# WAVE-BREAKING AND GLOBAL EXISTENCE FOR A GENERALIZED TWO-COMPONENT CAMASSA-HOLM SYSTEM

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ABSTRACT. In this paper we study a generalized two-component Camassa-Holm system which can be derived from the theory of shallow water waves moving over a linear shear flow. This new system also generalizes a class of dispersive waves in cylindrical compressible hyperelastic rods. We show that this new system can still exhibit the wave-breaking phenomenon. We also determine the exact blow-up rate of such solutions. In addition, we establish a sufficient condition for global solutions.

## 1. INTRODUCTION

An interesting phenonmenon in water wave channels is the appearance of waves with length much greater than the depth of the water. Various models have been proposed in understanding the long wave, shallow water problem. In the context of nonlinear elasticity, similar dynamics arise in the study of an elastic rod whose diameter is much smaller than the axial length scale. Such systems describe in a relatively simple way the competition between nonlinear and dispersive effects. The best known model is the Korteweg-de Vries (KdV) equation (see [50] and [44] for a physical derivation).

In 1993, Camassa and Holm [7] proposed a following new equation (CH) for shallow water waves:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.1}$$

It is a nonlinear dispersive wave equation that models the propagation of unidirectional irrotational shallow water waves over a flat bed [7, 18, 31, 33], as well as water waves moving over an underlying shear flow [34]. The CH equation (1.1) also arises in the study of a certain non-Newtonian fluids [5] and also models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods [24]. The CH equation (1.1) was first obtained by Fokas and Fuchssteiner [27, 28] as a bi-Hamiltonian generalization of KdV. The novelty of Canmassa and Holm's work was the physical derivation of (1.1) and the discovery that the solitary wave solutions to this equation are solitons.

The CH equation (1.1) has caught a lot of attention in recent years due to two remarkable features. The first is the presence of solutions in the form of peaked solitary waves or "peakons" [7, 2, 39]: $u(t,x) = ce^{-|x-ct|}$, $c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative. The peakons replicate a feature that is characteristic for the waves of great height – waves of the largest amplitude that are exact solutions of the governing equations for water waves [13, 19, 49]. These peakons are shown to be stable [21, 22, 40]. It is worth mentioning that recently it was pointed out by Lakshmanan [38] that the Camassa-Holm equation could be relevant to the modeling of tsunami waves (see also the discussion in Constantin and Johnson [17], and Segur [46]).

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Another remarkable property of the CH equation is the presence of breaking waves (i.e. the solution remains bounded while its slope becomes unbounded in finite time [7, 10, 11, 12, 15, 42, 50]). In [3] and [4] the authors show that the solutions can be uniquely continued after breaking as either global conservative of global dissipative weak solution. It is noted that the KdV equation does not have wave breaking phenomena [37, 48]. Wave breaking is one of the most intriguing long-standing problems of water wave theory [50]. As mentioned by Whitham [50], it is intriguing to know which mathematical models for shallow water waves exhibit both phenomena of soliton interaction and wave breaking. It is found that the CH equation could be the first such equation and has the potential to become the new master equation for shallow water wave theory, modeling the soliton interaction of peaked traveling waves, wave breaking, admitting solutions as permanent waves, and being integrable Hammiltonian systems.

The Camassa-Holm equation also admits many integrable multi-component generalizations. The most popular one is

$$\begin{cases} m_t - Au_x + um_x + 2u_x m + \rho\rho_x = 0, \quad m = u - u_{xx}, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.2}$$

Notice that the CH equation can be obtained via the obvious reduction $\rho \equiv 0$ and $A = 0$. System (1.2) was derived first in [47], where $\rho(t,x)$ is related to the free surface elevation from equilibrium (or scalar density), and $A \geq 0$ characterizes a linear underlying shear flow. Recently, Constantin-Ivanov [16] and Ivanov [32] established a rigorous justification of the derivation of system (1.2). Mathematical properties of the system have been also studied further in many works, e.g. [1, 9, 26, 43, 45]. Chen, Liu and Zhang [9] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher, Lechtenfeld and Yin [26] investigated local well-posedness for the two-component Camassa-Holm system with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s \geq 2$ by applying Kato's theory [36], and provided some precise blow-up scenarios for strong solutions to the system. The local wellposedness is improved by Gui and Liu [29] to the Besov spaces (especially in the Sobolev space $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > 3/2$), and they showed that the finite time blow-up is determined by either the slope of the first component $u$ or the slope of the second component $\rho$ (also see [16, 26]). The blow-up criterion is made more precise in [41] where the authors showed that the wave-breaking in finite time only depends on the slope of $u$. In other words, the wave-breaking in $u$ must occur before that in $\rho$. This blow-up criterion is further improved in [30] to the lowest Sobolev spaces $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > 3/2$.

In this paper we follow Ivanov's modeling approach [32] and derive the following generalized two-component CH system

$$\begin{cases} m_t - Au_x + \sigma(2mu_x + um_x) + 3(1-\sigma)uu_x + \rho\rho_x = 0, \quad m = u - u_{xx}, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.3}$$

or equivalently, in terms of $u$ and $\rho$,

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.4}$$

with $u \to 0$, $\rho \to 1$ as $|x| \to \infty$. We see the appearance of a new free parameter $\sigma$. When $\sigma = 1$ it recovers the standard two-component CH system (1.2). In the case $\rho \equiv 0$, it becomes

$$u_t - u_{xxt} + 3uu_x = \sigma\left(2u_x u_{xx} + uu_{xxx}\right),$$

which models finite length, small amplitude radial deformation waves in cylindrical hyper-elastic rods (see [24]). System (1.3) has the following two Hamiltonians

$$H_1 = \frac{1}{2}\int_{\mathbb{R}} \left(mu + (\rho-1)^2\right)\, dx,$$
$$H_2 = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u(\rho-1) + u(\rho-1)^2 - Au^2\right)\, dx.$$

Similar as in [16, 26], we can use the method of Besov spaces together with the transport equation theory to show that system (1.3) is locally wellposed in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > 3/2$. Due to the Hamiltonian $H_1$, the horizontal velocity component $u$ is uniformly bounded by the Sobolev imbedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$.

The two equations for $u$ and $\rho$ are of a transport structure

$$\partial_t f + v\partial_x f = g.$$

It is well known that most of estimates are available when $v$ has enough regularity. Roughly speaking, the regularity of the initial data is expected to be preserved as soon as $v$ belongs to $L^1(0,T;\mathrm{Lip})$. More specifically, $u$ and $\rho$ are "transported" along directions of $\sigma u$ and $u$ respectively. Thus the solution can be estimated by in a Gronwall way involving $\|u_x\|_{L^\infty}$. Hence one can use these estimates to derive a criterion which says if

$$\int_0^T \|u_x(\tau)\|_{L^\infty} d\tau < \infty,$$

then solutions can be extended further in time, c.f. Theorem 3.3.

In concern of the finite time blow-up, notice that system (1.4) has two characteristics

$$\begin{cases} \dfrac{\partial q_1}{\partial t} = u(t,q_1), & 0 < t < T, \\ q_1(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \dfrac{\partial q_2}{\partial t} = \sigma u(t,q_2), & 0 < t < T, \\ q_2(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{1.6}$$

As disscussed in Section 3, these two characteristics are both increasing diffeomorphisms of $\mathbb{R}$. When $\sigma = 1$, these two characteristics $q_1$ and $q_2$ coincide, which suggests one to carry out the estimates along that trajectory. In fact one may use the invariance of the $\rho$ component associated to the transport equation to control $\|\rho\|_{L^\infty}$ and $\|\rho_x\|_{L^\infty}$ in terms of $\|u_x\|_{L^\infty}$ and construct wave-breaking solutions with certain initial profiles (see [16, 30, 41]). However when $\sigma \neq 1$ the analysis is a little different. Since there is no uniform characteristics, in order to obtain similar estimates, one needs more regularity assumptions. The way to resolve this issue is to employ the method of characteristics along a properly chosen $q_1$ which captures the maximum/minimum of $u_x$. In this way, the transport equation for $\rho$ is the same as before and $u_{xx}$ vanishes a.e. along such characteristics and hence there is no extra regularity needed. Moreover, using this method of characteristics together with a use of the conservation laws we have an improved estimate of $u_x$, c.f. Lemma 3.5, which says that $\sigma u_x$ is always uniformly bounded from above. Therefore we see that the only way wave-breaking can occur is that $\sigma u_x$ tends to $-\infty$ and hence we obtain a necessary and sufficient condition for the finite time blow-up:

$$\lim_{t\to T^-} \inf_{x\in\mathbb{R}} \sigma u_x(t,x) = -\infty.$$

We also study the problem of global existence of solutions. We use the method of Lyapunov functions introduced in [16]. We find a sufficient condition for global solutions which is determined only by a positive profile of the free surface component $\rho$ of the system, in the case $0 < \sigma < 2$. However the cases when $\sigma < 0$ or $\sigma \geq 2$ still remain open at this moment.

Our main results of the paper are Theorem 3.4 (Wave-breaking criterion), Theorem 3.9 (Wave-breaking), Theorem 4.1 (Blow-up rate), and Theorem 5.1 (Global exsitence).

The rest of the paper is organized as follows. Section 2 gives a new generalized two-component CH system using the same approach as in [32]. Section 3 concerns the wave-breaking of this new system. Theorem 3.3 states a wave-breaking criterion which says that the the wave-breaking only depends on the slope of $u$ not the slope of $\rho$. Theorem 3.4 improves the blow-up criterion with a more precise condition. Various results of wave-breaking are demonstrated in details. Section 4 is about the blow-up rate of strong solutions. Finally Section 5 provides a sufficient condition for global solutions.

## 2. Derivation of the model

Following Ivanov's approach in [32] (also see [16]), we consider the motion of an inviscid incompressible fluid with a constant density $\varrho$ governed by the Euler's equations

$$\begin{aligned}&\vec{v}_t + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{1}{\varrho}\nabla P + \vec{g},\\ &\nabla\cdot\vec{v} = 0,\end{aligned}$$

where $\vec{v}(t,x,y,z)$ is the velocity of the fluid, $P(t,x,y,z)$ is the pressure and $\vec{g} = (0,0,-g)$ is the gravity acceleration.

