STABILIZATION EFFECT OF ELASTICITY ON THREE-DIMENSIONAL COMPRESSIBLE VORTEX SHEETS

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Abstract. We are concerned with the vortex sheet solutions for the three-dimensional compressible isentropic elastic flows. This is a nonlinear hyperbolic problem with a characteristic free boundary. Compared with the analysis in two dimensions, this added dimension leads to more complicated frequency interactions between the effects of elasticity and the fluid velocity, making the stability analysis more challenging. Through a very delicate examination of the Lopatinskiĭ determinant of the linearized boundary value problem, necessary and sufficient conditions are established for the linear stability of the planar vortex sheet solutions. These conditions are closely related to the geometric properties of the elastic deformation gradient and provide the first stability criterion justifying the stabilization effect of elasticity on the compressible vortex sheets in the three-dimensional elastodynamics. In contrast to the two-dimensional isentropic elastic fluids, we find that the stability can only hold in the subsonic region for the three-dimensional vortex sheets.

1. Introduction

In this paper, we consider the vortex sheet solutions to the three-dimensional compressible inviscid flow in elastodynamics ([10, 17, 27]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \operatorname{div}(\rho\mathbf{F}\mathbf{F}^T), \\ (\rho\mathbf{F}_j)_t + \operatorname{div}(\rho\mathbf{F}_j\otimes\mathbf{u} - \mathbf{u}\otimes\rho\mathbf{F}_j) = 0, \end{cases} \tag{1.1}$$

where ρ denotes the density, $\mathbf{u} = (u, v, w) \in \mathbb{R}^3$ is the velocity, $\mathbf{F}_j$ is the jth column of the deformation gradient $\mathbf{F} = (F_{ij}) \in \mathbf{M}^{3\times 3}$ and $p = p(\rho)$ is a smooth strictly increasing function on $(0, \infty)$ denoting the pressure. The vortex sheet structures are piecewise smooth weak solutions to (1.1) with a discontinuity interface, across which there is no mass transfer but the tangential velocity experiences a jump.

Vortex sheets in compressible Euler flows are classical subjects in the study of gas dynamics which date back to 1950's in the works of Miles [34, 35] and Fejer-Miles [12]. A linear analysis performed in [35] indicates that the vortex sheets exhibit violent instability for Mach number $M < \sqrt{2}$ in two or three dimensions. It was until more than four decades later that Coulombel and Secchi proved, in the pioneer works [7] and [8], via a micro-local analysis and Nash-Moser technique, that the vortex sheets in two dimensions are linearly and (local-in-time) nonlinearly stable when Mach number $M > \sqrt{2}$. The initial

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data chosen in their works are small perturbation of a rectilinear vortex sheet and their definition of linear stability is in a sense similar to that of shock waves by Coulombel [5,7] and Majda [31,32]. Similar stability results of the two-dimensional vortex sheets were also obtained recently in [11,14,28] for the two-phase flows and relativistic flows.

As for three-dimensional Euler flows, the situation becomes much more complex and the results are quite limited. As observed in Miles [35], disturbances traveling at sufficiently large angles with respect to the undisturbed flow are unstable. Moreover, according to the normal mode analysis in [40], three-dimensional ideal compressible vortex sheets are always violently unstable, regardless of how large the Mach number is. For three-dimensional steady flows, on the other hand, Wang–Yu [44,47] proved the structural nonlinear stability under a supersonic stability condition, that is, the contact discontinuity is supersonic in one of the spatial directions, which could be regarded as time-like. Separating this time-like direction makes the problem two-dimensional-like, and hence the stability is consistent with [7,8,45]. For unsteady Euler flows in three dimensions, however, a growing mode can always be generated due to the increased degree of freedom. Therefore to stabilize the fluids, additional fields or viscosity are needed to neutralize and counterbalance the violent instability.

For the three-dimensional compressible magnetohydrodynamic (MHD) flows, Trakhinin [42] and Chen–Wang [4] proved independently the nonlinear stability of compressible current-vortex sheets, which indicates that non-paralleled magnetic fields stabilize the motion of three-dimensional current-vortex sheets. Both of these two results developed a nonlinear energy method and proposed a sufficient condition for the weak stability of planar current-vortex sheets.

