UNIQUE DETERMINATION OF STRATIFIED STEADY WATER WAVES FROM PRESSURE

ROBIN MING CHEN AND SAMUEL WALSH

Abstract. Consider a two-dimensional stratified solitary wave propagating through a body of water that is bounded below by an impermeable ocean bed. In this work, we study how such a wave can be recovered from data consisting of the wave speed, upstream and downstream density and velocity profile, and the trace of the pressure on the bed. In particular, we prove that this data uniquely determines the wave, both in the (real) analytic and Sobolev regimes.

1. Introduction

The work of the present paper originates from a very applied problem: tracking the movement of waves far at sea. One of the primary ways in which this is currently done is through the use of recording devices capable of reading the subsurface pressure in the ocean. Inherent to this approach are a number of highly nontrivial mathematical statements. For instance, we may reasonably ask: How much information can really be gleaned about the flow from just the trace of the pressure on the ocean bed? Is it possible to totally reconstruct a passing traveling wave from this data? Addressing some of these issues is the main objective of this paper. We are particularly concerned with the role density stratification may play and whether one can perform this reconstruction in a low regularity regime.

Let us first discuss the situation for constant density fluids. If one takes the linearized free surface Euler equations for a small-amplitude gravity wave, it is simple to show that the surface profile can be expressed in terms of the pressure trace; indeed, this mapping is a Fourier multiplier whose symbol is referred to as the (linear) *pressure transfer function* (see, e.g., [4, 18, 24, 25]). More concretely, for a sinusoidal wave with wavenumber k, the connection between the the dynamic pressure $\mathfrak{p}$ and the surface displacement η can be written as

$$\mathfrak{p} = \rho g \frac{\cosh(kh_m)}{\cosh(kd)}\eta, \tag{1.1}$$

where h_m is the height of the pressure meter above the bottom, d is the mean depth, and the dynamic pressure $\mathfrak{p}$ is defined to be the difference between the pressure and the hydrostatic pressure, that is,

$$\mathfrak{p} = P - P_{\mathrm{atm}} - \rho g(d - h_m).$$

For pressure data obtained from the sea-bed, $h_m = 0$ and hence (1.1) becomes

$$\mathfrak{p} = \frac{\rho g}{\cosh(kd)}\eta,$$

which, in the zero depth limit $d \to 0$, recovers the hydrostatic approximation

$$\mathfrak{p} = \rho g \eta. \tag{1.2}$$

The linear pressure transfer function (1.1) works very well in the deep water regime. However, it is also well known that nonlinear effects play an important role in the surface reconstruction problem for shallow water waves or for waves in the surf zone (see [3, 4, 29], for instance). Experiments suggest that the predictions of linear models such as (1.1) may diverge sharply from observation [5]. For waves of large or even moderate amplitude, the discrepancy can be significant [4, 29]. Another issue with the linear theory is that, due to its reliance on Fourier series techniques, it can only treat periodic traveling waves.

These limitations have motivated a recent push to obtain reconstruction formulas that account for nonlinear effects and which apply to solitary waves. Nonlinear nonlocal equations relating the trace of the dynamic pressure on the bed to the surface profile of a solitary wave were established by several groups of researchers (see [11, 14, 27]). Notably, these papers work directly from the free surface Euler equations without further approximation. In [9, 10], the authors obtained implicit but exact and tractable relations for periodic traveling waves and performed straightforward numerical procedures to derive the free surface from the pressure at the bed. In [12], Constantin used a strong maximum principle argument completely different from the aforementioned approach to derive some improved estimates for the wave height of periodic traveling waves.

The above body of work focuses on irrotational steady waves propagating through constant density water. Actual waves in the ocean, however, frequently exhibit a heterogeneous density due to salinity or temperature gradients. This density stratification generates vorticity within the flow as the effects of gravity and inertia are experienced differently by heavier and lighter fluid particles. Moreover, a heterogeneous density distribution can permit large-amplitude waves to propagate in the bulk even while the surface remains relatively undisturbed. These so-called *internal waves* have been observed in oceanic areas for many years (cf., e.g. [28], or the compendium in [23]).

The vorticity that naturally accompanies stratification poses a serious mathematical challenge: nearly all of the irrotational theory is built on tools (e.g., conformal mappings, nonlocal reformulations via Dirichlet–Neumann operators, variational principles, etc.) that do not have obvious analogues in the rotational regime. As a consequence, rigorous results on large-amplitude rotational or stratified waves have begun to appear in the literature only quite recently (see, for e.g. [13, 32, 37, 38, 7, 8]). Assuming the vorticity is constant throughout the fluid domain, which is the simplest model of a wave-current interaction, Ali and Kalisch [2] derived a relationship in the long-wave limit that gives a direct map between the surface elevation and the pressure at the bottom of the fluid and within the bulk of the fluid domain. Later Vasan and Oliveras [30] obtained an exact reconstruction formula in this regime without approximation.

At present, no one to our knowledge has discovered how to adapt these ideas to the case of an *arbitrary* vorticity distribution. Indeed, constant vorticity is a very special (though physically significant) case that rarely provides much insight into how to attack the general problem. However, it is worth noting that recently Matioc [26] derived a power series reconstruction formula given the horizontal velocity along an axis of even symmetry for Stokes waves with C^∞ vorticity.

Unique determinability with stratification. Despite the success in seeking exact formulas for the pressure transfer functions in the previously mentioned works, a fundamental question still awaits to be resolved: is this surface reconstruction from pressure problem mathematically well-defined? Namely, knowing the wave speed and the upstream/downstream wave profile, is the free surface uniquely determined by the trace of the pressure on the bed? In [20], Henry first came up with an affirmative answer for rotational solitary waves, provided that the vorticity and pressure are real analytic. The analyticity condition is needed because he employs a Cauchy–Kowalevski argument. Matioc [26] considered a similar problem where the horizontal velocity is specified along an axis of even symmetry for the wave and the vorticity is C^∞.

Our main results pertain to the question of unique determinability of a stratified steady wave from the trace of the pressure on the bed. Mathematically, stratification plays a somewhat similar role in the governing equation for steady waves as vorticity. This suggests that Henry's method can be adapted to analytically stratified flows. Following this strategy, we prove that, if two stratified traveling waves share the same wave speed, analytic streamline density function, and pressure trace on the bed, then they must coincide exactly; see Theorem 3.1 and Corollary 3.7.

However, analyticity is a very strong assumption that is particularly ill-suited to stratified waves. Water columns in the ocean are frequently observed to have nearly constant density outside of thin transition layers called pycnoclines. Thus, as it passes through a pycnocline, the density experiences something close to a jump discontinuity. Indeed, a common practice is to view these waves as being a layering of multiple immiscible fluids each with its own constant density. In this model, the density distribution is piece-wise analytic, allowing us to iterate the Cauchy–Kowalevski approach of Henry and determine all of the free surfaces (cf. Corollary 3.2).

The more interesting question in the stratified regime, therefore, is whether one has unique determinability for a continuous but not necessarily analytic density (e.g., the existence theory for periodic stratified waves in [32] is formulated for density of class $C^{1,\alpha}$, and for solitary waves [7, 8] assumes that density is $C^{2,\alpha}$). For this case, Henry's argument does not seem to apply and so an entirely new idea is needed. In Theorem 3.4, we confirm that unique determinability still holds in the Sobolev space setting. Our method is based on reformulating the problem as an elliptic equation on a strip which can be treated using the theory of strong unique continuation (cf. Theorem 3.3). In fact, this approach can be used for constant density rotational waves as well, hence Theorem 3.4 generalizes Henry's work to the Sobolev regularity regime (see Remark 3.6).

