$ is independent of $\mathring{\mathcal{F}}_{t_0}$.

It follows from (2) of Definition 2.2 that the random variable $\mathring{X}_n = N_{\mathring{\pi}_n}(t) - N_{\mathring{\pi}_n}(s)$ is independent of $N_{\mathring{\pi}_n}(t_0)$, hence we only need to prove that $\mathring{X}_n$ is independent of $\mathring{\rho}^n(r), \mathring{u}^n(r)$ and $\rho(r), u(r)$ for any $r \le t_0$.

Fix $r \in [0, t_0]$. Since the distributions of $(\mathring{W}_n, \mathring{\pi}_n, \mathring{\rho}^n, \mathring{u}^n)$ and $(W_n, \pi_n, \rho^n, u^n)$ are same, then $\mathcal{L}(\mathring{\rho}^n|_{[0,r]}, \mathring{u}^n|_{[0,r]}, \mathring{X}_n) = \mathcal{L}(\rho^n|_{[0,r]}, u^n|_{[0,r]}, X_n)$, where $X_n = N_{\pi_n}(t) - N_{\pi_n}(s)$ and $\mathcal{L}(h)$ denotes the distribution of h. Note that $\mathring{\pi}_n = \pi$ (see [9, Theorem D.1]), (ρ^n, u^n) is the solution of the stochastic approximation equations, then it is adapted to the σ-algebra generated by $\mathring{\pi}_n$. Therefore, $\rho^n|_{[0,r]}, u^n|_{[0,r]}$ are independent of X_n. From [9, Remark A.6] and $\mathcal{L}(\mathring{\rho}^n|_{[0,r]}, \mathring{u}^n|_{[0,r]}, \mathring{X}_n) = \mathcal{L}(\rho^n|_{[0,r]}, u^n|_{[0,r]}, X_n)$, one can deduce that $\mathring{\rho}^n|_{[0,r]}, \mathring{u}^n|_{[0,r]}$ are independent of $\mathring{X}_n$. By [9, Lemma 9.3], it holds that $\mathring{X}_n$ is independent of $\rho|_{[0,r]}, u|_{[0,r]}$. Since $\pi(\omega) = \mathring{\pi}_n(\omega)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$, then π is a time homogeneous Poisson random measure. □

Now we need to prove that $(\mathring{W}_{n_j}, \mathring{\pi}_{n_j}, \mathring{\rho}^{n_j}, \mathring{u}^{n_j})$ satisfies the equation (3.1), that is,

$$
\begin{aligned}
&\mathbb{P}^n[\mathring{\rho}^{n_j}\mathring{u}^{n_j}(t)] + \int_0^t \mathbb{P}^n\left[\mathrm{div}(\mathring{\rho}^{n_j}\mathring{u}^{n_j} \otimes \mathring{u}^{n_j}) - \nu\Delta\mathring{u}^{n_j}\right] ds \\
&\quad = \mathbb{P}^n[\mathring{\rho}^{n_j}\mathring{u}^{n_j}(0)] + \mathbb{P}^n\left[\int_0^t \mathring{\rho}^{n_j} f(s, \mathring{u}^{n_j}) ds + \int_0^t \mathring{\rho}^{n_j} g(s, \mathring{u}_{n_j}) d\mathring{W}_{n_j}\right] \\
&\qquad + \mathbb{P}^n \int_0^t \int_{|z|_Z<1} \mathring{\rho}^{n_j} F\left(\mathring{u}^{n_j}(x, s-), z\right) \tilde{\mathring{\pi}}_{n_j}(ds, dz) \\
&\qquad + \mathbb{P}^n \int_0^t \int_{|z|_Z\ge 1} \mathring{\rho}^{n_j} G\left(\mathring{u}^{n_j}(x, s-), z\right) \mathring{\pi}_{n_j}(ds, dz),
\end{aligned} \tag{4.9}
$$

where $\mathbb{P}^n : L^2(D) \to X_n$ is the projection onto X_n. To this end, we define

