SOLITARY WAVES OF THE ROTATION-MODIFIED KADOMTSEV-PETVIASHVILI EQUATION

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ABSTRACT. The rotation-modified Kadomtsev-Petviashvili equation describes small-amplitude, long internal waves propagating in one primary direction in a rotating frame of reference. The main investigation is the existence and properties of its solitary waves. The existence and non-existence results for the solitary waves are obtained, and their regularity and decay properties are established. Various characterizations are given for the ground states and their cylindrical symmetry is demonstrated. When the effects of rotation are weak, the energy minima constrained by constant momentum are shown to be nonlinearly stable. The weak rotation limit of solitary waves as the rotation parameter tends to zero is studied.

1. INTRODUCTION

The rotation-modified Kadomtsev-Petviashvili (RMKP) equation

$$(u_t - \beta u_{xxx} + (u^2)_x)_x + u_{yy} - \gamma u = 0 \tag{1.1}$$

describes [17, 18] small-amplitude, long internal waves in a rotating fluid propagating in one dominant direction with slow transverse effects, where the effects of rotation balance with weakly nonlinear and dispersive effects. Here, $u(t, x, y)$ represents the wave displacement, $t \in \mathbb{R}^+$ is a timelike variable, $x \in \mathbb{R}$ is a spatial variable in the dominant direction of wave propagation, and $y \in \mathbb{R}$ is a spatial variable in a direction transverse to the x-direction. The coefficient β determines the type of dispersion; In case $\beta < 0$ (negative dispersion), the equation models gravity surface waves in a shallow water channel and internal waves in the ocean, while in case $\beta > 0$ (positive dispersion) it models capillary surface waves or oblique magneto-acoustic waves in plasma. The parameter $\gamma > 0$ measures the effects of rotation and is proportional to the Coriolis force.

In case $\gamma = 0$, namely in the absence of rotation effects, (1.1) reduces to the (usual) Kadomtsev-Petviashvili equation [22]

$$(u_t - \beta u_{xxx} + (u^2)_x)_x + u_{yy} = 0, \tag{1.2}$$

method and in the absence of y-dependence, it reduces to the Ostrovsky equation [34]

$$(u_t - \beta u_{xxx} + (u^2)_x)_x - \gamma u = 0. \tag{1.3}$$

The equation (1.1) may be viewed as modified from the Kadomtsev-Petviashvili equation (1.2) to accommodate the effects of rotation, on one hand, and as extended

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from the Ostrovsky equation (1.3) with allowance for weak transverse effects*, on the other hand.

Existence and properties of localized traveling waves, commonly referred to as *solitary waves*, are important in general in the study of nonlinear dispersive equations. In particular, it is of great interest whether those plane solitary waves remain stable with respect to transverse perturbations and effects of rotation, and if not, what kind of localized two-dimensional structures may emerge. Our main results include the existence of solitary waves of (1.1) and their geometric properties, the existence and the symmetry property of ground states, conditions for stability of ground states, and the weak rotation limits. Particular emphasis is given to the effects of rotation. In many geophysical problems, the effects of rotation on the dynamics of fluid flows and wave motions are subtle and require special attention [31, 35].

To our knowledge, (1.1) has not been studied analytically although it has been studied by means of numerical computations and formal analysis in various geometrical settings. The purpose of the present work is to establish fundamental analytical results regarding (1.1). It is perhaps more interesting to study (1.1) in the setting of a bounded slab, i.e. $0 < y < b$ for some $b > 0$. Laboratory experiments [31] and numerical computations [1, 19] indicate that solitary-like waves are found in such a setting whose wave fronts are curved in a direction transverse to the wave propagation, which are accompanied by trailing Poincaré waves, and that these effects are known to become more pronounced as the effects of rotation are increased. We are planning to pursue this direction analytically in future.

Remark on the well-posedness. Prior to our development on solitary waves, we need to understand the (local) well-posedness of the Cauchy problem associated to (1.1). Without a local existence result in a suitable function space that contains solitary waves, the question of stability or instability has no clear significance. Upon the inspection of the equation, the local well-posedness is established in function spaces consisting of functions which are x-derivatives of L^2 functions. For k a positive integer, let

$$\dot{H}_x^{-k}(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2) : \xi^{-k}\hat{f}(\xi,\eta) \in L^2(\mathbb{R}^2)\}$$

equipped with the norm $\|f\|_{-k,x} = \|\xi^{-k}\hat{f}\|_{L^2(\mathbb{R}^2)}$. For $s \geqslant 0$ a real parameter, let

$$X_s = \{f \in H^s(\mathbb{R}^2) : (\xi^{-1}\hat{f})^\vee \in H^s(\mathbb{R}^2)\}$$

with its norm $\|f\|_{X_s} = \|f\|_{H^s(\mathbb{R}^2)} + \|(\xi^{-1}\hat{f})^\vee\|_{H^s(\mathbb{R}^2)}$. The operator ∂_x^{-k}, with k a positive integer, acts on $\dot{H}_x^{-k}(\mathbb{R}^2)$ and is defined via the Fourier transform as $(\partial_x^{-k}u)^\wedge(\xi,\eta) = (i\xi)^{-k}\hat{u}(\xi,\eta)$.

Using the parabolic regularization and a compactness argument then we establish the following local well-posedness result of (1.1).

Theorem 1.1 (Local well-posedness). *Let $\beta \neq 0$ and $\gamma > 0$. If $\phi \in X_s$ for $s > 2$, then there exist $T > 0$ and the unique solution $u = u(t) \in C([0,T]; X_s)$ to (1.1) with $u(0,x,y) = \phi(x,y)$ such that u_t is computed with respect to the topology of $H^{s-3}(\mathbb{R}^3)$. Moreover, the map $\phi \mapsto u(t)$ is continuous in the H^s-norm.*

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*In many of earth's lakes, sea straits, and costal regions, the transverse scale is not negligible when compared to the Rossby radius [31], indicating that the weak transverse effects may not be ignored.

The proof is discussed in the appendix.

It is standard to show that the solution $u(t)$ obtained in Theorem 1.1 satisfies the conservation laws

$$E(u(t)) = E(\phi) \quad \text{and} \quad V(u(t)) = V(\phi) \quad \text{for } t \in [0,T),$$

where

$$E(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}\beta u_x^2 + \frac{1}{3}u^3 + \frac{1}{2}(\partial_x^{-1}u_y)^2 + \frac{1}{2}\gamma(\partial_x^{-1}u)^2 \right) dxdy \tag{1.4}$$

and

$$V(u) = \frac{1}{2}\int_{\mathbb{R}^2} u^2 \; dxdy, \tag{1.5}$$

express, respectively, the energy and the momentum. and Moreover, the zero-mass condition† in the x-variable

$$\int_{-\infty}^{\infty} u(t;x,y)\; dx = 0$$

holds for $t \in (0,T)$. However this condition does not give an immediate L^1-integrability of $u(t;x,y)$.

The global well-posedness at this point is open. The conservation laws suggest that a natural space to establish global well-posedness is the "energy" space X_1. The proof of the above theorem is based on the rather general idea of the parabolic regularization, and it is proved in X_s for $s > 2$. To achieve well-posedness in lower X_1, therefore one must study closely the dispersive property and the behavior of the nonlinearity of (1.1).

We now define the function space in which solitary waves of (1.1) are constructed and their stability analysis will be performed. Denoted by X is the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ with the norm

$$\|u\|_X^2 = \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|\partial_x^{-1}u_y\|_{L^2}^2 + \|\partial_x^{-1}u\|_{L^2}^2,$$

where $\partial_x(C_0^\infty(\mathbb{R}^2))$ is the space of functions of the form $\partial_x\varphi$ with $\varphi \in C_0^\infty(\mathbb{R}^2)$. Denoted by Y, the function space for the solitary waves of (1.2), is analogously the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ with the norm

$$\|u\|_Y^2 = \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|\partial_x^{-1}u_y\|_{L^2}^2.$$

Note that [8, Remark 1.1] if $u \in Y$ and $\varphi \in L^q_{loc}(\mathbb{R}^2)$, $2 \leqslant q < +\infty$, with $\partial_x\varphi = u$ then $v = \partial_x\varphi \in L^2(\mathbb{R}^2)$ is well-defined and is denoted by $\partial_x^{-1}u_y$.

A *solitary wave* of (1.1) refers to as a traveling-wave solution of the form $u(x - ct, y)$, where $u \in X$ and $c \in \mathbb{R}$ is the speed of wave propagation. Alternatively, it is a solution $u = u(x,y)$ in X of the equation

$$-cu_x - \beta u_{xxx} + (u^2)_x + \partial_x^{-1}u_{yy} - \gamma\partial_x^{-1}u = 0 \tag{1.6}$$

for some $c \in \mathbb{R}$. Section 2 concerns the existence and the non-existence of solitary waves of (1.1). of c. In the negative-dispersion case ($\beta < 0$) which is relevant to (gravity) water waves, there are no solitary waves of (1.1) and any two-dimensional initial disturbance eventually disperses out. For $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ solitary waves

†For the Korteweg-de Vries (KdV) equation and many others, the total mass $\int u$ is an integral of motion, which may take an arbitrary constant (determined by the initial data). For (1.1), however, it is not only an integral of motion but also a constraint which requires that only initial data satisfying the zero-mass condition may be considered.

of (1.1) are obtained as an application of the concentration-compactness principle [26]. Solitary waves of (1.1) are shown to be smooth, and they decay algebraically as $(x^2 + y^2)^{-1}$ at infinity.

It is known in [8] that (1.2) with $\beta > 0$, which is referred to as the KP-I equation, has no solitary waves in Y if $c \geqslant 0$. Our result shows that (1.1) when $\beta > 0$ has solitary waves for $0 \leqslant c < 2\sqrt{\beta\gamma}$, indicating the effects of rotation shall not be ignored. Interpreted in the water-wave context [18], this means that when the initial soliton disperses out, a new pulse whose shape is close to a Kadomtsev-Petviashvili solitary wave forms from the leading edge of radiation so that some sort of recurrence phenomenon takes place.

A *ground state* of (1.1) is a minimizer for the functional

$$S(u) = E(u) - cV(u) \tag{1.7}$$

among all nontrivial solutions in X of (1.6). In what follows, $\mathcal{G}(c, \beta, \gamma)$ denotes the set of ground states for parameters c, β and γ. Section 3 then provides for $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ an existence proof of ground states of (1.1) and gives characterizations of $\mathcal{G}(c, \beta, \gamma)$. In particular, solitary waves of (1.1) obtained in Section 2 are shown to be exactly the ground states of (1.1). This alternative characterization of a ground state of (1.1) combined with the unique continuation principle [30] is then used in Section 4 to demonstrate the cylindrical symmetry of ground states of (1.1). Section 5 demonstrates for $\beta > 0$ and $c < 0$ that the weak rotation limit of ground states of (1.1) as $\gamma \to 0$ is a ground state of (1.2).

Finally, Section 6 is devoted to the stability property of the solitary waves of (1.1). Let us define the precise notion of orbital stability.

Definition 1.2. A set $S \subset X$ is said to be X-stable if for any $\epsilon > 0$ there exists, correspondingly, $\delta > 0$ such that for any $\phi \in X \bigcap X_s$, $s > 2$, with

$$\inf_{u \in S} \|\phi - u\|_X < \delta$$

the unique solution $u(t) \in C([0,T), X \bigcap X_s)$ of (1.1) with the initial condition $u(0) = \phi$ satisfies

$$\sup_{0 \leqslant t < T} \inf_{u \in S} \|u(t) - u\|_X < \epsilon,$$

where $T > 0$ is the maximal time of existence. Otherwise, S is said to be X-unstable.

A standard method of studying orbital stability for a general class of nonlinear Hamiltonian systems, which hinges on that solitary waves are critical points of some functional constructed with the help of invariant quantities of the evolution equation ([16], for instance), uses a certain hypothesis on the spectrum of the second derivative of such a functional. Adapted to our setting, a solitary wave u_c of (1.1) is a critical point of $d(c) = E(u_c) - cV(u_c)$, where c is the wave speed, and it is orbitally stable provided that $d''(c) > 0$. Unfortunately, the usual scaling and dilation technique does not lead to an explicit dependence of $d(c)$ on c here, and thus it is difficult to verify the convexity hypothesis. The difficulty, on the other hand, is avoided by showing directly that the solitary waves considered are global minimizers of energy constrained by constant momentum. Theorem 6.1 establishes that the set of energy minimizers is X-stable if the rotation parameter $\gamma > 0$ is sufficiently small. equations.

The existence and properties of solitary waves of the Kadomtsev-Petviashvili equation (1.2) and generalized Kadomtsev-Petviashvili equation, that is, (1.2) with power nonlinearity u^{p+1} in place of u^2, have been studied in many works including [7, 8, 9, 10, 23, 29]. A main difference of (1.1) from (1.2) from the point of view of the mathematical analysis of solitary waves is the lack of scaling properties. Specifically, in case $\gamma = 0$ in (1.6), one may assume that $c = -1$ since the scaling change $u_c(x, y) = |c|u(|c|^{1/2}x, |c|y)$ transforms (1.6) in u into the same equation in u_c with $c = -1$, whereas such a scaling property is not available if $\gamma > 0$. Instead, u_c solves (1.6) with γ replaced by $\gamma|c|^{-2}$, concentrating its energy on the rotation term. Consequently, the traditional method for stability based on a Lyapunov function ([16], for instance) is difficult to apply.

Our analysis of the existence and properties of solitary waves of (1.1) is closely related to that of the Ostrovsky equation (1.3) in [24, 25, 28, 37]. While solitary waves of the Korteweg-de Vries equation are quite different from those of the Kadomtsev-Petviashvili equaiton, in the presence of the effects of rotation, one-dimensional solitary waves share much in common geometric and dynamic properties with two-dimensional solitary waves.

The lack of scaling property also occurs in the Ostrovsky equation (1.3). It is an interesting future direction to develop a new method to study nonlinear stability for a class of equations without a good scaling property.

2. Existence of solitary waves

Our investigation is the existence and non-existence of solitary waves of (1.1) and their geometric properties. A non-existence proof of solitary waves for $\beta \leqslant 0$ uses Pohojaev type identities. An existence proof for $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ uses the concentration-compactness principle. These solitary waves are shown to be smooth and decay algebraically like $(x^2 + y^2)^{-1}$ at infinity.

Proposition 2.1 (Non-existence of solitary waves)**.** *If $\beta \leqslant 0$ then (1.6) does not admit any nontrivial solution $u \in H^1(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$ and $u_{xx}, u_{yy} \in L^2_{loc}(\mathbb{R}^2)$.*

Based on the Pohojaev type identities, the proof is similar to that of [8, Theorem 1.1] for the generalized Kadomtsev-Petviashvili equation, and hence is omitted.