2. Formulation

In this section, we present several formulations of the steady stratified water wave system, each tailored to one of the problems we later consider. Their equivalence is fairly straightforward to show (cf., e.g., [6, Lemma A.2] where it is proved for much weaker regularity solutions).

Notation. Before we proceed, let us first outline some conventions. For $k \in \mathbb{N}$ and $\alpha \in [0,1]$, and an open subset $U \subset \mathbb{R}^2$, we denote

$$C^{k+\alpha}(U) := \left\{ u \in C^k(U) : \|u\|_{C^{k+\alpha}} < \infty \right\}, \qquad C^{k+\alpha}(\overline{U}) := \left\{ u \in C^k(\overline{U}) : \|u\|_{C^{k+\alpha}} < \infty \right\},$$

which are Banach spaces under the $C^{k,\alpha}$ norm. Likewise, the set of real analytic functions with domain U will be written as $C^\omega(U)$.

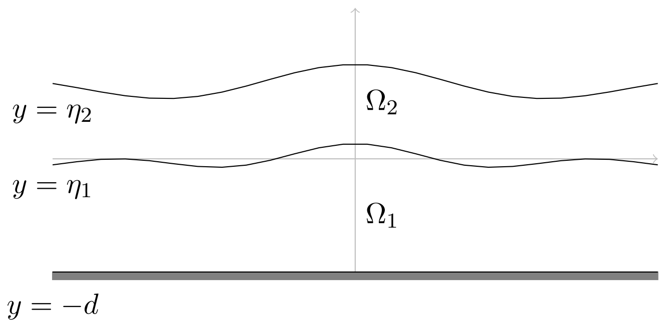

Figure 1. Example with two fluid layers, Ω_1 and Ω_2.

In what follows, we will often encounter functions defined on sets $U \subset \mathbb{R}^2$ of the form $U = U_1 \cup U_2$, for disjoint open sets U_1 and U_2 with $\partial U_1 \cap \partial U_2 \neq \emptyset$. In such cases, we may write $u \in C^{k,\alpha}(\overline{U_1}) \cap C^{k,\alpha}(\overline{U_2})$ to indicate that u restricted to U_i has a unique extension to $\overline{U_i}$ of class $C^{k,\alpha}(\overline{U_i})$, for $i = 1, 2$. Note that such a u need not be in $C^{k,\alpha}(\overline{U})$, however, as these extensions can disagree on $\partial U_1 \cap \partial U_2$.

For $k \in \mathbb{N}$ and $r \in [1, \infty]$, and supposing that U has Lipschitz continuous boundary, we define

$$\|u\|_{W^{k,r}(U)} := \sum_{|\beta| \leq k} \|\partial^\beta u\|_{L^r(U)},$$

and denote by $W^{k,r}(U)$ the closure of $C^\infty(\overline{U})$ under this norm.

Finally, we may add a subscript of "per" to any of these spaces to indicate that we are restricting to functions that are even and 2π-periodic in their first argument.

2.1. **Eulerian formulation.** Consider a two-dimensional traveling wave in a stratified water propagating to the right with speed $c > 0$ and acted upon by gravity. We work in a moving reference so that the resulting system is independent of time and the wave occupies the a priori unknown domain $\Omega \subset \mathbb{R}^2$. The flow is then described mathematically by a velocity field $(u, v) : \Omega \to \mathbb{R}^2$, a density function $\varrho : \Omega \to \mathbb{R}_+$, and a pressure $P : \Omega \to \mathbb{R}$.

Ocean waves typically exhibit thin transition layers where density varies sharply. For this, we begin by studying the *layered model* wherein Ω is partitioned into finitely many immiscible fluid regions

$$\Omega = \bigcup_{i=1}^N \Omega_i, \quad \Omega_i := \{(x, y) \in \Omega : \eta_{i-1}(x) < y < \eta_i(x)\} \tag{2.1}$$

where the density $\varrho_i := \varrho|_{\Omega_i}$ is smooth in each layer. Here we index so that $\eta_0 := -d$, $\eta_N := \eta$ and Ω_i lies beneath Ω_{i+1}, for $i = 1, \ldots N-1$; see Figure 2.1. Note that this is implicitly assuming that the free surfaces all have graph geometry. Also notice that the continuously stratified wave simply corresponds to the case when $N = 1$. We will use the convention that, for a quantity f with domain Ω, $f_i := f|_{\Omega_i}$.

Adapting the layer-wise model means in particular that the governing equations are satisfied in the classical sense in each Ω_i, but (u, v) and ϱ need not be continuous across the internal interfaces. By contrast, P will be continuous throughout the fluid domain. Here, smoothness can be measured in either the sense of Hölder continuity or Sobolev regularity.

As the governing equations for traveling water waves we use the incompressible Euler system, which consists of the conservation of mass,

$$(u-c)\varrho_x + v\varrho_y = 0, \qquad \text{in } \Omega_i, \tag{2.2a}$$

conservation of momentum

$$\begin{cases} \varrho(u-c)u_x + \varrho v u_y + P_x = 0 \\ \varrho(u-c)v_x + \varrho v v_y + P_y + g\varrho = 0 \end{cases} \qquad \text{in } \Omega_i, \tag{2.2b}$$

and incompressibility

$$u_x + v_y = 0, \qquad \text{in } \Omega_i. \tag{2.2c}$$

Here $g > 0$ the gravitational constant of acceleration. The density stratification will be assumed to be *stable*, meaning that

$$y \mapsto \varrho(\cdot, y) \text{ is non-increasing.} \tag{2.3}$$

The motion of the interfaces is driven by the kinematic boundary condition:

$$v = (u-c)\partial_x \eta_i, \qquad \text{on } \{y = \eta_i(x)\}. \tag{2.4a}$$

Note that this includes the air-sea interface. Also implicit above is the fact that

$$\frac{v_i}{u_i - c} = \frac{v_{i+1}}{u_{i+1} - c} \qquad \text{on } \{y = \eta_i(x)\},$$

which is equivalent to requiring that the normal component of the velocity be continuous over each interfaces. Since the ocean bed is impermeable, the normal velocity there must vanish:

$$v = 0, \qquad \text{on } \{y = -d\}. \tag{2.4b}$$

Finally, for the pressure we require

$$P = P_{\mathrm{atm}}, \qquad \text{on } \{y = \eta(x)\}, \tag{2.4c}$$

where P_{atm} is the (constant) atmospheric pressure.

It will be important for later reformulations to assume that there is no horizontal stagnation in the flow:

$$\sup_{\Omega}(u-c) < 0. \tag{2.4d}$$

Looking at (2.2b), it is clear that (2.4d) prohibits a certain degeneracy in the system. In particular, it guarantees that the level curves of the relative velocity field have graph geometry.

