

On the existence and qualitative theory of stratified solitary water waves

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Abstract

In this note, we announce new results on the existence of two-dimensional solitary waves moving through a body of density stratified water lying beneath air. The fluid domain is assumed to lie above an impenetrable flat ocean bed, while the interface between the air and water is a free boundary where the pressure is constant. We prove that, for any smooth choice of upstream velocity and density distribution, there exists a continuous curve of such solutions that includes large-amplitude waves that come arbitrarily close to having a (horizontal) stagnation point.

Additionally, we provide several results characterizing the qualitative features of solitary stratified waves. In part, these include: estimates on the Froude number, velocity, and pressure, some of which are new even for the constant density case; a proof of the nonexistence of monotone bores in this physical regime; and a theorem ensuring that all supercritical stratified solitary waves of elevation have an axis of even symmetry.

Résumé

Sur l'existence et théorie qualitative des ondes d'eau stratifiés solitaires.

Dans cette note, nous annonçons de nouveaux résultats sur l'existence des ondes de gravité solitaires en deux dimensions se déplaçant à travers un plan d'eau stratifié et situé sous l'air. Le domaine de fluide est limité vers le bas par un fond imperméable, tandis que l'interface entre l'eau et l'air est une frontière libre où la pression est constante. Nous montrons que, pour tout choix de profil de vitesse et de distribution de densité en amont, il existe une courbe continue de ces solutions qui comprend les ondes de surface de grande amplitude qui sont arbitrairement près d'avoir un point de stagnation horizontale.

En outre, nous fournissons plusieurs résultats concernant les caractéristiques qualitatives des ondes solitaires stratifiées. En partie, cela comprend : des estimations sur le nombre de Froude, la vitesse et la pression, dont certains sont nouveaux, même pour le cas de densité constante ; une preuve de non-existence des mascarets monotones dans ce régime ; et un théorème énonçant la parité des ondes stratifiées super-critiques d'élévation.

1. Introduction

Consider a two-dimensional traveling wave in water of heterogeneous density moving with constant speed c under the influence of gravity. We can eliminate time-dependence by switching to a moving reference frame, so that the wave occupies a steady domain $\Omega_\eta := \{(x, y) \in \mathbb{R}^2 : -d < y < \eta(x)\}$, where the a priori unknown function η is the free surface profile and $\{y = -d\}$ is an impermeable flat bed.

A stratified water wave is identified with the following mathematical data: a free surface profile η , density ϱ , velocity field (u, v) , and pressure P . They must collectively satisfy the incompressible steady Euler system

$$\begin{cases} (u - c)\varrho_x + v\varrho_y = 0 \\ \varrho(u - c)u_x + \varrho v u_y = -P_x \\ \varrho(u - c)v_x + \varrho v v_y = -P_y - g\varrho \\ u_x + v_y = 0 \end{cases} \quad \text{in } \Omega_\eta. \quad (1a)$$

Here $g > 0$ is the gravitational constant of acceleration. On the free boundary, we impose the kinematic and dynamic conditions,

$$\begin{cases} v = (u - c)\eta_x \\ P = P_{\text{atm}} \end{cases} \quad \text{on } y = \eta(x). \quad (1b)$$

where P_{atm} is the (constant) atmospheric pressure. The first of these couples the motion of the boundary to that of the fluid, while the second ensures that the pressure is continuous across the interface. Lastly, the normal velocity is required to vanish on the bed

$$v = 0 \quad \text{on } y = -d. \quad (1c)$$

We restrict our attention to waves possessing no points of horizontal stagnation:

$$u - c < 0 \quad \text{in } \overline{\Omega_\eta}. \quad (2)$$

This will be an important assumption with many implications. For example, (2) and the implicit function guarantee that the integral curves of the relative velocity field $(u - c, v)$, called the *streamlines*, extend from $x = -\infty$ to $x = +\infty$.

A *solitary wave* is a solution of the above system that is spatially localized:

$$(u - c, v) \rightarrow (-Fu^*, 0), \quad \varrho \rightarrow \hat{\varrho}, \quad \eta(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \text{ uniformly in } y. \quad (3)$$

Here $\hat{\varrho} = \hat{\varrho}(y)$ is a given upstream density profile, $u^* = u^*(y) > 0$ is a (scaled) asymptotic relative velocity, and $F > 0$ is a dimensionless parameter called the *Froude number*. It will turn out that there is a critical Froude number, F_{cr} , that plays an important role in determining the structure of solutions; we say that a solution with $F > F_{\text{cr}}$ is *supercritical*.