The next theorem establishes for $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ the existence of solitary waves of (1.1) in the space X. Compared with the Kadomtsev-Petviashvili equation (1.2) with $\beta > 0$, where solitary waves exist only for $c < 0$, (1.1) with $\gamma > 0$ possesses solitary waves for $0 \leqslant c < 2\sqrt{\beta\gamma}$, additionally.

Theorem 2.2. *If $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ then (1.6) possesses a nontrivial solution in X.*

Our approach is variational. Let

$$G(u) = G(u; c, \beta, \gamma) = \int_{\mathbb{R}^2} \left(-cu^2 + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy \tag{2.1}$$

and

$$K(u) = -\int_{\mathbb{R}^2} u^3 \, dxdy. \tag{2.2}$$

It is immediate to see that

$$S(u) = \frac{1}{2}G(u) - \frac{1}{3}K(u),$$

where the functional $S(u)$ is defined in (1.7). If the minimization problem to find

$$I_\lambda = I_\lambda(c,\beta,\gamma) = \inf\{G(u;c,\beta,\gamma) : u \in X,\ K(u) = \lambda\} \tag{2.3}$$

has a nontrivial solution $u \in X$ for some $\lambda > 0$ then it satisfies the corresponding Euler-Lagrange equation

$$-cu - \beta u_{xx} + \partial_x^{-2} u_{yy} - \gamma \partial_x^{-2} u = -\frac{3}{2}\theta u^2 \qquad \text{in} \quad X'$$

with a Lagrange multiplier θ, where X' is the dual space of X with respect to the L^2-duality and $\partial_x^{-2} u_{yy}$ and $\partial_x^{-2} u$ are elements in X' so that

$$\begin{aligned} < \partial_x^{-2} u_{yy}, f > &= (\partial_x^{-1} u_y, \partial_x^{-1} f_y)_{L^2}, \\ < \partial_x^{-2} u, f > &= (\partial_x^{-1} u, \partial_x^{-1} f)_{L^2} \end{aligned}$$

for any $f \in X$. Differentiating the above equation in the x-variable in $\mathcal{D}'(\mathbb{R}^2)$ and performing the scale change $\underline{u} = (3/2)\theta u$, one arrives at that $\underline{u}$ satisfies (1.6) in $\mathcal{D}'(\mathbb{R}^2)$. That is, $\underline{u}$ is a solitary wave of (1.1).

Our proof of the existence of a minimizer $u \in X$ for I_λ is based on the concentration-compactness principle [26, Lemma 1.1]. Let us list some relevant properties of I_λ.

First, in view of the homogeneity properties of $G(u)$ and $K(u)$ it follows that any minimizer u for I_λ is a minimizer for

$$I_1 = I_1(c,\beta,\gamma) = \inf\left\{\frac{G(u;c,\beta,\gamma)}{K(u)^{2/3}} : u \in X,\ K(u) > 0\right\} \tag{2.4}$$

as well. Subsequently, it follows the scaling property

$$I_\lambda = \lambda^{2/3} I_1. \tag{2.5}$$

The next lemma shows that I_λ is bounded from below for $\lambda > 0$.

Lemma 2.3. *If $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ then $I_\lambda > 0$ for any $\lambda > 0$.*

Proof. In case $0 < c < 2\sqrt{\beta\gamma}$, the inequality

$$\begin{aligned} G(u;c,\beta,\gamma) &= \int_{\mathbb{R}^2} \left(cu_x(\partial_x^{-1}u) + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy \\ &\geqslant \int_{\mathbb{R}^2} \left(\frac{c^2\delta}{4} u_x^2 + (\partial_x^{-1}u_y)^2 + \frac{\gamma^2\delta}{1+\gamma\delta}(\partial_x^{-1}u)^2\right) dxdy \end{aligned}$$

holds, where $\delta = 2\beta/c^2 - 1/(2\gamma) > 0$, and in case $c \leqslant 0$ the ineqaulity

$$G(u;c,\beta,\gamma) \geqslant \int_{\mathbb{R}^2} \left(\beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy$$

holds. Since $\|u\|_{L^2}^2 \leqslant \|u_x\|_{L^2}\|\partial_x^{-1}u\|_{L^2}$ the above two inequalities lead to for $\beta, \gamma > 0$ and $c < 2\sqrt{\beta\gamma}$ the coercivity condition

$$G(u;c,\beta,\gamma) \geqslant C_c\|u\|_X^2. \tag{2.6}$$

We observe that $G(u;c,\beta,\gamma) \leqslant C_b\|u\|_X^2$ for $C_b > 0$ and for all $\beta, \gamma > 0$. That is to say, for $\beta, \gamma > 0$ and $c < 2\sqrt{\beta\gamma}$ the functional $G(u;c,\beta,\gamma)$ is equivalent to $\|u\|_X^2$:

$$C_c\|u\|_X^2 \leqslant G(u;c,\beta,\gamma) \leqslant C_b\|u\|_X^2.$$

On the other hand, the imbedding theorem for anistropic Sobolev spaces [3, Lemma 2.1] yields that

$$\begin{aligned}\|u\|_{L^3}^3 \leqslant& C\|u\|_{L^2}^{3/2}(\|u\|_{L^2}+\|u_x\|_{L^2})\|\partial_x^{-1}u_y\|_{L^2}^{1/2}\\ \leqslant& C((\|\partial_x^{-1}u\|_{L^2}+\|u_x\|_{L^2}^2)^{5/4}\|\partial_x^{-1}u_y\|_{L^2}^{1/2}\\ &\quad+(\|\partial_x^{-1}u\|_{L^2}+\|u_x\|_{L^2}^2)^{3/4}\|u_x\|_{L^2}\|\partial_x^{-1}u_y\|_{L^2}^{1/2})\\ \leqslant& C(\|\partial_x^{-1}u\|_{L^2}^2+\|u_x\|_{L^2}^2+\|\partial_x^{-1}u_y\|_{L^2}^2)^{3/2}\\ \leqslant& C\|u\|_X^3,\end{aligned}$$

where $C>0$ is a generic constant independent of u. Therefore,

$$\lambda = K(u) \leqslant \|u\|_{L^3}^3 \leqslant C^3\|u\|_X^3 \leqslant C^3(G(u)/C_c)^{3/2},$$

and correspondingly, if $\lambda>0$ then

$$I_\lambda \geqslant C_c(\lambda/C^3)^{2/3} > 0.$$

This completes the proof. □

Finally, the scaling property (2.5) leads to the strict subadditivity condition

$$I_\lambda < I_{\lambda-\lambda'} + I_{\lambda'} \tag{2.7}$$

for all $\lambda' \in (0,\lambda)$.

Proof of Theorem 2.2. Let $\lambda>0$. We call $\{u_n\}$ a minimizing sequence if $G(u_n)\to I_\lambda$ and $K(u_n)\to\lambda$. Furthermore we can make $K(u_n)$ bounded away from zero uniformly. The result of Lemma 2.3 allows us to choose a minimizing sequence $\{u_n\}$ for (2.3). The coercivity condition (2.6) of $G(u)$ asserts that $\{u_n\}$ is bounded in the X-norm. Let us define

$$\rho_n = \rho(u_n) = \beta(\partial_x u_n)^2 + (\partial_x^{-1}\partial_y u_n)^2 + \gamma(\partial_x^{-1}u_n)^2, \tag{2.8}$$

and $\rho(u) = \beta(u_x)^2 + (\partial_x^{-1}u_y)^2 + (\partial_x^{-1}u)^2$ for $u \in X$. After extracting a subsequence we may assume that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} \rho_n \; dxdy = L > 0.$$

Furthermore, after normalization

$$u_n \mapsto \omega_n u_n, \quad \text{where} \quad \omega_n = \sqrt{\frac{L}{\int_{\mathbb{R}^2} \rho_n \; dxdy}} \to 1,$$

we get

$$\int_{\mathbb{R}^2} \rho(\omega_n u_n) \; dxdy = L \qquad \text{for all } n.$$

Then $K(\omega_n u_n) = \omega_n^3 K(u_n) \to \lambda$ and we can make $\omega_n > \kappa > 0$ uniformly. Hence it is still a minimizing sequence. We still denote the renormalized sequence $\{u_n\}$. By the concentration-compactness principle [26, Lemma 1.1] there are three possiblities for $\{\rho_n\}$:

(i) *Vanishing*: For any $R>0$

$$\lim_{n\to\infty}\sup_{(x,y)\in\mathbb{R}^2}\int_{(x,y)+B_R} \rho_n \; dxdy = 0, \tag{2.9}$$

(ii) *Dichotomy*: There exists $l \in (0, L)$ such that

$$\lim_{r\to\infty} Q(r) = l, \tag{2.10}$$

where

$$Q(r) = \lim_{n\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_r} \rho_n \, dxdy. \tag{2.11}$$

(iii) *Compactness*: There is a sequence $\{(x_n, y_n)\}$ in $\mathbb{R}^2$ such that for any $\epsilon > 0$ there exist $R = R(\epsilon) > 0$ and $n_0 \geqslant 1$ a positive integer such that

$$\int_{(x_n,y_n)+B_R} \rho_n \, dxdy \geqslant \int_{\mathbb{R}^2} \rho_n \, dxdy - \epsilon \tag{2.12}$$

We first assume that "vanishing" occurs. dxdy=0, the origin. The local version of the embedding theorem for anistropic Sobolev spaces [3, pp. 187] asserts that

$$\|u\|_{L^q((x,y)+B_1)} \leqslant C \left(\|u\|_{L^2((x,y)+B_1)} + \|u_x\|_{L^2((x,y)+B_1)} + \|\partial_x^{-1} u_y\|_{L^2((x,y)+B_1)}\right)$$

for $u \in X$ and $(x, y) \in \mathbb{R}^2$, where $2 \leqslant q \leqslant 6$. Correspondingly,

$$\int_{(x,y)+B_1} |u|^3 dxdy \leqslant C \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} \rho \, dxdy \right)^{3/2}$$

holds for all $u \in X$, where $C > 0$ is independent of $(x, y) \in \mathbb{R}^2$. We cover $\mathbb{R}^2$ by balls of unit radius in a way that each point in $\mathbb{R}^2$ is contained in at most three balls, and thus

$$\int_{\mathbb{R}^2} |u|^3 dxdy \leqslant 3C \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{(x,y)+B_1} \rho \, dxdy \right)^{1/2} \|u\|_X^2$$

holds for any $u \in X$. In view of (2.9) this implies $u_n \to 0$ in $L^3(\mathbb{R}^2)$, which however contradicts since $K(u_n)$ is bounded away from zero uniformly. Therefore, vanishing cannot occur.

Next, we assume "dichotomy" occurs. As is done in [26, Lemma 1.1], [8, Theorem 3.1] and [13, Theorem 5.1], for instance, let us split u_n into two functions u_n^1 and u_n^2 in X satisfying that for any $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ and $n_0 \geqslant 1$ a positive integer such that $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and

$$\left|G(u_n^1) + G(u_n^2) - G(u_n)\right| \leqslant \delta(\epsilon),$$

and

$$\left|K(u_n^1) + K(u_n^2) - \lambda\right| \leqslant \delta(\epsilon)$$

with supp $u_n^1 \subset B_{2R}((x_n, y_n))$, supp $u_n^2 \subset \mathbb{R}^2 \backslash B_{2R}((x_n, y_n))$, for some $(x_n, y_n) \in \mathbb{R}^2$ and some $R > 0$, supp $u_n^1 \bigcap$ supp $u_n^2 = \varnothing$, and dist(supp u_n^1, supp u_n^2) $\to \infty$ as $n \to \infty$. Taking subsequences if necessary, one may assume that

$$K(u_n^1) \to \lambda_1(\epsilon) \quad \text{and} \quad K(u_n^2) \to \lambda_2(\epsilon)$$

as $n \to \infty$.

Assume first that $\lambda_1(\epsilon) \to 0$ as $\epsilon \to 0$. Then, $\lambda_2(\epsilon) \to \lambda$. We use the coercivity of $G(u)$ as in the proof of Lemma 2.3 and the assumption of dichotomy to estimate

$$\begin{aligned} G(u_n^1) &\geqslant C_c \int_{\mathbb{R}^2} \Big(\beta(\partial_x u_n^1)^2 + (\partial_x^{-1}\partial_y u_n^1)^2 + \gamma(\partial_x^{-1}u_n^1)^2\Big)\, dxdy \\ &= C_c\Big(\int_{B_{2R}((x_n,y_n))} \rho_n\, dxdy + O(\epsilon)\Big) = C_c(l + O(\epsilon)). \end{aligned}$$

Therefore,

$$G(u_n) = G(u_n^1) + G(u_n^2) + O(\epsilon) \geqslant C_c(l + O(\epsilon)) + G(u_n^2) + O(\epsilon).$$

Letting $n \to \infty$ it follows that$I_\lambda \geqslant C_c l + I_\lambda > I_\lambda$, which is impossible. Therefore, dichotomy cannot occur.

If $\lambda_1(\epsilon) > 0$ and $\lambda_2(\epsilon) > 0$ as $\epsilon \to 0$, then

$$I_{\lambda_1} + I_{\lambda_2} \leqslant \liminf_{n\to\infty} G(u_n^1) + \liminf_{n\to\infty} G(u_n^2) \leqslant I_\lambda + O(\epsilon).$$

By letting ϵ tend to zero, this however leads to a contradiction since $I_\lambda = \lambda^{2/3} I_1$.

Finally, if $\lambda_1(\epsilon) > \lambda$ (or equivalently $\lambda_1(\epsilon) < 0$) we use the positivity of I_λ and the scaling property (2.5) to get

$$I_\lambda = \lambda^{2/3} I_1 \geqslant I_{\lambda_1} + O(\epsilon) = \lambda_1^{2/3} I_1 + O(\epsilon).$$

Once again sending $\epsilon \to 0$ yields the contradiction

$$I_\lambda \geqslant \lambda_1^{2/3} I_1 > \lambda^{2/3} I_1 = I_\lambda.$$

Therefore, dichotomy cannot occur.

