

A KATO-TYPE CRITERION FOR VANISHING VISCOSITY NEAR ONSAGER'S CRITICAL REGULARITY

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ABSTRACT. We consider a vanishing viscosity sequence of weak solutions for the three-dimensional Navier–Stokes equations of incompressible fluids in a bounded domain. In a seminal paper [26] Kato showed that for sufficiently regular solutions, the vanishing viscosity limit is equivalent to having vanishing viscous dissipation in a boundary layer of width proportional to the viscosity. We prove that Kato's criterion holds for the Hölder continuous solutions with the regularity index arbitrarily close to Onsager's critical exponent through a new boundary layer foliation and a global mollification technique.

1. INTRODUCTION

The motion of an incompressible viscous fluid with constant density is governed by the following Navier-Stokes equations:

$$\begin{cases} \partial_t u^\nu + \operatorname{div}(u^\nu \otimes u^\nu) + \nabla P^\nu = \nu \Delta u^\nu, \\ \operatorname{div} u^\nu = 0, \\ u^\nu(x, 0) = u_0^\nu(x), \end{cases} \quad (1.1)$$

where the constant $\nu > 0$ denotes the viscosity of the fluid, the unknown functions u^ν and P^ν are the velocity field and pressure, respectively. Here the superscript ν is used on all the unknowns to emphasize the dependence on the viscosity. The Navier-Stokes equations at zero viscosity $\nu = 0$ formally become the Euler equations:

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.2)$$

where u and P are the velocity and pressure of the inviscid fluid, respectively.

An important problem in the study of incompressible hydrodynamics is the vanishing viscosity limit from the Navier-Stokes equations (1.1) to the Euler equations (1.2), which is naturally associated with the physical phenomena of turbulence and of boundary layers. On domains without boundary, such a problem is well-understood: given a strong Euler solution $u^E \in C^1$, the Leray–Hopf solutions u^ν of (1.1) converge strongly in the energy space $L_t^\infty L_x^2$ to u as $\nu \rightarrow 0$; see, for example, [3].

In the presence of boundary, on the other hand, systems (1.1) and (1.2) considered in a bounded domain Ω are supplemented with the no-slip boundary condition $u|_{\partial\Omega} = 0$ and

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slip boundary condition $(u \cdot n)|_{\partial\Omega} = 0$, respectively, with n the unit outward normal of the boundary $\partial\Omega$. The mismatch of the boundary conditions leads to the phenomenon of boundary layer separation. Establishing the vanishing viscosity limit in the energy space $L_t^\infty L_x^2$ in this case is much less understood. A well-known result of Kato [26] states that, for a strong Euler solution u^E , the vanishing viscosity limit

$$u^\nu \rightarrow u^E \quad \text{in } L^\infty(0, T; L^2(\Omega))$$

holds if and only if

$$\nu \int_0^T \|\nabla u^\nu\|_{L^2(\Gamma_{c\nu})}^2 dt \rightarrow 0 \quad \text{as } \nu \rightarrow 0, \quad (1.3)$$

where $\Gamma_{c\nu}$ is a very thin boundary layer of width proportional to ν .

Kato's theory is by nature conditional. Many of the known results on strong inviscid limits are also conditioned on special properties of the solutions [1, 3, 11, 12, 27, 28, 38, 40]. Some unconditional strong convergence results do exist, but with additional assumptions on the data like real analyticity [7], vanishing of the initial vorticity near the boundary [34], or special symmetry of the flow [29, 32, 33, 36]. These results are for short time and for laminar flows close to a smooth Euler solution when there is no boundary layer separation or other characteristic turbulent behavior.

The vanishing viscosity for turbulent flows faces serious challenges and remains widely open. Little is known about the inviscid limit even when a strong Euler solution exists for a short time. Therefore it is natural to consider the weak Euler solutions for the vanishing viscosity limit. One type of such weak Euler solutions is the measure valued solutions [17]. In some recent works [14, 19] the authors describe sufficient conditions in terms of interior structure functions under which the weak ($L_{t,x}^2$) solutions u^E of the Euler equations can be obtained as weak $L_{t,x}^2$ limits of u^ν . In [13], the authors further extend the result of [14] to allow certain interior vorticity concentration.

The classical result of Kato [26] indicates that the anomalous energy dissipation leads to the failure of the inviscid limit to a strong (C^1) Euler solution; while the issue that weak Euler solutions may arise from the inviscid limit is closely related to Onsager's conjecture (see, for example [5, 19] and the references therein). It has been made a precise statement that the critical Onsager's Hölder regularity exponent is $1/3$, below which the Euler equations become non-conservative [4, 15, 16, 24, 25], while above $1/3$ the energy conservation can be justified [9, 10, 20, 22]. In the works of [2, 18] the authors derive sufficient conditions for C^α solutions under which the global viscous dissipation vanishes in the inviscid limit for Leray–Hopf solutions u^ν , with an emphasis on the behavior or solutions near the boundary. In particular in [18] a Kato-type criterion on the vanishing of the energy dissipation rate in a thin boundary layer of thickness $O(\nu^\beta)$ is proposed, among other regularity conditions, where $\beta = \frac{3}{4} + \epsilon$ near the critical Onsager threshold $\alpha = \frac{1}{3} + \epsilon$. Note that the boundary layer in the result of [18] is thicker than that of Kato [26].

The main goal of this paper is to bridge the gap between the original result of Kato [26] for strong C^1 Euler solutions and the result of [18] for weak C^α Euler solutions. Specifically, we will show that under certain ν -dependent assumptions on the family of solutions of (1.1), a Kato-type result with boundary layer of thickness $O(\nu)$ holds for weak Euler solutions up to Onsager-critical spatial regularity $\alpha = \frac{1}{3} +$. See Table 1 below.

	NS solution u^ν	Euler solution u^E	boundary layer
Kato [26]	Leray–Hopf	$C^1(\bar{\Omega})$	$O(\nu)$
Drivas–Nguyen [18]	$C_{\text{loc}}^{1/3+}(\Omega)$ with boundary regularity	$C_{\text{loc}}^{1/3+}(\Omega)$	$O(\nu^{3/4+})$
Our result	$C_{\text{loc}}^{1/3+}(\Omega)$ with boundary regularity	$C_{\text{loc}}^{1/3+}(\Omega)$	$O(\nu)$

TABLE 1. Comparison of results

Let us recall the classical existence results of Leray [31] and Hopf [23]. For a divergence-free function $u_0 \in L^2$, problem (1.1) has a weak solution $u \in C(0, T; L^2) \cap L^2(0, T; H^1(\Omega))$ in a bounded smooth domain Ω for any $T < \infty$. Additionally, u is divergence-free and the following energy inequality holds

$$\frac{1}{2} \int_{\Omega} |u^\nu|^2 dx + \nu \int_0^t \int_{\Omega} |\nabla u^\nu|^2 dx ds \leq \frac{1}{2} \int_{\Omega} |u_0^\nu|^2 dx, \quad a.e. \ t \in (0, T). \quad (1.4)$$

Such a weak solution is called *Leray–Hopf weak solution*.

Next we introduce some notation. For some (small) $h > 0$, we define

$$\Omega^h := \{x \in \Omega, \text{dist}(x, \partial\Omega) > h\} \quad \text{and} \quad \Gamma_h := \Omega \setminus \Omega^h. \quad (1.5)$$

Also, we introduce the Besov space $B_p^{\alpha, \infty}(\Omega)$ which consists of measurable functions with the norm

$$\|f\|_{B_p^{\alpha, \infty}(\Omega)} := \|f\|_{L^p(\Omega)} + \sup_{y \in \mathbb{R}^3} \frac{\|f(\cdot + y) - f(\cdot)\|_{L^p(\Omega \cap (\Omega - \{y\}))}}{|y|^\alpha}, \quad p \geq 1, \alpha \in (0, 1). \quad (1.6)$$

We further denote $H(\Omega)$ to be the completion in $L^2(\Omega)$ of the space $\{v \in C_c^\infty(\Omega; \mathbb{R}^3) : \text{div} v = 0\}$, and recall the following definition of the weak Euler solutions (see, for example [18]).