For viscoelastic fluids, there have been extensive works on various aspects from the mathematical modeling, theoretical analysis and applications [9,10,21,30,41]. It is commonly believed that viscoelasticity plays a notable stabilization role. Confirmation of such a stabilization effect can be found in examples of shear flows and vortex flows [1,3,26,36]. Moreover, examples of vortex sheet formation from unsteady shearing motions in certain viscoelastic fluids are constructed by Huigol [25,26] through considering the Rayleigh problem. On the other hand, when the viscosity is turned off, Hu–Wang [22] managed to construct a class of initial data that lead to the formation of singularity and the breakdown of classical solutions to system (1.1). In the case of partial dissipation, the global stability around a constant equilibrium for system (1.1) was established by Hu–Zhao [23,24]. The sensitivity of the stability of the vortex sheets with respect to the viscosity naturally leads to the question of the stabilization from solely the elastic component. Such a question was addressed in a series of recent works by Chen-Hu-Wang [38,39] and Chen-Hu-Wang-Wang-Yuan [37] in the two-dimensional setting. The linear stability was achieved in [38,39] through a sophisticated spectral analysis together with an upper triangulation scheme for the energy estimates. In [37] the nonlinear stability and local existence of elastic vortex sheets was established in the usual Sobolev spaces. The upper triangulation method has also been adapted in [14] in establishing nonlinear stability for two-dimensional vortex sheets in a relativistic compressible fluid.

It is worthwhile mentioning a few works on some variants of system (1.1). The local well-posedness theory for the incompressible counterpart was established in Li–Wang–Zhang [18] for the vortex sheet problem with a varying density and Hu–Huang [20] for a single phase free boundary problem with a constant density. The work of [18] verifies the elasticity stabilization on the Rayleigh–Taylor instability, while the result of [20] further assumes a Rayleigh–Taylor sign condition on the initial data.

Stability analysis in two-dimensional compressible elastodynamics (1.1) for discontinuity structures other than vortex sheets has been performed in Trakhinin–Morando–Trebeschi [2] and Chen–Secchi–Wang [15]. The former one provides a sufficient condition for the uniform stability of rectilinear shock waves by exploiting the symmetrization of the wave equation and using an energy method with no regularity loss for the solutions of the linearized problem with constant coefficients. The latter paper considers nonisentropic thermoelastic contact discontinuities for which the velocity is continuous across the discontinuity interface. Sufficient conditions for stability for such structures were derived, confirming the stabilization role of thermoelasticity.

The goal of this paper is to understand the stability problem of the vortex sheet structure for (1.1) in three spatial dimensions, attempting to push the stability analysis of two-dimensional vortex sheet flows in [37–39] forward to the more challenging three-dimensional case. As a first step, we consider the linear stability of the elastic vortex sheets. We will provide a *necessary and sufficient* condition for the neutral linear stability and instability of planar vortex sheets in the three-dimensional inviscid compressible isentropic elastic flows in the sense of [7] through discussing the Lopatinskiĭ determinant of the linearized boundary value problem. The new stability condition (3.9) we propose can be easily adapted to the two-dimensional elastic flows as in [38] and the three-dimensional Euler flows [40]. To the best of the authors' knowledge, this is the first (linear) stability result towards proving the local-in-time existence of stable nonplanar compressible vortex sheets in the three-dimensional elastic fluids.

To review some of the challenging features of the problem (see, for example, [38]), we know that the system has a characteristic free boundary, which fails to provide sufficient control on the trace of the characteristic parts of the solutions; see [7, 29, 33]. The uniform Kreiss-Lopatinskiĭ condition also fails to hold, which causes certain loss of tangential derivatives in the estimates of the solutions in terms of the source term on the right hand side of the linearized problem. Moreover, the elasticity exerts a more complicate distribution of roots for Lopatinskiĭ determinant, which leads to another difficulty in our analysis. As in [38], the standard Kreiss symmetrization technique cannot be adopted directly.

In addition to the above difficulties, recall that we are considering a genuine three-dimensional problem, which is very different from the two-dimensional flows [37–39] or the steady three-dimensional flows [44, 46]. The tangential velocities of the sheets of contact discontinuities now inherit two components, which could potentially host more directions for instability (and this is exactly the reason for the instability of three-dimensional Euler vortex sheets). On the Fourier side, the increase of physical dimension leads to an extra degree of freedom in frequency space, and hence the frequency interactions and resonances become much more complicated to track. On the other hand, we still hope to utilize the (subtle) enhancement of the elastic stabilization to compensate and deter the tendency of instability and thus restrict the growing mode from unstable perturbation.