The only remaining possibility is “compactness”. Hence from (2.12) we know

$$L = \int_{\mathbb{R}^2} \rho_n\, dxdy \geqslant \int_{(x_n,y_n)+B_R} \rho_n\, dxdy \geqslant \int_{\mathbb{R}^2} \rho_n\, dxdy - \epsilon \tag{2.13}$$

for all $n \geqslant n_0$. Accordingly,

$$\int_{B_R} (\partial_x^{-1} u_n)^2\, dxdy \geqslant \int_{\mathbb{R}^2} (\partial_x^{-1} u_n)^2\, dxdy - 2\epsilon \tag{2.14}$$

for $n \geqslant 1$ and $R > 0$ sufficiently large. Since $\{u_n\}$ is bounded in X, we may assume that $\{u_n(\cdot - x_n, \cdot - y_n)\}$ converges weakly in X to some $u \in X$. The weak lower semi-continuity of $G(u)$ then yields that

$$G(u) \leqslant \liminf_{n\to\infty} G(u_n) = I_\lambda.$$

Since X is imbedded in $L^2_{loc}(\mathbb{R}^2)$ compactly, we may assume up to a subsequence that $u_n(\cdot - x_n, \cdot - y_n)$ converges to u strongly in $L^2_{loc}(\mathbb{R}^2)$, and furthermore we have that $\partial_x^{-1} u_n(\cdot - x_n, \cdot - y_n)$ converges to $\partial_x^{-1} u$ strongly in $L^2_{loc}(\mathbb{R}^2)$. This, together with (2.14), yields that $\partial_x^{-1} u_n(\cdot - x_n, \cdot - y_n)$ converges to $\partial_x^{-1} u$ strongly in $L^2(\mathbb{R}^2)$. Since

$$\|u_n\|_{L^2}^2 = -\int_{\mathbb{R}^2} \partial_x^{-1} u_n \partial_x u_n\, dxdy,$$

subsequently, $u_n(\cdot - x_n, \cdot - y_n)$ converges to u strongly in $L^2(\mathbb{R}^2)$. Then, by interpolation and with the use of the embedding theorem [3, pp. 323] for anisotropic Sobolev spaces $X \subset L^6(\mathbb{R}^2)$, the convergence of $u_n(\cdot - x_n, \cdot - y_n)$ to u is strongly in $L^3(\mathbb{R}^2)$, and as such $K(u) = \lambda$. Therefore, u is a solution for I_λ of (2.3). This completes the proof. □

Remark 2.4 (Non-existence for $\beta > 0$ and $c \geqslant 2\sqrt{\beta\gamma}$). Theorem 2.1 and Theorem 2.2 leave the question of the existence of solitary waves of (1.1) for $\beta > 0$ and $c \geqslant 2\sqrt{\beta\gamma}$ unanswered. In the one-dimensional case, namely, the Ostrovsky equation (1.3), the non-existence of solitary waves under $\beta > 0$ and $c \geqslant 2\sqrt{\beta\gamma}$ is established in [15] via Rolle's theorem. We remark on the two-dimensional setting, that is, the non-existence of solitary waves of (1.1) for $\beta > 0$ and $c \geqslant 2\sqrt{\beta\gamma}$.

Let $u \in X$ be a nontrivial solution of (1.6). Theorem 2.5 below shows that $u \in H^\infty(\mathbb{R}^2)$, and in particular, $u \in C^1(\mathbb{R}^2)$ and $u, u_x, u_y \to 0$ as $x^2 + y^2 \to \infty$. Thus, the asymptotic state of u near infinity satisfies the linear equation

$$-cu_{xx} - \beta u_{xxxx} + u_{yy} - \gamma u = 0.$$

Consider the separation of variables $u(x, y) = U(x)V(y)$. Then

$$\frac{cU_{xx} + \beta U_{xxxx} + \gamma U}{U} = \frac{V_{yy}}{V} = \mu,$$

where $\mu \in \mathbb{R}$. Correspondingly, the characteristic equations for U and V are given as

$$\beta k_1^4 + ck_1^2 + (\gamma - \mu) = 0 \quad \text{and} \quad k_2^2 = \mu,$$

whose solutions are

$$k_1^2 = \frac{-c \pm \sqrt{c^2 - 4\beta(\gamma - \mu)}}{2\beta} \quad \text{and} \quad k_2 = \pm\sqrt{\mu},$$

respectively. Here, k_1 and k_2 are respectively variables for the characteristic equations for U and V.

Since $V(y) \to 0$ as $|y| \to \infty$ for every $x \in \mathbb{R}$ it follows that $\mu > 0$. If $\beta > 0$ and $c \geqslant 2\sqrt{\beta\gamma}$, however, all four roots for k_1 would be purely imaginary since $c^2 - 4\beta(\gamma - \mu) > c^2 - 4\beta\gamma > 0$, which would contradict since $U(x) \to 0$ as $|x| \to \infty$ for every $y \in \mathbb{R}$. Therefore, such solitary waves of (1.1) do not exist for $\beta > 0$ and $c \geqslant 2\sqrt{\beta\gamma}$.

the KP-I equation A generic solitary wave of (1.1) is not expected to be in the form of the product of a function of the x-variable only and a function of the y-variable only. Nevertheless, from the above arguments one may expect the nonexistence of solitary waves of (1.1) in case $c \geqslant 2\sqrt{\beta\gamma}$, although a rigorous proof is still missing at this moment.

Our next result concerns the regularity of solitary waves.

Proposition 2.5 (Smoothness of solitary waves)**.** *For $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ a solitary wave solution of* (1.1) *belongs to*

$$H^\infty(\mathbb{R}^2) = \bigcap_{k \geqslant 1} H^k(\mathbb{R}^2),$$

where k is a positive integer. In particular, $u \in C^{1+\alpha}(\mathbb{R}^2)$, where $\alpha \in (0, 1)$, and $u, u_x, u_y \to 0$ as $x^2 + y^2 \to \infty$.

The proof of the first assertion is very similar to that of [8, Theorem 4.1] for the generalized Kadomtsev-Petviashvili equation, and thus it is omitted.

We end the section with the algebraic decay property of solitary waves of (1.1).

Theorem 2.6 (Algebraic decay of solitary waves)**.** *For $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ any nontrivial solitary wave $u \in X$ of* (1.1) *satisfies $r^2 u \in L^\infty(\mathbb{R}^2)$, where $r^2 = x^2 + y^2$.*

The method is related to that of [4, Theorem 3.1.2] or [9, Theorem 3.1], studying the decay property of solutions to the convolution equation equivalent to (1.6).

Our first task of the proof of Theorem 2.6 is the following simple integral decay estimate.

Lemma 2.7. *Under the condition $\beta, \gamma > 0$ and $c < 2\sqrt{\beta\gamma}$, any solitary wave of* (1.1) *satisfies*

$$\int_{\mathbb{R}^2} r^2(u^2 + (u_y)^2 + u_{xx}^2)dxdy < +\infty, \tag{2.15}$$

where $r^2 = x^2 + y^2$.

Proof. The proof is similar to that of [9, Lemma 3.1]. The regularity of u may be justified by the standard truncation argument as in the proof of [9, Lemma 3.1], and thus we shall proceed formally.

Multiplication of (1.6) by x^2u and several applications of integration by parts yield that

$$\int_{\mathbb{R}^2} x^2(-cu_x^2 + \beta u_{xx}^2 + u_y^2 + \gamma u^2)\ dxdy = \int_{\mathbb{R}^2} \left(-cu^2 + 4\beta u_x^2 + \frac{4}{3}u^3\right)\ dxdy - 2\int_{\mathbb{R}^2} x^2uu_x^2\ dxdy.$$

Since the result of Theorem 2.5 is that $u \to 0$ as $r \to \infty$, it follows that for $0 < \epsilon < 2\sqrt{\beta\gamma} - c$ there exists $R > 0$ such that $r \geqslant R$ implies $|u| \leqslant \epsilon/2$, and as such

$$\int_{\mathbb{R}^2} x^2u_x^2u\ dxdy \leqslant C(R) + \frac{\epsilon}{2}\int_{\mathbb{R}^2} x^2u_x^2\ dxdy$$

for some $C(R) > 0$. Accordingly, the above equation reduces to

$$\int_{\mathbb{R}^2} x^2((-c-\epsilon)u_x^2+\beta u_{xx}^2+u_y^2+\gamma u^2)\ dxdy \leqslant 2C(R)+\int_{\mathbb{R}^2} \left(-cu^2 + 4\beta u_x^2 + \frac{4}{3}u^3\right)\ dxdy. \tag{2.16}$$

In case $0 < c + \epsilon < 2\sqrt{\beta\gamma}$, as is done in the proof of Lemma 2.3, the inequality

$$\begin{aligned}\int_{\mathbb{R}^2} &((-c-\epsilon)x^2u_x^2 + \beta x^2u_{xx}^2 + \gamma x^2u^2)\ dxdy \\ &= (-c-\epsilon)\int_{\mathbb{R}^2} u^2\ dxdy + \int_{\mathbb{R}^2} ((c+\epsilon)x^2uu_{xx} + \beta x^2u_{xx}^2 + \gamma x^2u^2)\ dxdy \\ &\geqslant (-c-\epsilon)\int_{\mathbb{R}^2} u^2\ dxdy + \int_{\mathbb{R}^2} x^2\left(\frac{(c+\epsilon)^2\delta}{4}u_{xx}^2 + \frac{\gamma^2\delta}{1+\gamma\delta}u^2\right)\ dxdy\end{aligned}$$

holds true, where $\delta = 2\beta/(c+\epsilon)^2 - 1/(2\gamma) > 0$. In case $0 \leqslant c + \epsilon$, then (2.16) reduces to

$$\int_{\mathbb{R}^2} x^2(\beta u_{xx}^2 + u_y^2 + \gamma u^2)\ dxdy \leqslant 2C(R) + \int_{\mathbb{R}^2} \left(-cu^2 + 4\beta u_x^2 + \frac{4}{3}u^3\right)\ dxdy.$$

Therefore, (2.16) becomes

$$C\int_{\mathbb{R}^2} x^2(u_{xx}^2 + u_y^2 + u^2)\ dxdy \leqslant 2C(R) + \int_{\mathbb{R}^2} \left((|c|+\epsilon)u^2 + 4\beta u_x^2 + \frac{4}{3}u^3\right)\ dxdy \tag{2.17}$$

for some $C(R) > 0$ and $C > 0$.

Similarly, multiplication of (1.6) by y^2u and integrations by parts yield that

$$\int_{\mathbb{R}^2} y^2(-cu_x^2+\beta u_{xx}^2+u_y^2+\gamma u^2)\ dxdy=\int_{\mathbb{R}^2} u^2\ dxdy-2\int_{\mathbb{R}^2} y^2uu_x^2\ dxdy.$$

The same calculations as above then imply that

$$C\int_{\mathbb{R}^2} y^2(u_{xx}^2+u_y^2+u^2)\ dxdy\leqslant 2C(R)+\int_{\mathbb{R}^2} u^2\ dxdy \tag{2.18}$$

holds true, where $C(R)$ and $C>0$ are as in (2.17). The assertion then follows by adding (2.17) and (2.18). □

Our next task is the analysis of the decay of solutions of the convolution equation

$$u=h*u^2, \tag{2.19}$$

where

$$\hat{h}(\xi,\eta)=\frac{-\xi^2}{-c\xi^2+\beta\xi^4+\eta^2+\gamma}.$$

Lemma 2.8. *The function h is bounded and decays algebraically as $(x^2+y^2)^{-1}$ at infinity. More precisely, $h, r^2h\in L^\infty(\mathbb{R}^2)$.*

Proof. It is immediate that $\|h\|_{L^\infty}\leqslant C\|\hat{h}\|_{L^1}\leqslant C'$. On the other hand, it is straightforward to show that $\partial_\xi\hat{h},\partial_\eta\hat{h}=O((\xi^2+\eta^2)^{-3/2})$ and $\partial_\xi^2\hat{h},\partial_\eta^2\hat{h}=O((\xi^2+\eta^2)^{-2})$ as $\xi^2+\eta^2\to\infty$. Since

$$|r^2h(x,y)|\leqslant C\int_{\mathbb{R}^2}|\partial_\xi^2\hat{h}(\xi,\eta)|+|\partial_\eta^2\hat{h}(\xi,\eta)|d\xi d\eta,$$

the assertion follows. □

We are now in a position of the proof of Theorem 2.6.

Proof of Theorem 2.6. In view of (2.19), one writes

$$\begin{aligned}|(x^2+y^2)u(x,y)|\leqslant &C\Big|\int_{\mathbb{R}^2}|((x-x')^2+(y-y')^2)h(x-x',y-y')||u^2(x',y')\ dx'dy'\Big|\\ &+C\Big|\int_{\mathbb{R}^2}|h(x-x',y-y')|((x')^2+(y')^2)u^2(x',y')\ dx'dy'\Big|.\end{aligned}$$

By Lemma 2.8 and the fact that $u\in L^2(\mathbb{R}^2)$, the first term in the right-side is bounded independently of x and y. By Lemma 2.8 and Lemma 2.7 then the last term in the right-side is bounded independently of x and y. This completes the proof. □

If $\gamma=0$ the optimal algebraic decay rate as $(x^2+y^2)^{-1}$ at infinity is sharp. Indeed, the lump solution of the KP-I equation

$$u(x-ct,y)=\frac{4c(1+c/3(x-ct)^2+c^2/3y^2)}{(1-c/3(x-ct)^2+c^2/3y^2)^2}$$

exhibits the precise decay rate of $(x^2+y^2)^{-1}$ at infinity. The optimal decay rate of the solitary waves of (1.1) in the presence of rotation $\gamma>0$ remains open‡.

‡Laboratory experiments [31] and numerical computations [1] indicate that solitary waves of (1.1) in a bounded slab in the y-direction, that is, $0<y<b$ for some b, decays exponentially in the y-direction.

3. The characterization of ground states

A *ground state* of (1.6) is a solitary wave of (1.1) which minimizes the functional

$$S(u) = E(u) - c\, V(u)$$

among all nonzero solutions of (1.6), where $E(u)$ and $V(u)$ are defined in (1.4) and (1.5), respectively. Both $E(u)$ and $V(u)$ are conserved quantities associated to (1.1). Our goal in this section is to establish for $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ the existence of a ground state of (1.6) and to give its alternative characterizations. Recall that a solitary wave of (1.1) corresponds to a critical point of $S(u)$, that is, $S'(u) = 0$. Thus, the set of ground states may be characterized as

(3.1)
$$\mathcal{G}(c,\beta,\gamma) = \{u \in X : S'(u) = 0, S(u) \leqslant S(v) \quad \text{for all } v \in X \text{ satisfying } S'(v) = 0\}.$$

In Section 2 established is for $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ the existence of solitary waves of (1.1) as a minimizer for I_λ in (2.3).