The first equation in (1a) implies that the density is constant on streamlines. We will therefore fix $\hat{\varrho}$, which determines the value of ϱ along each streamline. To ensure that the solutions are physically realistic, we require that $\hat{\varrho}$ is positive and *stable* in the sense that it is non-increasing.

Finally, let us introduce some terminology for describing the qualitative features of these waves. A traveling wave is called *laminar* if all of its streamlines are parallel to the bed. A *wave of elevation* is a solitary wave where the height of each streamline above the bed attains its minimum value only at

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infinity. A traveling wave is *symmetric* provided that u and η are even in x while v is odd. We say a symmetric wave of elevation is *monotone* if the height of every streamline (except the bed) is strictly decreasing on either side of the crest line $\{x = 0\}$.

2. Statement of results and outline of the argument

Our results come in two distinct but interrelated parts. First, we give the following existence result.

Theorem 2.1 *Fix a Hölder exponent $\alpha \in (0, 1/2]$, wave speed $c > 0$, gravitational constant $g > 0$, depth $d > 0$, stable asymptotic density $\bar{\rho} \in C^{2+\alpha}([-d, 0], \mathbb{R}_+)$, and an asymptotic relative velocity $u^* \in C^{2+\alpha}([-d, 0], \mathbb{R}_+)$. There exists a continuous curve*

$$\mathcal{C} = \{(u(s), v(s), \eta(s), F(s)) : s \in (0, \infty)\}$$

of solitary wave solutions to (1)–(3) with the regularity

$$(u(s), v(s), \eta(s)) \in C^{2+\alpha}(\overline{\Omega_{\eta(s)}}) \times C^{2+\alpha}(\overline{\Omega_{\eta(s)}}) \times C^{3+\alpha}(\mathbb{R}),$$

and exhibiting the following properties.

(i) \mathcal{C} contains waves that are arbitrarily close to having (horizontal) stagnation points:

$$\liminf_{s \rightarrow \infty} \inf_{\Omega_{\eta(s)}} |c - u(s)| = 0. \quad (4)$$

(ii) The left endpoint of \mathcal{C} is a critical laminar flow,

$$\lim_{s \rightarrow 0} (u(s), v(s), \eta(s), F(s)) = (c - F_{\text{cr}} u^*, 0, 0, F_{\text{cr}}).$$

(iii) Every solution in \mathcal{C} is a wave of elevation that is symmetric, monotone, and supercritical.

This is the first large-amplitude existence theorem for solitary stratified waves with a free upper boundary. Observe also that u^* above is allowed to be an *arbitrary* smooth laminar profile, whereas all previous studies of heterogeneous solitary waves have assumed that the velocity is constant and purely horizontal at infinity (see, e.g., [1,3]). By allowing for a general u^* , we are able to treat traveling waves that exhibit a nontrivial wave-current interaction. Another strength of Theorem 2.1 is that the stagnation limit (4) can be approached arbitrarily close along the continuum. This is connected to the famous Stokes conjecture [13], which was originally made for periodic irrotational homogeneous waves. Stokes formally argued that there exists a family of such waves that terminates at an “extreme wave” that has a stagnation point at its crest. Later, Amick, Fraenkel, and Toland [2] showed rigorously that this does indeed occur. Theorem 2.1 gives the first construction of rotational solitary waves, even without stratification, where the stagnation limit (4) is known to hold for arbitrary u^* .

Let us now outline the ideas behind the proof of Theorem 2.1. Observe first of all that (1) is a free boundary problem. We therefore begin by making a change of coordinates that maps Ω_η to a fixed infinite strip $R \subset \mathbb{R}^2$. The governing equations become a scalar quasilinear elliptic PDE with fully nonlinear boundary conditions set on R . We will write this abstractly as

$$\mathcal{F}(w, F) = 0, \quad (5)$$

where $w \in C^{3+\alpha}(\overline{R})$ is a new unknown measuring the deviation of the streamlines from their asymptotic heights, and $\mathcal{F} : U \subset X \times \mathbb{R} \rightarrow Y$ is a real analytic mapping for some Banach spaces X and Y . Here U is an open subset of $X \times \mathbb{R}$ that ensures the waves are supercritical and (2) holds. Density stratification is manifested in (5) as a zeroth order term whose sign violates the hypotheses of the maximum principle.

As the maximum principle is relied upon at several key steps of the argument, this creates some serious technical problems.

Another major difficulty is the singularity of the bifurcation point: the linearized operator at the critical laminar flow $\mathcal{F}_w(0, F_{cr})$ is not Fredholm, which is related to the unboundedness of the domain. One cannot, therefore, construct small-amplitude waves as perturbations of the laminar flow via standard Lyapunov–Schmidt reduction techniques, as is done for periodic waves in [18], for example. Instead, we use spatial dynamics and the center manifold reduction method, essentially generalizing the work of Groves and Wahlén [8] on constant density rotational waves to the stratified regime.