Using the shallow water approximation and non-dimensionalization the above equations can be written as

$$\begin{aligned}&u_t + \varepsilon(uu_x + wu_z) = -p_x,\\ &\delta^2\left(w_t + \varepsilon(uw_x + ww_z)\right) = -p_z,\\ &u_x + w_z = 0,\\ &w = \eta_t + \varepsilon u\eta_x, \quad p = \eta \quad \text{on} \quad z = 1 + \varepsilon\eta,\\ &w = 0 \quad \text{on} \quad z = 0,\end{aligned}$$

where now $\vec{v} = (u,0,w)$, $p(x,z,t)$ is the pressure variable measuring the deviation from the hydrostatic pressure distribution, $\eta(t,x)$ is the deviation from the mean level $z = h$ of the water surface, and $\varepsilon = a/h, \delta = h/\lambda$ are the two dimensionless parameters with $a$ being the typical amplitude of the wave and $\lambda$ being the typical wavelength of the wave.

In the presence of an underlying shear flow, the horizontal velocity of the flow becomes $u + \tilde{U}(z)$. Taking the simplest case $\tilde{U}(z) = Az$ where $A > 0$ is a constant. Notice that the Burns condition [6] gives the shallow-water limit of the dispersion relation for the waves with vorticity and hence determines the speed of propagation of the linear waves [23, 35]. From Burns condition one has the following expression for the speed $c$ of the traveling waves in linear approximation:

$$c = \frac{1}{2}\left(A \pm \sqrt{4 + A^2}\right). \tag{2.1}$$

In the case of constant vorticity $\omega = A$, one obtain at the order of $O(\varepsilon, \delta^2)$ the following equations for $u_0$ and $\eta$, where $u_0$ is the leading order approximation for $u$.

$$\left(u_0 - \delta^2 \frac{1}{2} u_{0xx}\right)_t + \varepsilon u_0 u_{0x} + \eta_x - \delta^2 \frac{A}{3} u_{0xxx} = 0, \tag{2.2}$$

$$\eta_t + A\eta_x + \left[(1+\varepsilon\eta)u_0 + \varepsilon \frac{A}{2}\eta^2\right]_x - \delta^2 \frac{1}{6} u_{0xxx} = 0. \tag{2.3}$$

Introducing a new variable

$$\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\gamma u_{0xx},$$

for some constants $\alpha, \beta$ and $\gamma$ satisfying

$$\frac{\gamma}{\alpha} = \frac{1}{6(c-A)}$$

$$\alpha = 1 + \frac{Ac}{2} + \frac{\beta}{\alpha},$$

equations (2.2) and (2.3) become

$$\begin{cases} m_t + Am_x - Au_{0x} + \delta^2\Big(\frac{A}{6} + \kappa(A-c) - \frac{\gamma}{\alpha}\Big)u_{0xxx} \\ \qquad +\varepsilon\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)u_0u_{0x} + \frac{1}{2\varepsilon\alpha}(\rho^2)_x = 0, \\ \rho_t + A\rho_x + \alpha\varepsilon(\rho u_0)_x = 0, \end{cases} \tag{2.4}$$

where $m = u_0 - \delta^2(\frac{1}{2} + \kappa)u_{0xx}$. As for the first equation, one may choose $\kappa$ to be

$$\kappa = \frac{1}{A-c}\Big(\frac{\gamma}{\alpha} - \frac{A}{6}\Big) = \frac{1}{6(c-A)}\Big(A - \frac{1}{c-A}\Big),$$

so that the $u_{0xxx}$ term disappears. At the order of $O(1)$, we may break $u_0u_{0x}$ up as

$$u_0u_{0x} = s(2mu_{0x} + u_0m_x) + (1-3s)u_0u_{0x} + O(\delta^2),$$

for any $s \in \mathbb{R}$. Thus equation (2.4) can be written at the order $O(\varepsilon, \delta^2)$ as

$$m_t + Am_x - Au_{0x} + \varepsilon\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)s(2mu_{0x} + u_0m_x)$$
$$+ \varepsilon\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)(1-3s)u_0u_{0x} + \frac{1}{2\varepsilon\alpha}(\rho^2)_x = 0.$$

Using the same scaling as in [32]: $u_0 \to \frac{1}{\alpha\varepsilon}u_0,\ x \to \frac{\delta}{\sqrt{B}}x\ t \to \frac{\delta}{\sqrt{B}}t$, then (2.4) becomes

$$\begin{cases} m_t + Am_x - Au_{0x} + \frac{1}{\alpha}\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)s(2mu_{0x} + u_0m_x) \\ \qquad +\frac{1}{\alpha}\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big)(1-3s)u_0u_{0x} + \rho\rho_x = 0, \quad m = u_0 - u_{0xx}, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{cases}$$

Now if we choose

$$\frac{1}{3\alpha}\Big(1 - \frac{\alpha^2+2\beta}{\alpha}c^2\Big) = 1,$$

and denote $\sigma = 3s$. Then we arrive at

$$\begin{cases} m_t + Am_x - Au_{0x} + \sigma(2mu_{0x} + u_0m_x) + 3(1-\sigma)u_0u_{0x} + \rho\rho_x = 0, \\ \rho_t + A\rho_x + (\rho u_0)_x = 0. \end{cases} \tag{2.5}$$

Thus the constants $\alpha, \beta, \gamma$ and $c$ satisfy

$$\begin{aligned}
\alpha &= \frac{1}{3(1+c^2)} + \frac{c^2}{3}, \\
\beta &= \alpha^2 - \alpha\Big(1 + \frac{Ac}{2}\Big), \\
\gamma &= \frac{\alpha}{6(c-A)}, \\
c^2 &- Ac - 1 = 0.
\end{aligned}$$

With a further Galilean transformation $x \to x - ct,\ t \to t$, as used in [32] we can drop the terms $A\rho_x$ and $Am_x$ in (2.5) and hence obtain the generalized two-component Camassa-Holm system (1.3) or (1.4). Notice that we have the following two conservation laws for system (1.4)

$$E(u,\rho) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + (\rho-1)^2\right)\, dx, \tag{2.6}$$

$$F(u,\rho) = \frac{1}{2}\int_{\mathbb{R}} \left(u^3 + \sigma u u_x^2 + 2u(\rho-1) + u(\rho-1)^2 - Au^2\right)\, dx. \tag{2.7}$$

## 3. Formation of singularities for $\sigma \neq 0$

Applying transport equation theory combined with the method of Besov spaces, one may follow the similar argument as in [29] to obtain the following local well-posedness result for the system (1.4).

**Theorem 3.1.** *If $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$, $s > 3/2$, then there exists a maximal time $T = T(\|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}}) > 0$ and a unique solution $(u, \rho - 1)$ of* (1.4) *in $C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2})$ with $(u,\rho)|_{t=0} = (u_0, \rho_0)$. Moreover, the solution depends continuously on the initial data and $T$ is independent of $s$. In addition, the Hamiltonian $E(u,\rho)$ defined in* (2.6) *is independent of the existence time.*

As introduced in the introduction, we consider the following two associated Lagrangian scales of the two component CH system (1.4)

$$\begin{cases} \dfrac{\partial q_1}{\partial t} = u(t, q_1), & 0 < t < T, \\ q_1(0,x) = x, & x \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial q_2}{\partial t} = \sigma u(t, q_2), & 0 < t < T, \\ q_2(0,x) = x, & x \in \mathbb{R}, \end{cases}$$

where $u \in C^1([0,T), H^{s-1})$ is the first component of the solution $(u,\rho)$ to (1.4) with initial data $(u_0, \rho_0) \in H^s \times H^{s-1}$ with $s > 3/2$ and $T > 0$ is the maximal time of existence. Notice that when $\sigma = 1$ the two characteristics $q_1(t,x)$ and $q_2(t,x)$ are the same.

A direct calculation shows

$$q_{1,tx}(t,x) = u_x(t, q_1(t,x)) q_{1,x}(t,x)$$

and

$$q_{2,tx}(t,x) = \sigma u_x(t, q_2(t,x)) q_{2,x}(t,x).$$

Thus for $t > 0, x \in \mathbb{R}$

$$q_{1,x}(t,x) = e^{\int_0^t u_x(\tau, q_1(\tau,x))d\tau} > 0, \quad \text{and} \quad q_{2,x}(t,x) = e^{\int_0^t \sigma u_x(\tau, q_2(\tau,x))d\tau} > 0,$$

indicating that $q_1(t,\cdot) : \mathbb{R} \to \mathbb{R}$ and $q_2(t,\cdot) : \mathbb{R} \to \mathbb{R}$ are diffeomorphisms of the line for each $t \in [0,T)$. Hence the $L^\infty$ norm of any function $v(t,\cdot) \in L^\infty(\mathbb{R}), t \in [0,T)$ is preserved under the family of diffeomorphisms $q_1(t,\cdot)$ and $q_2(t,\cdot)$ with $t \in [0,T)$, i.e.,

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R})} = \|v(t, q_1(t,\cdot))\|_{L^\infty(\mathbb{R})} = \|v(t, q_2(t,\cdot))\|_{L^\infty(\mathbb{R})}, \quad t \in [0,T). \tag{3.1}$$

Similarly we have

$$\inf_{x\in\mathbb{R}} v(t,x) = \inf_{x\in\mathbb{R}} v(t, q_1(t,x)) = \inf_{x\in\mathbb{R}} v(t, q_2(t,x)), \quad t \in [0,T), \tag{3.2}$$

$$\sup_{x\in\mathbb{R}} v(t,x) = \sup_{x\in\mathbb{R}} v(t, q_1(t,x)) = \sup_{x\in\mathbb{R}} v(t, q_2(t,x)), \quad t \in [0,T). \tag{3.3}$$

Our system (1.4) can be written in the following "transport" type

$$\begin{cases} u_t + \sigma u u_x = -\partial_x p * \left(-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2\right), \\ \rho_t + u\rho_x = -u_x\rho, \end{cases} \tag{3.4}$$

where $p(x) := \frac{1}{2}e^{-|x|}, x \in \mathbb{R}$. We may use the following proposition derived in [30] to handle the regularity propogation of solutions to (1.4).