Let

$$P(u) = \int_{\mathbb{R}^2} \left(-cu^2 + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2 + u^3\right) dxdy. \tag{3.2}$$

Since $P(u) = G(u) - K(u)$ it is straightforward to see that

$$S(u) = \frac{1}{2}P(u) + \frac{1}{6}K(u) = \frac{1}{2}G(u) - \frac{1}{3}K(u),$$

where $G(u)$ and $K(u)$ are defined in (2.1) and (2.2), respectively. Note that $P(u) = 0$ for any solution of (1.6). The theorem below finds a ground state of (1.6) as a minimizer for $S(u)$ under the constraint that $P(u) = 0$. Our result is related to that in [27].

Theorem 3.1 (Existence of ground states)**.** *If $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$ then $\mathcal{G}(c,\beta,\gamma)$ is nonempty and $u^* \in \mathcal{G}(c,\beta,\gamma)$ if and only if $S(u^*)$ solves the minimization problem*

$$J = \inf\{S(u) : \ u \in X,\ u \neq 0,\ P(u) = 0\}. \tag{3.3}$$

(**??**).

Let us denote

$$J_1 = \inf\{S_1(u) : \ u \in X,\ u \neq 0,\ P(u) \leqslant 0\}, \tag{3.4}$$

where

$$S_1(u) = S(u) - \frac{1}{3}P(u) = \frac{1}{6}G(u).$$

For $u \in X$ with $P(u) = 0$ it is straightforward to see $J_1 \leqslant S_1(u) = S(u)$, and correspondingly, $J_1 \leqslant J$. We claim that $J \leqslant J_1$. Indeed, provided that $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$, any $u \in X$ with

$$P(u) = 2\int_{\mathbb{R}^2} \left(-cu^2 + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy + \int_{\mathbb{R}^2} u^3\ dxdy \leqslant 0$$

satisfies

(3.5)
$$P(bu) = b^2\int_{\mathbb{R}^2} \left(-cu^2 + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy + b^3\int_{\mathbb{R}^2} u^3\ dxdy > 0$$

for $b \in (0,1)$ sufficiently small. We choose $b_0 \in (0,1)$ such that $P(b_0 u) = 0$, and furthermore

$$
\begin{aligned}
J \leqslant S(b_0 u) &= \frac{b_0^2}{2}\int_{\mathbb{R}^2} \left(-cu^2 + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy + \frac{b_0^3}{3}\int_{\mathbb{R}^2} u^3 \; dxdy \\
&\leqslant (\frac{b_0^2}{2} - \frac{b_0^3}{3})\int_{\mathbb{R}^2} \left(-cu^2 + \beta u_x^2 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2\right) dxdy \\
&\leqslant \frac{b_0^2}{6} G(u) \leqslant S_1(u).
\end{aligned}
$$

This proves the claim. Therefore, $J = J_1$.

Proof of Theorem 3.1. Our approach is to show the existence of a minimizer for J_1 in (3.4). The coercivity condition of $G(u) = 6S_1(u)$ allows us to choose a minimizing sequence $\{u_n\}$ for $S_1(u)$ satisfying

$$
P(u_n) \leqslant 0 \quad \text{for all } n \quad \text{and} \quad S_1(u_n) = (1/6)G(u_n) \to J_1 \quad \text{as } n \to \infty. \tag{3.6}
$$

Since $\{u_n\}$ is bounded in X a subsequence, still denoted by $\{u_n\}$, converges weakly to some $u^* \in X$. Our goal is then to show that $S_1(u^*) = (1/6)G(u^*) = J_1$ and $P(u^*) = 0$.

The proof is divided into several steps. **The first step** is to show that

$$
\inf_n \|u_n\|_{L^3}^3 > 0.
$$

Suppose on the contrary that a subsequence, still denoted by $\{u_n\}$, of the minimizing sequence satisfying (3.6) has $u_n \neq 0$ yet $\|u_n\|_{L^3}^3 \to 0$ as $n \to \infty$. Since $P(u_n) \leqslant 0$ it follows that

(3.7)

$$
\begin{aligned}
G(u_n) = \int_{\mathbb{R}^2} \big(- cu_n^2 + \beta(\partial_x u_n)^2 + (\partial_x^{-1}\partial_y u_n)^2 + \gamma(\partial_x^{-1}u_n)^2\big) dxdy \\
\leqslant -\int_{\mathbb{R}^2} u_n^3 \; dxdy \leqslant \|u_n\|_{L^3}^3 \to 0
\end{aligned}
$$

and $G(u^*) \leqslant 0$. On the other hand, a Sobolev embedding theorem [3, pp. 323] for anistropic Sobolev spaces asserts that

$$
\begin{aligned}
\|u_n\|_{L^3}^3 \leqslant& C(\|\partial_x^{-1}u_n\|_{L^2}^2 + \|\partial_x^{-1}\partial_y u_n\|_{L^2}^2 + \|\partial_x u_n\|_{L^2}^2)^{3/2} \\
\leqslant& C'(-c\|u_n\|_{L^2}^2 + \beta\|\partial_x u_n\|_{L^2}^2 + \|\partial_x^{-1}\partial_y u_n\|_{L^2}^2 + \gamma\|\partial_x^{-1}u_n\|_{L^2}^2)^{3/2} \\
=& C' G^{3/2}(u_n)
\end{aligned}
$$

holds, where $C, C' > 0$ depend only on c, β and γ. The first inequality is obtained in the proof in Lemma 2.3, and the second inequality uses that $\beta > 0$ and $c < 2\sqrt{\beta\gamma}$. (See again the proof of Lemma 2.3.) This, combined with (3.7) leads to

$$
G(u_n)(1 - C'G^{1/2}(u_n)) \leqslant 0,
$$

and subsequently, $G(u^*) \geqslant (C')^{-2} > 0$. A contradiction then proves the assertion.

The next step is to show that $u^* \neq 0$ almost everywhere in $\mathbb{R}^2$. The result of the previous step is that $\inf_n \|u_n\|_{L^3}^3 = \alpha > 0$. It is straightforward that

$$\begin{aligned}\alpha \leqslant \|u_n\|_{L^3}^3 &= \int_{|u_n|\leqslant\epsilon} |u_n|^3 \, dxdy + \int_{\epsilon<|u_n|<1/\epsilon} |u_n|^3 \, dxdy + \int_{|u_n|\geqslant 1/\epsilon} |u_n|^3 \, dxdy \\ &\leqslant \epsilon \int_{|u_n|\leqslant\epsilon} |u_n|^2 \, dxdy + (1/\epsilon)^3 \text{meas}(\{|u_n| > \epsilon\}) + \epsilon \int_{|u_n|\geqslant 1/\epsilon} |u_n|^4 \, dxdy \\ &\leqslant \epsilon \|u_n\|_X^2 + (1/\epsilon)^3 \text{meas}(\{|u_n| > \epsilon\}) + C\epsilon \|u_n\|_X^4 \\ &\leqslant C'\epsilon + (1/\epsilon)^3 \text{meas}(\{|u_n| > \epsilon\})\end{aligned}$$

holds for any $\epsilon > 0$, where $C, C' > 0$ are independent of u. We choose $\epsilon < \alpha/C'$ sufficiently small so that

$$\text{meas}(\{|u_n| > \epsilon\}) > \epsilon^3(\alpha - C'\epsilon) = \delta > 0.$$

Since $\|u_n\|_X$ is bounded, Lemma 4 in [29] applies to asserts that

$$\text{meas}(B \cap \{|u^*| > \epsilon/2\}) > \delta_0 > 0$$

for some $\delta_0 > 0$, where B is a ball in $\mathbb{R}^2$ of unit radius. This proves the assertion.

The third step is to prove that $P(u^*) \leqslant 0$ and $S(u^*) = \frac{1}{6}G(u^*) = J_1$. Since $\{u_n\}$ is bounded in X and u_n converges to u^* as $n \to \infty$ almost everywhere in $\mathbb{R}^2$, the refinement due to Brézis and Lieb [11] of Fatou's lemma applies to $G(u_n)$ and $K(u_n)$ to assert that

$$\begin{aligned}G(u_n) - G(u_n - u^*) - G(u^*) \to 0, \\ K(u_n) - K(u_n - u^*) - K(u^*) \to 0\end{aligned}$$

as $n \to \infty$. Correspondingly,

$$P(u_n) - P(u_n - u^*) - P(u^*) \to 0$$

as $n \to \infty$.

Suppose that $P(u^*) > 0$. Since $P(u_n) \leqslant 0$ for all n it follows from the above convergence that $P(u_n - u^*) \leqslant 0$ as $n \to \infty$. By (3.4), subsequently, it follows that $\frac{1}{6}G(u_n - u^*) \geqslant J_1$. On the other hand, since $\frac{1}{6}G(u_n) \to J_1$ as $n \to \infty$ the above convergence asserts that $G(u^*) \leqslant 0$. Correspondingly, the coercivity condition (2.6) of $G(u)$ implies that $\|u^*\|_X \leqslant 0$. This in turn implies that $u^* = 0$ almost everywhere in $\mathbb{R}^2$, which contradicts the result obtained in the previous step. Therefore, $P(u^*) \leqslant 0$, and by the weak lower semi-continuity of $G(u)$ it follows that $\frac{1}{6}G(u^*) = J_1$.

The fourth step of the proof is to show that $P(u^*) = 0$. Suppose on the contrary that $P(u^*) < 0$. Note that $P(\delta u^*) > 0$ for $\delta \in (0, 1)$ sufficiently small. By continuity of $P(u)$ then $P(\delta_0 u^*) = 0$ for some $\delta_0 \in (0, 1)$. Accordingly.

$$J_1 \leqslant \frac{1}{6}G(\delta_0 u^*) = \frac{1}{6}\delta_0^2 G(u^*) < \frac{1}{6}(G^*) = J_1,$$

A contradiction then proves the assertion.

Our next task is to show that $u^* \in \mathcal{G}(c, \beta, \gamma)$, that is, $S'(u^*) = 0$ and $S(u^*) \leqslant S(u)$ for all $u \in X$ satisfying $S'(u) = 0$. We recall that $J = J_1$. Since $S_1(u^*) = S(u^*) - \frac{1}{3}P(u^*) = S(u^*)$ it follows that

$$S(u^*) = J = \inf\{S(u) : \ u \in X, \ u \neq 0, \ P(u) = 0\}. \tag{3.8}$$

In other words, u^* is a minimizer for J. Thus, it satisfies the Euler-Lagrange equation

$$S'(u^*) + \theta P'(u^*) = 0$$

for some $\theta \in \mathbb{R}$. Taking the L^2-inner product of this equation with u^* it reduces to $\theta(P'(u^*), u^*)_{L^2} = 0$. Indeed, $(S'(u^*), u^*)_{L^2} = P(u^*) = 0$. On the other hand,

$$\begin{aligned}(P'(u^*), u^*)_{L^2}\\ &= 2\int_{\mathbb{R}^2} \left(-c(u^*)^2 + \beta(\partial_x u^*)^2 + (\partial_x^{-1}\partial_y u^*)^2 + \gamma(\partial_x^{-1}u^*)^2\right)\,dxdy + 3\int_{\mathbb{R}^2} (u^*)^3\,dxdy\\ &= 2G(u^*) - 3K(u^*) = -G(u^*) < 0.\end{aligned}$$

Therefore, $\theta = 0$ and in turn $S'(u^*) = 0$. Moreover, since $(S'(u), u)_{L^2} = P(u)$ for any $u \in X$, the equation (3.8) asserts that $S(u^*) \leqslant S(u)$ for any solitary wave $u \in X$ of (1.1).

Our last step is to show that a ground state of (1.6) is a minimizer for J in (3.3). Let $\bar{u} \in X$ satisfy $\bar{u} \neq 0$, $S'(\bar{u}) = 0$ and $S(\bar{u}) \leqslant S(u)$ for any $u \in X$ satisfying $S'(u) = 0$. Since $S'(u) = 0$ implies $P(u) = (S'(u), u)_{L^2} = 0$, it follows that $S(\bar{u}) \leqslant S(u)$ for any $u \in X$ with $P(u) = 0$. That is, u is a minimizer for J. This completes the proof. □

The proposition below gives other characterizations of a ground state of (1.6), which will be useful in future consideration.

Proposition 3.2 (Characterization of ground states)**.** *There is a positive real number λ^* such that the following statements are equivalent:*

(i) $K(u^*) = \lambda^*$ *and* u^* *is a minimizer for* I_{λ^*} *in* (2.3)*;*
(ii) u^* *is a ground state;*
(iii) $P(u^*) = 0$ *and* $K(u^*) = \inf\{K(u) :\ u \in X, u \neq 0, P(u) = 0\}$*;*
(iv) $P(u^*) = 0 = \inf\{P(u) :\ u \in X, u \neq 0, K(u) = K(u^*)\}$*.*

Proof. Our first task is to show that **(i) implies (iii)**. Let $K(u^*) = \lambda^*$ and $u^* \in X$ is a minimizer for I_{λ^*}. Note that $P(u^*) = 0$ and $G(u^*) = V(u^*) = \lambda^*$. Let $u \in X$ be such that $u \neq 0$ and $P(u) = 0$. Let

$$b = \left(\frac{K(u^*)}{K(u)}\right)^{1/3}.$$

Note that $P(u) = 0$ implies that $G(u) = V(u) > 0$ unless $u \equiv 0$. Our goal is to show that $b \leqslant 1$.

Straightforward calculations yield that $P(bu) = b^2G(u) - b^3K(u) = b^2(1-b)G(u)$. Since $K(bu) = b^3K(u) = K(u^*) = \lambda^*$, it follows that $G(u^*) \leqslant G(bu)$, and subsequently,

$$\begin{aligned}0 = P(u^*) =& G(u^*) - K(u^*)\\ \leqslant& G(bu) - K(bu) = P(bu) = b^2(1-b)G(u).\end{aligned}$$

Therefore, $b \leqslant 1$, and the assertion follows.

The result of Theorem 3.1 says that **(ii) is equivalent to (iii)**.

Next, our task is to show that **(ii) implies (i)**. Let u^* is a ground state of (1.6). Since $P(u^*) = G(u^*) - V(u^*) = 0$ and $S(u^*) = \frac{1}{2}G(u^*) - \frac{1}{3}V(u^*)$ it follows that u^* minimizes $G(u)$ among solutions of (1.6). Let $\lambda^* = K(u^*) = G(u^*)$.

Let u_1 be a minimizer for I_{λ^*}. That is, $K(u_1) = \lambda^*$ and $G(u_1) = I_{\lambda^*}$ minimizes $G(u)$ among $K(u) = \lambda^*$. In particular,

$$I_{\lambda^*} = G(u_1) \leqslant G(u^*) = \lambda.$$

From the variational consideration, u_1 satisfies that

$$-cu_1 - \beta(u_1)_{xx} + \partial_x^{-1}(u_1)_{yy} - \gamma\partial_x^{-2}u_1 = -\theta u_1^2$$

for some $\theta \in \mathbb{R}$. Multiplication of the above by u_1 and integration by parts then yield that $G(u_1) = \theta\lambda^*$. Since $G(u_1) = I_{\lambda^*}$, this implies $\theta \leqslant 1$.