In the vortex sheet configuration, under a Galilean boost and an appropriate scaling, one may consider the constant background state to be such that the velocity $\mathbf{u} = (u, 0, 0)$, and the third row of $\mathbf{F}$ to be zero. In the spectral analysis, we find through a detailed examination of the Lopatinskiĭ determinant (cf. Lemma 3.3) that the validity of the

Lopatinskiĭ condition relies on the competition between the projected fluid velocity and the projected "elastic" sound speed (see (3.27))

$$|\mathbf{u}\cdot\mathbf{s}|^2 \quad \text{v.s.} \quad \sum_{j=1}^{3}|\mathbf{F}_j\cdot\mathbf{s}|^2,$$

where $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ are column vectors of $\mathbf{F}$ and $\mathbf{s}$ is a unit vector indicating the direction of projection. The analysis in the two-dimensional situation [38] seems to suggest two stability regions

$$|\mathbf{u}\cdot\mathbf{s}|^2 \leq \sum_{j=1}^{3}|\mathbf{F}_j\cdot\mathbf{s}|^2 \text{ (subsonic)} \quad \text{and} \quad |\mathbf{u}\cdot\mathbf{s}|^2 \geq \sum_{j=1}^{3}|\mathbf{F}_j\cdot\mathbf{s}|^2 + 2c^2 \text{ (supersonic)} \tag{1.2}$$

where $c = \sqrt{p'(\rho)}$ is the standard sound speed, and the region in between indicating instability. However, in three spatial dimensions, one needs to verify (1.2) along all directions $\mathbf{s}$ in order to obtain the stability. It is possible that the subsonic region might degenerate along a certain direction $\mathbf{s}$, and the supersonic threshold may blow up to infinity. It turns out that the latter case always happens, and hence the elasticity stabilization can only take place in the subsonic zone, depending crucially on the *geometric* property of the deformation gradient $\mathbf{F}$. In fact, a necessary and sufficient condition for the generation of the stable subsonic region is

$$\exists\ i,j \in \{1,2,3\},\ i \neq j \quad \text{such that} \quad \mathbf{F}_i \times \mathbf{F}_j \neq 0. \tag{1.3}$$

Or, in terms of the row vectors $\mathrm{F}_1, \mathrm{F}_2$ of $\mathbf{F}$ (recall that the third row of $\mathbf{F}$ is zero),

$$\mathrm{F}_1 \times \mathrm{F}_2 \neq 0, \quad \text{or, equivalently,} \quad \mathrm{rank}\mathbf{F} = 2, \tag{1.4}$$

cf. Theorem 3.1 (i), (ii). This is in sharp contrast to the case of two-dimensional elastic vortex sheets, where a stable supersonic region exists [38], and is consistent with the case of three-dimensional Euler vortex sheets [40]; see Remark 3.5.

To finally close the estimates for stability in the subsonic regime, we follow the upper triangularization method of [38] to separate only the outgoing modes from the system at all points in the Fourier space. This allows us to conveniently conclude the triviality of the outgoing modes in the homogeneous system, and the estimate for the incoming modes can be derived directly from the Lopatinskiĭ determinant. Similar to the two-dimensional case, there exist a special class of states within the stable subsonic region where the Lopatinskiĭ determinant exhibits higher order of degeneracy at such states. This results in a weaker stability at those states in the sense that there is an additional loss of tangential derivatives.

The rest of the paper is organized as follows. In Section 2, we present the mathematical formulation for the three-dimensional vortex sheets and introduce some weighted Sobolev spaces. In Section 3, we first introduce the boundary-fixing transformation, and linearize the system around a given constant solution. Motivated by [43], the formulation of boundary conditions we derive is different from the one in [38]. In Section 3.1, we state our main result on the stability and instability criteria, and the energy estimates of solutions to the linearized problem are obtained. In Section 3.2, we perform some preliminary reductions to transform the problem into a system of ordinary differential equations. By decomposing the system we find that the linearized problem has different boundary conditions from the two-dimensional unsteady elastic flows. We further derive an estimate for the front with an order-one degeneracy. In Section 3.3, we consider the normal mode analysis. In Section 3.4, we perform the upper triangularization technique of the system in the spirit of [38] to separate only the outgoing modes from the system at all points in the frequency space. In Section 3.5, we present a delicate analysis of the Lopatinskiĭ determinant and derive the estimates on the neighborhood of the zeros of the Lopatinskiĭ determinant. In Section 3.6, we separate different modes and the estimates from the Lopatinskiĭ determinant to achieve the energy estimates and then complete the proof of the main theorem.