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. We say the pair (u, P) is a weak Euler solution to (1.2) on $\Omega \times (0, T)$ if $u \in C_w(0, T; H(\Omega))$, $P \in L_{\text{loc}}^1(\Omega \times (0, T))$ and for all test vector fields $\varphi \in C_0^\infty(\Omega \times (0, T))$ it holds that

$$\int_0^T \int_{\Omega} (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi + P \nabla \cdot \varphi) dx dt = 0.$$

Our main result is stated in the following theorem.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. Let $\{u^\nu\}_{\nu > 0}$ be a sequence of Leray–Hopf weak solutions to (1.1) with initial data u_0^ν and suppose that $u_0^\nu \rightarrow u_0$ in $L^2(\Omega)$ as $\nu \rightarrow 0$. Assume in addition that

$$u^\nu \text{ is uniformly in } \nu \text{ bounded in } L^3(0, T; B_3^{\alpha, \infty}(\Omega^\nu)) \text{ for some } \alpha \in \left(\frac{1}{3}, 1\right), \quad (1.7)$$

$$\begin{cases} u^\nu \text{ is uniformly in } \nu \text{ bounded in } L^4(0, T; L^\infty(\Gamma_{4\nu})), \\ P^\nu \text{ is uniformly in } \nu \text{ bounded in } L^2(0, T; L^\infty(\Gamma_{4\nu})). \end{cases} \quad (1.8)$$

Then, if

$$\lim_{\nu \rightarrow 0} \nu \int_0^T \int_{\Gamma_{4\nu}} |\nabla u^\nu|^2 dx dt = 0, \quad (1.9)$$

we have that the global viscous dissipation vanishes, i.e.,

$$\lim_{\nu \rightarrow 0} \nu \int_0^T \int_{\Omega} |\nabla u^\nu|^2 dx dt = 0, \quad (1.10)$$

and moreover, u^ν converges locally in $L^3(0, T; L^3(\Omega))$, up to some subsequence, to a weak solution of the Euler equations (1.2).

Remark 1.1. Condition (1.9) recovers Kato's criterion (1.3), but now in the framework of weak solutions with C^α regularity, where α can be taken arbitrarily close to Onsager's critical exponent, cf. (1.7).

Remark 1.2. As pointed out in [18], violation of conditions (1.7) – (1.9) is responsible for global dissipation to persist in the vanishing viscosity limit. More precisely, violation of (1.7) corresponds to a failure of uniform interior regularity, which is required for anomalous dissipation in domains without boundaries; see, for example [10]. On the other hand, conditions (1.8)–(1.9) provide new mechanisms for anomalous dissipation in wall-bounded flows. In fact several numerical investigations [37,39] have been carefully performed, where formation and shedding of vortex sheet of width ν at the boundary is confirmed, suggesting the possible failure of the Kato-type near-wall dissipation criterion (1.9). On the other hand, in [6] Cadot et al. conduct experiments for flows in Taylor-Couette cells with smooth walls, and found that for sufficiently large Reynolds number, the energy dissipation in the boundary layer decreases with Reynolds number which satisfies (1.9), while the energy dissipation in the bulk becomes dominant and stabilizes to a constant.

Remark 1.3. Also pointed out in [18], the sufficient conditions (1.7)–(1.9) may still seem stronger than necessary, especially when a smooth background Euler solution exists. Instead, in this case the uniform equicontinuity at the boundary alone is enough to conclude the convergence [18, Theorem 4]. A result like that could give a more precise description of what sort of objects can cause anomalous dissipation at the boundary. It would be interesting to know if under the weak regularity setting the vanishing of the near-wall dissipation (1.9) can be replaced by such a type of condition, although this is not what we pursue here in this paper.

The basic idea in [18] is separation and regularization: applying a cut-off function to separate the boundary from the interior domain, and mollifying the interior velocity field. This introduces two length scales: the thickness h of the boundary layer and the mollification scale ϵ . With this localization, the resolved dissipation is bounded by

$$\nu \epsilon^{2(\alpha-1)} \int_0^t \|u^\nu\|_{B_3^{\alpha,\infty}(\Omega^h)}^2;$$

see [18, Section 2.1]. Recalling the natural constraint that $\epsilon \leq h$, imposing appropriate interior regularity assumption on the solution, and setting $h \sim \nu^\beta$, the above estimate translates to $\nu^{1+2\beta(\alpha-1)}$. In order for this to vanish at the inviscid limit $\nu \rightarrow 0$ one has to require that

$$1 + 2\beta(\alpha - 1) > 0, \quad (1.11)$$

which, at Onsager's critical regularity $\alpha = \frac{1}{3}+$, returns to $\beta = \frac{3}{4}+$.

The obvious obstacle in the above approach to get to the $O(\nu)$ boundary layer thickness lies in the strong constraint between the two scales ϵ and h . In other words, if one can find a way to "free up" the choice for ϵ so as to improve the mollification, then it is reasonable to hope to obtain a thinner boundary layer.

Motivated by a recent work of the authors [8], where a global mollification was introduced, we design a new localization technique with additional treatment near the boundary. In particular, we will start with a boundary layer of the type as in [18] and perform a further foliation within that boundary layer, mollify the solution with different scales in each leaf of the foliation, and then glue everything together by a partition of unity. Such a new type of mollification generates additional cancellation effects in estimating the resolved dissipation, allowing one to reach the $O(\nu)$ boundary layer. Moreover, using the same idea, we also show that as the solution becomes more regular (corresponding to increasing α), the regularity requirement (1.8) near the boundary can be relaxed, and the the boundary layer in (1.9) is allowed to be thinner, cf. Theorem 5.1.

The rest of the paper is organized as follows. In Section 2 we briefly recall some needed analytical results, and introduce the boundary layer foliation. In Section 3 we define the mollification and use that to regularize the system. By testing the resolved system with suitable test functions we prove the balance of the resolved energy, from which we proceed in Section 4 to give the proof of Theorem 1.1. Finally in Section 5 we extend the result of Theorem 1.1 to the case when solutions are more regular.

2. PRELIMINARIES

In this section, we recall some commutator and pressure estimates, and introduce the boundary layer foliation.

2.1. Commutator estimates. Recall the standard mollification for the function f

$$\bar{f}_\epsilon(x) := \int_{B_\epsilon(0)} f(x-y)\eta_\epsilon(y)dy, \quad \forall x \in \Omega^\epsilon, \quad (2.1)$$

with η_ϵ being the standard mollifier of width ϵ . A straightforward calculation gives

$$\nabla \bar{f}_\epsilon(x) = - \int_{B_\epsilon(0)} \eta_\epsilon(y) \nabla_y (f(x-y) - f(x)) dy = \frac{1}{\epsilon} \int_{B_1(0)} \nabla \eta(y) (f(x-\epsilon y) - f(x)) dy,$$

and hence, for $f \in B_r^{\alpha, \infty}$ with $r \in [1, \infty]$,

$$\|\nabla \bar{f}_\epsilon\|_{L^r(\Omega^\epsilon)} \leq \epsilon^{\alpha-1} \|f\|_{B_r^{\alpha, \infty}(\Omega)}. \quad (2.2)$$

Similarly,

$$\|\bar{f}_\epsilon - f\|_{L^r(\Omega^\epsilon)} \leq C\epsilon^\alpha \|f\|_{B_r^{\alpha, \infty}(\Omega)}. \quad (2.3)$$

Lemma 2.1. *Let $f \in B_{r_1}^{\alpha, \infty}(\Omega)$, $g \in B_{r_2}^{\alpha, \infty}(\Omega)$, and let $1 \leq r$, $r_1, r_2 < \infty$, $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$. Then there exists some $C > 0$ such that the following holds*

$$\|(\overline{f \otimes g})_\epsilon - \bar{f}_\epsilon \otimes \bar{g}_\epsilon\|_{L^r(\Omega^\epsilon)} \leq C\epsilon^{2\alpha} \|f\|_{B_{r_1}^{\alpha, \infty}(\Omega)} \|g\|_{B_{r_2}^{\alpha, \infty}(\Omega)}. \quad (2.4)$$

Proof. Inequality (2.4) is nothing but the commutator estimate in [10]. Here we give an outline of the proof.