On the other hand, since $u_2 = \theta u_1$ is a solution of (1.6), one obtains

$$\theta^2 G(u_1) = G(u_2) \leqslant G(u^*).$$

Since $G(u_1) = \theta\lambda^*$ and $G(u^*) = \lambda^*$, this implies $\theta \geqslant 1$. Therefore, $\theta = 1$ and $G(u^*) = I_{\lambda^*}$. This proves the assertion.

Our next task is to prove that **(iii) implies (iv)**. Let $u \in X, u \neq 0$ with $K(u) = K(u^*)$, where $u^* \in X$ satisfies (iv). Our goal is to prove that $P(u) \geqslant 0$. Suppose on the contrary that $P(u) < 0$. Note from (3.5) that $P(bu) > 0$ for $b \in (0,1)$ sufficiently small. Correspondingly, $K(u) > 0$ must hold and $P(b_0u) = 0$ for some $b_0 \in (0,1)$. This however contradicts (iii) since

$$V(b_0u) = b_0^3 K(u) < K(u) = K(u^*).$$

Therefore, $P(u) \geqslant 0$. The assertion then follows since $P(u^*) = 0$.

Finally, our task is to show that **(iv) implies (iii)**. Let $u \in X, u \neq 0$ with $P(u) = 0$. Our goal is to show that $K(u) \geqslant K(u^*)$, where $u^* \in X$ satisfies (iv). Assume the opposite inequality. Similarly as in the previous argument, a scaling consideration dictates that $\int_{\mathbb{R}^2} u^3\, dxdy < 0$ and $P(bu) < 0$ for $b > 1$. We may choose $b_0 > 1$ such that $V(b_0u) = b_0^3K(u) = K(u^*)$. This contradicts (iii) since $P(b_0u) < 0 = P(u^*)$. This completes the proof. □

The result of Proposition 3.2 says that minima for I_λ in (2.3) are exactly the ground states of (1.6).

4. Symmetry property of the ground states

Our goal in this section is to prove that any ground state of (1.6) is cylindrically symmetric, in the sense that it has radial symmetry with respect to the transverse coordinate, that is, the y-direction, up to a translation of the origin.

Theorem 4.1 (Cylindrical symmetry of ground states)**.** *Any ground state u^* of* (1.6) *is symmetric in the y-direction up to a translation of the origin of the coordinate in y.*

The proof relies on *the unique continuation principle* due Lopes [30], which we present in a form suitable for our purposes.

Lemma 4.2. *Let $a, b, c \in L^\infty(\mathbb{R}^2)$ and let u satisfy*

$$u_{yy} - u_{xxxx} = a(x,y)u + b(x,y)u_x + c(x,y)u_{xx} \qquad \text{in } \mathbb{R}^2$$

and $u, u_y, u_{xy}, u_{xx}, u_{xxx} \in L^2(\mathbb{R}^2)$. If u vanishes on a half-plane in $\mathbb{R}^2$ then it vanishes everywhere in $\mathbb{R}^2$.

The proof is found, for instance, in the appendix of [9].

Proof of Theorem 4.1. Our proof is very similar to that of [9, Theorem 2.1] for the generalized Kadomtsev-Petviashivili equation although our characterization of the ground states of (1.6) is different from that for the generalized Kadomtsev-Petviashvili equation.

Since u^* is a ground state of (1.6), in view of its characterization in Proposiion 3.2 it follows that $\int_{\mathbb{R}^2}(u^*)^3\ dxdy<0$. By continuity we may choose $b\in\mathbb{R}$ such that

$$\int_{\Delta^+}(u^*)^3\ dxdy=\int_{\Delta^-}(u^*)^3\ dxdy=\frac{1}{2}\int_{\mathbb{R}^2}(u^*)^3\ dxdy, \tag{4.1}$$

where $\Delta^+=\{(x,y)\in\mathbb{R}^2: y>b\}$ and $\Delta^-=\{(x,y)\in\mathbb{R}^2: y<b\}$ denote the half planes delimited by the horizontal line $y=b$.

Let us define the function u^+ as $u^+=u^*$ in Δ^+ and u^+ is symmetric with respect to $y=b$. We claim that $u^+\in X$. Indeed, if $\varphi\in L^2_{loc}(\mathbb{R}^2)$ satisfies $\varphi_x=u^*$ and $\varphi_y=\partial_x^{-1}u^*_y$ and if

$$\varphi^+(x,y)=\begin{cases}\varphi(x,y) & \text{if}\quad y>b,\\ \varphi(x,2b-y) & \text{if}\quad y<b,\end{cases}$$

then $\varphi^+_x=u^+$ and $\int_{\mathbb{R}^2}(\varphi^+_y)^2\ dxdy=2\int_{\Delta^+}\varphi_y^2\ dxdy<+\infty$. From a density argument it then follows that $\partial_x^{-1}u^+_y=\varphi^+_y$ and $\partial_x^{-1}u^+\in L^2(\mathbb{R}^2)$. This proves the claim. Moreover, from (4.1) it follows that $\int_{\mathbb{R}^2}(u^+)^3\ dxdy=\int_{\mathbb{R}^2}(u^*)^3\ dxdy$. Similarly, if $u^-=u^*$ in Δ^- and $u^-$is symmetric with respect to $y=b$ then $u^-\in X$ and $\int_{\mathbb{R}^2}(u^-)^3\ dxdy=\int_{\mathbb{R}^2}(u^*)^3\ dxdy$. Hence, it follows from Proposition 3.2 that $P(u^+),P(u^-)\geqslant 0$.

On the other hand, it is readily seen that

$$P(u^+)+P(u^-)=2P(u^*)=0,$$

and as such u^+ and u^- are both ground states of (1.6). Accordingly, u^*,u^+, and u^- satisfy

$$-cu_{xx}-\beta u_{xxxx}+u_{yy}-\gamma u=(u^2)_{xx}\qquad\text{in}\quad\mathbb{R}^2.$$

Finally, since $u^+=u^*$ in Δ^+ and $u^-=u^*$ in Δ^-, the unique continuation principle Lemma 4.2 applies to u^+-u^* and u^--u^* assert $u^*=u^+=u^-$. That is to say, u^* is symmetric with respect to $y=b$. This completes the proof. ☐

5. Weak rotation limit as $\gamma\to 0$

Our investigation in this section concerns the behavior of solitary waves of (1.1) as the rotation parameter γ tends to zero.

In case $\gamma=0$, (1.1) formally reduces to the Kadomtsev-Petviashvili equation (1.2). A solitary wave of (1.2) then refers to as a solution $u\in Y$ of the equation

$$-cu_x-\beta u_{xxx}+(u^2)_x+\partial_x^{-1}u_{yy}=0. \tag{5.1}$$

Recall that Y is the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ with the norm

$$\|u\|_Y^2=\|u\|_{L^2}^2+\|u_x\|_{L^2}^2+\|\partial_x^{-1}u_y\|_{L^2}^2.$$

The result of Theorem 2.2 is that for each $\gamma>0$ and for $\beta>0$ and $c<2\sqrt{\beta\gamma}$ there exists a solitary wave of (1.1) in X, and the result of [8] that for each $\beta>0$ and $c<0$ there exists a solitary wave of (1.2) in the function space Y. A natural question is then for $\beta > 0$ and $c < 0$ whether the solitary waves of (1.1) converge to that of (1.2) as $\gamma \to 0+$. The answer is YES.

The following theorem is a precise statement of the weak rotation limit.

Theorem 5.1 (Weak rotation limit)**.** *For $\beta > 0$ and $c < 0$ fixed, let a sequence $\{\gamma_n\}$ be such that $\gamma_n \to 0+$ as $n \to \infty$, and let $u_n \in X$ be a solitary wave solution of* (1.1) *with $\gamma = \gamma_n$. There exist a solitary wave $u_0 \in Y$ of* (1.2)*, a subsequence, still denoted as $\{\gamma_n\}$, and a sequence in $\mathbb{R}^2$ of translations $\{(x_n, y_n)\}$ such that*

$$u_n(\cdot - x_n, \cdot - y_n) \to u_0 \quad \text{in } \; Y$$

as $n \to \infty$. That is, u_0 is the limit in Y of solitary waves $\{u_n\}$ of (1.1) *as $\gamma \to 0+$.*

The proof uses the variational characterization of solitary waves of (1.1) and (1.2).

For $\beta, \gamma > 0$ and $c < 2\sqrt{\beta\gamma}$, let us define

$$d(c, \beta, \gamma) = S(u) = E(u) - cV(u) \tag{5.2}$$

where $u \in X$ is a ground state of (1.6). Since a solitary wave of (1.1) satisfies $P(u) = G(u; c, \beta, \gamma) - K(u) = 0$ it follows that

$$d(c, \beta, \gamma) = \frac{1}{2}G(u; c, \beta, \gamma) - \frac{1}{3}K(u) = \frac{1}{6}G(u; c, \beta, \gamma).$$

Furthermore, in view of the minimization problem for $I_1(c, \beta, \gamma)$ in (2.4) it follows that

$$d(c, \beta, \gamma) = \frac{1}{6}G(u; c, \beta, \gamma) = \frac{1}{6}K(u) = \frac{1}{6}I_1^3(c, \beta, \gamma).$$

That is, the function d is well-defined, independent of the choice of the ground state. The above equations characterize the set of ground states of (1.6) as

$$\mathcal{G}(c, \beta, \gamma) = \left\{ u \in X : d(c, \beta, \gamma) = \frac{1}{6}I_1^3(c, \beta, \gamma) = \frac{1}{6}G(u; c, \beta, \gamma) = \frac{1}{6}K(u) \right\}. \tag{5.3}$$

Theorem 2.2 then can be stated in terms of $d(c, \beta, \gamma)$ as the relative compactness, up to translation, of a minimizing sequence for $I_1(c, \beta, \gamma)$.

($C\gamma$) If $\{u_n\}$ in X satisfies

$$\lim_{n\to\infty} G(u_n; c, \beta, \gamma) = \lim_{n\to\infty} K(u_n) = 6d(c, \beta, \gamma),$$

then there exist $u \in \mathcal{G}(c, \beta, \gamma)$, a subsequence, still renamed $\{u_n\}$, and a sequence $\{(x_n, y_n)\}$ in $\mathbb{R}^2$ such that $u_n(\cdot - x_n, \cdot - y_n) \to u$ strongly in X as $n \to \infty$.

Similarly, we may state the existence of solitary waves of (1.2) in terms of $d(c, \beta, 0)$ and the relative compactness in Y. Indeed, a solitary wave solution of the KP-I equation, (1.2) with $\beta > 0$, is obtained in [8] by the same variational approach as devised for the solitary waves of (1.1) in Section 2, but with $\gamma = 0$ and in the space Y. That is, it achieves the minimum

$$I_1(c, \beta, 0) = \inf \left\{ \frac{G(u; c, \beta, 0)}{K(u)^{2/3}} : u \in Y,\, K(u) \neq 0 \right\},$$

where

$$G(u; c, \beta, 0) = \int_{\mathbb{R}^2} (-cu^2 + \beta u_x^2 + (\partial_x^{-1} u_y)^2)\, dxdy$$

and $K(u)$ is defined in (2.2). Note that if $\beta > 0$ and $c < 0$ then

$$C_c\|u\|_Y^2 \leqslant G(u; c, \beta, 0) \leqslant C_b\|u\|_Y^2$$

for some $C_c, C_b > 0$. Let us then extend the definition of $d(c, \beta, \gamma)$ and say

$$d(c, \beta, 0) = \frac{1}{6}G(u; c, \beta, 0) = \frac{1}{6}K(u) = \frac{1}{6}I_1^3(c, \beta, 0),$$

where $u \in Y$ is a solitary wave of (1.2). One may repeat the argument for $\gamma > 0$ to assert that $d(c, \beta, 0)$ is well-defined, independent of the choice of the ground state. It is known [9], moreover, that the minima for $I_1(c, \beta, 0)$ are exactly the ground states of (5.1). The set of ground states of (5.1) is thus characterized as

$$\mathcal{G}(c, \beta, 0) = \left\{ u \in Y : d(c, \beta, 0) = \frac{1}{6}I_1^3(c, \beta, 0) = \frac{1}{6}G(u; c, \beta, 0) = \frac{1}{6}K(u) \right\}, \tag{5.4}$$

and the existence theorem in [8] is stated as is done for the ground states of (1.6), as the relative compactness of a minimizing sequence for $I_1(c, \beta, 0)$ up to translations.

$(C0)$ If $\{u_n\}$ in X satisfies

$$\lim_{n\to\infty} G(u_n; c, \beta, 0) = \lim_{n\to\infty} K(u_n) = 6d(c, \beta, 0),$$

then there exist $u \in \mathcal{G}(c, \beta, 0)$, a subsequence, still renamed $\{u_n\}$), and a sequence $\{(x_n, y_n)\}$ in $\mathbb{R}^2$ such that $u_n(\cdot - x_n, \cdot - y_n) \to u$ strongly in Y as $n \to \infty$.

The next lemma establishes the continuity and monotonicity of the function $d(c, \beta, \gamma)$.

Lemma 5.2. *In the domain $\beta > 0$, $\gamma > 0$ and $c < 2\sqrt{\gamma\beta}$, the function $d(c, \beta, \gamma)$ is continuous, and $d(c, \beta, \gamma)$ is strictly increasing in γ and β and is strictly decreasing in c.*

Proof. For $\beta > 0$, $\gamma_1, \gamma_2 > 0$ and $c \in \mathbb{R}$ such that $\gamma_1 > \gamma_2 > c_+^2/4\beta$, where $c_+ = \max\{0, c\}$ (thus, $c_+ < 2\sqrt{\beta\gamma_j}$, $j = 1, 2$), let u_1 and u_2 be ground states of (1.6) corresponding to $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively. It is straightforward that

$$\begin{aligned}
I_1(c, \beta, \gamma_2) &\leqslant \frac{G(u_1; c, \beta, \gamma_2)}{K(u_1)^{2/3}} \\
&= \frac{G(u_1; c, \beta, \gamma_1) + (\gamma_2 - \gamma_1)\int_{\mathbb{R}^2}(\partial_x^{-1}u_1)^2\,dxdy}{K(u_1)^{2/3}} \\
&= \frac{G(u_1; c, \beta, \gamma_1)}{K(u_1)^{2/3}} + (\gamma_2 - \gamma_1)\frac{\int_{\mathbb{R}^2}(\partial_x^{-1}u_1)^2\,dxdy}{K(u_1)^{2/3}} \\
&= I_1(c, \beta, \gamma_1) + (\gamma_2 - \gamma_1)\frac{\int_{\mathbb{R}^2}(\partial_x^{-1}u_1)^2\,dxdy}{K(u_1)^{2/3}} \\
&< I_1(c, \beta, \gamma_1).
\end{aligned}$$

That means, $I_1(c, \beta, \gamma)$ is strictly increasing in γ. By (5.3), correspondingly, $d(c, \beta, \gamma)$ is strictly increasing in γ.