2. Formulation And Notation

In this section, we will first present the derivation of the vortex sheet problem from the elastodynamic equations (1.1), and then introduce some weighted Sobolev spaces which will be used in our stability estimates.

2.1. **Statement for the vortex sheet problem.** Let us recall the definition of vortex sheet solutions to (1.1). Let $U(t,x_1,x_2,x_3)=(\rho,\mathbf{u},\mathbf{F})(t,x_1,x_2,x_3)$ be a solution to system (1.1) which is piecewise smooth on the both sides of a smooth hypersurface

$$\Gamma=\{x_3=\psi(t,x_1,x_2)\}.$$

Denote $\partial_i=\partial_{x_i}, i=1,2,3$, for the partial derivatives, normal vector $\nu=(-\partial_1\psi,-\partial_2\psi,1)$ on Γ and

$$U(t,x_1,x_2,x_3)=\begin{cases}U^+(t,x_1,x_2,x_3), & \text{when } x_3>\psi(t,x_1,x_2),\\ U^-(t,x_1,x_2,x_3), & \text{when } x_3<\psi(t,x_1,x_2),\end{cases} \tag{2.1}$$

where $U^\pm=(\rho^\pm,\mathbf{u}^\pm,\mathbf{F}^\pm)(t,x_1,x_2,x_3)$. The solution U satisfies the Rankine-Hugoniot jump relations at each point on Γ :

$$\begin{aligned}&\partial_t\psi[\rho]-[\rho\mathbf{u}\cdot\nu]=0,\\ &\partial_t\psi[\rho\mathbf{u}]-[(\rho\mathbf{u}\cdot\nu)\mathbf{u}]-[p]\nu+[\rho\mathbf{F}\mathbf{F}^T\nu]=0,\\ &\partial_t\psi[\rho\mathbf{F}_j]-[(\mathbf{u}\cdot\nu)\rho\mathbf{F}_j]+[(\rho\mathbf{F}_j\cdot\nu)\mathbf{u}]=0,\end{aligned} \tag{2.2}$$

where we write $[f]$ as the jump of the quantity f crossing the hypersurface Γ. For a vortex sheet (contact discontinuity), we require

$$[\mathbf{u}\cdot\nu]=0,\ [\mathbf{u}]\neq 0 \text{ and } \psi_t=\mathbf{u}^\pm\cdot\nu\Big|_\Gamma. \tag{2.3}$$

The first condition in (2.2) is automatically satisfied. Combining the remaining two conditions in (2.2), we obtain

$$-[p]\nu+[\rho\mathbf{F}\mathbf{F}^T\nu]=0, \tag{2.4}$$

$$[(\rho\mathbf{F}_j\cdot\nu)\mathbf{u}]=0. \tag{2.5}$$

From (2.3) and (2.5) we derive that $\partial_t\psi[\rho\mathbf{F}_j\cdot\nu]=0$. Since $\partial_t\psi\neq 0$, we get $[\rho\mathbf{F}_j\cdot\nu]=0$, and then from (2.5) we further have $\rho\mathbf{F}_j^\pm\cdot\nu=0$. Then (2.4) infers that $[p]=0$. Therefore the jump conditions reduce to

$$\rho^+=\rho^-,\quad \psi_t=\mathbf{u}^+\cdot\nu=\mathbf{u}^-\cdot\nu. \tag{2.6}$$

To flatten and fix the free boundary Γ, we need to introduce the function $\Phi(t,x_1,x_2,x_3)$ to set the variable transformation $\Phi^\pm(t,x_1,x_2,x_3)$ as follows. We first consider the class of functions $\Phi(t,x_1,x_2,x_3)$ such that $\inf\{\partial_3\Phi\}>0$, and $\Phi(t,x_1,x_2,0)=\psi(t,x_1,x_2)$. Then we define

$$U_\sharp^\pm=(\rho_\sharp^\pm,\mathbf{u}_\sharp^\pm,\mathbf{F}_\sharp^\pm)(t,x_1,x_2,x_3):=(\rho,\mathbf{u},\mathbf{F})(t,x_1,x_2,\Phi(t,x_1,x_2,\pm x_3)),$$