Let $\delta f(x, y) := f(x - y) - f(x)$. By (2.1) we compute for every $x \in \Omega^\epsilon$,

$$\begin{aligned} & \overline{(f \otimes g)_\epsilon} - \overline{f_\epsilon} \otimes \overline{g_\epsilon} \\ &= \int_{B_\epsilon(x)} \delta f(x, y) \otimes \delta g(x, y) \eta_\epsilon(y) dy - \left(\int_{B_\epsilon(x)} \delta f(x, y) \eta_\epsilon(y) dy \right) \otimes \left(\int_{B_\epsilon(x)} \delta g(x, y) \eta_\epsilon(y) dy \right) \\ &\leq \left(\int_{B_\epsilon(x)} |\delta f(x, y)|^{r_1} \eta_\epsilon(y) dy \right)^{\frac{1}{r_1}} \left(\int_{B_\epsilon(x)} |\delta g(x, y)|^{r_2} \eta_\epsilon(y) dy \right)^{\frac{1}{r_2}} + |\overline{f_\epsilon} - f| |\overline{g_\epsilon} - g|. \end{aligned} \quad (2.5)$$

Integrating (2.5) over Ω^ϵ gives

$$\begin{aligned} & \int_{\Omega^\epsilon} \left| \overline{(f \otimes g)_\epsilon} - \overline{f_\epsilon} \otimes \overline{g_\epsilon} \right|^r dx \\ &\leq C \left(\int_{\mathbb{R}^3} \eta_\epsilon(y) \int_{\Omega^\epsilon} |\delta f(x, y)|^{r_1} dx dy \right)^{\frac{r}{r_1}} \left(\int_{\mathbb{R}^3} \eta_\epsilon(y) \int_{\Omega^\epsilon} |\delta g(x, y)|^{r_2} dx dy \right)^{\frac{r}{r_2}} \\ &\quad + C \|\overline{f_\epsilon} - f\|_{L^{r_1}}^r \|\overline{g_\epsilon} - g\|_{L^{r_2}}^r. \end{aligned}$$

This together with (1.6) and (2.3) conclude the desired (2.4). \square

2.2. Pressure estimates. The pressure P^ν appeared in (1.1) can be deduced from the velocity u^ν via the Poisson equation

$$-\Delta P^\nu = \operatorname{div} \operatorname{div}(u^\nu \otimes u^\nu).$$

From [21, Lemma 2], we have the following lemmas.

Lemma 2.2. *Let $p \in (1, \infty)$. Assume that $u^\nu \in L^{2p}(\Omega)$ and $P^\nu|_{\partial\Omega} \in L^p(\partial\Omega)$. Then the pressure $P^\nu \in L^p(\Omega)$. In addition, the following estimate holds,*

$$\|P^\nu\|_{L^p(\Omega)} \leq C \left(\|P^\nu|_{\partial\Omega}\|_{L^p(\partial\Omega)} + \|u^\nu\|_{L^{2p}(\Omega)}^2 \right). \quad (2.6)$$

Lemma 2.3 (Hardy-type embedding [30]). *Let $p \in [1, \infty)$ and $f \in W_0^{1,p}(\Omega)$. Then*

$$\left\| \frac{f(x)}{\operatorname{dist}(x, \partial\Omega)} \right\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)},$$

where C depends on p and Ω .

2.3. Boundary layer foliation. For $\alpha \in (\frac{1}{3}, \frac{5}{6})$, we define the following sequence

$$\beta_0^* = 0 \quad \text{and} \quad \beta_n^* = \frac{1}{2(1-\alpha)} \left(1 + \frac{1}{3} \beta_{n-1}^* \right), \quad n = 1, 2, 3, \dots \quad (2.7)$$

Clearly, $\{\beta_n^*\}$ is bounded and strictly increasing, and

$$\beta_\infty^* := \lim_{n \rightarrow \infty} \beta_n^* = \frac{3}{5-6\alpha} > 1 \quad \text{if} \quad \frac{1}{3} < \alpha < \frac{5}{6}.$$

Hence there exists some finite number

$$N := \begin{cases} N(\alpha), & \alpha \in \left(\frac{1}{3}, \frac{1}{2} \right], \\ 1, & \alpha \in \left(\frac{1}{2}, \frac{5}{6} \right), \end{cases} \quad (2.8)$$

such that

$$0 = \beta_0^* < \beta_1^* < \beta_2^* < \cdots < \beta_{N-1}^* \leq 1 < \beta_N^*. \quad (2.9)$$

In light of (2.7)–(2.9), we define, for any $\alpha \in (\frac{1}{3}, 1)$, an increasing sequence $\{\beta_n\}_{n=1}^N$ such that

$$0 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_{N-1} < \beta_N := 1 \quad \text{and} \quad (2.10)$$

$$\beta_n < \frac{1}{2(1-\alpha)} \left(1 + \frac{1}{3}\beta_{n-1} \right), \quad (2.11)$$

where N is given in (2.8).

The purpose of introducing the sequence $\{\beta_n\}$ is to design the following decomposition of the boundary layer. Note that, when $n = 1$, (2.11) reads $\beta_1 < \frac{1}{2(1-\alpha)}$, which agrees with (1.11). Next we decompose the inner region of Ω as

$$V_1 := \Omega^{2\nu^{\beta_1}}, \quad V_n := \Omega^{2\nu^{\beta_n}} - \Omega^{2\nu^{\beta_{n-1}+2\nu^{\beta_n}}} \quad \text{when } 2 \leq n \leq N; \quad (2.12)$$

see Figure 1.

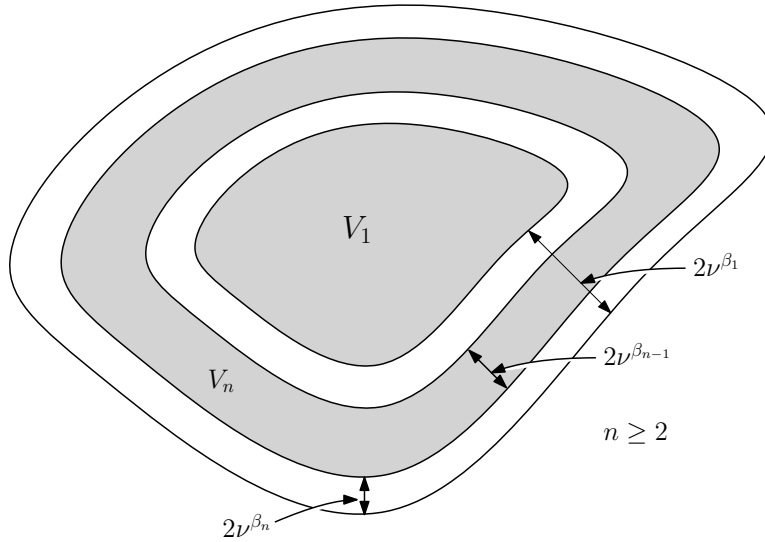


FIGURE 1. Decomposition of Ω

It is easy to check that, for ν small enough,

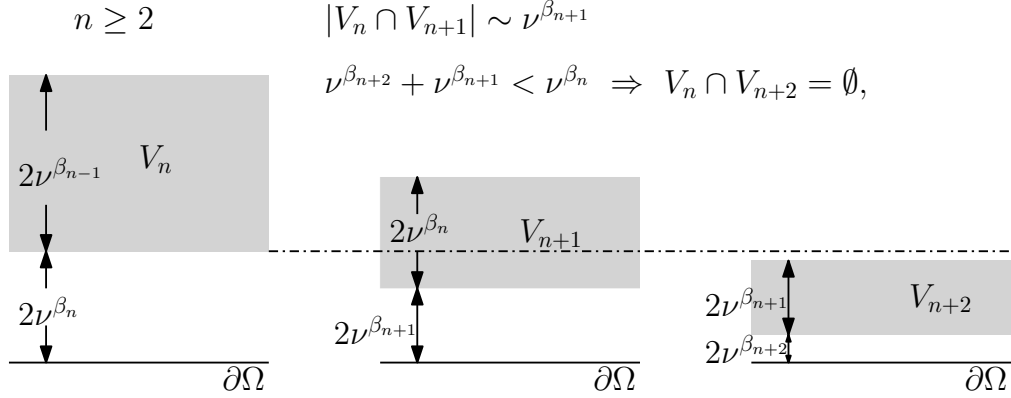
$$\text{meas}(V_n) \leq C\nu^{\beta_{n-1}}, \quad \text{meas}(V_k \cap V_m) \leq \begin{cases} C\nu^{\beta_{\max\{k,m\}}} & \text{if } |k-m| = 1, \\ 0 & \text{if } |k-m| > 1; \end{cases} \quad (2.13)$$

see Figure 2.

With the above decomposition, we have the following proposition.