The same calculation as above yields that

$$I_1(c, \beta, \gamma_1) \leqslant I_1(c, \beta, \gamma_2) + (\gamma_1 - \gamma_2)\frac{\int_{\mathbb{R}^2}(\partial_x^{-1}u_2)^2\,dxdy}{K(u_2)^{2/3}}.$$

Since

$$K(u_2) = G(u_2; c, \beta, \gamma_2) \geqslant C_c \int_{\mathbb{R}^2} (\partial_x^{-1} u_2)^2 \, dxdy,$$

where $C_c > 0$ is chosen as in Lemma 2.3, it follows that

$$|I_1(c, \beta, \gamma_1) - I_1(c, \beta, \gamma_2)| \leqslant C_c^{-1} I_1(c, \beta, \gamma_2)(\gamma_1 - \gamma_2).$$

That is, $I_1(c, \beta, \gamma)$ is locally Lipschitz continuous in γ. Correspondingly, $d(c, \beta, \gamma)$ is locally Lipschitz continuous in γ.

Finally, similar arguments as above are employed to show that $I_1(c, \beta, \gamma)$ is strictly decreasing in c and strictly increasing in β, and is locally Lipschitz continuous in c and β. Then, (5.2) proves the assertions. □

It is readily seen that $X \subset Y$. Furthermore, X is dense in Y.

Lemma 5.3. *The space X is dense in Y.*

Proof. For any $u \in Y$ and $\delta > 0$, let us define u_δ as

$$\hat{u}_\delta(\xi, \eta) = \begin{cases} \hat{u}(\xi, \eta) & \text{for} \quad |\xi| > \delta, \\ 0 & \text{for} \quad |\xi| \le \delta. \end{cases}$$

By Parseval's identity follow that

$$\|\partial_x^{-1} u_\delta\|_{L^2}^2 = \|\xi^{-1} \hat{u}_\delta\|_{L^2}^2 = \iint_{|\xi| > \delta} \xi^{-2} |\hat{u}(\xi, \eta)|^2 d\xi d\eta < \delta^{-2} \|u\|_{L^2}^2 < +\infty$$

and that

$$\|\partial_x^{-1} \partial_y u_\delta\|_{L^2}^2 = \|\xi^{-1} \eta \hat{u}_\delta\|_{L^2}^2 = \iint_{|\xi| > \delta} \xi^{-2} \eta^2 |\hat{u}(\xi, \eta)|^2 d\xi d\eta < \|\partial_x^{-1} u_y\|_{L^2} < +\infty.$$

Since

$$\|u_\delta\|_{L^2} \leqslant \|u\|_{L^2} < +\infty \quad \text{and} \quad \|\partial_x u_\delta\|_{L^2} \leqslant \|u_x\|_{L^2} < +\infty$$

it follows that $u_\delta \in X$. In view of the definition of u_δ and $u \in Y$ then the inequality

$$\begin{aligned} \|u_\delta - u\|_Y^2 &= \int_{|\xi| < \delta} \Big((1 + \xi^2) \|\hat{u}(\xi, \cdot)\|_{L_y(\mathbb{R})}^2 + \xi^{-2} \|\hat{u}_y(\xi, \cdot)\|_{L_y(\mathbb{R})}^2 \Big) \, d\xi \\ &\le \|u\|_Y^2 < +\infty \end{aligned}$$

holds true. Hence from continuity we may choose $\delta > 0$ sufficiently small so that

$$\|u_\delta - u\|_Y^2 = \int_{|\xi| < \delta} \Big((1 + \xi^2) \|\hat{u}(\xi, \cdot)\|_{L_y(\mathbb{R})}^2 + \xi^{-2} \|\hat{u}_y(\xi, \cdot)\|_{L_y(\mathbb{R})}^2 \Big) \, d\xi < \epsilon,$$

which completes the proof. □

Proof of Theorem 5.1. For $\beta > 0$ and $c < 0$ let $\{u_n\}$ be a sequence in X of the ground states of (1.6) with $\gamma = \gamma_n$, where $\gamma_n \to 0+$ as $n \to \infty$. It is immediate that

$$G(u_n; c, \beta, \gamma_n) = K(u_n) = I_1^3(c, \beta, \gamma_n) = d(c, \beta, \gamma_n)$$

holds for each n. Below we prove the continuity of $I_1(c,\beta,\gamma)$ at $\gamma=0$, that is, $\lim_{\gamma\to0^+} I_1(c,\beta,\gamma)=I_1(\beta,c,0)$. Then it follows that

$$\begin{aligned}G(u_n;c,\beta,0)&=G(u_n;c,\beta,\gamma_n)-\gamma_n\int_{\mathbb{R}^2}(\partial_x^{-1}u_n)^2\,dxdy\\&\leqslant G(u_n;c,\beta,\gamma_n)\\&=I_1^3(c,\beta,\gamma_n)\to I_1^3(c,\beta,0)=6d(c,\beta,0)\end{aligned}$$

and

$$K(u_n)=I_1^3(c,\beta,\gamma_n)\to I_1^3(c,\beta,0)=6d(c,\beta,0).$$

The assertion then follows from (C0).

We now claim that $\lim_{\gamma\to0^+} I_1(c,\beta,\gamma)=I_1(\beta,c,0)$. By the monotonicity of $I_1(c,\beta,\gamma)$ in γ, it suffices to show that $I_1(c,\beta,\gamma_n)\to I_1(c,\beta,0)$ for some sequence $\{\gamma_n\}$ with $\gamma_n\to0$ as $n\to\infty$. Let $u\in Y$ is a ground state of (5.1). For each n a positive integer it follows from Lemma 5.3 that there is a function $u_n\in X$ with $\|u_n-u\|_Y<1/n$. Let

$$\gamma_n=\min\left(\frac{1}{n},\frac{1}{n}\left(\int_{\mathbb{R}^2}(\partial_x^{-1}u_n)^2\,dxdy\right)^{-1}\right),$$

and as such

$$\begin{aligned}I_1(c,\beta,\gamma_n)&\leqslant\frac{G(u_n;c,\beta,\gamma_n)}{K(u_n)^{2/3}}\\&=\frac{G(u_n;c,\beta,0)+\gamma_n\int_{\mathbb{R}^2}(\partial_x^{-1}u_n)^2\,dxdy}{K(u_n)^{2/3}}\\&\leqslant\frac{G(u_n;c,\beta,0)+1/n}{K(u_n)^{2/3}}.\end{aligned}$$

Since both $G(u;c,\beta,0)$ and $K(u)$ are continuous functions on Y, therefore it follows that

$$\lim_{n\to\infty}I_1(c,\beta,\gamma_n)\leqslant\frac{G(u;c,\beta,0)}{K(u)^{2/3}}=I_1(c,\beta,0).$$

On the other hand, since $I_1(c,\beta,\gamma)$ is strictly increasing in γ, it follows that

$$\lim_{n\to\infty}I_1(c,\beta,\gamma_n)=I_1(c,\beta,0)$$

This proves the claim. The proof is complete. □

Remark 5.4 (Stability of ground states in terms of $d(c)$). The function $d(c,\beta,\gamma)$ may serve as a Lyapunov function in the study of stability of ground states of (1.6) (see [16], for instance). Indeed, it is standard to show that for $\beta>0$ and $c<2\sqrt{\beta\gamma}$ the set of ground states of (1.6) is X-stable provided that $d''(c)>0$. Since $d(c)$ is strictly increasing in c, we may write $c(u)=d^{-1}(K(u)/6)$, where u is a ground state of (1.6). The proof then relies on the expansion near a ground state $u\in\mathcal{G}(c,\beta,\gamma)$ that

$$E(v)-E(u)-c(v)(V(v)-V(u))\geqslant\frac{1}{4}d''(c)|c(v)-c|^2$$

for $v\in X$ in the ϵ-tube, $\inf_{u\in\mathcal{G}(c,\beta,\gamma)}\|v-u\|_X<\epsilon$. However, it is difficult to establish the convexity of $d''(c)$ since it is difficult to describe $d(c)$ explicitly in terms of c.

6. Stability of the set of ground states

This section is devoted to the orbital stability of the ground states of (1.6) in the sense in Definition 1.2.

As is discuss in Remark 5.4, a method of studying orbital stability of solitary waves, based on the fact that ground states are characterized as

$$d(c,\beta,\gamma) := S(u;c,\beta,\gamma) = \frac{1}{6}G(u;c,\beta,\gamma) = \frac{1}{6}I_1^3(c,\beta,\gamma),$$

has difficulties since the usual scaling and dilation technique does not give the description of the function $d(c,\beta,\gamma)$ corresponding to (1.1) explicitly in terms of the wave speed c. On the other hand, this technical part may be avoided if it is possible to show directly that the ground states considered are global minima of energy constrained by constant momentum, which is the approach taken here. We show in case of weak effects of rotation, i.e. $\gamma > 0$ small that the ground state solutions of (1.6) with $c < 0$ are characterized as energy minimizers constrained by constant momentum and study its implications for nonlinear stability. The smallness of $\gamma > 0$ is required to ensure that the energy of the rotation terms is dominated by the energy of the Kadomtsev-Petrivashvili terms. See Remark 6.5.

For $\beta, \gamma > 0$ and $q > 0$, let us consider the minimization problem

$$j_q(\beta,\gamma) = \inf\{E(u;\beta,\gamma) : u \in X,\ V(u) = q\}. \tag{6.1}$$

Let

$$\Sigma_q(\beta,\gamma) = \{u \in X : E(u;\beta,\gamma) = j_q(\beta,\gamma),\ V(u) = q\} \tag{6.2}$$

denote the set of minimizers for $j_q(\beta,\gamma)$.

The Euler-Lagrange equation for the constrained minimization problem for $\Sigma_q(\beta,\gamma)$ is

$$-\beta u_{xx} + u^2 + \partial_x^2 u_{yy} - \gamma\partial_x^2 u = \theta u \qquad \text{in} \quad X'$$

for some Lagrange multiplier $\theta \in \mathbb{R}$, where X' is the dual space of X with respect to the L^2-duality, and $\partial_x^{-2}u_{yy}$ and $\partial_x^{-2}u$ are elements of X'. In view of (1.6) this says that if $u \in \Sigma_q(\beta,\gamma)$ then u is a solitary wave of (1.1) with the wave speed θ.

Multiplication of the above equation by u and integration by parts yield that

$$\int_{\mathbb{R}^2} (\beta u_x^2 + u^3 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2)\ dxdy = \theta \int_{\mathbb{R}^2} u^2\ dxdy.$$

On the other hand, we shall show below in Lemma 6.3 that

$$E(u;\beta,\gamma) = \int_{\mathbb{R}^2} \left(\frac{\beta}{2}u_x^2 + \frac{1}{3}u^3 + \frac{1}{2}(\partial_x^{-1}u_y)^2 + \frac{\gamma}{2}(\partial_x^{-1}u)^2\right) dxdy < 0$$

for any $q = \frac{1}{2}\int_{\mathbb{R}^2} u^2\ dxdy > 0$, if $\gamma > 0$ is sufficiently small. In particular, a constrained energy minimizer must satisfy $\int_{\mathbb{R}^2} u^3\ dxdy < 0$. Accordingly,

$$\int_{\mathbb{R}^2} (\beta u_x^2 + u^3 + (\partial_x^{-1}u_y)^2 + \gamma(\partial_x^{-1}u)^2)\ dxdy = 2E(u;\beta,\gamma) + \frac{1}{3}\int_{\mathbb{R}^2} u^3\ dxdy < 0,$$

and therefore, $\theta < 0$. That is, for $\beta > 0$ and $\gamma > 0$ sufficiently small, $u \in \Sigma_q(\beta,\gamma)$ is a ground state of (1.6) for some wave speed $c < 0$. In the absence of the effects of rotation, $\gamma = 0$, a scaling argument asserts that the speed of wave propagation c in the solitary wave problem (1.6) may be set to be $c = -1$. Then, the set of minima for j_q is exactly equal to the set of ground states. In contrast, with a nontrivial rotation effects, it is not known that a ground state with $c<0$ is indeed a minimizer for j_q.

Our main result in this section is that $\Sigma_q(\beta,\gamma)$ is X-stable, provided that $\gamma>0$ is small.

Theorem 6.1 (Orbital stability of energy minimizers)**.** *For $\beta>0$ and $q>0$ there exists $\gamma_0>0$ sufficiently small such that for $0<\gamma<\gamma_0$, the set $\Sigma_q(\beta,\gamma)$ defined in* (6.2) *is X-stable.*

Note that if u is in the set Σ_q then so is its translate $u(\cdot-x,\cdot-y)$ with $(x,y)\in\mathbb{R}^2$. Theorem 6.1 then says in the light of Definition 1.2 that if $\phi\in X\cap X_s$ is close to $u\in\Sigma_q$ in X then the solution of the Cauchy problem for (1.1) with initial value ϕ remains close to the orbit of u modulo translations.

The proof of Theorem 6.1 uses that energy minimizers constrained by constant momentum is a ground state of (1.6). This characterization of global energy minimizers as ground states and its implication for stability were first used for the standing waves of the subcritical nonlinear Schrödinger equation [12] and ground states of the generalized Kadomtsev-Petvishvili equation [10]. A main difference from [12, 10] is that (1.1) lacks scaling properties and thus one may not obtain a description of the minimizer for j_q explicitly in terms of q.

Proposition 6.2. *For $q>0$ and $\beta>0$ there exists $\gamma_0>0$ sufficiently small such that for $0<\gamma<\gamma_0$, the followings hold true.*

(a) *The minimization problem of $j_q(\beta,\gamma)$ in* (6.1) *has a solution.*

(b) *Any minimizing sequence $\{u_n\}$ for $j_q(\beta,\gamma)$, i.e.*

$$V(u_n)=q \quad \textit{for all } n \quad \textit{and} \quad E(u_n;\beta,\gamma)\to j_q \quad \textit{as } n\to\infty,$$

is relatively compact in X up to translations. That is, there exist a sequence of translation vectors $\{(x_n,y_n)\}$ and $u\in\Sigma_q$ such that a subsequence of $u_n(\cdot-x_n,\cdot-y_n)$ converges strongly in X to u.