**Proposition 3.2.** *Let* $0 < s < 1$. *Suppose that* $f_0 \in H^s$, $g \in L^1([0,T]; H^s)$, $v, v_x \in L^1([0,T]; L^\infty)$ *and that* $f \in L^\infty([0,T]; H^s) \cap C([0,T]; \mathcal{S}')$ *solves the one-dimensional linear transport equation*

$$\begin{cases} f_t + v f_x = g, \\ f(0,x) = f_0(x). \end{cases} \tag{3.5}$$

*Then* $f \in C([0,T]; H^s)$. *More precisely, there exists a constant* $C$ *depending only on* $s$ *such that the following estimate holds:*

$$\|f(t)\|_{H^s} \le \|f_0\|_{H^s} + C\left(\int_0^t \|g(\tau)\|_{H^s} d\tau + \int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau\right). \tag{3.6}$$

*Hence*

$$\|f(t)\|_{H^s} \le e^{CV(t)}\left(\|f_0\|_{H^s} + C\int_0^t \|g(\tau)\|_{H^s} d\tau\right), \tag{3.7}$$

*where* $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|v_x(\tau)\|_{L^\infty})\, d\tau$.

The above proposition was proved using Littlewood-Paley analysis for the transport equation and Moser-type estimates. Using this result, and performing the same argument as in [30] we can obtain the following blowup criterion. The proof is very similar to that of Theorem 4.1 in [30], and hence is omitted.

**Theorem 3.3.** *Let* $\sigma \neq 0$ *and* $(u,\rho)$ *be the solution of* (1.4) *with initial data* $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, *and* $T$ *the maximal time of existence. Then*

$$T < \infty \quad \Rightarrow \quad \int_0^T \|u_x(\tau)\|_{L^\infty} d\tau = \infty. \tag{3.8}$$

Next we state the necessary and sufficient condition for the blowup of solutions.

**Theorem 3.4** (Wave-breaking criterion for $\sigma \neq 0$)**.** *Let $\sigma \neq 0$ and $(u, \rho)$ be the solution of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and $T$ the maximal time of existence. Then the solution blows up in finite time if and only if*

$$\lim_{t \to T^-} \inf_{x \in \mathbb{R}} \sigma u_x(t, x) = -\infty. \tag{3.9}$$

To prove this wave-breaking criterion, we use the following lemma to show that indeed $\sigma u_x$ is uniformly bounded from above.

**Lemma 3.5.** *Let $\sigma \neq 0$ and $(u, \rho)$ be the solution of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and $T$ the maximal time of existence. Then*

(1) *If $\sigma > 0$ then*

$$\sup_{x \in \mathbb{R}} u_x(t, x) \leq \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}. \tag{3.10}$$

(2) *If $\sigma < 0$, then*

$$\inf_{x \in \mathbb{R}} u_x(t, x) \geq -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}}. \tag{3.11}$$

*The constants above are defined as follows.*

$$C_0 = \|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}, \tag{3.12}$$

$$C_1 = \sqrt{\frac{1 + A^2 + |\sigma| + 2|3 - \sigma|}{2}} C_0, \tag{3.13}$$

$$C_2 = \sqrt{2 + \frac{5 + A^2 - 2\sigma}{2}} C_0, \quad \textit{for} \quad \sigma < 0. \tag{3.14}$$

*Remark* 3.6. It was shown in [30] and [41] that when $\sigma = 1$ and $u_x$ is bounded from below then $u_x$ grows at most exponentially. Such a result can be easily extended to the case when $\sigma > 0$. However we show here that indeed $\sigma u_x$ is always bounded from above uniformly by a constant, even if we don't know whether it is bounded from below. This result of the uniform upper bound of $u_x$ is also discussed in the Camassa-Holm case $A = 0, \rho \equiv 0$ and $\sigma = 1$ in [11] and [41], which says that the solution of the Camassa-Holm equation with initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$ always has a uniform upper bound. In the two-component Camassa-Holm system, it was proved in [41] that $u_x$ is uniformly bounded from above on the set $[0, T) \times \Lambda$, where $\Lambda = \{x \in \mathbb{R} : \ \rho_0(x) = 0\}$, the zero level set of $\rho_0$. Hence here we improve the estimate to all of $\mathbb{R}$.

The approach we take here is the method of characteristics. Applying the following Lemma 3.7 we may carry out the estimates along the characteristics $q_1(t, x)$ which captures $\sup_{x \in \mathbb{R}} u_x(t, x)$ and $\inf_{x \in \mathbb{R}} u_x(t, x)$.

**Lemma 3.7.** *(*[14]*) Let $T > 0$ and $v \in C^1\left([0, T); H^1(\mathbb{R})\right)$. Then for every $t \in [0, T)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x \in \mathbb{R}} \left[v_x(t, x)\right] = v_x\left(t, \xi(t)\right).$$

*The function $m(t)$ is absolutely continuous on $(0, T)$ with*

$$\frac{dm(t)}{dt} = v_{tx}\left(t, \xi(t)\right) \quad \textit{a.e. on } (0, T).$$

*Proof of Lemma 3.5.* The local wellposedness theorem and a density argument implies that it suffices to prove the desired estimates for $s \geq 3$. Thus we take $s = 3$ in the proof. Also we may assume that

$$u_0 \not\equiv 0. \tag{3.15}$$

Otherwise the results become trivial. Note that if $p(x) := \frac{1}{2}e^{-|x|}, x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$. Hence we can rewrite the first equation in (1.4) as

$$u_t + \sigma u u_x + \partial_x p * \left( -Au + \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2}\rho^2 \right) = 0. \tag{3.16}$$

Differentiating the above with respect to $x$ and using the identity $-\partial_x^2 p * f = f - p * f$ we obtain

$$u_{tx} + \sigma u u_{xx} + \frac{\sigma}{2} u_x^2 = \frac{1}{2}\rho^2 + \frac{3-\sigma}{2} u^2 + A\partial_x^2 p * u - p * \left( \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2}\rho^2 \right). \tag{3.17}$$

(1) When $\sigma > 0$, using Lemma 3.7 and the fact that

$$\sup_{x\in\mathbb{R}} \left[ v_x(t,x) \right] = - \inf_{x\in\mathbb{R}} \left[ -v_x(t,x) \right],$$

we can consider $\bar{m}(t)$ and $\eta(t)$ as follows

$$\bar{m}(t) := u_x\left(t, \eta(t)\right) = \sup_{x\in\mathbb{R}} \left( u_x(t,x) \right), \quad t \in [0,T). \tag{3.18}$$

Hence

$$u_{xx}\left(t, \eta(t)\right) = 0, \quad \text{a.e.} \quad t \in [0,T). \tag{3.19}$$

Take the trajectory $q_1(t,x)$ defined in (1.5). Then we know that $q_1(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for every $t \in [0,T)$. Therefore there exists $x_1(t) \in \mathbb{R}$ such that

$$q_1\left(t, x_1(t)\right) = \eta(t) \quad t \in [0,T). \tag{3.20}$$

Now let

$$\bar{\zeta}(t) = \rho(t, q_1(t,x_1)), \quad t \in [0,T). \tag{3.21}$$

Therefore along this trajectory $q_1(t,x_1)$ equation (3.17) and the second equation of (1.4) become

$$\begin{aligned} \bar{m}'(t) &= -\frac{\sigma}{2}\bar{m}^2 + \frac{1}{2}\bar{\zeta}^2 + f(t, q_1(t,x_1)), \\ \bar{\zeta}'(t) &= -\bar{\zeta}\bar{m}, \end{aligned} \tag{3.22}$$

for $t \in [0,T)$, where $'$ denotes the derivative with respect to $t$ and $f(t,q(t,x))$ is given by

$$f = \frac{3-\sigma}{2} u^2 + A\partial_x^2 p * u - p * \left( \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2}\rho^2 \right). \tag{3.23}$$

We first derive the upper and lower bounds for $f$ for later use in getting the wave-breaking result. Using that $\partial_x^2 p * u = \partial_x p * \partial_x u$, we have

$$\begin{aligned} f &= \frac{3-\sigma}{2} u^2 + A\partial_x p * \partial_x u - p * \left( \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 \right) - \frac{1}{2} p * 1 - p * (\rho - 1) \\ &\quad - \frac{1}{2} p * (\rho-1)^2 \\ &\leq \frac{3-\sigma}{2} u^2 + A|p_x * u_x| + \left| p * \left( \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 \right) \right| - \frac{1}{2} + |p * (\rho - 1)|. \end{aligned}$$

Since

$$A|p_x * u_x| \leq A\|p_x\|_{L^2}\|u_x\|_{L^2} = \frac{1}{2}A\|u_x\|_{L^2} \leq \frac{1}{4} + \frac{1}{4}A^2\|u_x\|_{L^2}^2, \tag{3.24}$$

$$|p * (\rho - 1)| \leq \|p\|_{L^2}\|\rho - 1\|_{L^2} = \frac{1}{2}\|\rho - 1\|_{L^2} \leq \frac{1}{4} + \frac{1}{4}\|\rho - 1\|_{L^2}^2, \tag{3.25}$$

$$\frac{3-\sigma}{2}u^2 \leq \frac{|3-\sigma|}{4}\int_{\mathbb{R}}(u^2 + u_x^2)\,dx, \tag{3.26}$$

$$\left|p * \left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2\right)\right| \leq \frac{1}{2}\left\|\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2\right\|_{L^1} \leq \int_{\mathbb{R}}\left(\frac{|3-\sigma|}{4}u^2 + \frac{|\sigma|}{4}u_x^2\right)dx, \tag{3.27}$$