$$
\begin{aligned}
\xi^n(t) = &\mathbb{P}^n[\rho^n u^n(t) - \rho^n u^n(0)] + \int_0^t \mathbb{P}^n\left[\mathrm{div}(\rho^n u^n \otimes u^n) - \nu\Delta u^n\right] ds \\
&- \mathbb{P}^n\left[\int_0^t \rho^n f(s, u^n) ds + \int_0^t \rho^n g(s, u^n) dW_n\right] \\
&- \mathbb{P}^n \int_0^t \int_{|z|_Z<1} \rho^n F\left(u^n(x, s-), z\right) \tilde{\pi}_n(ds, dz) \\
&- \mathbb{P}^n \int_0^t \int_{|z|_Z\ge 1} \rho^n G\left(u^n(x, s-), z\right) \pi_n(ds, dz),
\end{aligned}
$$

and

$$Z^n = \int_0^T \|\xi^n(t)\|^2_{H^{-1}(D)} dt.$$

Of course

$$Z^n = 0, \; P - a.s.$$

Let

$$
\begin{aligned}
\mathring{\xi}^{n_j}(t) = &\mathbb{P}^n[\mathring{\rho}^{n_j}\mathring{u}^{n_j}(t) - \mathring{\rho}^{n_j}\mathring{u}^{n_j}(0)] + \int_0^t \mathbb{P}^n\left[\mathrm{div}(\mathring{\rho}^{n^j}\mathring{u}^{n^j} \otimes \mathring{u}^{n_j}) - \nu\Delta\mathring{u}^{n_j}\right] ds \\
&- \mathbb{P}^n\left[\int_0^t \mathring{\rho}^{n_j} f(s, \mathring{u}^{n_j}) ds + \int_0^t \mathring{\rho}^{n_j} g(s, \mathring{u}^{n_j}) d\mathring{W}_{n_j}\right]
\end{aligned}
$$

$$- \mathbb{P}^n \int_0^t \int_{|z|_Z<1} \mathring{\rho}^{n_j} F\left(\mathring{u}^{n_j}(x,s-),z\right) \tilde{\mathring{\pi}}_{n_j}(ds,dz)$$
$$- \mathbb{P}^n \int_0^t \int_{|z|_Z\geq 1} \mathring{\rho}^{n_j} G\left(\mathring{u}^{n_j}(x,s-),z\right) \mathring{\pi}_{n_j}(ds,dz).$$

and

$$\mathring{Z}^{n_j} = \int_0^T \|\mathring{\xi}^{n_j}(t)\|^2_{H^{-1}(D)} dt.$$

We have the following proposition:

Proposition 4.3. $\mathring{Z}^{n_j} = 0$ $\mathring{P}-$a.s., that is, $(\mathring{W}_{n_j}, \mathring{\pi}_{n_j}, \mathring{\rho}^{n_j}, \mathring{u}^{n_j})$ satisfies the equation (3.1).

Proof. Here the difficulty comes from the presence of the stochastic integrals in Z^n. By Theorem 2.4 and Corollary 2.5 in [10], we can infer that

$$\mathcal{L}(\rho^n, u^n, \xi^n, W_n, \pi_n) = \mathcal{L}(\mathring{\rho}^{n_j}, \mathring{u}^{n_j}, \mathring{\xi}^{n_j}, \mathring{W}_{n_j}, \mathring{\pi}_{n_j}). \tag{4.10}$$

Note that $\mathring{Z}^{n_j}$ is continuous as a function of $\mathring{\xi}^{n_j}$ if $\mathring{u}^{n_j}$ belongs to a finite-dimensional subspace of $H^1(D)$. In view of (4.10) and the continuity of $\mathring{Z}^{n_j}$, one deduces that the distribution of $\mathring{Z}^{n_j}$ is equal to the distribution of Z^{n_j} on $\mathbb{R}_+$, that is,

$$\mathring{\mathbb{E}}\phi(\mathring{Z}^{n_j}) = \mathbb{E}\phi(Z^{n_j}), \tag{4.11}$$

for any $\phi \in C_b(\mathbb{R}_+)$, where $C_b(X)$ is the space of continuous bounded functions defined on X. Now, let $\varepsilon > 0$ be an arbitrary but fixed number and $\phi_\varepsilon \in C_b(\mathbb{R}_+)$ defined by

$$\phi_\varepsilon = \begin{cases} \frac{y}{\varepsilon}, \ 0 \leq y < \varepsilon; \\ 1, \ y \geq \varepsilon. \end{cases}$$