Proposition 2.1. *Let $\{\xi_n\}_{n=1}^N$ be a C^1 partition of unity subordinate to $\{V_n\}_{n=1}^N$ such that*

$$\text{spt } \xi_n \subset V_n, \quad 0 \leq \xi_n \leq 1, \quad \sum_{n=1}^N \xi_n = 1. \quad (2.14)$$

FIGURE 2. Foliation of $\partial\Omega$

Then, it follows that, for $0 \leq n \leq N$,

$$\nabla (\xi_n + \xi_{n+1})^2 = 0 \quad \text{if } x \in V_n \cap V_{n+1}, \quad \text{and} \quad \nabla \xi_n = 0 \quad \text{if } x \in V_n \setminus \cup_{i \neq n} V_i, \quad (2.15)$$

where we define

$$V_{N+1} := \left(\bigcup_{n=1}^N V_n \right)^c \quad \text{and} \quad V_0 := \emptyset. \quad (2.16)$$

Proof. From (2.14) and (2.13) we know that

$$(\xi_n + \xi_{n+1})|_{V_n \cap V_{n+1}} \equiv 1, \quad \xi_n|_{V_n \setminus \cup_{i \neq n} V_i} \equiv 1.$$

Therefore (2.15) follows trivially by differentiating the above. \square

3. REGULARIZATION AND RESOLVED ENERGY BALANCE

In this section, we define the mollification and regularize the system, and then derive the balance of the resolved energy.

For $n = 1, \dots, N$, define

$$\bar{f}_n(x, t) := \begin{cases} \int \eta_{\nu^{\beta_n}}(x-y) f(y, t) dy, & x \in V_n \cap V_{n+1}^c, \\ \int \eta_{\nu^{\beta_{n+1}}}(x-y) f(y, t) dy, & x \in V_n \cap V_{n+1}. \end{cases} \quad (3.1)$$

From (2.13) we know that

$$\bar{f}_n = \bar{f}_{n+1} \quad \text{on} \quad V_n \cap V_{n+1}. \quad (3.2)$$

From (2.2)-(2.3), it holds that, for $p \in [1, \infty)$,

$$\|\nabla \bar{f}_n\|_{L^p(V_n)} \leq \begin{cases} C\nu^{\beta_n(\alpha-1)} \|f\|_{B_p^{\alpha, \infty}(\Omega^\nu)}, & x \in V_n \cap V_{n+1}^c, \\ C\nu^{\beta_{n+1}(\alpha-1)} \|f\|_{B_p^{\alpha, \infty}(\Omega^\nu)}, & x \in V_n \cap V_{n+1}, \end{cases} \quad (3.3)$$

and

$$\|\overline{f}_n - f\|_{L^p(V_n)} \leq \begin{cases} C\nu^{\alpha\beta_n} \|f\|_{B_p^{\alpha,\infty}(\Omega^\nu)}, & x \in V_n \cap V_{n+1}^c, \\ C\nu^{\alpha\beta_{n+1}} \|f\|_{B_p^{\alpha,\infty}(\Omega^\nu)}, & x \in V_n \cap V_{n+1}. \end{cases} \quad (3.4)$$

With (3.1), we deduce from (1.1) that

$$\partial_t \overline{u}_n^\nu + \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n + \nabla \overline{P}_n^\nu = \nu \Delta \overline{u}_n^\nu, \quad \forall x \in V_n \quad (n = 1, \dots, N).$$

Multiplying it by ξ_n and summing up imply that

$$\partial_t \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right) + \sum_{n=1}^N \xi_n \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n + \sum_{n=1}^N \xi_n \nabla \overline{P}_n^\nu = \nu \sum_{n=1}^N \xi_n \Delta \overline{u}_n^\nu, \quad \forall x \in \Omega^{2\nu}. \quad (3.5)$$

To deal with the boundary contribution, we introduce a smooth cut-off function $\theta(x)$ in Ω such that,

$$0 \leq \theta(x) \leq 1, \quad \theta(x) = 1 \text{ if } x \in \Omega^{4\nu}, \quad \theta(x) = 0 \text{ if } x \notin \Omega^{2\nu}, \quad \text{and } |\nabla \theta| \leq 4\nu^{-1}. \quad (3.6)$$

Next we wish to test (3.5) by $\theta(x) \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right)$ to derive the resolved energy balance. However, this test function fails to be solenoidal, and hence cannot be used as a legitimate test field for Leray-Hopf solutions. Therefore, to make our argument work we must appeal to the following theorem:

Theorem 3.1 (Theorem 1, [35]). *Assume that Ω is an open, bounded domain with C^2 boundary $\partial\Omega$, and u is a Leray-Hopf solution of (1.1). Then there exists a pressure field $P \in L^r(0, T; W^{1,s}(\Omega))$ with*

$$\frac{3}{s} + \frac{2}{r} = 4, \quad \frac{4}{3} < s < \frac{3}{2}, \quad (3.7)$$

such that for all $\varphi \in C_0^\infty((0, T) \times \Omega)$,

$$\int_0^T \int_\Omega (u \cdot \partial_t \varphi + u \otimes u : \nabla \varphi + P \operatorname{div} \varphi + \nu u \cdot \Delta \varphi) \, dx dt = 0.$$

This way we can multiply (3.5) by $\theta(x) \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right)$ and integrate over $\Omega \times [0, T]$, leading to

$$\begin{aligned} & \frac{1}{2} \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right)^2 (x, T) \, dx - \frac{1}{2} \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right)^2 (x, 0) \, dx \\ &= \int_0^T \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right) \left(\sum_{n=1}^N \xi_n \nu \Delta \overline{u}_n^\nu - \sum_{n=1}^N \xi_n \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n - \sum_{n=1}^N \xi_n \nabla \overline{P}_n^\nu \right) \, dx dt. \end{aligned} \quad (3.8)$$

The main result of this section is the following.

Proposition 3.1 (Resolved energy balance). *Under the same hypotheses as in Theorem 1.1, it holds that*

$$\lim_{\nu \rightarrow 0} \int_0^T \int_\Omega \theta \left(\sum_{n=1}^N \xi_n \overline{u}_n^\nu \right) \left(\sum_{n=1}^N \xi_n \nu \Delta \overline{u}_n^\nu - \sum_{n=1}^N \xi_n \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n - \sum_{n=1}^N \xi_n \nabla \overline{P}_n^\nu \right) \, dx dt = 0. \quad (3.9)$$

Proposition 3.1 is a direct consequence of Lemmas 3.1 – 3.3 below.

Lemma 3.1 (Resolved dissipation). *Under the same hypotheses as in Theorem 1.1, we have*

$$\lim_{\nu \rightarrow 0} \int_0^T \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{u_n^\nu} \right) \left(\sum_{n=1}^N \xi_n \nu \Delta \overline{u_n^\nu} \right) dx dt = 0. \quad (3.10)$$

Proof. Owing to (2.13) and (2.14), we find

$$\xi_k \xi_m = 0 \quad \text{if } |k - m| \geq 2. \quad (3.11)$$

Integration by parts then gives

$$\begin{aligned} & \nu \int_0^T \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{u_n^\nu} \right) \left(\sum_{n=1}^N \xi_n \Delta \overline{u_n^\nu} \right) dx dt = \nu \sum_{k,m=1}^N \int_0^T \int_{\Omega} \theta \xi_k \xi_m \overline{u_k^\nu} \Delta \overline{u_m^\nu} dx dt \\ & = -\nu \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \theta \xi_k \xi_m \nabla \overline{u_k^\nu} \nabla \overline{u_m^\nu} dx dt - \nu \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \nabla \theta \xi_k \xi_m \overline{u_k^\nu} \nabla \overline{u_m^\nu} dx dt \\ & \quad - \nu \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \theta \nabla (\xi_k \xi_m) \overline{u_k^\nu} \nabla \overline{u_m^\nu} dx dt. \end{aligned} \quad (3.12)$$

The terms on the right side of (3.12) are treated as follows. First, it follows from (1.7), (2.13), (2.14), (3.3) that, if $|k - m| = 0$,

$$\begin{aligned} & \left| \nu \int_0^T \int_{\Omega} \theta \xi_k^2 \nabla \overline{u_k^\nu} \nabla \overline{u_k^\nu} dx dt \right| = \left| \nu \int_0^T \left(\int_{V_k \cap V_{k+1}} + \int_{V_k \cap V_{k+1}^c} \right) \theta \xi_k^2 \nabla \overline{u_k^\nu} \nabla \overline{u_k^\nu} dx dt \right| \\ & \leq C\nu \int_0^T \left(\|\xi_k^2\|_{L^3(V_k \cap V_{k+1})} \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_{k+1})}^2 + \|\xi_k^2\|_{L^3(V_k \cap V_{k+1}^c)} \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_{k+1}^c)}^2 \right) dt \\ & \leq C\nu \left(\nu^{\frac{1}{3}\beta_{k+1}} \int_0^T \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_{k+1})}^2 dt + \nu^{\frac{1}{3}\beta_{k-1}} \int_0^T \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_{k+1}^c)}^2 dt \right) \\ & \leq C\nu \left(\nu^{\frac{1}{3}\beta_{k+1} + 2\beta_{k+1}(\alpha-1)} + \nu^{\frac{1}{3}\beta_{k-1} + 2\beta_k(\alpha-1)} \right) \int_0^T \|u^\nu\|_{B_3^{\alpha,\infty}(\Omega^\nu)}^2 dt \\ & \leq C\nu^{1 + \frac{1}{3}\beta_{k-1} + 2\beta_k(\alpha-1)}. \end{aligned} \quad (3.13)$$