Therefore we obtain the upper bound of $f$

$$f \leq \frac{1}{4}\|\rho - 1\|_{L^2}^2 + \frac{|3-\sigma|}{2}\|u\|_{L^2}^2 + \frac{A^2 + |3-\sigma| + |\sigma|}{4}\|u_x\|_{L^2} \tag{3.28}$$

$$\leq \frac{1 + A^2 + |\sigma| + 2|3-\sigma|}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 = \frac{1}{2}C_1^2. \tag{3.29}$$

Now we turn to the lower bound of $f$. Similar as before, we get

$$\begin{aligned}
-f &\leq \frac{\sigma-3}{2}u^2 + A|p_x * u_x| + \left|p * \left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2\right)\right| + \frac{1}{2} + |p * (\rho - 1)| \\
&\quad + \frac{1}{2}p * (\rho - 1)^2. \\
&\leq 1 + \frac{1}{2}\|\rho - 1\|_{L^2}^2 + \frac{|3-\sigma|}{2}\|u\|_{L^2}^2 + \frac{A^2 + |3-\sigma| + |\sigma|}{4}\|u_x\|_{L^2} \\
&\leq 1 + \frac{2 + A^2 + |\sigma| + 2|3-\sigma|}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2.
\end{aligned} \tag{3.30}$$

When $\sigma < 0$, we have a finer estimate

$$\begin{aligned}
-f &\leq 1 + \frac{1}{2}\|\rho - 1\|_{L^2}^2 + \frac{3-\sigma}{4}\|u\|_{L^2}^2 + \frac{A^2 - \sigma}{4}\|u_x\|_{L^2}^2 \\
&\leq 1 + \frac{5 + A^2 - 2\sigma}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2 = \frac{1}{2}C_2^2.
\end{aligned} \tag{3.31}$$

Combining (3.29) and (3.30) we obtain

$$|f| \leq 1 + \frac{2 + A^2 + |\sigma| + 2|3-\sigma|}{4}\|(u_0, \rho_0 - 1)\|_{H^1 \times L^2}^2. \tag{3.32}$$

Since now $s \geq 3$, we have $u \in C_0^1(\mathbb{R})$. Therefore

$$\inf_{x \in \mathbb{R}} u_x(t, x) \leq 0, \quad \sup_{x \in \mathbb{R}} u_x(t, x) \geq 0, \quad t \in [0, T). \tag{3.33}$$

Hence $\bar{m}(t) > 0$ for $t \in [0, T)$. From the second equation of (3.22) we obtain that

$$\bar{\zeta}(t) = \bar{\zeta}(0)e^{-\int_0^t \bar{m}(\tau)d\tau}. \tag{3.34}$$

Hence

$$|\rho(t, q_1(t, x_1))| = |\bar{\zeta}(t)| \leq |\bar{\zeta}(0)| \leq \|\rho_0\|_{L^\infty}.$$

For any given $x \in \mathbb{R}$, define

$$P_1(t) = \bar{m}(t) - \|u_{0,x}\|_{L^\infty} - \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}.$$

Notice that $P_1(t)$ is a $C^1$-differentiable function in $[0,T)$ and satisfies

$$P_1(0) \le \bar{m}(0) - \|u_{0,x}\|_{L^\infty} \le 0.$$

We will show that

$$P_1(t) \le 0, \quad \text{for } t \in [0,T). \tag{3.35}$$

If not, then suppose there is a $t_0 \in [0,T)$ such that $P_1(t_0) > 0$. Define

$$t_1 = \max\{t < t_0 : \ P_1(t) = 0\}.$$

Then $P_1(t_1) = 0$ and $P_1'(t_1) \ge 0$, or equivalently,

$$\bar{m}(t_1) = \|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}},$$
$$\bar{m}'(t_1) \ge 0.$$

On the other hand, we have

$$\begin{aligned}\bar{m}'(t_1) &= -\frac{\sigma}{2}\bar{m}^2(t_1) + \frac{1}{2}\bar{\zeta}^2(t_1) + f(t_1, q(t_1,x)) \\ &\le -\frac{\sigma}{2}\left[\|u_{0,x}\|_{L^\infty} + \sqrt{\frac{\|\rho_0\|_{L^\infty}^2 + C_1^2}{\sigma}}\right]^2 + \frac{1}{2}\|\rho_0\|_{L^\infty}^2 + \frac{1}{2}C_1^2 < 0,\end{aligned}$$

a contradiction. Therefore $P_1(t) \le 0$, for $t \in [0,T)$. Since $x$ is chosen arbitrarily, we obtain (3.10).

(2) To derive a lower bound for $u_x$ in the case of $\sigma < 0$, we consider the functions $m(t)$ and $\xi(t)$ as in Lemma 3.7

$$m(t) := u_x\left(t, \xi(t)\right) = \inf_{x\in\mathbb{R}}\left(u_x(t,x)\right), \quad t \in [0,T). \tag{3.36}$$

Hence

$$u_{xx}\left(t,\xi(t)\right) = 0 \quad \text{a.e. } t \in [0,T). \tag{3.37}$$

Similar as before, we take the characteristic $q_1(t,x)$ defined in (1.5) and choose $x_2(t) \in \mathbb{R}$ such that

$$q_1\left(t, x_2(t)\right) = \xi(t) \quad t \in [0,T). \tag{3.38}$$

Let

$$\zeta(t) = \rho\left(t, q_1(t,x_2)\right), \quad t \in [0,T). \tag{3.39}$$

Hence along this trajectory $q_1(t,x_2)$ equation (3.17) and the second equation of (1.4) become

$$\begin{aligned}m'(t) &= -\frac{\sigma}{2}m^2 + \frac{1}{2}\zeta^2 + f(t, q_1(t,x_2)), \\ \zeta'(t) &= -\zeta m,\end{aligned} \tag{3.40}$$

we can define now for any given $x \in \mathbb{R}$

$$P_2(t) = m(t) + \|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}}.$$

Then $P_2(t)$ is also $C^1$-differentiable on $[0,T)$ and satisfies

$$P_2(0) \ge m(0) + \|u_{0,x}\|_{L^\infty} \ge 0.$$

We now claim that

$$P_2(t) \geq 0, \quad \text{for } t \in [0,T). \tag{3.41}$$

If not, then suppose there is a $\bar{t}_0 \in [0,T)$ such that $P_2(\bar{t}_0) < 0$. Define

$$t_2 = \max\{t < \bar{t}_0 : \ P_2(t) = 0\}.$$

Then $P_2(t_2) = 0$ and $P_2'(t_2) \leq 0$, or equivalently,

$$\begin{aligned} &m(t_2) = -\|u_{0,x}\|_{L^\infty} - \frac{C_2}{\sqrt{-\sigma}}, \\ &m'(t_2) \leq 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} m'(t_2) &= -\frac{\sigma}{2}m^2(t_2) + \frac{1}{2}\zeta^2(t_2) + f(t_2, q(t_2,x)) \\ &\geq -\frac{\sigma}{2}\left(\|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}}\right)^2 - \frac{1}{2}C_2^2 > 0, \end{aligned}$$

a contradiction. Therefore $P_2(t) \geq 0$, for $t \in [0,T)$. Since $x$ is chosen arbitrarily, we obtain (3.11).

□

In fact if $\sigma u_x$ is bounded from below, we may obtained the following estimates for $\|\rho\|_{L^\infty(\mathbb{R})}$.

**Proposition 3.8.** *Let $\sigma \neq 0$ and $(u,\rho)$ be the solution of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and $T$ the maximal time of existence. If there is an $M \geq 0$ such that*

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}} \sigma u_x \geq -M, \tag{3.42}$$

*Then*

(1) *If $\sigma > 0$ then*

$$\|\rho(t,\cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{Mt/\sigma}. \tag{3.43}$$

(2) *If $\sigma < 0$, then*

$$\|\rho(t,\cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{Nt}, \tag{3.44}$$

*where $N = \|u_{0,x}\|_{L^\infty} + \frac{C_2}{\sqrt{-\sigma}}$ and $C_2$ is given in* (3.14).

*Proof.* (1) We define for any given $x \in \mathbb{R}$

$$U(t) = u_x(t, q_1(t,x)), \quad \gamma(t) = \rho(t, q_1(t,x)), \quad x \in \mathbb{R}, t \in [0,T). \tag{3.45}$$

Then the $\rho$ equation of system (1.4) becomes

$$\gamma'(t) = -\gamma U. \tag{3.46}$$

Thus

$$\gamma(t) = \gamma(0)e^{-\int_0^t U(\tau)d\tau}. \tag{3.47}$$

From the assumption (3.42) and $\sigma > 0$ we see

$$U(t) \geq -\frac{M}{\sigma}, \quad t \in [0,T).$$

Hence

$$|\rho(t,q_1(t,x))| = |\gamma(t)| \le |\gamma(0)| e^{\int_0^t -U(\tau)d\tau} \le |\gamma(0)| e^{Mt/\sigma}$$
$$\le \|\rho_0\|_{L^\infty} e^{Mt/\sigma},$$

which, together with (3.1), leads to (3.43).

(2) To obtain (3.44), we perform a similar argument as before. Using (3.45), (3.47) and the lower bound (3.11) we have

$$|\rho(t,q_1(t,x))| = |\gamma(t)| \le e^{\int_0^t -U(\tau)d\tau} |\gamma(0)| \le \|\rho_0\|_{L^\infty} e^{Nt},$$

which, combining with (3.1), proves to (3.44). □

*Proof of Theorem 3.4.* Assume that $T < \infty$ and (3.9) is not valid. Then there is some positive number $M > 0$ such that

$$\sigma u_x(t,x) \ge -M, \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

It now follows from Lemma 3.5 that

$$|u_x(t,x)| \le C. \tag{3.48}$$

where $C = C(A, M, \sigma, \|(u_0, \rho_0 - 1)\|_{H^s \times H^{s-1}})$. Therefore Theorem 3.3 implies that the maximal existence time $T = \infty$, which contradicts the assumption that $T < \infty$.

Conversely, the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s > 1/2$ implies that if (3.9) holds, the corresponding solution blows up in finite time, which completes the proof of Theorem 3.4.