for $x_3\geq 0$. In the following argument, we drop the index $\sharp$ for notation simplicity. Define $\Phi^\pm(t,x_1,x_2,x_3);=\Phi(t,x_1,x_2,\pm x_3)$. Inspired by [7, 13], it is natural to require $\Phi^\pm$ satisfying the eikonal equation

$$\partial_t\Phi^\pm+u^\pm\partial_1\Phi^\pm+v^\pm\partial_2\Phi^\pm-w^\pm=0,$$

for $x_3\geq 0$. This condition simplifies the expression of the nonlinear problem in the fixed domain and guarantees the constant rank property of boundary matrix in the whole domain. Through this variable transformation, equations (1.1) become

$$\begin{aligned}&\partial_tU^\pm+A_1(U^\pm)\partial_1U^\pm+A_2(U^\pm)\partial_2U^\pm\\&\quad+\frac{1}{\partial_3\Phi^\pm}[A_3(U^\pm)-\partial_t\Phi^\pm I-\partial_1\Phi^\pm A_1(U^\pm)-\partial_2\Phi^\pm A_2(U^\pm)]\partial_3U^\pm=0,\end{aligned}\tag{2.7}$$

for $x_3>0$ with free boundary $x_3=0$, where

$$A_1(U):=\begin{bmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{p'}{\rho} & u & 0 & 0 & -F_{11} & 0 & 0 & -F_{12} & 0 & 0 & -F_{13} & 0 & 0\\ 0 & 0 & u & 0 & 0 & -F_{11} & 0 & 0 & -F_{12} & 0 & 0 & -F_{13} & 0\\ 0 & 0 & 0 & u & 0 & 0 & -F_{11} & 0 & 0 & -F_{12} & 0 & 0 & -F_{13}\\ 0 & -F_{11} & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -F_{11} & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -F_{11} & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -F_{12} & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -F_{12} & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -F_{12} & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0\\ 0 & -F_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0\\ 0 & 0 & -F_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0\\ 0 & 0 & 0 & -F_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u \end{bmatrix},$$

$$A_2(U):=\begin{bmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & v & 0 & 0 & -F_{21} & 0 & 0 & -F_{22} & 0 & 0 & -F_{23} & 0 & 0\\ \frac{p'}{\rho} & 0 & v & 0 & 0 & -F_{21} & 0 & 0 & -F_{22} & 0 & 0 & -F_{23} & 0\\ 0 & 0 & 0 & v & 0 & 0 & -F_{21} & 0 & 0 & -F_{22} & 0 & 0 & -F_{23}\\ 0 & -F_{21} & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -F_{21} & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -F_{21} & 0 & 0 & v & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -F_{22} & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -F_{22} & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -F_{22} & 0 & 0 & 0 & 0 & 0 & v & 0 & 0 & 0\\ 0 & -F_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0 & 0\\ 0 & 0 & -F_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & 0\\ 0 & 0 & 0 & -F_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix},\tag{2.8}$$

$$\text{and }A_3(U):=\begin{bmatrix} w & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & w & 0 & 0 & -F_{31} & 0 & 0 & -F_{32} & 0 & 0 & -F_{33} & 0 & 0\\ 0 & 0 & w & 0 & 0 & -F_{31} & 0 & 0 & -F_{32} & 0 & 0 & -F_{33} & 0\\ \frac{p'}{\rho} & 0 & 0 & w & 0 & 0 & -F_{31} & 0 & 0 & -F_{32} & 0 & 0 & -F_{33}\\ 0 & -F_{31} & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -F_{31} & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -F_{31} & 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -F_{32} & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -F_{32} & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -F_{32} & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0\\ 0 & -F_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0\\ 0 & 0 & -F_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0\\ 0 & 0 & 0 & -F_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{bmatrix}.$$

We can write

$$\begin{cases}\mathcal{L}(U^{\pm},\Phi^{\pm})=0, & \text{if } x_3>0,\\ \mathcal{B}(U^{\pm},\psi)=0, & \text{if } x_3=0,\\ (U^{\pm},\psi)|_{t=0}=(U_0^{\pm},\psi_0),\end{cases}$$

where

$$\begin{aligned}&\mathcal{L}(U,\Phi)=L(U,\Phi)U,\\ &L(U,\Phi):=\partial_t+A_1(U)\partial_1+A_2(U)\partial_2+\tilde{A}_3(U,\Phi)\partial_3,\\ &\tilde{A}_3(U,\Phi)=\frac{1}{\partial_3\Phi}[A_3(U)-\partial_t\Phi I-\partial_1\Phi A_1(U)-\partial_2\Phi A_2(U)],\end{aligned}\tag{2.9}$$

$$\mathcal{B}(U^{\pm},\psi)=\begin{bmatrix}(u^+-u^-)\partial_1\psi+(v^+-v^-)\partial_2\psi-(w^+-w^-)\\ \partial_t\psi+u^+\partial_1\psi+v^+\partial_2\psi-w^+\\ \rho^+-\rho^-\end{bmatrix}.\tag{2.10}$$