From this analysis we obtain a curve of small-amplitude solitary waves bifurcating from the critical laminar flow. To prove Theorem 2.1, we continue this curve to the global curve \mathcal{C} using an adaptation of the analytic global bifurcation theory of Dancer [6] as generalized by Buffoni and Toland [4]. However, one cannot directly apply this machinery because it fundamentally requires that closed and bounded subsets of $\mathcal{F}^{-1}(0)$ be compact, and also that \mathcal{F}_w is Fredholm of index 0 along \mathcal{C} . It is not at all clear that this will hold for an elliptic PDE posed on an unbounded domain.

With that in mind, making a careful reading of [4], we first prove a new abstract global bifurcation result that applies to systems of the form (5) for which \mathcal{F}_w may not be Fredholm at the bifurcation point and $\mathcal{F}^{-1}(0)$ may fail to be locally compact. These relaxed hypotheses come at the price of additional possibilities for the global behavior of the solution set.

Theorem 2.2 *Let \mathcal{X}, \mathcal{Y} be Banach spaces, \mathcal{I} an open interval, and let $\mathcal{U} \subset \mathcal{X}$ be an open set with $0 \in \partial\mathcal{U}$. For an analytic map $\mathcal{G} : \mathcal{U} \times \mathcal{I} \rightarrow \mathcal{Y}$, consider the set of solutions $\mathcal{Z} := \mathcal{G}^{-1}(0)$. Assume that: (i) $\mathcal{G}_x(x, \lambda)$ is Fredholm with index 0 for any $(x, \lambda) \in \mathcal{Z}$, and (ii) there exists a continuous curve $\mathcal{C}_{loc} = \{(\tilde{x}(s), \tilde{\lambda}(s)) : s \in (0, 1)\} \subset \mathcal{Z}$ such that*

$$\lim_{s \searrow 0} \tilde{x}(s) = 0, \quad \mathcal{G}_x(x, \lambda) : \mathcal{X} \rightarrow \mathcal{Y} \text{ is invertible for all } (x, \lambda) \in \mathcal{C}_{loc}.$$

Then there is a continuous path $\mathcal{C} = \{(x(s), \lambda(s)) \in \mathcal{U} \times \mathcal{I} : s \in (0, \infty)\}$ of solutions extending \mathcal{C}_{loc} along which one of the following alternatives must hold:

(A1) *as $s \rightarrow \infty$,*

$$N(s) := \|x(s)\|_{\mathcal{X}} + \frac{1}{\text{dist}(x(s), \partial\mathcal{U})} + |\lambda(s)| + \frac{1}{\text{dist}(\lambda(s), \partial\mathcal{I})} \rightarrow \infty; \text{ or} \quad (6)$$

(A2) *there exists a sequence $s_n \rightarrow \infty$ such that $\sup_n N(s_n) < \infty$ but $\{x(s_n)\}$ has no subsequences converging in \mathcal{X} .*

The objective is now to winnow down the alternatives, especially (A2) which as stated above has no obvious physical significance. For this we need to exploit some specific structure of the problem at hand, which leads naturally to our second category of results: the qualitative theory of solitary stratified waves. We emphasize, however, that the theorems we obtain are of independent interest. Some of them are actually quite large improvements on the prior theory for rotational waves with constant density.

Generalizing the strategy underlying [19,21], we prove a loss of compactness lemma characterizing precisely the way in which (A2) must occur when Theorem 2.2 is applied to a quasilinear elliptic equation that is set on an infinite cylinder and invariant under axial translations. For the case of the stratified water wave problem, this lemma says the following: suppose that $\{(w_n, F_n)\}$ is a uniformly bounded sequence of solutions to (5) that are monotone and even but not precompact. Then, there is a translated subsequence that converges locally to a monotone front-type solution that has distinct limits as $x \rightarrow \pm\infty$.

To make use of the lemma, we must confirm that the waves we construct are monotone and even. This we get as a consequence of the following more general theorem.

Theorem 2.3 *Let (u, v, η, F) be a supercritical wave of elevation that solves (1)–(2) with $\|u\|_{C^2(\Omega_\eta)}$, $\|v\|_{C^2(\Omega_\eta)}$, $\|\eta\|_{C^3(\mathbb{R})} < \infty$, and*

$$(u, v) \rightarrow (\hat{u}, 0), \quad (u_y, v_y) \rightarrow (\hat{u}_y, 0) \quad \text{uniformly as } x \rightarrow +\infty \text{ (or as } x \rightarrow -\infty).$$

Then the wave is necessarily even, and the height of each streamline above the bed decreases strictly monotonically as $x \rightarrow \infty$ to the right of the crest.