□

Now we give the following series of theorems that provide some cases for wave breaking in finite time.

**Theorem 3.9.** *Let $\sigma \neq 0$ and $(u, \rho)$ be the solution of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and $T$ the maximal time of existence.*

(i) *When $\sigma > 0$, assume that there is some $x_0 \in \mathbb{R}$ such that*

$$\rho_0(x_0) = 0, u_{0,x}(x_0) = \inf_{x \in \mathbb{R}} u_{0,x}(x),$$

*and one of the following two conditions holds,*

$$\|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2} < \frac{1}{2\left(1 + 2A^2 + \sigma + 2|3 - \sigma|\right)}, \tag{3.49}$$

$$u_{0,x}(x_0) < -\frac{C_1}{\sqrt{\sigma}}, \tag{3.50}$$

*where $C_1$ is defined in* (3.13). *Then the corresponding solution to system* (1.4) *blows up in finite time in the following sense: there exists a $T_1$ with*

$$0 < T_1 \le \frac{2}{\sigma} + \frac{8 + 8|u_{0,x}(x_0)|}{1 - 2\left(1 + 2A^2 + \sigma + 2|3 - \sigma|\right) \|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}}, \tag{3.51}$$

$$0 < T_1 \le -\frac{2}{\sigma u_{0,x}(x_0) + \sqrt{-\sigma^{3/2} C_1 u_{0,x}(x_0)}}, \tag{3.52}$$

*respectively, such that*

$$\liminf_{t \to T_1^-} \left( \inf_{x \in \mathbb{R}} u_x(t,x) \right) = -\infty.$$

(ii) *When $\sigma < 0$, assume that there is some $x_0 \in \mathbb{R}$ such that*

$$u_{0,x}(x_0) > \frac{C_2}{\sqrt{-\sigma}}, \tag{3.53}$$

*where $C_2$ is defined in* (3.14). *Then the corresponding solution to system* (1.4) *blows up in finite time in the following sense: there exists a $T_2$ with*

$$0 < T_2 \leq -\frac{2}{\sigma u_{0,x}(x_0) - \sqrt{(-\sigma)^{3/2} C_2 u_{0,x}(x_0)}} \tag{3.54}$$

*such that*

$$\liminf_{t \to T_2^-} \left( \sup_{x \in \mathbb{R}} u_x(t,x) \right) = \infty.$$

*Remark.* If $\sigma = 1$, then the assumption $u_0(x_0) = \inf_{x \in \mathbb{R}} u_{0,x}(x)$ is unnecessary and (3.49) and (3.50) can be improved [41].

*Proof.* (i) When $\sigma > 0$, similar to the proof of Lemma 3.5, it suffices to consider $s \geq 3$. We consider along the trajectory $q_1(t, x_2)$ defined in (1.5) and (3.38). In this way, we can write the transport equation of $\rho$ in (1.4) along the trajectory of $q_1(t, x_2)$ as

$$\frac{d\rho\left(t, \xi(t)\right)}{dt} = -\rho\left(t, \xi(t)\right) u_x\left(t, \xi(t)\right). \tag{3.55}$$

From the assumption of the theorem we see

$$m(0) = u_x\left(0, \xi(0)\right) = \inf_{x \in \mathbb{R}} u_{0,x}(x) = u_{0,x}(x_0).$$

Hence we can choose $\xi(0) = x_0$ and then $\rho_0\left(\xi(0)\right) = \rho_0(x_0) = 0$. Thus from (3.55) we see that

$$\rho\left(t, \xi(t)\right) = 0 \quad \text{for every } t \in [0, T). \tag{3.56}$$

Differentiating equation (3.16) with respect to $x$, evaluating the result at $x = \xi(t)$ and using (3.37) and (3.56) we deduce from Lemma 3.7 that

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2} m^2(t) + \frac{3-\sigma}{2} u^2\left(t, \xi(t)\right) + A\left(p_x * u_x\right)\left(t, \xi(t)\right) \\ &\quad - p * \left( \frac{3-\sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1}{2}\rho^2 \right)\left(t, \xi(t)\right) \\ &= -\frac{\sigma}{2} m^2(t) + f\left(t, q_1(t, x_1)\right). \end{aligned} \tag{3.57}$$

In the case of (3.49), we modify the estimate (3.24) to be

$$A|p_x * u_x|(t,x) \leq \frac{1}{2} A \left\|u_x\right\|_{L^2} \leq \frac{1}{8} + \frac{1}{2} A^2 \left\|u_x\right\|_{L^2}^2 \tag{3.58}$$

for $(t,x) \in [0,T) \times \mathbb{R}$. The same process to (3.28) leads to

$$\begin{aligned} f(t,x) &\leq -\frac{1}{2} + \frac{3}{8} + \frac{2A^2 + \sigma + |3-\sigma|}{4} \left\|u_x\right\|_{L^2}^2 + \frac{|3-\sigma|}{2} \left\|u\right\|_{L^2}^2 + \frac{1}{4} \left\|\rho - 1\right\|_{L^2}^2 \\ &= -\frac{1}{8} + \frac{1 + 2A^2 + \sigma + 2|3-\sigma|}{4} \left\|(u_0, \rho_0 - 1)\right\|_{H^1 \times L^2}^2 \\ &:= -c_1 < 0, \end{aligned}$$

for $(t,x) \in [0,T) \times \mathbb{R}$. The last inequality is due to assumption (3.49). Then

$$m'(t) \leq -\frac{\sigma}{2} m^2(t) - c_1 \leq -c_1 < 0, \quad t \in [0,T), \tag{3.59}$$

so $m(t)$ is strictly decreasing in $[0, T)$.

If the solution $(u, \rho)$ to (1.4) exists globally in time, i.e. $T = \infty$, we will show that this leads to a contradiction. Let

$$t_1 = \frac{1 + |u_{0,x}(x_0)|}{c_1}.$$

Integrating (4.4) over $[0, t_1]$ yields

$$m(t_1) = m(0) + \int_0^{t_1} m'(t)dt \leq |u_{0,x}(x_0)| - c_1 t_1 = -1.$$

Hence we know

$$m(t) \leq -1 \quad \text{ for } t \in [t_1, T). \tag{3.60}$$

Also we know from (4.4) that $m'(t) \leq -\frac{\sigma}{2} m^2(t)$ on $[t_1, T)$, which leads to

$$-\frac{d}{dt}\left(\frac{1}{m(t)}\right) \leq -\frac{\sigma}{2}, \quad t \in [t_1, T).$$

Integrating both sides and knowing that $m(t_1) \leq -1$ yield

$$-\frac{1}{m(t)} - 1 \leq -\frac{1}{m(t)} + \frac{1}{m(t_1)} \leq -\frac{\sigma}{2}(t - t_1), \quad t \in [t_1, T).$$

Therefore

$$m(t) \leq \frac{2}{\sigma(t - t_1) - 2} \to -\infty, \quad \text{ as } t \to t_1 + \frac{2}{\sigma}.$$

Thus $T \leq t_1 + \frac{2}{\sigma}$, which is a contradiction to $T = \infty$. This proves that $T < \infty$ and completes the proof of (3.51).

Next, to prove (3.52), using the upper bound of $f$ in (3.29) we see that

$$m'(t) \leq -\frac{\sigma}{2} m^2(t) + \frac{1}{2} C_1^2 \quad t \in [0, T).$$

By assumption (3.50), $m(0) = u_{0,x}(x_0) < -C_1/\sqrt{\sigma}$, we see that $m'(0) < 0$ and $m(t)$ is strictly decreasing over $[0, T)$. Set

$$\delta = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{C_1}{-u_{0,x}(x_0)\sqrt{\sigma}}} \in \left(0, \frac{1}{2}\right).$$

Using that $m(t) < m(0) = u_{0,x}(x_0) < 0$, we obtain

$$\begin{aligned} m'(t) &\leq -\frac{\sigma}{2} m^2(t) + \frac{1}{2} C_1^2 \leq -\frac{\sigma}{2} m^2(t) \left[1 - (1 - 2\delta)^4\right] \\ &\leq -\delta \sigma m^2(t), \quad t \in [0, T). \end{aligned}$$

A similar argument as in the previous case yields

$$m(t) \leq \frac{u_{0,x}(x_0)}{1 + \delta \sigma u_{0,x}(x_0) t} \to -\infty \quad \text{ as } t \to -\frac{1}{\delta \sigma u_{0,x}(x_0)}.$$

Hence

$$T \leq -\frac{1}{\delta \sigma u_{0,x}(x_0)},$$

which proves (3.52).

(ii) Similarly as in (i), we consider the functions $\bar{m}(t)$ and $\eta(t)$ as defined in (3.18), and we take the trajectory $q_1(t, x_1)$ with $x_1$ defined in (3.20). Then we have

$$\bar{m}'(t) = -\frac{\sigma}{2}\bar{m}^2(t) + \frac{1}{2}\rho^2\left(t, \eta(t)\right) + f\left(t, q_1(t, x_1)\right) \geq -\frac{\sigma}{2}\bar{m}^2(t) + f\left(t, q_1(t, x_1)\right). \tag{3.61}$$

Using the lower bound of $f$ as in (3.31) we have

$$\bar{m}'(t) \geq -\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}C_2^2 \quad t \in [0, T).$$

By assumption (3.53), $\bar{m}(0) \geq u_{0,x}(x_0) > C_2/\sqrt{-\sigma}$, we see that $\bar{m}'(0) > 0$ and $\bar{m}(t)$ is strictly increasing over $[0, T)$. Set

$$\theta = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{C_2}{u_{0,x}(x_0)\sqrt{-\sigma}}} \in \left(\frac{1}{2}, 1\right).$$

Using that $\bar{m}(t) > \bar{m}(0) \geq u_{0,x}(x_0) > 0$, we obtain

$$\begin{aligned}\bar{m}'(t) &\geq -\frac{\sigma}{2}m^2(t) - \frac{1}{2}C_2^2 \geq -\frac{\sigma}{2}m^2(t)\left[1 - (2\theta - 1)^4\right] \\ &\geq -\theta\sigma m^2(t), \quad t \in [0, T).\end{aligned}$$