A solitary wave is a solution of the above system that is localized in space in the sense that it satisfies the following asymptotic conditions

$$(u, v) \to (\mathring{u}, 0), \;\; \eta_i \to \mathring{\eta}_i, \;\; \varrho \to \mathring{\varrho} \qquad \text{as } |x| \to \infty, \text{ uniformly in } y. \tag{2.5}$$

Here $\mathring{u} = \mathring{u}(y)$ and $\mathring{\varrho} = \mathring{\varrho}(y)$ are given function that describe the upstream velocity profile and density distribution, respectively, while $\mathring{\eta}_i$ are the asymptotic heights for the interface streamlines. Naturally, we require that $-d < \mathring{\eta}_1 < \ldots < \mathring{\eta}_N = 0$.

Problem 2.1 (Eulerian formulation)**.** Let a wave speed $c > 0$, volumetric mass flux $p_0 < 0$, and mean ocean depth $d > 0$ be given. Fix $\alpha \in (0,1)$.

A periodic solution of the traveling water wave problem consists of a velocity field (u, v), density ϱ, pressure P, and interface profiles $\eta_1, \ldots, \eta_N$ with the regularity

$$u, v, \varrho \in C^{0,\alpha}_{\mathrm{per}}(\overline{\Omega_i}) \cap C^{1,\alpha}_{\mathrm{per}}(\Omega_i), \quad \eta_i \in C^{2,\alpha}_{\mathrm{per}}(\mathbb{R}), \quad P \in C^{0,\alpha}(\overline{\Omega}) \cap C^{1,\alpha}_{\mathrm{per}}(\Omega_i), \tag{2.6}$$

and collectively satisfying (2.2)–(2.4).

Given upstream heights $-d < \mathring{\eta}_1 < \ldots < \mathring{\eta}_N := 0$, upstream velocity profile $\mathring{u} : [-d, 0] \to \mathbb{R}$ and upstream density distribution $\mathring{\varrho} : [-d, 0] \to \mathbb{R}$ with $\mathring{u}_i, \mathring{\varrho}_i \in C^{1,\alpha}([\mathring{\eta}_{i-1}, \mathring{\eta}_i])$, a solitary wave solution consists of a velocity field (u, v), density ϱ, pressure P, and interface profiles $\eta_1, \ldots, \eta_N$ with the regularity

$$u, v, \varrho \in C^{0,\alpha}(\overline{\Omega_i}) \cap C^{1,\alpha}(\Omega_i), \quad \eta_i \in C^{2,\alpha}(\mathbb{R}), \quad P \in C^{0,\alpha}(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega_i}), \tag{2.7}$$

and collectively satisfying (2.2)–(2.5).

2.2. **Stream function formulation.** Observe that the conservation of mass (2.2a) and incompressibility (2.2c) ensure that we may define a function $\psi = \psi(x, y)$ by

$$\psi_x = -\sqrt{\varrho} v, \qquad \psi_y = \sqrt{\varrho}(u - c) \qquad \text{in } \Omega_i. \tag{2.8}$$

This is the pseudo (relative) stream function for the flow, but throughout this paper we will simply refer to it as the *stream function*. From (2.8) and (2.4d), we see that the no stagnation condition translates to

$$\psi_y < 0 \qquad \text{in } \Omega_i. \tag{2.9}$$

The level sets of ψ are called the *streamlines*; by (2.8) and (2.2a), it is clear that they are precisely the integral curves of the relative velocity field. Thus, in particular, (2.4a)–(2.4b) directly imply that the free surface, internal interfaces, and ocean bed are all streamlines. Since (2.8) only determines ψ up to a constant in each Ω_i, we may take ψ to be continuous in $\overline{\Omega}$, and set $\psi = 0$ on the air–sea interface. Then $\psi = -p_0$ on the bed $\{y = -d\}$, where p_0 is the *(relative) pseudo-volumetric mass flux*:

$$p_0 := \int_{-d}^{\eta(x)} \sqrt{\varrho(x, y)}\, [u(x, y) - c] \; dy, \tag{2.10}$$

which is a (strictly negative) constant. Likewise, we define

$$p_i := \int_{-d}^{\eta_i(x)} \sqrt{\varrho(x, y)}\, [u(x, y) - c] \; dy, \tag{2.11}$$

so that $\{y = \eta_i(x)\}$ coincides with the level set $\{\psi = -p_i\}$.

The conservation of mass (2.2a) implies that there exists a function $\rho : [p_0, 0] \to \mathbb{R}^+$ such that

$$\varrho(x, y) = \rho(-\psi(x, y)) \qquad \text{in } \Omega_i. \tag{2.12}$$

We refer to ρ as the *streamline density function*. In the periodic setting, we will always think of ρ as being given. For solitary waves, as we explain below, one can determine ρ by examining the flow upstream or downstream.

Conservation of energy can be expressed via Bernoulli's theorem, which states that the quantity

$$E := P + \frac{\varrho}{2}\left((u-c)^2 + v^2\right) + g\varrho y, \tag{2.13}$$

is constant along streamlines (notice, though, that E is continuous in Ω_i but typically not across the interior interfaces). This allows us to define the so-called *Bernoulli function* β by

$$\frac{dE}{d\psi}(x,y) = -\beta(\psi(x,y)), \qquad \text{in } \Omega_i. \tag{2.14}$$

The dynamics in the interior of each layer is then captured by Yih's equation [39]

$$\Delta\psi - gy\rho'(-\psi) + \beta(\psi) = 0 \qquad \text{in } \Omega_i. \tag{2.15a}$$

Note that if in some layer Ω_i the flow is irrotational and ρ_i is a constant, then (2.15a) reduces to Laplace's equation for ψ.

Lastly, for the stream function formulation to be equivalent to the full Euler system, it is necessary that E satisfies certain jump conditions on the free surfaces. On the air–sea interface, $P = P_{\mathrm{atm}}$ and hence (2.13) gives immediately that

$$|\nabla\psi|^2 + 2g\varrho_N(y+d) = Q, \qquad \text{on } \{y = \eta(x)\}, \tag{2.15b}$$

where $Q = 2(E + \varrho d)|_{y=\eta}$. Likewise, for the interior interfaces we use the fact that P is continuous in $\overline{\Omega}$ when evaluating E from above and below on $\{y = \eta_i(x)\}$. This results in

$$|\nabla\psi_{i+1}|^2 - |\nabla\psi_i|^2 + 2g(\rho_{i+1} - \rho_i)(\eta_i + d) = Q_i \qquad \text{on } \{y = \eta_i(x)\}, \tag{2.15c}$$

with $Q_i := 2(E_{i+1} + g\rho_{i+1} - E_i - g\rho_i)|_{y=\eta_i}$.