The symmetry of steady water waves has been a very active subject of research. The first results in this direction are due to Craig and Sternburg [5], who used a moving-plane method (cf. [7,10]) to establish the even symmetry of solitary waves in the irrotational regime. For stratified flows, Maia [11] obtained a symmetry result for channel flows with uniform velocity at infinity, and Walsh considered the case of continuously stratified periodic waves [17]. A notable feature of Theorem 2.3 is that it only imposes asymptotic conditions upstream (or downstream), while typical moving-plane arguments for solitary waves require that the solution decays in both directions in order to obtain symmetry.

Next, consider the front-type solutions described in the loss of compactness lemma. In the context of water waves, they are referred to as *bores*. Numerical computations of bores have been carried out in various regimes (see, e.g., [15]), and there are rigorous proofs of their existence in multi-fluid channel flows (see, e.g., [14]). However, with a free upper surface, no bores can exist having the property that the asymptotic height of all streamlines upstream lie at or below their asymptotic height downstream:

Theorem 2.4 *Suppose that (u, v, η) is a solution of (1)–(2) which is a bore in the sense that*

$$(u(x, \cdot), v(x, \cdot), \eta(x)) \rightarrow (\hat{u}_\pm(\cdot), 0, \eta_\pm), \quad \text{as } x \rightarrow \pm\infty$$

pointwise, where $\eta_\pm > -d$ are constants and $\hat{u}_\pm \in C^1([-d, \eta_\pm])$. If the limiting height of each streamline at $x = -\infty$ is no greater (or no less) than the limiting height of the same streamline at $x = \infty$, then $\eta_+ = \eta_-$ and $\hat{u}_+ \equiv \hat{u}_-$.

In fact, this theorem generalizes even to the case of multiple fluid flows. To the best of our knowledge, the nonexistence of monotone bores with a free upper surface has never been previously observed, which is somewhat surprising given how thoroughly bores have been studied in channel flows, for example.

Together, Theorem 2.3, Theorem 2.4, and the loss of compactness lemma rule out alternative (A2). To complete the proof of Theorem 2.1, we must show that the remaining alternative (A1) implies that the extreme wave limit (4) occurs. This can be inferred from the following new estimates for the pressure, velocity field, and Froude number.

Theorem 2.5 *Let (u, v, η, F) be a solution of (1)–(3).*

(i) *The Froude number has the following upper bound*

$$F \leq \frac{1}{\pi} \frac{gd}{\min(u^*)^2} \frac{\max \varrho}{\min \varrho} \frac{\sqrt{gd}}{\inf_{\{x=0\}}(c-u)}.$$

(ii) *If $F = F_{\text{cr}}$, then $(u, v, \eta) = (c - F_{\text{cr}}u^*, 0, 0)$.*

(iii) *If $F \geq F_0 \geq F_{\text{cr}}$, then the pressure and velocity field obey the bounds*

$$P - P_{\text{atm}} + MF\psi \geq 0, \quad (u - c)^2 + v^2 \leq CF^2 \quad \text{in } \overline{\Omega_\eta},$$

where the constants C and M depend only on u^ , $\hat{\varrho}$, g , d , and F_0 . Here ψ is the pseudo stream function defined uniquely by $\nabla^\perp \psi = \sqrt{\varrho}(u - c, v)$ and $\psi|_{y=\eta} = 0$.*

Let us make some remarks. Part (i) is the first upper bound of the Froude number for rotational solitary waves — with or without stratification — that makes no additional assumptions on the shear profile u^* ; in [20], Wheeler established upper bounds on F that are independent of $\inf_{\Omega_\eta}(c - u)$, but imposed additional requirements on u^* . Thanks to Theorem 2.5, we can avoid making similar restrictions

on u^* in Theorem 2.1. In the special case of homogeneous irrotational waves, the argument leading to Theorem 2.5 can be modified to recover the estimates formally derived by Starr [12], and later rigorously proved by Keady and Pritchard [9].

Part (ii) serves as a type of lower bound for the Froude number; it says that a curve of supercritical waves cannot limit to a subcritical wave without first encountering the critical laminar flow.

Finally, in part (iii) we provide a lower bound on P and an upper bound on (u, v) in terms of the given quantities. To our knowledge, these are the only estimates of this type for stratified steady waves; Varvaruca obtained analogous estimates for constant density waves in [16]. Here the situation is more delicate because the elliptic problem satisfied by the pressure has a zeroth order term of indeterminate sign.

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