Therefore

$$\bar{m}(t) \geq \frac{u_{0,x}(x_0)}{1 + \theta\sigma u_{0,x}(x_0)t} \to \infty \quad \text{as } t \to -\frac{1}{\theta\sigma u_{0,x}(x_0)}.$$

Hence

$$T \leq -\frac{1}{\theta\sigma u_{0,x}(x_0)},$$

which proves (3.54). □

**Corollary 3.10.** *If $\sigma = 3$ and $A = 0$ then all solutions of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), s > \frac{3}{2}$ satisfying $u_0 \not\equiv 0$ and $\rho_0(x_0) = 0$ for some $x_0 \in \mathbb{R}$, blow up in finite time.*

*Proof.* Let $T > 0$ be the maximal existence time of such a solution. Define

$$m(t) := u_x\left(t, \xi(t)\right) = \inf_{x \in \mathbb{R}}\left(u_x(t, x)\right), \quad t \in [0, T),$$

and consider along the trajectory $q_1(t, x)$. Taking $x = x_1(t)$ as in (3.20) with $\xi(0) = x_0$ so that $\rho\left(t, \xi(t)\right) = 0$, we obtain from Lemma 3.7 and (3.57) that

$$m'(t) + \frac{3}{2}m^2(t) \leq 0 \quad \text{a.e. on } [0, T). \tag{3.62}$$

But $m(0) < 0$ and $m(t)$ is absolutely continuous, so $m(t)$ is strictly decreasing. Solving the above differential inequality we have

$$m(t) \leq \frac{2m(0)}{2 + 3m(0)t} \to -\infty, \quad \text{as } t \to -\frac{2}{3m(0)}.$$

Therefore $T \leq -\frac{2}{3m(0)}$. □

The following theorem provides another condition for blowup of $u_x$.

**Theorem 3.11.** *Let $0<\sigma\le 3$ and $(u,\rho)$ be the solution of* (1.4) *with initial data $(u_0,\rho_0-1)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$, $s>\frac{3}{2}$, and $T$ the maximal time of existence. Assume that*

$$u_0 \text{ is odd},\ \rho_0 \text{ is even},\ u_{0,x}(0)<0,\ \text{ and } \rho_0(0)=0. \tag{3.63}$$

*Then the corresponding solution to the system* (1.4) *blows up in finite time. More precisely, there exists a $T_0$ with $0<T_0\le -\frac{2}{\sigma u_{0,x}(0)}$ such that*

$$\liminf_{t\to T_0^-}\left(\inf_{x\in\mathbb{R}} u_x(t,x)\right)=-\infty.$$

*Proof.* Similar to the proof of Lemma 3.5, it suffices to consider $s\ge 3$. Since $u_0$ is odd and $\rho_0$ is even, the corresponding solution $(u(t,x),\rho(t,x))$ satisfies that $u(t,x)$ is odd and $\rho(t,x)$ is even with respect to $x$ for given $0<t<T$. Hence, $u(t,0)=0$ and $\rho_x(t,0)=0$. Thanks to the transport equation of $\rho$ in (1.4), we have

$$\begin{cases} \rho_t(t,0)+\rho(t,0)u_x(t,0)=0,\\ \rho(0,0)=0. \end{cases}$$

Thus $\rho(t,0)=0$. Evaluating (3.17) at $(t,0)$ and denoting $M(t)=u_x(t,0)$ we obtain

$$M'(t)+\frac{\sigma}{2}M^2(t)=A\left(\partial_x^2 p*u\right)(t,0)-p*\left(\frac{3-\sigma}{2}u^2+\frac{\sigma}{2}u_x^2+\frac{1}{2}\rho^2\right)(t,0). \tag{3.64}$$

Notice that $u(t,x)$ is odd and $p(x)$ is even, so

$$\left(\partial_x^2 p*u\right)(t,0)=0.$$

Using $0<\sigma\le 3$,

$$M'(t)+\frac{\sigma}{2}M^2(t)\le 0. \tag{3.65}$$

Hence

$$M(t)\le M(0)=u_{0,x}(0)<0,\qquad \text{for } t\in[0,T),$$

and

$$-\frac{1}{M(t)}+\frac{1}{M(0)}\le -\frac{\sigma}{2}t,$$

and then

$$u_x(t,0)=M(t)\le \frac{2M(0)}{2+\sigma M(0)t}\to-\infty,\quad \text{as } t\to -\frac{2}{\sigma M(0)} \tag{3.66}$$

which indicates that the maximal existence time $T\le -\frac{2}{\sigma u_{0,x}(0)}$ and hence completes the proof of the theorem. □

## 4. Blow-up rate

We now address the question of the blow-up rate of the slope to a breaking wave for system (1.4).

**Theorem 4.1.** *Let $\sigma\neq 0$. If $T<\infty$ is the blow-up time of the solution $(u,\rho)$ to* (1.4) *with initial data $(u_0,\rho_0-1)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$ with $s>3/2$ satisfying the assumption of Theorem 3.9. Then*

$$\lim_{t\to T^-}\left[\left(\inf_{x\in\mathbb{R}} u_x(t,x)\right)(T-t)\right]=-\frac{2}{\sigma},\quad \textit{for}\quad \sigma>0, \tag{4.1}$$

$$\lim_{t\to T^-}\left[\left(\sup_{x\in\mathbb{R}} u_x(t,x)\right)(T-t)\right]=-\frac{2}{\sigma},\quad \textit{for}\quad \sigma<0, \tag{4.2}$$

*Proof.* We may again assume $s = 3$ to prove the theorem.

In the case $\sigma > 0$, from (3.57) we have

$$m'(t) = -\frac{\sigma}{2} m^2(t) + f\left(t, q_1(t, x_1)\right).$$

Using (3.32) and denote

$$K = 1 + \frac{2 + A^2 + |\sigma| + 2|3 - \sigma|}{4} \|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}, \tag{4.3}$$

we know

$$-\frac{\sigma}{2} m^2(t) - K \leq m'(t) \leq -\frac{\sigma}{2} m^2(t) + K. \tag{4.4}$$

Choose $0 < \epsilon < \sigma/2$. Since $m(t) \to -\infty$ as $t \to T^-$, we can find $t_0 \in (0, T)$ such that

$$m(t_0) < -\sqrt{2\sigma K + \frac{K}{\epsilon}}.$$

Since $m(t)$ is absolutely continuous on $[0, T)$. It is then inferred from the above differential inequlity that $m(t)$ is strictly decreasing on $[t_0, T)$ and hence

$$m(t) < -\sqrt{2\sigma K + \frac{K}{\epsilon}} < -\sqrt{\frac{K}{\epsilon}}, \quad t \in [t_0, T).$$

Then (4.4) implies that

$$\frac{\sigma}{2} - \epsilon < \frac{d}{dt}\left(\frac{1}{m(t)}\right) < \frac{\sigma}{2} + \epsilon, \quad \text{a.e.} \quad t \in [t_0, T).$$

Integrating the above relation on $(t, T)$ with $t \in [t_0, T)$ and noticing that $m(t) \to -\infty$ as $t \to T^-$, we obtain

$$\left(\frac{\sigma}{2} - \epsilon\right)(T - t) \leq -\frac{1}{m(t)} \leq \left(\frac{\sigma}{2} + \epsilon\right)(T - t).$$

Since $\epsilon \in (0, \sigma/2)$ is arbitrary, in view of the definition of $m(t)$, the above inequality implies (4.1).

When $\sigma < 0$, from (3.61) we have

$$\bar{m}'(t) \geq -\frac{\sigma}{2} \bar{m}^2(t) - K,$$

where $K$ is defined in (4.3). Since $\bar{m}(t) \to \infty$ as $t \to T^-$, we can choose a $t_0 \in (0, T)$ such that

$$\bar{m}(t_0) > \sqrt{-2\sigma K}.$$

Therefore we have $\bar{m}(t)$ is strictly increasing on $[t_0, T)$ and

$$\bar{m}(t) > \bar{m}(t_0) > \sqrt{-2\sigma K} > 0.$$

Using the transport equation for $\rho$ we have that

$$\rho'\left(t, \eta(t)\right) = -\bar{m}(t) \rho\left(t, \eta(t)\right).$$

Hence

$$\rho\left(t, \eta(t)\right) = \rho\left(t_0, \eta(t_0)\right) e^{-\int_{t_0}^t \bar{m}(\tau) d\tau}, \quad t \in [t_0, T).$$

Then

$$\rho^2\left(t, \eta(t)\right) \leq \rho^2\left(t_0, \eta(t_0)\right), \quad t \in [t_0, T).$$

Therefore using (3.61) again we have

$$-\frac{\sigma}{2}\bar{m}^2(t) - \frac{1}{2}\rho^2\left(t_0, \eta(t_0)\right) - K \leq \bar{m}'(t) \leq -\frac{\sigma}{2}\bar{m}^2(t) + \frac{1}{2}\rho^2\left(t_0, \eta(t_0)\right) + K. \tag{4.5}$$

Now let

$$\bar{K} = \frac{1}{2}\rho^2\left(t_0, \eta(t_0)\right) + K$$

and choose $0 < \epsilon < -\sigma/2$. We can pick a $t_1 \in [t_0, T)$ such that

$$\bar{m}(t_1) > \sqrt{-2\sigma\bar{K} + \frac{\bar{K}}{\epsilon}}.$$

Then

$$\bar{m}(t) > \bar{m}(t_1) > \sqrt{-2\sigma\bar{K} + \frac{\bar{K}}{\epsilon}} > \sqrt{\frac{\bar{K}}{\epsilon}}.$$

Hence (4.5) implies that

$$\frac{\sigma}{2} - \epsilon < \frac{d}{dt}\left(\frac{1}{\bar{m}(t)}\right) < \frac{\sigma}{2} + \epsilon, \quad \text{a.e.} \quad t \in [t_1, T).$$

Integrating the above relation on $(t, T)$ with $t \in [t_1, T)$ and noticing that $\bar{m}(t) \to \infty$ as $t \to T^-$, we obtain

$$\left(\frac{\sigma}{2} - \epsilon\right)(T - t) \leq -\frac{1}{\bar{m}(t)} \leq \left(\frac{\sigma}{2} + \epsilon\right)(T - t).$$

Since $\epsilon \in (0, -\sigma/2)$ is arbitrary, in view of the definition of $\bar{m}(t)$, the above inequality implies (4.2). $\square$

## 5. Global existence

In this section we provide a sufficient condition for the global solution of system (1.4) in the case when $0 < \sigma < 2$.