One can check that

$$\mathring{P}(\mathring{Z}^{n_j} \geq \varepsilon) = \int_{\mathring{\Omega}} 1_{[\varepsilon,\infty]} \mathring{Z}^{n_j} d\mathring{P} \leq \int_{\mathring{\Omega}} 1_{[0,\varepsilon]} \frac{\mathring{Z}^{n_j}}{\varepsilon} d\mathring{P} + \int_{\mathring{\Omega}} 1_{[\varepsilon,\infty]} \mathring{Z}^{n_j} d\mathring{P}.$$

Hence by the definition of $\mathring{\mathbb{E}}(\mathring{Z}^{n_j})$, we can infer that

$$\mathring{P}(\mathring{Z}^{n_j} \geq \varepsilon) \leq \mathring{\mathbb{E}}\phi_\varepsilon(\mathring{Z}^{n_j}),$$

which, together with (4.11) implies that

$$\mathring{P}(\mathring{Z}^{n_j} \geq \varepsilon) \leq \mathbb{E}\phi_\varepsilon(Z^{n_j}).$$

By the fact that $(\rho^n, u^n, W_n, \pi_n)$ satisfies the Galerkin equation, from the above inequality, it holds that

$$\mathring{P}(\mathring{Z}^{n_j} \geq \varepsilon) \leq \mathbb{E}\phi_\varepsilon(Z^{n_j}) = 0, \tag{4.12}$$

for any $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, from (4.12), we can infer that

$$\mathring{Z}^{n_j} = 0 \ \ \mathring{P}-\text{a.s.}. \tag{4.13}$$

It follows from (4.13) that $(\mathring{W}_{n_j}, \mathring{\pi}_{n_j}, \mathring{\rho}^{n_j}, \mathring{u}^{n_j})$ satisfies the equation (3.1). □

Now, we want to pass to the limit directly. To this end, we need the following proposition and lemma (see [41, Chapter 3]).

Proposition 4.4 (Uniform integrability). If there exists a nonnegative measurable function f in $\mathbb{R}^+$, such that $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \infty$ and $\sup_{\alpha\in\Gamma} \mathbb{E}[f(|X_\alpha|)] < \infty$. Then $\{X_\alpha, \alpha \in \Gamma\}$ are uniformly integrable.

Lemma 4.2 (Vitali's convergence Theorem)**.** *Suppose* $p \in [1, \infty)$, $\{X_n\} \in L^p$ *and* $\{X_n\}$ *converges to* X *in probability. Then the following are equivalent:*

(1) $X_n \overset{L^p}{\to} X$*;*
(2) the variables $|X_n|^p, n \in \mathbb{N}$ *are uniformly integrable;*
(3) $\mathbb{E}(|X_n|^p) \to \mathbb{E}(|X|^p)$.

Let us denote the subsequence $(\mathring{\rho}^{n_j}, \mathring{u}^{n_j}), j \geq 1$ by $(\mathring{\rho}^n, \mathring{u}^n)$. Since $(\mathring{\rho}^n, \mathring{u}^n)$ and (ρ^n, u^n) has the same distribution, thus by (3.44) and (3.45), we have

$$\sup_n \mathring{\mathbb{E}} \left(\sup_{0 \leq s \leq T} \left\| \sqrt{\mathring{\rho}^n} \mathring{u}^n \right\|_{L^2(D)}^{2p} \right) \leq C, \tag{4.14}$$

$$\sup_n \mathring{\mathbb{E}} \left(\int_0^T \|\nabla \mathring{u}^n(s)\|_{L^2(D)}^2 \, ds \right) \leq C. \tag{4.15}$$

Proof of Theorem 1.1. In order to prove Theorem 1.1, we will break the limits into deterministic and stochastic parts. First, we pass the limits of the deterministic parts and finally pass the limits of the stochastic parts.