If $|k - m| = 1$,

$$\begin{aligned} & \left| -\nu \int_0^T \int_{\Omega} \theta \xi_k \xi_m \nabla \overline{u_k^\nu} \nabla \overline{u_m^\nu} dx dt \right| \\ & \leq C\nu \int_0^T \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_m)} \|\nabla \overline{u_m^\nu}\|_{L^3(V_k \cap V_m)} \|\xi_k \xi_m\|_{L^3(V_k \cap V_m)} dt \\ & \leq C\nu^{1 + \beta_{\max\{k,m\}}(\frac{1}{3} + 2(\alpha-1))} \int_0^T \|u^\nu\|_{B_3^{\alpha,\infty}(\Omega^\nu)}^2 dt \\ & \leq C\nu^{1 + \beta_{\max\{k,m\}}(2\alpha - \frac{5}{3})}. \end{aligned} \quad (3.14)$$

Thanks to (1.7) and (2.10), we know that

$$1 + \frac{1}{3}\beta_{k-1} + 2\beta_k(\alpha - 1) > 0, \quad \text{and} \quad 1 + \beta_{\max\{k,m\}} \left(2\alpha - \frac{5}{3}\right) > 1 - \beta_{\max\{k,m\}} \geq 0.$$

Hence, from (3.13)-(3.14) we conclude

$$\begin{aligned} & \lim_{\nu \rightarrow 0} \left| \sum_{|k-m| \leq 1} \nu \int_0^T \int_{\Omega} \theta \xi_k \xi_m \overline{\nabla u_k^\nu} \overline{\nabla u_m^\nu} dx dt \right| \\ & \leq \lim_{\nu \rightarrow 0} \left(\sum_{|k-m|=0} + \sum_{|k-m|=1} \right) \left| \nu \int_0^T \int_{\Omega} \theta \xi_k \xi_m \overline{\nabla u_k^\nu} \overline{\nabla u_m^\nu} dx dt \right| \\ & \leq C \lim_{\nu \rightarrow 0} \sum_{|k-m| \leq 1} \left(\nu^{1+\frac{1}{3}\beta_{k-1}+2\beta_k(\alpha-1)} + \nu^{1+\beta_{\max\{k,m\}}(2\alpha-\frac{5}{3})} \right) = 0. \end{aligned} \quad (3.15)$$

Second, observe from (3.6) that $\theta = 0$ if $x \notin \Omega^{2\nu} \cap \Gamma_{4\nu}$. Then, utilizing (3.4), (1.7), (3.6), and the Hardy-type inequality, it follows that, for small ν ,

$$\begin{aligned} & \sum_{|k-m| \leq 1} \nu \int_0^T \int_{\Omega} \nabla \theta \xi_k \xi_m \overline{u_k^\nu} \overline{\nabla u_m^\nu} dx dt = \nu \int_0^T \int_{\Omega^{2\nu} \cap \Gamma_{4\nu}} \nabla \theta \xi_N^2 \overline{u_N^\nu} \overline{\nabla u_N^\nu} dx dt \\ & \leq \left(\nu \int_0^T \int_{\Omega^{2\nu} \cap \Gamma_{4\nu}} |\nabla \overline{u_N^\nu}|^2 dx dt \right)^{\frac{1}{2}} \left(\nu \int_0^T \|\nabla \theta\|_{L^6(\Omega^{2\nu} \cap \Gamma_{4\nu})}^2 \|\overline{u_N^\nu} - u^\nu\|_{L^3(V_N)}^2 + \|u^\nu \nabla \theta\|_{L^2(\Gamma_{4\nu})}^2 \right)^{\frac{1}{2}} \\ & \leq \left(\nu \int_0^T \int_{\Omega} |\nabla \overline{u_N^\nu}|^2 dx dt \right)^{\frac{1}{2}} \left(\nu \int_0^T \nu^{2\alpha-\frac{5}{3}} \|u^\nu\|_{B_3^\alpha(\Omega^\nu)}^2 + \|\nabla u^\nu\|_{L^2(\Gamma_{4\nu})}^2 \right)^{\frac{1}{2}} \\ & \leq C \left(\nu^{1+(2\alpha-\frac{5}{3})} + \nu \int_0^T \int_{\Gamma_{4\nu}} |\nabla \overline{u_N^\nu}|^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (3.16)$$

Thanks to (1.7) and (1.9), we take $\nu \rightarrow 0$ in (3.16) to get

$$\lim_{\nu \rightarrow 0} \left| \sum_{|k-m| \leq 1} \nu \int_0^T \int_{\Omega} \nabla \theta \xi_k \xi_m \overline{u_k^\nu} \overline{\nabla u_m^\nu} dx dt \right| = 0. \quad (3.17)$$

Finally, thanks to Proposition 2.1 and (3.2), we infer that

$$\begin{aligned} & \sum_{|k-m| \leq 1} \nu \int_0^T \int_{\Omega} \theta \nabla(\xi_k \xi_m) \overline{u_k^\nu} \overline{\nabla u_m^\nu} dx dt \\ & = \sum_{k=1}^N \nu \int_0^T \left(\int_{V_k \cap V_{k-1}} + \int_{V_k \cap V_{k+1}} \right) \theta \nabla(\xi_k^2) \overline{u_k^\nu} \overline{\nabla u_k^\nu} dx dt \\ & \quad + \sum_{k=1}^{N-1} \nu \int_0^T \int_{V_k \cap V_{k+1}} \theta \nabla(\xi_k \xi_{k+1}) (\overline{u_{k+1}^\nu} \overline{\nabla u_k^\nu} + \overline{u_k^\nu} \overline{\nabla u_{k+1}^\nu}) dx dt \\ & = \sum_{k=1}^N \nu \int_0^T \left(\int_{V_k \cap V_{k-1}} \theta \nabla(\xi_k^2) \overline{u_k^\nu} \overline{\nabla u_k^\nu} dx + \int_{V_k \cap V_{k+1}} \theta \nabla(\xi_k^2) \overline{u_{k+1}^\nu} \overline{\nabla u_{k+1}^\nu} dx \right) dt \end{aligned} \quad (3.18)$$

$$\begin{aligned}
& + \sum_{k=1}^{N-1} \nu \int_0^T \int_{V_k \cap V_{k+1}} \theta \nabla (2\xi_k \xi_{k+1}) \overline{u_{k+1}^\nu} \nabla \overline{u_{k+1}^\nu} dx dt \\
& = \sum_{k=1}^{N-1} \nu \int_0^T \int_{V_k \cap V_{k+1}} \theta \nabla (\xi_k^2 + \xi_{k+1}^2 + 2\xi_k \xi_{k+1}) \overline{u_{k+1}^\nu} \nabla \overline{u_{k+1}^\nu} dx dt = 0,
\end{aligned}$$

where the second equality is due to (3.2), and the third equality comes from relabeling and (2.15).