**Theorem 5.1.** *Let $0 < \sigma < 2$ and $(u, \rho)$ be the solution of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and $T$ the maximal time of existence. If*

$$\inf_{x\in\mathbb{R}} \rho_0(x) > 0, \tag{5.1}$$

*then $T = +\infty$ and the solution $(u, \rho)$ is global.*

We need the following lemmas to prove the above theorem.

**Lemma 5.2.** *Let $0 < \sigma < 2$ and $(u, \rho)$ be the solution of* (1.4) *with initial data $(u_0, \rho_0 - 1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, $s > 3/2$, and $T$ the maximal time of existence. Assume that $\inf_{x\in\mathbb{R}} \rho_0(x) > 0$.*

*(i) If $0 < \sigma \leq 1$ then*

$$\left|\inf_{x\in\mathbb{R}} u_x(t, x)\right| \leq \frac{1}{\inf_{x\in\mathbb{R}} \rho_0(x)} C_4 e^{C_3 t}, \tag{5.2}$$

$$\left|\sup_{x\in\mathbb{R}} u_x(t, x)\right| \leq \frac{1}{\inf_{x\in\mathbb{R}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3 t}{2-\sigma}}, \quad \text{for } t \in [0, T). \tag{5.3}$$

*(ii) If $1 \leq \sigma < 2$ then*

$$\left|\inf_{x\in\mathbb{R}} u_x(t, x)\right| \leq \frac{1}{\inf_{x\in\mathbb{R}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3 t}{2-\sigma}}, \tag{5.4}$$

$$\left|\sup_{x\in\mathbb{R}} u_x(t, x)\right| \leq \frac{1}{\inf_{x\in\mathbb{R}} \rho_0(x)} C_4 e^{C_3 t}, \quad \text{for } t \in [0, T). \tag{5.5}$$

*The constants $C_3$ and $C_4$ are defined as follows where*

$$C_3 = 2 + \frac{2 + A^2 + |\sigma| + 2|3-\sigma|}{4}\|(u_0, \rho_0 - 1)\|^2_{H^1 \times L^2}. \tag{5.6}$$
$$C_4 = 1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty}. \tag{5.7}$$

*Proof.* Similar as before, a density argument indicates that it suffices to prove the desired results for $s \geq 3$. Thus we have

$$\inf_{x\in\mathbb{R}} u_x(t,x) < 0, \quad \sup_{x\in\mathbb{R}} u_x(t,x) > 0, \quad t \in [0,T).$$

as before.

(i) First we will derive an estimate for $|\inf_{x\in\mathbb{R}} u_x(t,x)|$. Define $m(t)$ and $\xi(t)$ as in (3.36) and consider along the characteristcs $q_1(t, x_1(t))$ as in (1.5) and (3.20). Thus from (3.33),

$$m(t) \leq 0 \quad \text{for} \quad t \in [0,T). \tag{5.8}$$

Letting $\zeta(t) = \rho(t, \xi(t))$ and evaluating (3.17) and the second equation of system (1.4) at $(t, \xi(t))$ we have

$$\begin{aligned} m'(t) &= -\frac{\sigma}{2}m^2(t) + \frac{1}{2}\zeta^2(t) + f(t, q_1(t,x_1)), \\ \zeta'(t) &= -\zeta m, \end{aligned} \tag{5.9}$$

for $t \in [0,T)$ where $f$ is defined in (3.23). The second equation above implies that $\zeta(t)$ and $\zeta(0)$ are of the same sign.

Now we want to construct a Lyapunov function for our system, as in [20]. Since here we have a free parameter $\sigma$, we couldn't find a uniform Lyapunov function. Instead, we will split the case $0 < \sigma \leq 1$ and the case $1 < \sigma < 2$. From the assumption of the theorem we know that $\zeta(0) = \rho(0, \xi(0)) > 0$.

When $0 < \sigma \leq 1$ we define the following Lyapunov function

$$w_1(t) = \zeta(0)\zeta(t) + \frac{\zeta(0)}{\zeta(t)}\left(1 + m^2(t)\right), \tag{5.10}$$

which is always positive for $t \in [0,T)$. Differentiating $w_1(t)$ and using (5.9) we obtain

$$\begin{aligned} w_1'(t) &= \zeta(0)\zeta' - \frac{\zeta(0)}{\zeta^2}(1+m^2)\zeta' + \frac{2}{\zeta}\zeta(0)mm' \qquad (5.11) \\ &= \frac{2\zeta(0)m}{\zeta}\left[\frac{1-\sigma}{2}m^2 + \frac{1}{2} + f(t, q_1(t,x_1))\right] \\ &\leq \frac{\zeta(0)}{\zeta}(1+m^2)\left[|f(t, q_1(t,x_1))| + \frac{1}{2}\right] \\ &\leq C_3 w_1(t), \end{aligned}$$

where we have used (5.8) and the bound (3.32) for $f$. Hence

$$\begin{aligned} w_1(t) &\leq w_1(0)e^{C_3 t} = \left[\zeta^2(0) + 1 + m^2(0)\right]e^{C_3 t} \\ &\leq \left(1 + \|u_{0,x}\|^2_{L^\infty} + \|\rho_0\|^2_{L^\infty}\right)e^{C_3 t} = C_4 e^{C_3 t}. \end{aligned} \tag{5.12}$$

Recalling that $\zeta(t)$ and $\zeta(0)$ are of the same sign, we have

$$\zeta(0)\zeta(t) \leq w_1(t), \quad \text{and} \quad |\zeta(0)|\,|m(t)| \leq w_1(t).$$

Then from (5.12) we have

$$\left|\inf_{x\in\mathbb{R}} u_x(t,x)\right| = |m(t)| \le \frac{w_1(t)}{\zeta(0)} \le \frac{1}{\inf_{x\in\mathbb{R}}\rho_0(x)} C_4 e^{C_3 t}, \quad \text{for } \ t\in[0,T),$$

which proves (5.2).

If $1\le\sigma<2$, we may define the Lyapunov function to be

$$w_2(t) = \zeta^\sigma(0)\frac{\zeta^2(t)+1+m^2(t)}{\zeta^\sigma(t)}. \tag{5.13}$$

Then a quick computation shows that

$$\begin{aligned} w_2'(t) &= \frac{2\zeta^\sigma(0)m}{\zeta^\sigma}\left[\frac{\sigma-1}{2}\zeta^2+\frac{\sigma}{2}+f\left(t,q_1(t,x_1)\right)\right] \\ &\le \frac{\zeta^\sigma(0)}{\zeta^\sigma}(1+m^2)\left[\left|f\left(t,q_1(t,x_1)\right)\right|+\frac{\sigma}{2}\right] \\ &\le C_3 w_2(t). \end{aligned} \tag{5.14}$$

Thus

$$\begin{aligned} w_2(t) &\le w_2(0)e^{C_3t} = \left[\zeta^2(0)+1+m^2(0)\right]e^{C_3t} \\ &\le \left(1+\|u_{0,x}\|^2_{L^\infty}+\|\rho_0\|^2_{L^\infty}\right)e^{C_3t} = C_4e^{C_3t}. \end{aligned} \tag{5.15}$$

Applying Young's inequality $ab\le a^p/p+b^q/q$ to (5.13) with

$$p=\frac{2}{\sigma}, \quad \text{and} \quad q=\frac{2}{2-\sigma}$$

we have

$$\begin{aligned} \frac{w_2(t)}{\zeta^\sigma(0)} &= \left[\zeta^{\frac{\sigma(2-\sigma)}{2}}\right]^{2/\sigma} + \left[\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\right]^{2/(2-\sigma)} \\ &\ge \frac{\sigma}{2}\left[\zeta^{\frac{\sigma(2-\sigma)}{2}}\right]^{2/\sigma} + \frac{2-\sigma}{2}\left[\frac{(1+m^2)^{\frac{2-\sigma}{2}}}{\zeta^{\frac{\sigma(2-\sigma)}{2}}}\right]^{2/(2-\sigma)} \\ &\ge (1+m^2)^{\frac{2-\sigma}{2}} \ge |m(t)|^{2-\sigma}. \end{aligned}$$

Therefore

$$\left|\inf_{x\in\mathbb{R}} u_x(t,x)\right| \le \left[\frac{w_2(t)}{\zeta^\sigma(0)}\right]^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in\mathbb{R}}\rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3t}{2-\sigma}}, \quad t\in[0,T),$$

which proves (5.4).