For solitary waves, the form at infinity strongly constrains the global structure. First, observe that (2.5) implies that

$$\nabla^\perp\psi = \sqrt{\varrho}(u-c, v) \to \sqrt{\mathring{\varrho}}(\mathring{u} - c, 0),\ \ \eta_i \to \mathring{\eta}_i \qquad \text{as } |x| \to \infty, \text{ uniformly in } y. \tag{2.16}$$

Using this to evaluate the upstream limits of (2.15b) and (2.15c) allows us to determine Q and Q_i directly in terms of the other asymptotic quantities:

$$\begin{aligned} Q &= \mathring{\varrho}(0)(c - \mathring{u}(0))^2 + 2g\mathring{\varrho}(0)d \\ Q_i &= \sqrt{\mathring{\varrho}_{i+1}(\eta_i)}(c - \mathring{u}_{i+1}(\eta_i))^2 - \sqrt{\mathring{\varrho}_i(\eta_i)}(c - \mathring{u}_i(\eta_i))^2 + 2g(\mathring{\varrho}_{i+1}(\eta_i) - \mathring{\varrho}_i(\eta_i))(\eta_i + d). \end{aligned} \tag{2.17}$$

Moreover, given $\mathring{u}$ and $\mathring{\varrho}$, one can uniquely determine the Bernoulli function β and streamline density function ρ. This is a consequence of the fact that (2.16) decides the upstream height of every streamline. That is, if we denote the upstream limit of the stream function by $\mathring{\psi} = \mathring{\psi}(y)$, and let $\mathring{h}(p)$ be the limiting height above the bed of the streamline $\{\psi = -p\}$, by definition we have

$$\mathring{\psi}(\mathring{h}(p) - d) = -p \qquad \text{for all } p \in [p_0, 0], \qquad h(p_0) = 0. \tag{2.18}$$

Using the implicit function theorem, this furnishes a unique $\mathring{h}$ with $\mathring{h} \in C^{0,\alpha}([p_0,0])$ and $\mathring{h}_i \in C^{2,\alpha}((p_{i-1},p_i)) \cap C^{1,\alpha}([p_{i-1},p_i])$. But then it becomes clear that ρ and $\mathring{\varrho}$ are related by

$$\rho(p) = \mathring{\varrho}(\mathring{h}(p) - d). \tag{2.19}$$

Indeed, one can just as easily think of ρ and say $\mathring{u}$ as being the given asymptotic data. Similarly, evaluating E upstream, it is possible to explicitly compute that β must take the form

$$\beta(-p) = \rho_p(p)\left(g(\mathring{h}(p)-d) + \frac{1}{2}(\mathring{u}(\mathring{h}(p)-d)-c)^2\right) + \rho(p)(\mathring{u}(\mathring{h}(p)-d)-c)\mathring{u}'(\mathring{h}(p)-d). \tag{2.20}$$

This is derived, for example, in [8, Remark 2.1]. Finally, the wave speed, depth, upstream velocity, and upstream density are all coupled together by the requirement that

$$\mathring{h}(0) = d. \tag{2.21}$$

Remark 2.2. It is clear that there are many alternative ways to formulate the solitary wave problem depending on what asymptotic data we wish to take as given. For example, letting $\mathring{h}$ and ρ be fixed, we can define Q_i using (2.17), $\mathring{u}$ by (2.18), and $\mathring{\varrho}$ by (2.19).

Remark 2.3. It is worthwhile to look at the special case where $\mathring{u} = 0$, which simplifies many of the above expressions. For instance, (2.18) can be solved explicitly:

$$\mathring{h}(p) = \int_{p_0}^{p} \frac{1}{c\sqrt{\rho(s)}}\,ds. \tag{2.22}$$

The coupling of the wave speed and density (2.21) then takes the form

$$d = \int_{p_0}^{0} \frac{1}{c\sqrt{\rho(s)}}\,ds, \tag{2.23}$$

while the constants Q, Q_i are

$$Q = \varrho_N c^2 + 2g\varrho_N d, \quad Q_i = (\varrho_{i+1} - \varrho_i)\left(c^2 + 2g\mathring{h}\right). \tag{2.24}$$

Finally, from (2.20), we see that β in this setting is simply

$$\beta(-p) = \rho'(p)\left[\frac{1}{2}c^2 + g(\mathring{h} - d)\right]. \tag{2.25}$$

This is a well known fact about solitary stratified waves, but an elementary derivation is given in [6]. Notice that (2.25) suggests that, for solitary waves, it is more natural to prescribe ρ and $\mathring{h}$.

2.3. **Height function formulation.** The stream function formulation has the advantage of being a scalar equation, but it is still posed in an a priori unknown domain. For that reason, it is useful to consider changes of variables that map Ω to a fixed domain. Of course, one should expect this to come at the cost of additional nonlinearity in the resulting PDE. In the absence of stagnation (2.4d), a natural choice is to adopt semi-Lagrangian variables:

$$(x,y) \mapsto (q,p) := (x, -\psi).$$

These coordinates have a long history in the literature on existence of stratified water waves, dating back to their introduction by Dubreil-Jacotin [16]. Note that, because the internal interfaces and free surface are level sets of ψ, each layer Ω_i maps to a strip $\mathcal{R}_i := \{(q,p) : p \in (p_{i-1}, p_i)\}$. Similarly, $\mathcal{R} := \bigcup_i \mathcal{R}$ is the image of Ω.

In place of ψ, we work with the new unknown

$$h = h(q,p) := y + d,$$

which denotes the height above the bed of the point with x-coordinate q that lies on the streamline $\{\psi = -p\}$. Notice that h is continuous in $\overline{\mathcal{R}}$, though its derivatives will in general not be. One can show that Yih's equation (2.15a) is equivalent to the quasilinear elliptic problem

$$(1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_q h_p h_{pq} - g(h-d)\rho_p h_p^3 = -h_p^3 \beta(-p) \qquad \text{in } \mathcal{R}_i. \tag{2.26a}$$

The Bernoulli condition on the top (2.15b) likewise becomes

$$1 + h_q^2 + h_p^2(2g\rho h - Q) = 0 \qquad \text{on } \{p = 0\}, \tag{2.26b}$$

while the jump conditions on the internal interfaces translate to

$$\frac{1+(\partial_q h_{i+1})^2}{(\partial_p h_{i+1})^2} - \frac{1+(\partial_q h_i)^2}{(\partial_p h_i)^2} + 2g(\rho_{i+1} - \rho_i)h = Q_i \qquad \text{on } \{p = p_i\}. \tag{2.26c}$$

Here we are adapting our previous convention by writing $h_i := h|_{\mathcal{R}_i}$. The no stagnation condition (2.4d) expressed in terms of h becomes

$$h_p > 0 \qquad \text{in } \overline{\mathcal{R}}, \tag{2.26d}$$

while, simply from its definition, it is clear that

$$h = 0 \qquad \text{on } \{p = p_0\}. \tag{2.26e}$$

The derivation of this system can be found in many places (see, e.g., [32, 35]).

In the solitary wave setting, the behavior upstream is completely captured by the limiting height function $\mathring{h}$ defined in the previous subsection:

$$h \to \mathring{h} \qquad \text{as } |q| \to \infty, \text{ uniformly in } p. \tag{2.27}$$

The constants Q and Q_i can be computed by evaluating (2.26b) and (2.26c) at $q = \pm\infty$:

$$Q = \mathring{h}_p(0)^{-2} + 2g\rho(0)d, \qquad Q_i = \frac{1}{(\partial_p \mathring{h}_{i+1})(p_i)^2} - \frac{1}{(\partial_p \mathring{h}_i)(p_i)^2} + 2g(\rho_{i+1}(p_i) - \rho_i(p_i))h(p_i). \tag{2.28}$$

Just as before, there is a strong interrelation between β, ρ, $\mathring{h}$, $\mathring{u}$, and $\mathring{\varrho}$. As we are working in the semi-Lagrangian variables, the most natural choice when approaching (2.26a) is to think of ρ and $\mathring{h}$ in the solitary wave case, or ρ, β, Q, and Q_i for periodic waves.