As a result of (3.12), (3.15), (3.17) and (3.18), we conclude (3.10). \square

Lemma 3.2 (Bulk energy flux). *Under the same hypotheses as in Theorem 1.1, we have*

$$\lim_{\nu \rightarrow 0} \int_0^T \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{u_n^\nu} \right) \left(\sum_{n=1}^N \xi_n \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n \right) dx dt = 0. \quad (3.19)$$

Proof. By (3.11), integration by parts leads to

$$\begin{aligned}
& \int_0^T \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{u_n^\nu} \right) \left(\sum_{n=1}^N \xi_n \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n \right) dx dt \\
& = \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \theta \xi_k \xi_m \overline{u_k^\nu} \operatorname{div}(\overline{u^\nu \otimes u^\nu})_m dx dt \\
& = \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu \otimes u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : \nabla (\theta \xi_k \xi_m \overline{u_k^\nu}) dx dt \\
& \quad - \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \overline{u_m^\nu \otimes u_m^\nu} : \nabla (\theta \xi_k \xi_m \overline{u_k^\nu}) dx dt.
\end{aligned} \quad (3.20)$$

We claim

$$\lim_{\nu \rightarrow 0} \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu \otimes u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : \nabla (\theta \xi_k \xi_m \overline{u_k^\nu}) dx dt = 0. \quad (3.21)$$

In fact, we notice from (1.7), (1.8), (3.6), (1.4), (3.4), and the Hardy-type inequality that

$$\begin{aligned}
& \left| \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu \otimes u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : (\nabla \theta) \xi_k \xi_m \overline{u_k^\nu} dx dt \right| \\
& \leq C \left(\int_0^T \int_{\Gamma_{2\nu} \cap \Omega^\nu} |\overline{u_N^\nu}|^4 dx dt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Gamma_{4\nu} \cap \Omega^{2\nu}} |\nabla \theta (\overline{u_N^\nu} - u^\nu + u^\nu)|^2 dx dt \right)^{\frac{1}{2}} \\
& \leq C \left(\nu \int_0^T \|u^\nu\|_{L^\infty(\Gamma_{4\nu})}^4 dt \right)^{\frac{1}{2}} \left(\int_0^T \nu^{-\frac{5}{3}} \|\overline{u_N^\nu} - u^\nu\|_{L^3}^2 dt + \int_0^T \int_{\Gamma_{4\nu}} |\nabla \theta u^\nu|^2 dx dt \right)^{\frac{1}{2}} \\
& \leq C \nu^{\frac{1}{2}} \left(\int_0^T \nu^{(2\alpha - \frac{5}{3})} \|u^\nu\|_{B_3^{\alpha, \infty}(\Omega^\nu)}^2 dt + \int_0^T \int_{\Gamma_{4\nu}} |\nabla u^\nu|^2 dx dt \right)^{\frac{1}{2}} \quad (3.22)
\end{aligned}$$

$$\leq C \left(\nu^{1+(2\alpha-\frac{5}{3})} + \nu \int_0^T \int_{\Gamma_{4\nu}} |\nabla u^\nu|^2 dxdt \right)^{\frac{1}{2}}.$$

This, together with (1.7) and (1.9), implies that

$$\lim_{\nu \rightarrow 0} \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu} \otimes \overline{u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : (\nabla \theta) \xi_k \xi_m \overline{u_k^\nu} dxdt = 0. \quad (3.23)$$

Next, from (2.14), (1.7), (3.3), and Lemma 2.1 we have

$$\begin{aligned} & \sum_k \int_0^T \int_{\Omega} \left(\overline{u_k^\nu} \otimes \overline{u_k^\nu} - \overline{(u^\nu \otimes u^\nu)_k} \right) : \theta \xi_k^2 \nabla \overline{u_k^\nu} dxdt \\ & \leq C \sum_k \int_0^T \left\| \overline{u_k^\nu} \otimes \overline{u_k^\nu} - \overline{(u^\nu \otimes u^\nu)_k} \right\|_{L^{\frac{3}{2}}(V_k \cap V_{k+1})} \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_{k+1})} dt \\ & \quad + C \sum_k \int_0^T \left\| \overline{u_k^\nu} \otimes \overline{u_k^\nu} - \overline{(u^\nu \otimes u^\nu)_k} \right\|_{L^{\frac{3}{2}}(V_k \cap V_{k+1}^c)} \|\nabla \overline{u_k^\nu}\|_{L^3(V_k \cap V_{k+1}^c)} dt \\ & \leq C \sum_k \int_0^T \left(\nu^{2\beta_{k+1}\alpha + \beta_{k+1}(\alpha-1)} + \nu^{2\beta_k\alpha + \beta_k(\alpha-1)} \right) \|u^\nu\|_{B_3^{\alpha, \infty}(\Omega^\nu)}^3 dt \\ & \leq C \sum_k \nu^{\beta_k(3\alpha-1)}. \end{aligned}$$

Similarly,

$$\left| \sum_{|k-m|=1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu} \otimes \overline{u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : \theta \xi_k \xi_m \nabla \overline{u_k^\nu} dxdt \right| \leq C \sum_{|k-m|=1} \nu^{\beta_{\max\{m,k\}}(3\alpha-1)}.$$

The above two inequalities and (1.7) guarantee that

$$\lim_{\nu \rightarrow 0} \left| \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu} \otimes \overline{u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : \theta \xi_k \xi_m \nabla \overline{u_k^\nu} dxdt \right| = 0. \quad (3.24)$$

By Proposition 2.1, the same deduction as (3.18) yields that

$$\begin{aligned} & \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu} \otimes \overline{u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : \theta \nabla (\xi_k \xi_m) \overline{u_k^\nu} dxdt \\ & = \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} \left(\overline{u_{k+1}^\nu} \otimes \overline{u_{k+1}^\nu} - \overline{(u^\nu \otimes u^\nu)_{k+1}} \right) : \theta \nabla (\xi_k^2 + \xi_{k+1}^2 + 2\xi_k \xi_{k+1}) dxdt \\ & = 0. \end{aligned} \quad (3.25)$$

As a result of (3.23)–(3.25), we conclude (3.21).

It remains to control the last integral appeared in (3.20). By the fact $\operatorname{div} \overline{u_n^\nu} = 0$, we deduce that, if $|k-m|=0$,

$$\sum_{|k-m|=0} \int_0^T \int_{\Omega} \overline{u_m^\nu} \otimes \overline{u_m^\nu} : \nabla (\theta \xi_k \xi_m \overline{u_k^\nu}) dxdt = \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{\Omega} |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla (\theta \xi_k^2) dxdt$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k} |\overline{u_k^\nu}|^2 \xi_k^2 \overline{u_k^\nu} \cdot \nabla \theta \, dx dt + \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k} \theta |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla (\xi_k^2) \, dx dt \\
&= \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k} |\overline{u_k^\nu}|^2 \xi_k^2 \overline{u_k^\nu} \cdot \nabla \theta \, dx dt + \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_{k-1} \cap V_k} \theta |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla (\xi_k^2) \, dx dt \\
&\quad + \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k \cap V_{k+1}} \theta |\overline{u_{k+1}^\nu}|^2 \overline{u_{k+1}^\nu} \cdot \nabla (\xi_k^2) \, dx dt \tag{3.26} \\
&= \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k} |\overline{u_k^\nu}|^2 \xi_k^2 \overline{u_k^\nu} \cdot \nabla \theta \, dx dt + \frac{1}{2} \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} \theta |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla (\xi_k^2 + \xi_{k+1}^2) \, dx dt \\
&\quad + \frac{1}{2} \int_0^T \left(\int_{V_0 \cap V_1} \theta |\overline{u_1^\nu}|^2 \overline{u_1^\nu} \cdot \nabla (\xi_1^2) \, dx + \int_{V_N \cap V_{N+1}} \theta |\overline{u_N^\nu}|^2 \overline{u_N^\nu} \cdot \nabla (\xi_N^2) \, dx \right) dt \\
&= \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k} |\overline{u_k^\nu}|^2 \xi_k^2 \overline{u_k^\nu} \cdot \nabla \theta \, dx dt - \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} \theta |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla (\xi_k \xi_{k+1}) \, dx dt,
\end{aligned}$$

where the third equality is due to (3.2) and (2.15); and if $|k - m| = 1$,

$$\begin{aligned}
&\sum_{|k-m|=1} \int_0^T \int_{\Omega} \overline{u_m^\nu} \otimes \overline{u_m^\nu} : \nabla (\theta \xi_k \xi_m \overline{u_k^\nu}) \, dx dt \\
&= \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} \overline{u_{k+1}^\nu} \otimes \overline{u_{k+1}^\nu} : \nabla (2\theta \xi_k \xi_{k+1} \overline{u_{k+1}^\nu}) \, dx dt \\
&= \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} |\overline{u_{k+1}^\nu}|^2 \overline{u_{k+1}^\nu} \cdot \nabla (\theta \xi_k \xi_{k+1}) \, dx dt \tag{3.27} \\
&= \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} \left(|\overline{u_{k+1}^\nu}|^2 \overline{u_{k+1}^\nu} \cdot \nabla \theta (\xi_k \xi_{k+1}) + \theta |\overline{u_{k+1}^\nu}|^2 \overline{u_{k+1}^\nu} \cdot \nabla (\xi_k \xi_{k+1}) \right) \, dx dt \\
&= \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} \left(|\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla \theta (\xi_k \xi_{k+1}) + \theta |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla (\xi_k \xi_{k+1}) \right) \, dx dt.
\end{aligned}$$