(ii) Next we try to control $|\sup_{x\in\mathbb{R}} u_x(t,x)|$. Similarly as before, we consider $\bar m(t)$, $\eta(t)$, $q_1\left(t,x_2(t)\right)$ as in (3.18) and (3.38). Then (5.9) becomes

$$\begin{aligned} \bar m'(t) &= -\frac{\sigma}{2}\bar m^2(t)+\frac{1}{2}\bar\zeta^2(t)+f\left(t,q_1(t,x_2)\right), \\ \bar\zeta'(t) &= -\bar\zeta\bar m, \end{aligned} \tag{5.16}$$

for $t\in[0,T)$, where $\bar\zeta(t)=\rho\left(t,\eta(t)\right)$. From (3.33) we have

$$\bar m(t)\ge 0, \quad \text{for} \quad t\in[0,T). \tag{5.17}$$

When $0<\sigma\le 1$, the corresponding Lyapunov function is

$$\bar w_1(t) = \bar\zeta^\sigma(0)\frac{\bar\zeta^2(t)+1+\bar m^2(t)}{\bar\zeta^\sigma(t)}. \tag{5.18}$$

Then from (5.14) and (5.17) we see that

$$\bar{w}_1'(t) \le C_3 \bar{w}_1(t), \quad \text{then} \quad \bar{w}_1(t) \le C_4 e^{C_3 t}.$$

Hence by the similar argument as before we get

$$\frac{\bar{w}_1(t)}{\bar{\zeta}^\sigma(0)} \ge |\bar{m}(t)|^{2-\sigma}.$$

Therefore

$$\left|\sup_{x\in\mathbb{R}} u_x(t,x)\right| \le \left[\frac{\bar{w}_1(t)}{\bar{\zeta}^\sigma(0)}\right]^{\frac{1}{2-\sigma}} \le \frac{1}{\inf_{x\in\mathbb{R}} \rho_0^{\frac{\sigma}{2-\sigma}}(x)} C_4^{\frac{1}{2-\sigma}} e^{\frac{C_3 t}{2-\sigma}}, \quad t \in [0,T),$$

which is (5.3).

When $1 \le \sigma < 2$, consider the Lyapunov function

$$\bar{w}_2(t) = \bar{\zeta}(0)\bar{\zeta}(t) + \frac{\bar{\zeta}(0)}{\bar{\zeta}(t)}\left(1 + \bar{m}^2(t)\right). \tag{5.19}$$

From (5.11) and (5.17),

$$\bar{w}_2'(t) \le C_3 \bar{w}_1(t), \quad \text{then} \quad \bar{w}_1(t) \le C_4 e^{C_3 t}.$$

Thus

$$\left|\sup_{x\in\mathbb{R}} u_x(t,x)\right| = |\bar{m}(t)| \le \frac{\bar{w}_1(t)}{\bar{\zeta}(0)} \le \frac{1}{\inf_{x\in\mathbb{R}} \rho_0(x)} C_4 e^{C_3 t}, \quad \text{for } t \in [0,T),$$

which proves (5.5). □

*Remark* 5.3. In fact when $\sigma \le 1$, $w_1(t)$ can always serve as a Lyapunov function to control $|\inf_{x\in\mathbb{R}} u_x(t,x)|$ to give the estimate (5.2). Similarly when $\sigma \ge 1$, we can always use $\bar{w}_2$ to estimate $|\sup_{x\in\mathbb{R}} u_x(t,x)|$ as in (5.5).

*Proof of Theorem 5.1.* Assume on the contrary that $T < \infty$ and the solution blows up in finite time. It then follows from Theorem 3.3 that

$$\int_0^T |u_x(t,x)|_{L^\infty} \, dt = \infty. \tag{5.20}$$

However from the assumptions of the theorem and Lemma 5.2 we have

$$|u_x(t,x)| < \infty$$

for all $(t,x) \in [0,T) \times \mathbb{R}$, a contradiction to (5.20). Thus $T = +\infty$ and the solution $(u, \rho)$ is global.

□

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## REFERENCES


[1] H. Aratyn, J. F. Gomes, and A. H. Zimerman, *On a negative flow of the AKNS hierarchy and its relation to a two-Component Camassa-Holm equation*, Symmetry, Integrability and Geom. Methods Appl., **2** (2006), paper 070, 12pp.

[2] R. Beals, D. Sattinger and J. Szmigielski, *Multi-peakons and a theorem of Stieltjes*, Inverse Problems **15** (1999), L1–L4.

[3] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Ration. Mech. Anal. **183** (2007), 215–239.

[4] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl. **5** (2007), 1–27.

[5] V. Busuioc, *On second grade fluids with vanishing viscosity*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 1241–1246.

[6] J.C. Burns, *Long waves on running water*, Proc. Cambridge Philos. Soc. **49** (1953) 695–706.

[7] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.

[8] R. Camassa, D. Holm and J. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech. **31** (1994), 1–33.

[9] M. Chen, S. Liu, and Y. Zhang, *A 2-Component generalization of the Camassa-Holm equation and its solutions*, Lett. Math. Phys., **75** (2006), 1–15.

[10] A. Constantin, *On the Cauchy problem for the periodic Camassa-Holm equation*, J. Differential Equations, **141** (1997), 218–235.

[11] A. Constantin, *Global existence of solutions and breaking waves for a shallow water equation: A geometric approach*, Ann. Inst. Fourier (Grenoble), **50** (2000), 321–362.

[12] A. Constantin, *On the blow-up of solutions of a periodic shallow water equation*, J. Nonlinear Sci., **10** (2000), 391–399.

[13] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math., **166** (2006), 523–535.

[14] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math., **181** (1998), 229–243.

[15] A. Constantin and J. Escher, *Well-posedness, global existence and blow-up phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math., **51** (1998), 475–504.

[16] A. Constantin and R. Ivanov, *On the integrable two-component Camassa-Holm shallow water system*, Phys. Lett. A, **372** (2008), 7129–7132.

[17] A. Constantin and R. S. Johnson, *Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis*, Fluid Dynam. Res., **40** (2008), 175–211.

[18] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal., **192** (2009), 165–186.

[19] A. Constantin and H. P. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., **52** (1999), 949–982.

[20] A. Constantin and R.I. Ivanov, *On an integrable two-component Camassa-Holm shallow water system*, Phys. Lett. A, **372** (2008), 7129–7132.

[21] A. Constantin and W.A. Strauss, *Stability of peakons*, Comm. Pure Appl. Math., 53 (2000), 603–610.

[22] A. Constantin and W.A. Strauss, *Stability of the Camassa-Holm solitons*, J. Nonlinear Sci. **12** (2002), 415–422.

[23] A. Constantin and W.A. Strauss, *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math. **57** (2004), 481–527.

[24] H.-H. Dai, *Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod*, Acta Mech., **127** (1998), 193–207.

[25] H.-H. Dai and Y. Huo, *Solitary shock waves and other travelling waves in a general compressible hyperelastic rod*, Proc. R. Soc. (London), **456** (2000), 331–363.

[26] J. Escher, O. Lechtenfeld and Z. Yin, *Well-posedness and blow-up phenom- ena for the 2-Component Camassa-Holm equation*, Discrete Contin. Dyn. Syst., **19** (2007), 493–513.

[27] A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäcklund transformation and hereditary symmetries*, Phys. D, **4** (1981), 47–66.

[28] B. Fuchssteiner, *Some tricks from the symmety-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation*, Phys. D, **95** (1996), 296–343.

[29] G. Gui and Y. Liu, *On the Cauchy problem for the two-component Camassa-Holm system*, Math. Z., (2010), DOI: 10.1007/s00209-009-0660-2.

[30] G. Gui and Y. Liu, *On the global existence and wave-breaking criteria for the two-component Camassa-Holm system*, J. Funct. Anal. **258** (2010), 4251-4278.
[31] D. Ionescu-Kruse, *Variational derivation of the Camassa-Holm shallow water equation*, J. Nonlinear Math. Phys. **14** (2007), 303–312.
[32] R. Ivanov, *Two-component integrable systems modelling shallow water waves:The constant vorticity case*, Wave Motion **46** (2009), 389–396.
[33] R.S. Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid Mech. **455** (2002), 63–82.
[34] R.S. Johnson, *The Camassa-Holm equation for water waves moving over a shear flow*, Fluid Dynam. Res. **33** (2003), 97–111.
[35] R.S. Johnson, *On solutions of the Burns condition (which determines the speed of propagation of linear long waves on a shear flow with or without a critical layer)*, Geophys. Astrophys. Fluid Dynam. **57** (1991), 115–133.
[36] T. Kato, *Quasi-linear equations of evolution, with applications to partial diÂőer- ential equations, in Spectral Theory and Differential Equations*, Lecture Notes in Math., Vol. 448, Springer Verlag, Berlin, (1975), 25–70.
[37] C. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., 46 (1993), 527–620.
[38] M. Lakshmanan, *Integrable nonlinear wave equations and possible connections to tsunami dynamics, Tsunami and nonliner waves*, (Ed. A. Kundu) Springer, Berlin, (2007), 31–49.
[39] J. Lenells, *Traveling wave solutions of the Camassa-Holm equation*, J. Differential Equations, **217** (2005), 393–430.
[40] J. Lenells, *Stability of periodic peakons*, Int. Math. Res. Not., **10** (2004), 485–499.
[41] Y. Liu and P. Zhang, *Stability of solitary waves and wave-breaking phenomena for the two-component Camassa-Holm system*, Int. Math. Res. Not., **211** (2009), 41 pages.
[42] H. P. Mckean, *Breakdown of a shallow water equation*, Asian J. Math., **2** (1998), 867–874.
[43] O. Mustafa, *On smooth traveling waves of an integrable two-component Camassa-Holm shallow water system*, Wave Motion **46** (2009), 397–402.
[44] G.A. Nariboli, *Nonlinear longitudinal dispersive waves in elastic rods*, J. Math. Phys. Sci., **4** (1970), 64–73.
[45] Z. Popowwicz, *A 2-component or $N = 2$ supersymmetric Camassa-Holm equa- tion*, Phys. Lett. A, **354** (2006), 110–114.
[46] H. Segur, *Waves in shallow water, with emphasis on the tsunami of 2004*, Tsunami and nonlinear waves (Ed. A. Kundu), Springer, Berlin, (2007), 3–29.
[47] A. Shabat, L. Martínez Alonso, *On the prolongation of a hierarchy of hydrodynamic chains*, Proceedings of the NATO advanced research workshop, Cadiz, Spain 2002, NATO Science Series, Kluwer Academic Publishers, Dordrecht, (2004), 263–280.
[48] T. Tao, *Low-regularity global solutions to nonlinear dispersive equations*, Surveys in Analysis and Operator Theory, Canberra, 2001, in: Proc. Centre Math. Appl. Austral. Nat. Univ., Austral. Nat. Univ., Canberra, **40** (2002), 19–48.
[49] J. F. Toland, *Stokes waves*, Topol. Methods Nonlinear Anal., **7** (1996), 1–48.
[50] G.B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1980.

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