Taking the limits of deterministic parts: Note that $\mathring{\rho}^n \in L^4(\mathring{\Omega}; L^\infty(0,T; L^\infty(D)))$, then

$$\mathring{\rho}^n \to \rho \ \text{ weakly star in } \ L^2(\mathring{\Omega}; L^\infty(0,T; L^\infty(D))), \tag{4.16}$$

and

$$\mathring{\mathbb{E}} \sup_{t \in [0,T]} \|\mathring{\rho}^n\|_{W^{-1,\infty}(D)}^4 \leq C. \tag{4.17}$$

This together with Proposition 4.4, (4.2) and Vitali's convergence Theorem implies

$$\mathring{\rho}^n \to \rho \ \text{ strongly in } \ L^2(\mathring{\Omega}; L^\infty(0,T; W^{-1,\infty}(D))). \tag{4.18}$$

By (4.15) we can infer that the sequence $\mathring{u}^n$ contains a subsequence, still denoted by $\mathring{u}^n$, that satisfies

$$\mathring{u}^n \to u \ \text{ weakly in } \ L^2(\mathring{\Omega}; L^2(0,T; H^1(D))). \tag{4.19}$$

Similar, by the fact $\mathring{\rho}^n(\omega) \in L^\infty(0,T; L^\infty(D))$, in view of (4.14), it holds that

$$\mathring{u}^n \to u \ \text{ weakly star in } \ L^4(\mathring{\Omega}; L^\infty(0,T; L^2(D))). \tag{4.20}$$

Let us consider the positive nondecreasing function $f(x) = x^2$ in Proposition 4.4. The function $f(x)$ obviously satisfies $\lim\limits_{x \to \infty} \frac{f(x)}{x} = \infty$. Thanks to the estimate $\mathring{\mathbb{E}} \sup_{t \in [0,T]} \|\mathring{u}^n\|_{L^2(D)}^4 \leq C$, we have that $\sup_{n \geq 1} \mathring{\mathbb{E}} f(\|\mathring{u}^n\|_{L^2(0,T;L^2(D))}^2) \leq C$. By Proposition 4.4, we see that the family $\left\{ \|\mathring{u}^n\|_{L^2(0,T;L^2(D))}^2 : n \in \mathbb{N} \right\}$ is uniformly integrable with respect to the probability measure. From Vitali's convergence Theorem and (4.2), one deduces that

$$\mathring{u}^n \to u \ \text{ strongly in } \ L^2(\mathring{\Omega}; L^2(0,T; L^2(D))). \tag{4.21}$$

Next, from (4.14), it holds that

$$\mathring{\rho}^n \mathring{u}^n \to h \ \text{ weakly star in } \ L^2(\mathring{\Omega}; L^\infty(0,T; L^2(D))). \tag{4.22}$$

By (4.14), we have $\mathring{\mathbb{E}} \sup_{t \in [0,T]} \|\mathring{\rho}^n \mathring{u}^n\|_{L^2(D)}^4 \leq C$. This yields that $\mathring{\mathbb{E}} \sup_{t \in [0,T]} \|\mathring{\rho}^n \mathring{u}^n\|_{W^{-\alpha,2}(D)}^4 \leq C$. It follows from (4.2) and Vitali's convergence Theorem that

$$\mathring{\rho}^n \mathring{u}^n \to h \ \text{ strongly in } \ L^2(\mathring{\Omega}; L^2(0,T; W^{-\alpha,2}(D))). \tag{4.23}$$

From the fact that $\|fg\|_{W^{-1,6}(D)} \leq C \|f\|_{W^{-1,\infty}(D)} \|g\|_{H^1(D)}$ in Lemma 2.1, (4.18) and (4.19), then one has

$$\mathring{\rho}^n \mathring{u}^n \to \rho u \ \text{ weakly in } \ L^2(\mathring{\Omega}; L^2(0,T; W^{-1,6}(D))). \tag{4.24}$$