Combining the above calculations together and applying Lemma 2.3 we have

$$\begin{aligned}
&\sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \overline{u_m^\nu} \otimes \overline{u_m^\nu} : \nabla (\theta \xi_k \xi_m \overline{u_k^\nu}) \, dx dt \\
&= \frac{1}{2} \sum_{k=1}^N \int_0^T \int_{V_k} |\overline{u_k^\nu}|^2 \xi_k^2 \overline{u_k^\nu} \cdot \nabla \theta \, dx dt + \sum_{k=1}^{N-1} \int_0^T \int_{V_k \cap V_{k+1}} |\overline{u_k^\nu}|^2 \overline{u_k^\nu} \cdot \nabla \theta (\xi_k \xi_{k+1}) \, dx dt \\
&\leq C \int_0^T \int_{\Gamma_{2\nu} \cap \Omega^\nu} |\overline{u_k^\nu}|^2 |\overline{u_k^\nu} \cdot \nabla \theta| \, dx dt \tag{3.28}
\end{aligned}$$

$$\leq C \left(\nu \int_0^T \|u^\nu\|_{L^\infty(\Gamma_{4\nu})} dt \right)^{1/2} \left(\int_0^T \int_{\Gamma_{4\nu}} |\nabla u^\nu|^2 dx dt \right)^{1/2}.$$

From (1.8) and (1.9) we conclude that

$$\lim_{\nu \rightarrow 0} \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \overline{u_m^\nu} \otimes \overline{u_m^\nu} : \nabla(\theta \xi_k \xi_m \overline{u_k^\nu}) dx dt = 0. \quad (3.29)$$

Taking (3.20)–(3.21), and (3.29) into account, we complete the proof of Lemma 3.2. \square

Lemma 3.3. *Under the same hypotheses as in Theorem 1.1, we have*

$$\lim_{\nu \rightarrow 0} \int_0^T \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{u_n^\nu} \right) \left(\sum_{n=1}^N \xi_n \nabla \overline{P_n^\nu} \right) dx dt = 0. \quad (3.30)$$

Proof. The proof is a slight modification of that in Lemma 3.2, and hence we omit it here. \square

4. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1.

4.1. Vanishing of global dissipation. We aim to prove the validity of (1.10). The combination of (1.4) with (3.8) generates

$$\begin{aligned} 0 &\leq 2\nu \int_0^T \int_{\Omega} |\nabla u^\nu|^2 \\ &\leq \left[\int_{\Omega} |u_0^\nu|^2 - \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{(u_0^\nu)_n} \right)^2 \right] + \left[\int_{\Omega} \theta \left| \sum_{n=1}^N \xi_n \overline{u_n^\nu} \right|^2 - \int_{\Omega} |u^\nu|^2 \right] \\ &\quad + 2 \int_0^T \int_{\Omega} \theta \left(\sum_{n=1}^N \xi_n \overline{u_n^\nu} \right) \left(\sum_{n=1}^N \xi_n \nu \Delta \overline{u_n^\nu} - \sum_{n=1}^N \xi_n \operatorname{div}(\overline{u^\nu \otimes u^\nu})_n - \sum_{n=1}^N \xi_n \nabla \overline{P_n^\nu} \right) \\ &=: I + II + III. \end{aligned} \quad (4.1)$$

First, from Proposition 3.1 it follows that

$$\lim_{\nu \rightarrow 0} III = 0. \quad (4.2)$$

Next, basic properties of mollifier $\eta_{\nu\beta}$ ensure that

$$\int_{\Omega} \theta \xi_1^2 |\overline{u_1^\nu}|^2 dx \leq \int_{V_1} |\overline{u_1^\nu}|^2 \leq \int_{\Omega} |u^\nu|^2.$$

Then,

$$\begin{aligned} II &= \sum_{2 < k+m} \int_{\Omega} \theta \xi_k \xi_m \overline{u_k^\nu} \overline{u_m^\nu} + \int_{\Omega} \theta \xi_1^2 \overline{u_1^\nu}^2 - \int_{\Omega} |u^\nu|^2 \\ &\leq \sum_{2 < k+m} \int_{\Omega} \theta \xi_k \xi_m \overline{u_k^\nu} \overline{u_m^\nu} \\ &\leq C \left(\int_{V_k \cap V_m} |\overline{u_k^\nu}|^2 \right)^{\frac{1}{2}} \left(\int_{V_k \cap V_m} |\overline{u_m^\nu}|^2 \right)^{\frac{1}{2}} \leq C \int_{\Gamma_{2\nu\beta_1}} |u^\nu|^2 dx, \end{aligned} \quad (4.3)$$

and hence the uniform bound of u^ν in $L^\infty(0, T; L^2)$ implies that

$$\lim_{\nu \rightarrow 0} II \leq C \lim_{\nu \rightarrow 0} \int_{\Gamma_{2\nu\beta_1}} |u^\nu|^2 dx = 0. \quad (4.4)$$

Finally, since

$$\begin{aligned} & \left| \int_{\Omega} |u_0|^2 - \int_{\Omega} \theta \xi_1^2 |\overline{(u_0^\nu)_1}|^2 \right| = \left| \int_{\Omega} |u_0|^2 - \int_{\Omega^{\nu\beta_1}} \xi_1^2 |\overline{(u_0^\nu)_1}|^2 \right| \\ & \leq \int_{\Omega \setminus \Omega^{\nu\beta_1}} |u_0|^2 + \int_{\Omega^{\nu\beta_1}} (1 - \xi_1^2) |\overline{(u_0^\nu)_1}|^2 + \left| \int_{\Omega^{\nu\beta_1}} (|u_0|^2 - |\overline{(u_0^\nu)_1}|^2) \right| \\ & \leq 2 \|u_0\|_{L^2(\Gamma_{2\nu\beta_1})} + C \|u_0\|_{L^2(\Omega)} \|u_0 - \overline{(u_0^\nu)_1}\|_{L^2(\Omega^{\nu\beta_1})} \\ & \leq 2 \|u_0\|_{L^2(\Gamma_{2\nu\beta_1})} + C \left(\|u_0 - \overline{(u_0)_1}\|_{L^2(\Omega^{\nu\beta_1})} + \|u_0 - u_0^\nu\|_{L^2(\Omega^{\nu\beta_1})} \right) \\ & \rightarrow 0 \quad \text{as } \nu \rightarrow 0, \end{aligned}$$

then the strong convergence of u_0^ν to u_0 in L^2 guarantees that

$$\lim_{\nu \rightarrow 0} \left| \int_{\Omega} |u_0^\nu|^2 - \int_{\Omega} \theta \xi_1^2 |\overline{(u_0^\nu)_1}|^2 \right| = 0.$$

This allows us to deduce that

$$\begin{aligned} \lim_{\nu \rightarrow 0} |I| &= \lim_{\nu \rightarrow 0} \left| \int_{\Omega} |u_0^\nu|^2 - \sum_{k,m=1}^N \int_{\Omega} \theta \xi_k \xi_m \overline{(u_0^\nu)_k} \overline{(u_0^\nu)_m} \right| \\ &\leq \lim_{\nu \rightarrow 0} \left| \int_{\Omega} |u_0^\nu|^2 - \int_{\Omega} \theta \xi_1^2 |\overline{(u_0^\nu)_1}|^2 \right| + \sum_{2 < k+m} \lim_{\nu \rightarrow 0} \left| \int_{\Omega} \theta \xi_k \xi_m \overline{(u_0^\nu)_k} \overline{(u_0^\nu)_m} \right| = 0, \end{aligned} \quad (4.5)$$

where in the last equality we have used

$$\sum_{2 < k+m} \lim_{\nu \rightarrow 0} \left| \int_{\Omega} \theta \xi_k \xi_m \overline{(u_0^\nu)_k} \overline{(u_0^\nu)_m} \right| = 0,$$

which comes from (4.3)–(4.4). As a result of (4.1)–(4.2) and (4.4)–(4.5), we obtain the desired (1.10).