(c) $\lim_{n\to\infty}\inf_{u\in\Sigma_q(\beta,\gamma)}\|u_n-u\|_X=0$*, where $\{u_n\}$ is a minimizing sequence for $j_q(\beta,\gamma)$.*

The proofs of (a) and (b) of Proposition 6.2 use a modified concentration-compactness lemma [26], for which we will need several preliminary results. Our first preliminary result is that $-\infty<j_q(\beta,\gamma)<0$ provided that $\gamma>0$ is small.

Lemma 6.3. *For $q>0$ and $\beta>0$, there exists $\gamma_0>0$ sufficiently small such that if $0<\gamma<\gamma_0$ then $-\infty<j_q(\beta,\gamma)<0$.*

Proof. We first show that $j_q(\beta)>-\infty$. It is straightforward to see that

$$\begin{aligned}\|u\|_{L^3}^3 &\leqslant C\|u\|_{L^2}^{3/2}(\|u\|_{L^2}+\|u_x\|_{L^2})\|\partial_x^{-1}u_y\|_{L^2}^{1/2}\\ &\leqslant C_\epsilon(\|u\|_{L^2}^{10/3}+\|u\|_{L^2}^6)+\epsilon(\|u_x\|_{L^2}^2+\|\partial_x^{-1}u_y\|_{L^2}^2)\end{aligned}$$

for any $u\in X$, for $\epsilon>0$ small, where $C,C_\epsilon>0$ are independent of u. The first inequality uses the embedding theorem for the anistropic Sobolev spaces, and the second inequality uses Young's inequality. Let $u\in X$ satisfy $V(u)=q$. One can then deduce that

$$\begin{aligned}E(u;\beta,\gamma) &= E(u;\beta,\gamma)+\beta V(u)-\beta V(u)\\ &= \int_{\mathbb{R}^2}\Big(\frac{\beta}{2}u_x^2+\frac{\beta}{2}u^2+\frac{1}{2}(\partial_x^{-1}u_y)^2+\frac{\gamma}{2}(\partial_x^{-1}u)^2+\frac{1}{3}u^3\Big)\,dxdy-\beta q\\ &\geqslant \frac{1}{2}\min(\beta,\gamma)\|u\|_{L^2}^2-\frac{1}{3}\epsilon\|u\|_{L^2}^2-C_\epsilon(\|u\|_{L^2}^{10/3}+\|u\|_{L^2}^6)-\beta q\\ &\geqslant \frac{1}{4}q-C_{\beta,\gamma}((2q)^{5/3}+(2q)^3)-\beta q>-\infty.\end{aligned}$$

Our next task is to show that $j_q(\beta,\gamma)<0$ when γ is small. Let $u_1\in\mathcal{G}(c_1,\beta,1)$ for $c_1<0$. Since $P(u_1)=0$ it follows that

$$-\int_{\mathbb{R}^2}(u_1)^3\;dxdy=\int_{\mathbb{R}^2}\Big(-c_1(u_1)^2+\beta(\partial_x u_1)^2+(\partial_x^{-1}u_1)^2+(\partial_x^{-1}\partial_y u_1)^2\Big)\;dxdy>0.$$

Now let us choose a constant $a>0$ so that $w=au_1$ satisfies that

$$V(w)=a^2V(u_1)=q.$$

That is, $a=\sqrt{q/V(u_1)}$. Note that $\int_{\mathbb{R}^2}\;w^3\;dxdy=a^3\int_{\mathbb{R}^2}\;(u_1)^3\;dxdy<0$. For $b>0$ let us define $w_b\in X$ as

$$w_b(x,y)=\delta w(b^{5/8}x,b^{11/8}y).$$

It is straightforward that $V(w_b)=V(w)=q$ for all $b>0$. Moreover,

$$\begin{aligned}E(w_b;\beta,\gamma) &= \int_{\mathbb{R}^2}\;\Big(\frac{\beta}{2}b^{5/4}w_x^2+\frac{\gamma}{2}b^{-5/4}(\partial_x^{-1}w)^2+\frac{1}{2}b^{3/2}(\partial_x^{-1}w_y)^2+\frac{1}{3}bw^3\Big)\;dxdy\\ &= b\int_{\mathbb{R}^2}\;\Big(\frac{\beta}{2}b^{1/4}w_x^2+\frac{1}{2}b^{1/4}(\partial_x^{-1}w)^2+\frac{1}{2}b^{1/2}(\partial_x^{-1}w_y)^2+\frac{1}{3}w^3\Big)\;dxdy\\ &= ba^2\int_{\mathbb{R}^2}\;\Big(\frac{\beta}{2}b^{1/4}(\partial_x u_1)^2+\frac{1}{2}b^{1/4}(\partial_x^{-1}u_1)^2+\frac{1}{2}b^{1/2}(\partial_x^{-1}\partial_y u_1)^2+\frac{1}{3}au_1^3\Big)\;dxdy,\end{aligned}$$

provided that $b=\gamma^{2/5}>0$. By taking $\gamma_0>0$ sufficiently small ($\gamma_0<(q/V(u_1))^{5/2}/3$, for instance) for each $0<\gamma<\gamma_0$ it follows that $E(w_b;\beta,\gamma)<0$. Consequently, $j_q(\beta,\gamma)<0$. This completes the proof. ☐

The smallness of γ is required in the proof to ensure that $j_q(\beta,\gamma)<0$. More precisely, energy due to the Kadomtsev-Petviashvili considerations must dominate the energy due to rotation effects.

The next lemma establishes the subadditivity property of j_q.

Lemma 6.4. *Any $q_1,q_2>0$ with $q_1+q_2=q$ satisfy*

$$j_q<j_{q_1}+j_{q_2}. \tag{6.3}$$

Proof. We claim that for any $0<q'<q$ it follows that

$$j_{q'}\geqslant\Big(\frac{q'}{q}\Big)^{5/3}j_q. \tag{6.4}$$

To see this, let $u \in X$ be such that $V(u) = q$, and let us define $w(x, y) = b^2 u(bx, y)$, where $b > 0$. A straightforward calculation yields that

$$E(w) = b^5 \int_{\mathbb{R}^2} \Big(\frac{\beta}{2}u_x^2 + \frac{\gamma}{2}b^{-4}(\partial_x^{-1}u)^2 + \frac{1}{2}b^{-4}(\partial_x^{-1}u_y)^2 + \frac{1}{3}u^3\Big)\ dxdy,$$
$$V(w) = \frac{1}{2}b^3 \int_{\mathbb{R}^2} u^2\ dxdy = b^3 q.$$

We choose $b = (q'/q)^{1/3}$, and as such $b < 1$. Accordingly, $V(w) = q'$ and

$$E(w) > b^5 E(u) = \Big(\frac{q'}{q}\Big)^{5/3} E(u).$$

This implies (6.4).

As a consequence, for any $q_1, q_2 > 0$ satisfying $q_1 + q_2 = q$ it follows that

$$j_{q_1} + j_{q_2} \geqslant \Big(\Big(\frac{q_1}{q}\Big)^{5/3} + \Big(\frac{q_2}{q}\Big)^{5/3}\Big) j_q > j_q.$$

This completes the proof. □

Remark 6.5. We may extend our definition of $j_q(\beta, \gamma)$ to the case when $\gamma = 0$ as

$$j_q(\beta, 0) = \inf\{E(u; \beta, 0) : u \in Y,\ V(u) = q\},$$

where

$$E(u; \beta, 0) = \int_{\mathbb{R}^2} \left(\frac{\beta}{2}u_x^2 + \frac{1}{2}(\partial_x^{-1}u_y)^2 + \frac{1}{3}u^3\right) dxdy.$$

It is then straightforward to see that for $u_b(x, y) = b^2 u(bx, b^2 y)$ the energy and momentum are scaled as

$$E(u_b; \beta, 0) = b^3 E(u; \beta, 0) \quad \text{and} \quad V(u_b) = bV(u).$$

Hence, if $-\infty < E(u, \beta, 0) < 0$ for some q then $-\infty < E(u; \beta, 0) < 0$ for all q, and thus Lemma 6.3 becomes immediate. Moreover, $j_q(\beta, 0) = q^3 j_1(\beta, 0)$, and consequently, it follows that $-\infty < j_q(\beta, 0) < 0$ for any $q > 0$ and that $j_q < j_{q_1} + j_{q_2}$ for any $q_1, q_2 > 0$ satisfying $q_1 + q_2 = q$. See the proof of [10, Lemma 2.2]. Thus, the application of the concentration-compactness lemma is standard.

By allowing the effects of rotation, on the other hand, one breaks down the scaling property of the Kadomtsev-Petviashvili equation (1.2). More precisely, $V(u_b) = bV(u)$ yet

$$E(u_b; \beta, \gamma) = b^3 \int_{\mathbb{R}^2} \left(\frac{\beta}{2}u_x^2 + \frac{1}{2}(\partial_x^{-1}u_y)^2 + \frac{1}{3}u^3\right)\ dxdy + b^{-1} \int_{\mathbb{R}^2} \frac{\gamma}{2}(\partial_x^{-1}u)^2\ dxdy.$$

In other words, as $b \to 0$ the L^2-norm $V(u_b)$ shrinks while the energy $E(u_b; \beta, \gamma)$ grows unboundedly by concentrating it on the rotation term, unless the rotation coefficient γ shrinks accordingly so that the energy due to rotation effects is not dominant.

Lemma 6.6. *For $q > 0$, $\beta > 0$ and $0 < \gamma < \gamma_0$, where $\gamma_0 > 0$ is obtained in the proof of Lemma 6.3, let $\{u_n\}$ is a minimizing sequence for j_q. That is,*

$$V(u_n) = q \quad \text{for all } n, \qquad E(u_n; \beta, \gamma) \to j_q \quad \text{as } n \to \infty.$$

(a) *$\|u_n\|_X$ is bounded for n.*
(b) *$\|u_n\|_{L^3} \geqslant \delta_0 > 0$ for all sufficiently large n for some δ_0.*
(c) *For a subsequence, still denoted by $\{u_n\}$, it follows $\lim_{n\to\infty} \|u_n\|_X^2 = \alpha > 0$.*

Proof. (a) In view of the coercivity property of $G(u;0,\beta,\gamma)$ in (2.6) the inequalities

$$\begin{aligned}\|u_n\|_X^2 &= \int_{\mathbb{R}^2} \Big((\partial_x u_n)^2 + (\partial_x^{-1}\partial_y u_n)^2 + (\partial_x^{-1}u_n)^2\Big)\, dxdy + 2q \\ &\leqslant C_c^{-1} G(u_n;0,\beta,\gamma) + 2V(u_n) \\ &= 2C_c^{-1}\Big(E(u_n;\beta,\gamma) - \frac{1}{3}\int_{\mathbb{R}^2} u_n^3\, dxdy\Big) + 2V(u_n) \\ &\leqslant 2C_c^{-1}\sup_n E(u_n;\beta,\gamma) + 2q \\ &\qquad + C\|u_n\|_{L^2}^{3/2}(\|u_n\|_{L^2} + \|\partial_x u_n\|_{L^2})\|\partial_x^{-1}\partial_y u_n\|_{L^2}^{1/2} \\ &\leqslant C'(1 + \|u_n\|_X^{3/2})\end{aligned}$$

hold, where $C_c > 0$ is given in the proof of Lemma 2.3 and $C' > 0$ depend only on β, γ and q, j_q. This proves the assertion.

(b) Suppose on the contrary that

$$\liminf_{n\to\infty} \|u_n\|_{L^3} = 0.$$

This would imply

$$\begin{aligned}j_q(\beta,\gamma) &= \lim_{n\to\infty}\int_{\mathbb{R}^2} \Big(\frac{\beta}{2}(\partial_x u_n)^2 + \frac{\gamma}{2}(\partial_x^{-1}u_n)^2 + \frac{1}{2}(\partial_x^{-1}\partial_y u_n)^2 + \frac{1}{3}u_n^3\Big)\, dxdy \\ &\geqslant -\frac{1}{3}\liminf_{n\to\infty}\|u_n\|_{L^3}^3 = 0,\end{aligned}$$

which however contradicts Lemma 6.3.

(c) By (a) it follows that there exists a subsequence, still denoted $\{u_n\}$, such that $\lim_{n\to\infty}\|u_n\|_X = \alpha$. By (b) then it follows that

$$0 < \delta_0^3 \leqslant \|u_n\|_{L^3}^3 \leqslant C_0\|u_n\|_{L^2}^{3/2}(\|u_n\|_{L^2} + \|\partial_x u_n\|_{L^2})\|\partial_x^{-1}\partial_y u_n\|_{L^2}^{1/2} \leqslant C_0\|u_n\|_X^3.$$

Therefore $\alpha > 0$. This completes the proof. □

Proof of Proposition 6.2. The proof of (a) and (b) uses the concentration-compactness lemma as is done in the proof of Theorem 2.2, and thus we only provide its sketch.

Let $\{u_n\}$ be a minimizing sequence for $j_q(\beta,\gamma)$. The results of Lemma 6.6 (a) and (c) are that $\{u_n\}$ is bounded in X and that there is a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\|_X^2 \to \alpha > 0$ as $n \to \infty$. We apply the concentration-compactness lemma [26] to

$$\bar\rho_n = (\partial_x u_n)^2 + u_n^2 + (\partial_x^{-1}\partial_y u_n)^2 + (\partial_x^{-1}u_n)^2. \tag{6.5}$$

Note that $\int_{\mathbb{R}^2}\bar\rho_n\, dxdy = \|u_n\|_X^2$.

First, "vanishing" is ruled out since in this case the embedding theorem for anistropic Sobolev spaces [3] applies to assert u_n tends to zero as $n \to \infty$ in $L^q(\mathbb{R}^2)$ for $2 < q < 6$ while Lemma 6.6 says that $\|u_n\|_{L^3} > \delta_0 > 0$ for n sufficiently large.