To summarize, we have the following problem statement.

Problem 2.4 (Height equation formulation)**.** Fix $\alpha \in (0,1)$, ocean depth $d > 0$, and mass fluxes $p_0, \ldots, p_N$. Let a streamline density function $\rho : [p_0, 0] \to \mathbb{R}_+$ be given with $\rho_i \in C^{1,\alpha}([p_{i-1}, p_i])$.

For a given Bernoulli function β with $\beta_i \in C^{0,\alpha}([|p_i|, |p_{i-1}|])$, and constants Q, Q_i, a periodic solution of the height equation formulation of the traveling water wave problem consists of a function h with the regularity

$$h \in C^{0,\alpha}_{\mathrm{per}}(\overline{\mathcal{R}}), \qquad h_i \in C^{1,\alpha}_{\mathrm{per}}(\overline{\mathcal{R}_i}) \cap C^{2,\alpha}_{\mathrm{per}}(\mathcal{R}_i) \tag{2.29}$$

that satisfies (2.26).

For a given upstream height $\mathring{h}$ with $\mathring{h} \in C^{0,\alpha}([p_0, 0])$ and $\mathring{h}_i \in C^{1,\alpha}([p_{i-1}, p_i]) \cap C^{2,\alpha}((p_{i-1}, p_i))$, a solution of the height equation formulation of the solitary water wave problem consists of a function h with the regularity

$$h \in C^{0,\alpha}(\overline{\mathcal{R}}), \qquad h_i \in C^{1,\alpha}(\overline{\mathcal{R}_i}) \cap C^{2,\alpha}(\mathcal{R}_i) \tag{2.30}$$

that satisfies (2.26) along with the asymptotic condition (2.27), where β is determined by (2.20), while Q and Q_i are defined via (2.28).

3. Determinability

The objective of this section is to establish that the pressure reconstruction problem is mathematically well-defined. That is, we prove in various regimes that the wave speed, streamline density function, Bernoulli function, and the trace of the pressure on the bed together uniquely determine a steady stratified water wave. In particular, among all solitary waves with a given form at infinity, to each pressure trace there corresponds precisely one surface profile.

The main insight underlying the argument is that, due to Bernoulli's law, the trace of the pressure on the bed gives Neumann data for the height function. Since we already know that the height is 0 on the bed, together this amounts to a complete set of Cauchy data.

To see this, first observe that, in the semi-Lagrangian coordinates, Bernoulli's law (2.13) becomes

$$E = P + \frac{1 + h_q^2}{2h_p^2} + g\rho(h - d). \tag{3.1}$$

Evaluating this on the flat bed and using the definition of the Bernoulli function β (2.14) leads to

$$E|_{y=-d} = E|_{\psi=p_0} = E|_{\psi=0} - \int_{p_0}^{0} \beta(-p)\, dp. \tag{3.2}$$

As before, the form of the flow in the far field (2.5) allows us to compute that the Bernoulli constant $E|_{\psi=0}$ is

$$E|_{\psi=0} = E|_{y=\eta} = P_{\mathrm{atm}} + \frac{1}{2}\rho(0)(c - \mathring{u}(0))^2. \tag{3.3}$$

Therefore, from (2.26e) and (3.1)–(3.3), we infer that

$$P(q, p_0) + \frac{1}{2h_p^2(q, p_0)} - g\rho(p_0)d = P_{\mathrm{atm}} + \frac{1}{2}\rho(0)(c - \mathring{u}(0))^2 - \int_{p_0}^{0} \beta(-p)\, dp.$$

Finally, the absence of stagnation (2.4d) implies $h_p > 0$, so we conclude

$$h_p(q, p_0) = \left[2\left(P_{\mathrm{atm}} + \frac{1}{2}\rho(0)(c - \mathring{u}(0))^2 + g\rho(p_0)d - \int_{p_0}^{0} \beta(-p)\, dp - P(q, p_0)\right)\right]^{-1/2}. \tag{3.4}$$

Thus the normal derivative of h on $\{p = p_0\}$ is known as soon as the streamline density ρ, Bernoulli function β, limiting downstream/upstream horizontal velocity on the free surface, and the pressure data on the sea-bed are specified.

3.1. **Analytic setting.** We have seen that, treating the pressure trace as given, (2.26e) and (3.4) amount to Cauchy data. It is therefore useful to transform the elliptic height equation (2.26) into a first-order system that can be approached via Cauchy–Kowalevski. Define

$$F := \frac{h_q}{h_p}, \quad G := \frac{1}{h_p}, \quad H := h. \tag{3.5}$$

From (2.4d) and the identity $1/h_p = \sqrt{\rho}(c-u)$ we know that these quantities are well-defined. We can then reformulate the height equation (2.26) as

$$\begin{cases} F_p = \dfrac{F}{G}\left[\dfrac{GF_q + FG_q + G\beta(-p) - Gg\rho_p(H-d)}{F^2+G^2}\right] - \dfrac{G_q}{G}, \\ G_p = \dfrac{GF_q + FG_q + G\beta(-p) - Gg\rho_p(H-d)}{F^2+G^2}, \\ H_p = \dfrac{1}{G}, \end{cases} \tag{3.6}$$

with the following "initial" conditions posed at $p = p_0$

$$F(q, p_0) = H(q, p_0) = 0.$$

Here $G(\cdot, p_0)$ is uniquely determined by the pressure trace $P(\cdot, p_0)$ via (3.4).

Theorem 3.1. *Let (u, v, η, P) with regularity given in (2.7) be a solitary wave solution to Problem 2.1 with given real analytic $\mathring{\varrho}, \mathring{u}$ and wave speed c. Assume that the wave speed exceeds the horizontal fluid velocity $u < c$ throughout the fluid. Then u, v, η are real analytic. Moreover η is uniquely determined by the pressure function P on the flat bed in the sense that if $\eta, \tilde{\eta}$ are two solutions with the same pressure trace on $\{y = -d\}$, then $\eta \equiv \tilde{\eta}$.*

Proof. From the equivalence between the Eulerian and the height function formulations, it suffices to show that the corresponding height function $h \in C^{2,\alpha}(\overline{\mathcal{R}})$ to equation (2.26) is real analytic and is uniquely determined by the bottom pressure. The Cauchy–Kowalevski theorem applied to system (3.6) implies the existence and uniqueness of F, G, H, which are real analytic in some open neighborhood $\mathcal{N}$ of the bed $\{p = p_0\}$. Hence, in view of (3.5), h, h_p, h_q are also real analytic in $\mathcal{N}$. On the other hand, under the regularity hypotheses above, all solutions of the height equation (2.26) are real analytic in $\overline{\mathcal{R}}$ (cf. [36, Theorem 5.1]). Therefore the unique continuation of real analytic functions ensures the uniqueness of the solution h to (2.26) with Neumann data on the bottom given by the pressure from (3.4). $\square$

The above determinability result can be easily extended to layer-wise smooth flows. Note that the interior interfaces are streamlines, hence they correspond to straight lines $\{p = p_i\}$ in the semi-Lagrangian variables. The first-order system (3.6) will hold in each strip $\mathcal{R}_i$. Beginning in the strip directly above the bed, we can argue as in Theorem 3.1 and Corollary 3.7 to show that $h \in C^\omega(\overline{\mathcal{R}_1})$ and is determined there uniquely by the prescribed data. By construction, h is continuous over the interface and indeed $q \mapsto h(q, \cdot)$ is real analytic. On the other hand, the jump condition (2.26c) allows us to determine $\partial_p h_2(\cdot, p_1)$. Thus we have a full set of Cauchy data for the first-order problem in $\mathcal{R}_2$. Applying Cauchy–Kowalevski once more allows us to conclude $h \in C^\omega(\overline{\mathcal{R}_2})$. Iterating this procedure, we get the following result.