This together with (4.23) implies that

$$h = \rho u. \tag{4.25}$$

It follows form (3.53) that

$$\mathring{\rho}^n\mathring{u}^n\mathring{u}^n \to \bar{h} \ \text{ weakly in } \ L^2(\mathring{\Omega}; L^{4/3}(0,T;L^2(D))). \tag{4.26}$$

Similarly, by the fact that $\|fg\|_{W^{-1,3}(D)} \leq C\|f\|_{W^{-1,6}(D)}\|g\|_{H^1(D)}$ in Lemma 2.1, and (4.19), (4.23) and (4.25), we can infer that

$$\mathring{\rho}^n\mathring{u}^n\mathring{u}^n \to \rho uu \ \text{ weakly in } \ L^2(\mathring{\Omega}; L^1(0,T;W^{-1,3}(D))). \tag{4.27}$$

Then $\bar{h} = \rho uu$. By (4.21), the continuity of f and g, Vitali's convergence Theorem imply that

$$f(t,\mathring{u}^n) \to f(t,u) \ \text{ strongly in } \ L^2(\mathring{\Omega}; L^2(0,T;L^2(D))), \tag{4.28}$$

and

$$g(t,\mathring{u}^n) \to g(t,u) \ \text{ strongly in } \ L^2(\mathring{\Omega}; L^2(0,T;L^2(D))). \tag{4.29}$$

Taking the limits of stochastic parts: First, we show that

$$\int_0^t \mathring{\rho}^n g(s,\mathring{u}^n(s))d\mathring{W}_n(s) \to \int_0^t \rho g(s,u(s))dW(s) \ \text{ weakly in } \ L^2(\mathring{\Omega}; L^2(0,T;L^2(D))). \tag{4.30}$$

To deal with the stochastic integral, we introduce the function:

$$\tilde{G}_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t J\left(\frac{t-s}{\varepsilon}\right)\tilde{G}(s)ds, \tag{4.31}$$

where J is the standard mollifier, $\tilde{G}(s) = \rho(s)g(s,u(s))$. Set $\tilde{G}^n(s) := \mathring{\rho}^n(s)g(s,\mathring{u}^n(s))$. Note that

$$\mathring{\mathbb{E}}\int_0^T \left\|\tilde{G}_\varepsilon(t)\right\|^2_{L^2(D)} dt \leq \mathring{\mathbb{E}}\int_0^T \left\|\tilde{G}(t)\right\|^2_{L^2(D)} dt, \tag{4.32}$$

and

$$\tilde{G}_\varepsilon(t) \to \tilde{G}(t) \ \text{ in } \ L^2(0,T;L^2(D)). \tag{4.33}$$

Since $\int_0^t \tilde{G}^n(s)d\mathring{W}_n(s) \in L^2(\mathring{\Omega}; L^2(D))$, then for $\forall\phi \in L^2(\mathring{\Omega}; L^2(D))$, there exists a ξ such that

$$\mathring{\mathbb{E}}\left\langle \phi, \int_0^t \tilde{G}^n(s)d\mathring{W}_n(s)\right\rangle \to \mathring{\mathbb{E}}\left\langle \phi,\xi\right\rangle. \tag{4.34}$$

Next, we will show that $\xi = \int_0^t \tilde{G}(s)dW(s)$. Integrating by parts, we obtain

$$\int_0^t \tilde{G}^n_\varepsilon(s)d\mathring{W}_n(s) = \tilde{G}^n_\varepsilon(t)\mathring{W}_n(t) - \int_0^t [\tilde{G}^n_\varepsilon(s)]_s\mathring{W}_n(s)ds. \tag{4.35}$$

Letting $n\to\infty$ in (4.35), in virtue of (4.2) and (4.29), we can obtain

$$\int_0^t \tilde{G}^n_\varepsilon(s)d\mathring{W}_n(s) \rightharpoonup \tilde{G}_\varepsilon(t)W(t) - \int_0^t [\tilde{G}_\varepsilon(s)]_s W(s)ds = \int_0^t \tilde{G}_\varepsilon(s)dW(s). \tag{4.36}$$