4.2. Convergence to Euler solutions. Under the assumptions in Theorem 1.1 and Lemma 2.2, there exist some functions (u, P) such that, upon to some subsequence,

$$u^\nu \rightharpoonup u \text{ in } L^3(0, T; B_3^{\sigma, \infty}(\Omega^\nu)) \cap L^\infty(0, T; L^2(\Omega)), \quad P^\nu \rightharpoonup P \text{ in } L^{\frac{3}{2}}(0, T; L^{\frac{3}{2}}(\Omega)). \quad (4.6)$$

Thanks to (4.6) and (1.4), we have

$$\partial_t u^\nu = \nu \Delta u^\nu - \nabla P^\nu - \operatorname{div}(u^\nu \otimes u^\nu) \in L^{\frac{3}{2}}(0, T; W^{-1, \frac{3}{2}}(\Omega)),$$

and moreover,

$$u^\nu \rightarrow u \text{ in } L^3(0, T; L^3(\Omega^\nu)) \cap C([0, T], L_{weak}^2(\Omega)), \quad (4.7)$$

owing to the compactness results. In addition, it follows from (1.4) that, as $\nu \rightarrow 0$,

$$\left| \int_0^T \int_{\Omega} \nu \nabla u^\nu \cdot \nabla \varphi \right| \leq C \nu^{\frac{1}{2}} \left(\nu \int_0^T \|\nabla u^\nu\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \rightarrow 0. \quad (4.8)$$

Having (4.6)–(4.8) in hand, we easily check that u solves Euler equations (1.2) in $\Omega \times (0, T)$.

5. BOUNDARY LAYERS FOR SMOOTHER SOLUTIONS

The boundary layer $\Gamma_{4\nu}$ in (1.9) of Theorem 1.1 in fact holds for all $\alpha > \frac{1}{3}$. The emphasis of the previous analysis lies in determining the boundary layer for solutions near the critical Onsager's regularity. On the other hand, when the solutions are more regular, the hypotheses in Theorem 1.1 can be relaxed, and the boundary layer can be even thinner, as is shown in the following theorem.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary. Let $\{u^\nu\}_{\nu>0}$ be a sequence of Leray–Hopf weak solutions to (1.1) with initial data u_0^ν and suppose that $u_0^\nu \rightarrow u_0$ in $L^2(\Omega)$ as $\nu \rightarrow 0$. Assume in addition that (1.7) holds, and*

$$\begin{cases} u^\nu \text{ is uniformly in } \nu \text{ bounded in } L^4(0, T; L^p(\Gamma_{4\nu})), \\ P^\nu \text{ is uniformly in } \nu \text{ bounded in } L^2\left(0, T; L^{\frac{p}{2}}(\Gamma_{4\nu})\right), \end{cases} \quad (5.1)$$

with $p > \frac{6}{3\alpha-1}$. Let $a > 1$ be such that

$$a < \frac{3}{5-6\alpha}, \text{ when } \frac{1}{3} < \alpha < \frac{5}{6}; \quad a < \infty, \text{ when } \frac{5}{6} \leq \alpha < 1. \quad (5.2)$$

If

$$\lim_{\nu \rightarrow 0} \nu \int_0^T \int_{\Gamma_{4\nu^a}} |\nabla u^\nu|^2 dx dt = 0, \quad (5.3)$$

then, the global viscous dissipation vanishes, i.e., (1.10) holds true. Moreover, u^ν converges locally in $L^3(0, T; L^3(\Omega))$, up to a subsequence, to a weak solution of Euler equations (1.2).

Remark 5.1. As $\alpha \rightarrow \frac{1}{3}^+$, Theorem 5.1 recovers Theorem 1.1. But as α increases, we can relax the regularity requirement on the boundary, cf. (5.1), and the thickness of the boundary layer becomes ν^a with $a > 1$. In particular, as $\alpha \rightarrow 1^-$, the thickness becomes arbitrarily small.

Proof. The idea of the proof is very similar to the one explained before, and hence we will only focus on the ingredients different from those in the proof of Theorem 1.1.

First of all, we modify the construction of the increasing and finite sequence $\{\beta_n\}_{n=1}^N$ in Section 2.3 as

$$0 = \beta_0 < \beta_1 < \cdots < \beta_{N-1} \leq 1 < \beta_N < \begin{cases} \frac{3}{5-6\sigma}, & \text{if } \alpha \in \left(\frac{1}{3}, \frac{5}{6}\right), \\ \infty, & \text{if } \alpha \in \left[\frac{5}{6}, 1\right), \end{cases} \quad (5.4)$$

and

$$\beta_n < \frac{1}{2(1-\alpha)} \left(1 + \frac{1}{3}\beta_{n-1}\right).$$

In fact, here we consider the case of $\beta_N = a > 1$ which satisfies (5.4), in stead of $\beta_N = 1$ defined in (2.10).

The ‘‘peeled-off’’ set V_{N+1} in (2.16) now becomes

$$\Gamma_{\nu^a} := V_{N+1} = \left(\bigcup_{n=1}^N V_n \right)^c. \quad (5.5)$$

In addition, the near boundary layer cut-off function θ in (3.6) is modified as

$$0 \leq \theta(x) \leq 1, \quad \theta(x) = 1 \text{ if } x \in \Omega^{2\nu^a}, \quad \theta(x) = 0 \text{ if } x \notin \Omega^{\nu^a}, \quad |\nabla\theta| \leq 2\nu^{-a}. \quad (5.6)$$

With the above preparations, to complete the proof of Theorem 5.1, we only need to check the following:

(a) **Inequality (3.16) in Lemma 3.1.**

In Theorem 5.1, it can be treated as

$$\begin{aligned} & \sum_{|k-m| \leq 1} \nu \int_0^T \int_{\Omega} \nabla \theta \xi_k \xi_m \overline{u_k^\nu} \nabla \overline{u_m^\nu} \\ &= \nu \int_0^T \int_{\Omega^{2\nu^a} \cap \Gamma_{4\nu^a}} \nabla \theta \xi_N^2 \overline{u_N^\nu} \nabla \overline{u_N^\nu} \\ &\leq \left(\nu \int_0^T \int_{\Omega} |\nabla \overline{u_N^\nu}|^2 \right)^{\frac{1}{2}} \left(\nu \int_0^T \nu^{a(2\alpha - \frac{5}{3})} \|u^\nu\|_{B_3^{\alpha, \infty}(\Omega^{\nu^a})}^2 + \|\nabla u^\nu\|_{L^2(\Gamma_{2\nu^a})}^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\nu^{1+a(2\alpha - \frac{5}{3})} + \nu \int_0^T \int_{\Gamma_{4\nu^a}} |\nabla \overline{u_N^\nu}|^2 \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned} \quad (5.7)$$

provided

$$1 + a \left(2\alpha - \frac{5}{3} \right) > 0,$$

which is valid owing to (5.4).

(b) **Inequality (3.22) in Lemma 3.2.**

In Theorem 5.1, we estimate (3.22) as the following:

$$\begin{aligned} & \left| \sum_{|k-m| \leq 1} \int_0^T \int_{\Omega} \left(\overline{u_m^\nu} \otimes \overline{u_m^\nu} - \overline{(u^\nu \otimes u^\nu)_m} \right) : \nabla \theta \xi_k \xi_m \overline{u_k^\nu} \right| \\ &\leq C \left(\int_0^T \int_{\Gamma_{4\nu^a} \cap \Omega^{2\nu^a}} |\overline{u_N^\nu}|^4 \right)^{\frac{1}{2}} \left(\int_0^T \nu^{-\frac{5}{3}a} \|\overline{u_N^\nu} - u^\nu\|_{L^3}^2 + \int_0^T \int_{\Gamma_{4\nu^a}} |\nabla \theta u^\nu|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\nu^{a(1 - \frac{4}{p})} \int_0^T \|u^\nu\|_{L^p(\Gamma_{8\nu^a})}^4 \right)^{\frac{1}{2}} \left(\int_0^T \nu^{a(2\alpha - \frac{5}{3})} \|u^\nu\|_{B_3^{\sigma, \infty}(\Omega^{\nu^a})}^2 + \int_0^T \int_{\Gamma_{4\nu^a}} |\nabla u^\nu|^2 \right)^{\frac{1}{2}} \\ &\leq C(T) \nu^{\frac{1}{2}a(1 - \frac{4}{p} + 2\alpha - \frac{5}{3})} + C\nu^{\frac{1}{2}(a(1 - \frac{4}{p}) - 1)} \left(\nu \int_0^T \int_{\Gamma_{4\nu^a}} |\nabla u^\nu|^2 \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned} \quad (5.8)$$

provided

$$a \left(1 - \frac{4}{p} + 2\alpha - \frac{5}{3} \right) > 0 \quad \text{and} \quad a \left(1 - \frac{4}{p} \right) - 1 \geq 0,$$

which holds true due to (5.2) and (5.4).

(c) **Inequality (3.28) in Lemma 3.2.**

This can be achieved from a similar argument to that in deriving (5.8). \square

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