Corollary 3.2. *Fix the wave speed c. Let (u, v, η, P) with*

$$u_i, v_i, P_i \in C^{1,\alpha}(\overline{\Omega}_i), \quad \eta_i \in C^{2,\alpha}(\mathbb{R})$$

be a solitary wave solution to Problem 2.1 with given $\mathring{\varrho}, \mathring{u} \in C^\omega([\eta_{i-1}, \eta_i])$. Assume that the wave speed exceeds the horizontal fluid velocity $u < c$ throughout the fluid. Then u_i, v_i, η_i are real analytic and η_i is uniquely determined by the pressure function P on the flat bed.

3.2. **Sobolev setting.** In this section, we considerably strengthen the above results by relaxing the analyticity requirement to a more physically reasonable degree of smoothness. Our argument is built off of the following result on the strong unique continuation for a general elliptic equation set in the plane.

Theorem 3.3. [1, Theorem 1] *Given a bounded connected open set $\Omega \subset \mathbb{R}^2$. Let $u \in W^{1,2}(\Omega)$ be a weak solution to the elliptic equation*

$$Lu := (a^{ij} u_{x_i} + a^j u)_{x_j} + b^i u_{x_i} + cu = 0 \quad \text{in } \Omega,$$

where

(i) *L is uniformly elliptic in the sense that there exists a $K > 0$ such that*

$$0 < K^{-1}|\zeta|^2 \leq a^{ij}(x)\zeta_i\zeta_j \leq K|\zeta|^2, \qquad \text{for all } \zeta \in \mathbb{R}^2, \text{and for a.e. } x \in \Omega; \text{and}$$

(ii) *$a^{ij} \in L^\infty(\Omega)$, a^i, $b^i \in L^q(\Omega)$, $c \in L^{q/2}(\Omega)$ for some $q > 2$.*

Then the operator L has the strong unique continuation property in the sense that if a solution u has a zero of infinite order at a point $x_0 \in \Omega$, i.e., there exists $\delta > 0$ such that for every integer $N \geq 0$

$$\int_{B_r(x_0)} |u|^2 \, dx \leq c_N r^N, \qquad \text{for all } r < \delta,$$

then $u \equiv 0$ in Ω.

With this result in place, we can now state and prove our main theorem on determinability.

Theorem 3.4. *Fix the wave speed c. Let $r > 2$ and (u, v, η, P) with*

$$u_i, v_i, P_i \in W^{2,r}(\overline{\Omega}_i), \quad \text{and} \quad \eta_i \in W^{3-\frac{1}{r},r}(\mathbb{R})$$

be a solution of Problem 2.1 representing a solitary water wave with given $\mathring{\varrho}_i, \mathring{u}_i \in W^{2,r}([\mathring{\eta}_{i-1}, \mathring{\eta}_i])$. Then η is uniquely determined by the trace of the pressure P on the bed.

Again the above theorem can be proved through considering the height equation formulation (2.26). Note that as discussed in the previous subsection, $h_p(\cdot, p_0)$ is determined by $P(\cdot, p_0)$ through (3.4), and thus Theorem 3.4 is equivalent to the following statement about unique determinability of h from the pressure trace (see [6, Lemma A.2]).

Theorem 3.5. *Let $r > 2$ and $h \in W^{1,r}(\overline{\mathcal{R}})$ with $h_i \in W^{3,r}(\overline{\mathcal{R}}_i)$ be a solution of (2.26). Assume that the stable streamline density stratification $\rho_i \in W^{2,r}([p_{i-1}, p_i])$ and the Bernoulli function $\beta_i \in W^{1,r}([|p_i|, |p_{i-1}|])$, and $\partial_p h_i > 0$. Then h is uniquely determined by the normal derivative h_p on the flat bed.*

Proof. Let h and $\tilde{h}$ be two solutions to (2.26) with identical Neumann data on the bed:

$$h_p(\cdot,p_0)=\tilde{h}_p(\cdot,p_0) \qquad \text{on } \mathbb{R}.$$

The difference $u:=h-\tilde{h}$ satisfies an elliptic equation (see also [31])

$$\mathscr{L}u:=(a^{ij}u_{x_i})_{x_j}+b^iu_{x_i}+cu=0 \quad \text{in } \mathcal{R},$$

where

$$\begin{aligned}
a^{11}&:=1+h_q^2, \quad a^{12}=a^{21}=-h_ph_q, \quad a^{22}=h_p^2,\\
b^1&:=\tilde{h}_{pp}(h_q+\tilde{h}_q)-2h_p\tilde{h}_{pq}-h_ph_{pq}+h_qh_{pp},\\
b^2&:=\tilde{h}_{qq}(h_p+\tilde{h}_p)-2\tilde{h}_q\tilde{h}_{pq}+\beta(-p)(h_p^2+h_p\tilde{h}_p+\tilde{h}_p^2)-g\rho_p(\tilde{h}-d(\tilde{h}))(h_p^2+h_p\tilde{h}_p+\tilde{h}_p^2)\\
&\quad -h_qh_{pq}+h_ph_{qq}\\
c&:=-g\rho_ph_p^3,
\end{aligned}$$

and

$$u=u_p=0, \quad \text{on } \{p=p_0\}.$$

The goal is to show that u vanishes identically in the entire strip $\mathcal{R}$. We will achieve this through an induction process.

First, we prove that $u_1(=u|_{\mathcal{R}_1})\equiv 0$ on the bottom strip $\mathcal{R}_1$. The above boundary condition guarantees that u can be extended to a $W^{1,r}$ function defined on the lower half-plane $\{p<p_0\}$ that vanishes in $\{p<p_0\}$. Also, since $h_1,\tilde{h}_1\in W^{3,r}(\mathcal{R}_1)$, from the trace theorem (see [19]) and the fractional Sobolev embedding theorem (cf., e.g., [15, Theorem 8.2]) we know that the traces

$$\nabla^2h_1,\nabla^2\tilde{h}_1\in W^{1/2,r}(\{p=p_0\})\subset C^{0,\frac{r-2}{2r}}(\{p=p_0\}). \tag{3.7}$$

Similarly, we can make sense of the traces $\partial_p\rho_1$ and β_1 on $\{p=0\}$ and $\{p=p_0\}$. This way, the functions $h_1,\tilde{h}_1,\rho_1$ and β_1 can all be extended below $\mathcal{R}_1$ while maintaining their regularity and ensuring $\partial_ph_1>0$ in the strong sense. Therefore, from the divergence structure of the operator $\mathscr{L}$, we see that the extended u_1 also satisfies $\mathscr{L}u_1=0$. This, combined with Sobolev embedding, implies that the coefficients in $\mathscr{L}$ satisfy the the conditions of Theorem 3.3, and hence $\mathscr{L}$ has the strong unique continuation property. Note that Theorem 3.3 is a local result. Thus we may apply it on any finite portion of $\mathcal{R}_1$ to show that the restriction of u_1 there vanishes identically. It follows that $u_1\equiv 0$ on the entire $\mathcal{R}_1$.