Here the symbol "$\rightharpoonup$" denotes weak convergence. Since

$$\begin{aligned}\mathring{\mathbb{E}}\left\|\int_0^t \tilde{G}^n_\varepsilon(s)d\mathring{W}_n(s)\right\|^2_{L^2(D)} &= \mathring{\mathbb{E}}\left(\int_0^t \|\tilde{G}^n_\varepsilon(s)\|^2_{L^2(D)}ds\right)\\ &\leq \mathring{\mathbb{E}}\left(\int_0^t \|\mathring{\rho}^n g(s,\mathring{u}^n(s))\|^2_{L^2(D)}ds\right) \leq C,\end{aligned} \tag{4.37}$$

then it follows from Remark 2.1 that

$$\int_0^t \tilde{G}^n_\varepsilon(s)d\mathring{W}_n(s) \rightharpoonup \int_0^t \tilde{G}_\varepsilon(s)dW(s) \ \text{ in } \ L^2(\mathring{\Omega}; L^2(D)). \tag{4.38}$$

That is, $\forall\phi \in L^2(\mathring{\Omega}; L^2(D))$, we have

$$\mathring{\mathbb{E}}\left\langle \phi, \int_0^t \tilde{G}^n_\varepsilon(s)d\mathring{W}_n(s)\right\rangle \to \mathring{\mathbb{E}}\left\langle \phi, \int_0^t \tilde{G}_\varepsilon(s)dW(s)\right\rangle. \tag{4.39}$$

Note that

$$\begin{aligned}
&\mathring{\mathbb{E}}\left\langle \phi, \int_0^t \tilde{G}^n(s)d\mathring{W}_n(s)\right\rangle - \mathring{\mathbb{E}}\left\langle \phi, \int_0^t \tilde{G}(s)dW(s)\right\rangle \\
&= \mathring{\mathbb{E}}\left\langle \phi, \int_0^t [\tilde{G}^n(s) - \tilde{G}^n_\varepsilon(s)]d\mathring{W}_n(s)\right\rangle \\
&\quad + \mathring{\mathbb{E}}\left\langle \phi, \int_0^t \tilde{G}^n_\varepsilon(s)d\mathring{W}_n(s) - \int_0^t \tilde{G}_\varepsilon(s)dW(s)\right\rangle \\
&\quad + \mathring{\mathbb{E}}\left\langle \phi, \int_0^t [\tilde{G}_\varepsilon(s) - \tilde{G}(s)]dW(s)\right\rangle \\
&:= H_1 + H_2 + H_3.
\end{aligned} \tag{4.40}$$

For the first term H_1, the Cauchy-Schwarz inequality, (4.29) and (4.32) yield that

$$\begin{aligned}
&\mathring{\mathbb{E}}\left\langle \phi, \int_0^t [\tilde{G}^n(s) - \tilde{G}^n_\varepsilon(s)]d\mathring{W}_n(s)\right\rangle \\
&\leq \left(\mathring{\mathbb{E}}\|\phi\|^2_{L^2(D)}\right)^{\frac{1}{2}} \left(\mathring{\mathbb{E}}\left\|\int_0^t [\tilde{G}^n(s) - \tilde{G}^n_\varepsilon(s)]d\mathring{W}_n(s)\right\|^2_{L^2(D)}\right)^{\frac{1}{2}} \\
&\leq \left(\mathring{\mathbb{E}}\|\phi\|^2_{L^2(D)}\right)^{\frac{1}{2}} \mathring{\mathbb{E}}\left(\int_0^t \left\|\tilde{G}^n(s) - \tilde{G}^n_\varepsilon(s)\right\|^2_{L^2(D)} ds\right)^{\frac{1}{2}} \to 0,
\end{aligned} \tag{4.41}$$

as $n \to \infty$ and $\varepsilon \to 0$. Similar as H_1, for the term H_3, it follows from the Cauchy-Schwarz inequality and (4.33) that $\mathring{\mathbb{E}}\left\langle \phi, \int_0^t [\tilde{G}_\varepsilon(s) - \tilde{G}(s)]dW(s)\right\rangle \to 0$ as $n \to \infty$ and $\varepsilon \to 0$. This together with (4.39), (4.40) and (4.41) implies that $\xi = \int_0^t \tilde{G}(s)dW(s)$. The proof of (4.30) is thus complete.