Next, "dichotomy" can be treated by the similar method as applied in Theorem 2.2. Using the same notation of the concentration function Q as defined in (2.11) we have that in the dichotomy case,

$$\lim_{r\to\infty} Q(r) = \eta \in (0,\alpha),$$

and hence there is a splitting of u_n into w_n and v_n satisfying that for any $\epsilon > 0$ there exist a $\delta(\epsilon) > 0$ (with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$), and an integer $n_0 > 0$ such that for $n \geqslant n_0$,

$$\begin{aligned}
&\|w_n + v_n - u_n\|_X \leqslant \delta(\epsilon),\\
&\Big|\|w_n\|_X^2 - \eta\Big| \leqslant \delta(\epsilon), \quad \Big|\|v_n\|_X^2 - (\alpha-\eta)\Big| \leqslant \delta(\epsilon),\\
&\Big|\|w_n\|_{L^2}^2 + \|v_n\|_{L^2}^2 - \|u_n\|_{L^2}^2\Big| \leqslant \delta(\epsilon),\\
&\Big|E(w_n) + E(v_n) - E(u_n)\Big| \leqslant \delta(\epsilon)
\end{aligned}$$

and

$$\text{supp} w_n \bigcap \text{supp} v_n = \emptyset, \quad \text{dist}(\text{supp} w_n,\ \text{supp} v_n) \to +\infty.$$

Taking a subsequence if necessary, we may assume

$$\lim_{n\to\infty} \|w_n\|_{L^2} = q_1(\epsilon), \quad \lim_{n\to\infty} \|v_n\|_{L^2} = q_2(\epsilon)$$

with $|q_1(\epsilon) + q_2(\epsilon) - q| \leqslant \delta(\epsilon)$. Applying the similar scaling argument in Theorem 2.2 we obtain that $\lim_{\epsilon\to 0} q_i(\epsilon) > 0$, for $i = 1, 2$ and

$$j_{q_1(\epsilon)} + j_{q_2(\epsilon)} \leqslant \liminf_{n\to\infty} E(w_n) + \liminf_{n\to\infty} E(v_n) \leqslant j_q + \delta(\epsilon).$$

Hence by letting $\epsilon \to 0$ we reach a contradiction with the subadditivity property of j_q in Lemma 6.4.

The only remaining possibility is then "compactness". That is, $\{u_n\}$ is relatively compact up to translations. As is shown in the proof of Theorem 2.2, this implies that a subsequence, still denoted by $\{u_n(\cdot - x_n, \cdot - y_n)\}$, converges weakly in X to some $u \in X$. Using the relative compactness of the injection $X \subset L^2_{loc}(\mathbb{R}^2)$, one then obtains that $u_n(\cdot - x_n, \cdot - y_n)$ converges to u strongly in $L^2(\mathbb{R}^2)$ and furthermore in $L^3(\mathbb{R}^2)$, which is a minimum of $j_q(\beta, \gamma)$. See the proof of Theorem 2.2 for details. It follows that it converges indeed to u strongly in X. This proves (a) and (b).

(c) We first claim that

$$\lim_{n\to\infty} \inf_{u\in\Sigma_q} \inf_{(x,y)\in\mathbb{R}^2} \|u_n(\cdot - x, \cdot - y) - u\|_X = 0.$$

Suppose on the contrary that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\inf_{u\in\Sigma_q} \inf_{(x,y)\in\mathbb{R}^2} \|u_{n_k}(\cdot - x, \cdot - y) - u\|_X \geqslant \delta > 0 \tag{6.6}$$

for all $k \geqslant 1$. By (a) it then follows that there exist a sequence $\{(x_k, y_k)\}$ in $\mathbb{R}^2$ and $u^* \in \Sigma_q(\beta, \gamma)$ such that

$$\lim_{k\to\infty} \|u_{n_k}(\cdot - x_k, \cdot - y_k) - u^*\|_X = 0,$$

which contradicts (6.6). This proves the claim.

Since $E(u)$ and $V(u)$ are translation invariant, $u \in \Sigma_q(\beta, \gamma)$ implies $u(\cdot - x, \cdot - y) \in \Sigma_q(\beta, \gamma)$ for any $(x, y) \in \mathbb{R}^2$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \inf_{u\in\Sigma_q} \|u_n - u\|_X &\leqslant \lim_{n\to\infty} \inf_{u\in\Sigma_q} \|u_n - u(\cdot - x, \cdot - y))\|_X\\
&= \lim_{n\to\infty} \inf_{u\in\Sigma_{q^*}} \|u_n(\cdot - x, \cdot - y) - u\|_X = 0.
\end{aligned}$$

This proves the assertion. □

Now we are ready to prove the stability of $\Sigma_q(\beta,\gamma)$.

Proof of Theorem 6.1. For simplicity of exposition, we drop β and γ dependence of E and Σ.

The assertion follows from Proposition 6.2 by using the classical argument which we repeat here for completeness. Suppose on the contrary that for some $u^*\in\Sigma_q$ there exist $\epsilon>0$, sequences $\{\phi_n\}$ in X and $\{t_n\}$ with $0\leqslant t_n<T$ such that $\|\phi_n-u^*\|_X\to 0$ as $n\to\infty$ yet

$$\inf_{u\in\Sigma_q}\|u_n(t_n)-u\|_X\geqslant\epsilon>0$$

for all $n\geqslant 1$, where $u_n(t)$ is the unique solution of the Cauchy problem associated to (1.1) on the time interval $[0,T)$ with the initial condition $u_n(0,x,y)=\phi_n(x,y)$.

Since $\phi_n\to u^*$ in X as $n\to\infty$ and since $E(u^*)=j_q$ and $V(u^*)=q$, it follows that

$$E(\phi_n)\to j_q \quad\text{and}\quad V(\phi_n)\to q$$

as $n\to\infty$. Moreover, Since $E(u)$ and $V(u)$ are conservation laws of (1.1), it follows that

$$E(u_n(t))=E(\phi_n)\to j_q \quad\text{and}\quad V(u_n(t))=V(\phi_n)\to q$$

as $n\to\infty$. In particular, $\{u_n(t_n)\}$ is bounded in the X-norm, say by $C>0$.

Let

$$\alpha_n=\Big(\frac{q}{V(\phi_n)}\Big)^{1/2},$$

and as such $\alpha_n\to 1$ as $n\to\infty$. Note that $V(\alpha_nu_n)=q$ for each n. Since $E(\alpha_nu_n)\to j_q$ as $n\to\infty$, it follows that $\{\alpha_nu_n\}$ is a minimizing sequence for j_q in (6.1). Proposition 6.2 (c) then asserts that for n sufficiently large there exists $u_n^*\in\Sigma_q$ such that $\|\alpha_nu_n-u_n^*\|_X<\epsilon/2$. It is straightforward that

$$\begin{aligned}\|u_n(t_n)-u_n^*\|_X&\leqslant\|u_n(t_n)-\alpha_nu_n\|_X+\|\alpha_nu_n-u_n^*\|_X\\&\leqslant|1-\alpha_n|\|u_n(t_n)\|_X+\epsilon/2<|1-\alpha_n|C+\epsilon/2.\end{aligned}$$

This however contradicts since $\inf_{u\in\Sigma_q}\|u_n(t_n)-u\|_X\geqslant\epsilon$. This completes the proof. ☐

Appendix A. Remarks on the well-posedness

This section concerns the existence and uniqueness of the solution to the Cauchy problem

$$\begin{cases}(u_t-\beta u_{xxx}+(u^2)_x)_x+u_{yy}-\gamma u=0\\ u(0,x,y)=\phi(x,y).\end{cases}\tag{A.1}$$

in the Sobolev space H^s, where $s>2$.

Our first step is the establishment of the local well-posedness of (A.1) in X_s for $s>2$. Recall that

$$X_s=\{u\in H^s(\mathbb{R}^2):(\xi^{-1}\hat{f})^\vee\in H^s(\mathbb{R}^2)\}.$$

Proof of Theorem 1.1. The proof uses the parabolic regularization in [21]. Let $\epsilon>0$ and let $\phi^\epsilon\in\partial_x(C_0^\infty(\mathbb{R}^2))$ converge to ϕ in X_s as $\epsilon\to 0$. We look at the regularized problem of (1.1)

$$\Big(u_t+\epsilon\Delta^2u_t-\beta u_{xxx}+(u^2)_x\Big)_x+u_{yy}-\gamma u=0.\tag{A.2}$$

It is standard that (A.2) has a unique solution $u^\epsilon \in C([0,T], H^k)$, where k is any fixed integer.

Integrating (A.2) in x once and taking the H^s scalar product with u^ϵ yields that

$$\frac{1}{2}\frac{d}{dt}\Big(\|u^\epsilon\|_{H^s}^2 + \epsilon\|\Delta u^\epsilon\|_{H^s}^2\Big) + 2(u^\epsilon u_x^\epsilon, u^\epsilon)_s = 0,$$

where $(\cdot,\cdot)_s$ denotes the H^s scalar product. Then from the standard commutator estimate we obtain that

$$\begin{aligned}\frac{d}{dt}\Big(\|u^\epsilon\|_{H^s}^2 + \epsilon\|\Delta u^\epsilon\|_{H^s}^2\Big) &\leqslant C\|\nabla u^\epsilon\|_{L^\infty}\|u^\epsilon\|_{H^s}^2 \\ &\leqslant C\|u^\epsilon\|_{H^{2+}}\|u^\epsilon\|_{H^s}^2.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\|\partial_x^{-1}u^\epsilon\|_{H^s}^2 + \epsilon\|\Delta\partial_x^{-1}u^\epsilon\|_{H^s}^2\Big) &\leqslant C\|u^\epsilon\|_{L^\infty}\|\partial_x^{-1}u^\epsilon\|_{H^s}\|u^\epsilon\|_{H^s} \\ &\leqslant C\|u^\epsilon\|_{H^{1+}}\|\partial_x^{-1}u^\epsilon\|_{H^s}\|u^\epsilon\|_{H^s}.\end{aligned}$$

Therefore, we eliminate the ϵ term and arrive at

$$\frac{d}{dt}\|u^\epsilon\|_{X_s}^2 \leqslant C\|u^\epsilon\|_{H^{2+}}\|u^\epsilon\|_{X^s}^2.$$

Thus, $T = T(\|\phi\|_{H^{2+}})$. A standard compactness argument shows that (1.1) possesses for each $\phi \in X_s$, where $s > 2$, a unique solution $u \in L^\infty([0,T], X_s)$. Further, by the regularization technique of [6] it follows that $u \in C([0,T], X_s)$ and the solution depends continuously on ϕ. □

Next, let us write down the corresponding integral equation for (1.1) as

$$u(t) = W(t)\phi - \int_0^t W(t-\tau)\partial_x\big(u^2(\tau)\big)\,d\tau, \tag{A.3}$$

where $W(t) = \exp(t(\beta\partial_x^3 - \partial_x^{-1}\partial_y^2 + \gamma\partial_x^{-1}))$ is the Fourier multiplier with symbol $\exp\left(-it\left(\beta\xi^3 + \dfrac{\eta^2+\gamma}{\xi}\right)\right)$.

The next theorem concerns the local well-posedness of (1.1) in $H^s(\mathbb{R}^2)$ as a solution to the integral equation (A.3).

Theorem A.1. *For $\phi \in H^s(\mathbb{R}^2)$, $s > 2$, the integral equation* (A.3) *has a unique solution $u(t) \in H^s(\mathbb{R}^2)$. Moreover, the map $\phi \mapsto u(t)$ is continuous in the H^s-norm.*

Proof. Given $\phi \in H^s(\mathbb{R}^2)$ with $s > 2$, one may find [32] a sequence $\{\phi^\epsilon\} \subset X_\infty$ such that ϕ^ϵ converges to ϕ in $H^s(\mathbb{R}^2)$. Theorem 1.1 then asserts that for each ϵ a unique solution $u^\epsilon \in C([0,T^\epsilon], X_\infty)$ of (A.1) exists with initial data ϕ^ϵ. As in the proof of Theorem 1.1 we know that $\{u^\epsilon\}$ is bounded in $L^\infty([0,T], H^s)$ where $T = T(\|\phi\|_{H^s})$.

On $[0,T]$ the limit function u^ϵ satisfies

$$u^\epsilon(t) = W(t)\phi^\epsilon - \int_0^t W(t-\tau)\partial_x\big(u^\epsilon(\tau)\big)^2\,d\tau := W(t)\phi^\epsilon - v^\epsilon(t).$$

Our goal is then to show that

$$v^\epsilon(t) \to v(t) = \int_0^t W(t-\tau)\partial_x(u(\tau))^2 d\tau$$

as $\epsilon \to 0$.

Since v^ϵ is bounded in $L^\infty([0,T], H^{s-1}(\mathbb{R}^2))$, from

$$v_t^\epsilon = \partial_x\big(u^\epsilon(t)\big)^2 - \int_0^t (\beta\partial_x^3 - \partial_x^{-1}\partial_y^2 + \gamma\partial_x^{-1})W(t-\tau)\partial_x\big(u^\epsilon(\tau)\big)^2\, d\tau$$

it follows that v_t^ϵ is bounded in $L^\infty([0,T], H^{s-4}(\mathbb{R}^2))$. Subsequently, the Aubin-Lions compactness theorem asserts that $v^\epsilon \to v$ as $\epsilon \to 0$ in $L^2_{loc}((0,T)\times\mathbb{R}^2)$.

Next, since $W(t)\phi^\epsilon \to W(t)\phi$ in $L^\infty((0,T), H^s(\mathbb{R}^2))$, it follows that $u^\epsilon \to u$ as $\epsilon \to 0$ in $L^2_{loc}((0,T)\times\mathbb{R}^2)$. Since $L^\infty((0,T), H^{s-1}(\mathbb{R}^2))$ is an algebra, then $(u^\epsilon)^2$ is bounded in $L^\infty((0,T), H^{s-1}(\mathbb{R}^2))$ and the above convergence of u^ϵ in $L^2_{loc}((0,T)\times\mathbb{R}^2)$ asserts that $(u^\epsilon)^2 \to u^2$ as $\epsilon \to 0$ weakly in $L^2((0,T), H^{s-1}(\mathbb{R}^2)$. Hence, for a fixed $t > 0$ it follows that $W(t-\tau)\partial_x\big(u^\epsilon(\tau)\big)^2 \to W(t-\tau)\partial_x\big(u(\tau)\big)^2$ as $\epsilon \to 0$ weakly in $L^2((0,T), H^{s-2}(\mathbb{R}^2))$, and accordingly,

$$\int_0^t W(t-\tau)\partial_x\big(u^\epsilon(\tau)\big)^2\, d\tau \to \int_0^t W(t-\tau)\partial_x\big(u(\tau)\big)^2\, d\tau \quad \text{weakly in} \quad H^{s-2}(\mathbb{R}^2).$$

In particular, the convergence is in sense of distributions. Therefore,

$$v(t) = \int_0^t W(t-\tau)\partial_x\big(u(\tau)\big)^2\, d\tau.$$

which completes the proof. □

The conservation laws

$$E(u(t)) = E(\phi) \quad \text{and} \quad V(u(t)) = V(\phi),$$

where E and V are defined as in (1.4) and (1.5) suggest that a natural space to establish the well-posedness of the Cauchy problem associated to (1.1) be the "energy" space X_1. Indeed, (1.1) is not completely integrable[§] [18] (although it is a Hamiltonian system) and its global well-posedness may not utilize higher Sobolev norms.

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