Now suppose $u_k\equiv 0$ on $\mathcal{R}_k$, for some $1\le k\le N$. We see that u_{k+1} satisfies the same equation $\mathscr{L}u_{k+1}=0$. In order to apply the previous argument, it suffices to show that u_{k+1} satisfies the homogeneous Dirichlet and Neumann conditions on $\{p=p_k\}$. This amounts to proving that h_{k+1} and $\tilde{h}_{k+1}$ share the same Dirichlet and Neumann data on $\{p=p_k\}$. From induction, $u_k\equiv 0$ on $\mathcal{R}_k$, and hence

$$h_k(\cdot,p_k)=\tilde{h}_k(\cdot,p_k), \quad \partial_ph_k(\cdot,p_k)=\partial_p\tilde{h}_k(\cdot,p_k).$$

Continuity of the h and $\tilde{h}$ implies that $h_{k+1}(\cdot,p_k)=\tilde{h}_{k+1}(\cdot,p_k)$, and thus

$$u_{k+1}=0, \quad \text{on } \ \{p=p_k\}.$$

From (2.26c) we see that

$$\frac{1+(\partial_q h_{k+1})^2}{(\partial_p h_{k+1})^2} = \frac{1+(\partial_q h_k)^2}{(\partial_p h_k)^2} - 2g(\rho_{k+1}-\rho_k)h + Q_k, \qquad \text{on } \{p=p_k\},$$

and the same holds for $\tilde{h}_{k+1}$. Further using the fact that h_q and $\tilde{h}_q$ are continuous, and $\partial_p h_{k+1}$, $\partial_p \tilde{h}_{k+1} > 0$ it follows that

$$\partial_p u_{k+1} = 0, \quad \text{on } \ \{p = p_k\}.$$

Therefore we can apply the unique continuation argument to conclude that $u_{k+1} \equiv 0$ in $\mathcal{R}_{k+1}$, and hence complete the proof. □

Remark 3.6. (i) Note that by Sobolev embedding, the height function $h_i \in C^{2,(r-2)/r}(\overline{\mathcal{R}}_i)$ and hence these are classical solutions of the height equation. Likewise, the velocity field and pressure in Theorem 3.4 also exhibit classical regularity in each layer. It is well-known that the height equation (2.26) can be written in divergence form which allows us to assume that $h \in W^{1,r}(\overline{\mathcal{R}})$, and this in particular guarantees that h is continuous in the whole domain. However, in (3.7) we use strongly the fact that $h_i \in W^{3,r}(\overline{\mathcal{R}}_i)$ when extending the height function over the lower boundary of $\mathcal{R}_i$, and therefore our method requires the $W^{3,r}$ regularity in each layer.

(ii) Theorem 3.5 easily applies to the case of a continuously stratified fluid: one can simply adapt the argument for the determinability of the bottom fluid region in the above proof for the continuously stratified case.

(iii) Indeed from a mathematical point of view, h can be uniquely determined by h_p on $\{p = p_i\}$ for any $1 \le i \le N-1$. What one can do is to first obtain the uniqueness of h in $\mathcal{R}_i$ by extending the solution below $\{p = p_i\}$ as before, and iterating the same argument to prove the uniqueness of h in $\{p \ge p_i\}$. Then apply the unique continuation in $\mathcal{R}_{i-1}$ by extending the difference $u_{i-1} = h_{i-1} - \tilde{h}_{i-1}$ above $\{p = p_i\}$ to prove that $u_{i-1} \equiv 0$. Then repeat the process downward till $\mathcal{R}_1$.

(iv) The same argument applies to the regime considered by Henry [20], namely constant density flow with a general vorticity distribution. In that case, the height equation is

$$\begin{cases} (1+h_q^2)h_{pp} + h_{qq}h_p^2 - 2h_q h_p h_{pq} = -h_p^3\gamma(-p), & p_0 < p < 0, \\ 1 + h_q^2 + h_p^2(2gh - Q) = 0, & p = 0, \\ h = 0, & p = p_0, \end{cases} \tag{3.8}$$

where γ is the vorticity function. Clearly γ plays a nearly identical role as β does in (2.26a), and so the arguments above go through with only minor alteration.

3.3. **Periodic waves.** To complete the theory we also consider the periodic setting. In contrast to the case of solitary waves, there has been a fairly robust existence theory for large-amplitude periodic steady stratified waves developed in recent years (cf. [32, 17, 22, 21, 33, 34]).

Since both the Cauchy–Kowalevski and the unique continuation theory are local, they can be applied just as well on a periodic domain. Thus, in order to adapt the argument of Theorem 3.1 and Theorem 3.4 to periodic traveling waves, it remains only to show that the pressure trace on the bed furnishes Neumann data for h there. Unfortunately, because this setting lacks an upstream state, we will need some additional information.

More precisely, observe that by combining (3.1) with (3.2) for a general Bernoulli function β, we obtain the identity

$$h_p(q, p_0) = \left[2\left(E|_{\psi=0} - \int_{p_0}^{0} \beta(-p)\,dp + g\rho(p_0)d - P(q, p_0)\right)\right]^{-1/2}. \tag{3.9}$$

So long as we know $E|_{\psi=0}$, we can proceed as before. This leads to the following results.

Corollary 3.7. *Let $(u, v, \eta, P) \in C^{1,\alpha}_{\mathrm{per}}(\overline{\Omega}) \times C^{1,\alpha}_{per}(\overline{\Omega}) \times C^{2,\alpha}_{\mathrm{per}}(\mathbb{R}) \times C^{1,\alpha}_{\mathrm{per}}(\overline{\Omega})$ be a periodic solution to Problem 2.1 with a stable streamline density function $\rho \in C^\omega([p_0, 0])$, Bernoulli function $\beta \in C^\omega([0, |p_0|])$, and with wave speed c. Then $u, v, \eta \in C^\omega_{\mathrm{per}}(\mathbb{R})$. Moreover, η is uniquely determined by the trace of the pressure P on the bed and the value of E on the free surface.*

Corollary 3.8. *Let $r > 2$ and (u, v, η, P) with*

$$u_i, v_i, P_i \in W^{2,r}_{\mathrm{per}}(\mathcal{R}_i) \cap W^{1,r}_{\mathrm{per}}(\overline{\mathcal{R}}_i), \quad \text{and} \quad \eta_i \in W^{3-\frac{1}{r},r}_{\mathrm{per}}(\mathbb{R})$$

be a periodic solution to Problem 2.1 with a stable streamline density stratification $\rho \in W^{2,r}([p_0, 0])$, a Bernoulli function $\beta \in W^{1,r}([0, |p_0|])$, and with the wave speed c. Then η is uniquely determined by the trace of the pressure P on the bed and the value of E on the free surface.

Remark 3.9. The above results should be appreciated from a theoretic viewpoint rather than a practical one. In field experiments it is quite difficult to record $E|_{\psi=0}$, though mathematically it can be determined, for instance, from any one of the following:

(i) the value of u at the crest and the height of the crest; or
(ii) the value of u at the trough and the height of the trough; or
(iii) the average value of the kinetic energy $\frac{\rho(0)}{2}|(u - c, v)|^2$ on the surface.