Next, we show that

$$\int_0^t \int_{|z|_Z<1} \mathbb{P}^n \left[\mathring{\rho}^n F\left(\mathring{u}^n(x,s-),z\right)\right] \tilde{\mathring{\pi}}_n(ds,dz) \rightharpoonup \int_0^t \int_{|z|_Z<1} \rho F\left(u(x,s-),z\right) \tilde{\pi}(ds,dz) \tag{4.42}$$

in $M^2(\mathring{\Omega}, [0,T], L^2(D))$, which is the space of all $\mathring{\mathfrak{F}}_t$-martingales M_t such that $\mathring{\mathbb{E}}\int_0^T \|M_t\|^2_{L^2(D)}\, dt < \infty$. From (4.21) and the continuity of $F\left(\mathring{u}^n(x,s-),z\right)$, we can infer that $\mathbb{P}_n\left[F\left(\mathring{u}^n(x,s-),z\right)\right]$ converges to $F\left(u(x,s-),z\right)$ in $L^2(Z,\mu;L^2(D))$ almost everywhere $(\mathring{\omega},s) \in \mathring{\Omega}\times[0,T]$. Thanks to the convergence (4.16), it holds that

$$\mathbb{P}_n\left[\mathring{\rho}^n F\left(\mathring{u}^n(x,s-),z\right)\right] \rightharpoonup \rho F\left(u(x,s-),z\right) \quad \text{in} \quad L^2(\mathring{\Omega}\times[0,T];L^2(Z,\mu;L^2(D))). \tag{4.43}$$

On the other hand, for any $\psi \in L^2(\mathring{\Omega}\times[0,T];L^2(Z,\mu;L^2(D)))$, one has

$$\begin{aligned}
&\int_0^t \int_{|z|_Z<1} \left\langle \mathbb{P}_n\left[\mathring{\rho}^n F\left(\mathring{u}^n(x,s-),z\right)\right], \psi\right\rangle \tilde{\mathring{\pi}}_n(ds,dz) - \int_0^t \int_{|z|_Z<1} \left\langle \rho F\left(u(x,s-),z\right), \psi\right\rangle \tilde{\pi}(ds,dz) \\
&= \int_0^t \int_{|z|_Z<1} \left\langle \mathbb{P}_n\left[\mathring{\rho}^n F\left(\mathring{u}^n(x,s-),z\right)\right], \psi\right\rangle (\tilde{\mathring{\pi}}_n - \tilde{\pi})(ds,dz) \\
&\quad + \int_0^t \int_{|z|_Z<1} \left\{\left\langle \mathbb{P}_n\left[\mathring{\rho}^n F\left(\mathring{u}^n(x,s-),z\right)\right] - \rho F\left(u(x,s-),z\right), \psi\right\rangle\right\} \tilde{\pi}(ds,dz).
\end{aligned} \tag{4.44}$$

Note that all the integrals in (4.44) are well-defined thanks to the discussion above. Since $\mathring{\pi}_n = \pi$ for any n almost surely, by [9, Proposition B.1], we can infer that the first term on the right-hand side of (4.44) goes to 0 as $n \to \infty$. It follows from the continuity of the stochastic integral (as linear functional from $M^2([0,T], L^2(Z,\mu;L^2(D)))$ into $M^2(\mathring{\Omega}\times[0,T];L^2(D)))$ and (4.43) that the second term on the right-hand side of (4.44) also converges to zero as $n \to \infty$. Similarly, thanks to $\int_{|z|_Z\geq 1}|z|^p\mu(dz)<\infty, \forall p\geq 1$, one deduces that $\int_0^t\int_{|z|_Z\geq 1}\mathbb{P}^n\left[\mathring{\rho}^n G\left(\mathring{u}^n(x,s-),z\right)\right]\mathring{\pi}_n(ds,dz)\rightharpoonup \int_0^t\int_{|z|_Z\geq 1}\rho G\left(u(x,s-),z\right)\pi(ds,dz)$. The proof of Theorem 1.1 is thus complete. ☐

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