For (i), we just evaluate Bernoulli's law at the crest (where $v = 0$). Likewise, for (ii), we evaluate Bernoulli at the trough. For (iii), we take the mean value on the surface and use the fact that η has mean 0.

Acknowledgements

The research of RMC is supported in part by National Science Foundation through DMS-1613375 and the Simons Foundation under Grant 354996. The research of SW is supported in part by the National Science Foundation through DMS-1514910.

The authors would like to thank the referees for the comments and suggestions which have improved the quality of the article.

References

[1] G. Alessandrini, *Strong unique continuation for general elliptic equations in 2D*, J. Math. Anal. Appl., 386 (2012), pp. 669–676.
[2] A. Ali and H. Kalisch, *Reconstruction of the pressure in long-wave models with constant vorticity*, European Journal of Mechanics-B/Fluids, 37 (2013), pp. 187–194.
[3] P. Bergan, A. Torum, and A. Tratteberg, *Wave measurements by a pressure type wave gauge*, Coastal Engineering Proceedings, 1 (1968).

[4] C. T. Bishop and M. A. Donelan, *Measuring waves with pressure transducers*, Coastal Engineering, 11 (1987), pp. 309–328.
[5] L. Cavaleri, *Wave measurement using pressure transducer*, Oceanologica Acta, 3 (1980), pp. 339–346.
[6] R. M. Chen and S. Walsh, *Continuous dependence on the density for stratified steady water waves*, Arch. Ration. Mech. Anal., 219 (2016), pp. 741–792.
[7] R. M. Chen, S. Walsh, and M. H. Wheeler, *On the existence and qualitative theory for stratified solitary water waves*, C. R. Acad. Sci. Paris Sér. I Math., 354 (2016), pp. 601–605.
[8] ———, *Existence and qualitative theory for stratified solitary water waves*, Ann. Inst. H. Poincaré Anal. Non Linéaire, (to appear).
[9] D. Clamond, *New exact relations for easy recovery of steady wave profiles from bottom pressure measurements*, Journal of Fluid Mechanics, 726 (2013), pp. 547–558.
[10] D. Clamond and A. Constantin, *Recovery of steady periodic wave profiles from pressure measurements at the bed*, J. Fluid Mech., 714 (2013), pp. 463–475.
[11] A. Constantin, *On the recovery of solitary wave profiles from pressure measurements*, J. Fluid Mech., 699 (2012), pp. 376–384.
[12] ———, *Estimating wave heights from pressure data at the bed*, Journal of Fluid Mechanics, 743 (2014), p. R2.
[13] A. Constantin and W. A. Strauss, *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math., 57 (2004), pp. 481–527.
[14] B. Deconinck, D. Henderson, K. Oliveras, and V. Vasan, *Recovering the water-wave surface from pressure measurements*, in Proceedings of 10th International Conference on WAVES, Vancouver, 2011.
[15] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), pp. 521–573.
[16] M. Dubreil-Jacotin, *Sur les theoremes d'existence relatifs aux ondes permanentes periodiques a deux dimensions dans les liquides heterogenes*, J. Math. Pures Appl., 16 (1937), pp. 43–67.
[17] J. Escher, A.-V. Matioc, and B.-V. Matioc, *On stratified steady periodic water waves with linear density distribution and stagnation points*, J. Differential Equations, 251 (2011), pp. 2932–2949.
[18] J. Escher and T. Schlurmann, *On the recovery of the free surface from the pressure within periodic traveling water waves*, Journal of Nonlinear Mathematical Physics, 15 (2008), pp. 50–57.
[19] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Math. Univ. Padova, 27 (1957), pp. 284–305.
[20] D. Henry, *On the pressure transfer function for solitary water waves with vorticity*, Math. Ann., 357 (2013), pp. 23–30.
[21] D. Henry and A.-V. Matioc, *Global bifurcation of capillary–gravity-stratified water waves*, Proc. Roy. Soc. Edinburgh Sect. A, 144 (2014), pp. 775–786.
[22] D. Henry and B.-V. Matioc, *On the existence of steady periodic capillary-gravity stratified water waves*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 12 (2013), pp. 955–974.
[23] C. R. Jackson, *Atlas of internal solitary waves.* (http://www.internalwaveatlas.com/), February 2004.
[24] B. Kinsman, *Wind waves: their generation and propagation on the ocean surface*, Courier Dover Publications, 1965.
[25] Y.-Y. Kuo and Y.-F. Chiu, *Transfer function between wave height and wave pressure for progressive waves*, Coastal Engineering, 23 (1994), pp. 81–93.
[26] B.-V. Matioc, *Recovery of Stokes waves from velocity measurements on an axis of symmetry*, J. Phys. A, 48 (2015), pp. 255501, 8.
[27] K. L. Oliveras, V. Vasan, B. Deconinck, and D. Henderson, *Recovering the water-wave profile from pressure measurements*, SIAM Journal on Applied Mathematics, 72 (2012), pp. 897–918.
[28] R. B. Perry and G. R. Schimke, *Large-amplitude internal waves observed off the northwest coast of sumatra*, Journal of Geophysical Research, 70 (1965), pp. 2319–2324.
[29] C.-H. Tsai, M.-C. Huang, F.-J. Young, Y.-C. Lin, and H.-W. Li, *On the recovery of surface wave by pressure transfer function*, Ocean Engineering, 32 (2005), pp. 1247–1259.
[30] V. Vasan and K. Oliveras, *Pressure beneath a traveling wave with constant vorticity*, Discrete Contin. Dyn. Syst., 34 (2014), pp. 3219–3239.
[31] S. Walsh, *Some criteria for the symmetry of stratified water waves*, Wave Motion, 46 (2009), pp. 350–362.

[32] ———, *Stratified and steady periodic water waves*, SIAM J. Math. Anal., 41 (2009), pp. 1054–1105.
[33] ———, *Steady stratified periodic gravity waves with surface tension I: Local bifurcation*, Discrete Contin. Dyn. Syst. Ser. A, 34 (2014), pp. 3287–3315.
[34] ———, *Steady stratified periodic gravity waves with surface tension II: Global bifurcation*, Discrete Contin. Dyn. Syst. Ser. A, 34 (2014), pp. 3241–3285.
[35] S. Walsh, O. Bühler, and J. Shatah, *Steady Water Waves in the Presence of Wind*, SIAM J. Math. Anal., 45 (2013), pp. 2182–2227.
[36] L.-J. Wang, *Regularity of traveling periodic stratified water waves with vorticity*, Nonlinear Anal., 81 (2013), pp. 247–263.
[37] M. H. Wheeler, *Large-amplitude solitary water waves with vorticity*, SIAM J. Math. Anal., 45 (2013), pp. 2937–2994.
[38] ———, *Solitary water waves of large amplitude generated by surface pressure*, Arch. Ration. Mech. Anal., 218 (2015), pp. 1131–1187.
[39] C.-S. Yih, *Dynamics of nonhomogeneous fluids*, The Macmillan Co., New York, 1965.

Robin Ming Chen
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

E-mail address: mingchen@pitt.edu

Samuel Walsh
Department of Mathematics, University of Missouri, Columbia, MO 65211

E-mail address: walshsa@missouri.edu