Remark 2.1. Note that by taking divergence of the third equations in (1.1), we end up with

$$\partial_t(\operatorname{div}(\rho\mathbf{F}_j))=0,\ \text{ for } j=1,2,3.$$

In column-wise components, we can write the intrinsic property (involution condition for the elastic flow, refer to [10]) as follows:

$$\operatorname{div}(\rho\mathbf{F}_j)=0,\ \text{ for } j=1,2,3.\tag{2.11}$$

The intrinsic property holds at any time throughout the flow if it is initially satisfied. In the discussion of derivation of Rankine Hugoniot condition, $\rho\mathbf{F}_j^{\pm}\cdot\nu=0$ can also be regarded as an intrinsic property.

From (2.6), the elastic components should satisfy the following equations, which are regarded as the restrictions on the initial data. We remark that if initially $\mathbf{F}_0^{\pm}\cdot\nu_0=0$, then $\mathbf{F}^{\pm}\cdot\nu=0$, since it satisfies the transport equation. Therefore, the following equations are satisfied naturally,

$$\begin{cases}(F_{11}^+-F_{11}^-)\partial_1\psi+(F_{21}^+-F_{21}^-)\partial_2\psi-(F_{31}^+-F_{31}^-)=0,\\ F_{11}^+\partial_1\psi+F_{21}^+\partial_2\psi-F_{31}^+=0,\\ (F_{12}^+-F_{12}^-)\partial_1\psi+(F_{22}^+-F_{22}^-)\partial_2\psi-(F_{32}^+-F_{32}^-)=0,\\ F_{12}^+\partial_1\psi+F_{22}^+\partial_2\psi-F_{32}^+=0,\\ (F_{13}^+-F_{13}^-)\partial_1\psi+(F_{23}^+-F_{23}^-)\partial_2\psi-(F_{33}^+-F_{33}^-)=0,\\ F_{13}^+\partial_1\psi+F_{23}^+\partial_2\psi-F_{33}^+=0,\end{cases}\tag{2.12}$$

where $\Phi^{\pm}=\psi,$ at $x_3=0.$

Remark 2.2. It is easy to check that the system (2.7) contains piecewise constant planar solutions.

Remark 2.3. The boundary matrix for the problem (2.9) is $\operatorname{diag}(\tilde{A}_3(U^+,\Phi^+),\tilde{A}_3(U^-,\Phi^-))$, which has constant rank on the whole closed half space $x_3\geq 0$. This matrix has two positive and two negative eigenvalues, and the remaining are zero eigenvalues. The boundary $x_3=0$ is characteristic and since one of the boundary conditions is needed to determine the function ψ, there should be three boundary conditions, see (2.10).

2.2. **Function spaces.** Now we introduce some necessary functional spaces, i.e., weighted Sobolev spaces in preparation for our main theorem. Let $\mathcal{D}'$ denote the distributions and define

$$H^s_\gamma(\mathbb{R}^3) := \{u(t,x_1,x_2) \in \mathcal{D}'(\mathbb{R}^3) : e^{-\gamma t}u(t,x_1,x_2) \in H^s(\mathbb{R}^3)\},$$
$$H^s_\gamma(\mathbb{R}^4_+) := \{v(t,x_1,x_2,x_3) \in \mathcal{D}'(\mathbb{R}^4_+) : e^{-\gamma t}v(t,x_1,x_2,x_3) \in H^s(\mathbb{R}^4_+)\},$$

for $s \in \mathbb{R}, \gamma \geq 1$, with equivalent norms

$$||u||_{H^s_\gamma(\mathbb{R}^3)} := ||e^{-\gamma t}u||_{H^s(\mathbb{R}^3)}, \quad ||v||_{H^s_\gamma(\mathbb{R}^4_+)} := ||e^{-\gamma t}v||_{H^s(\mathbb{R}^4_+)},$$

respectively, where

$$\mathbb{R}^4_+ := \{(t,x_1,x_2,x_3) \in \mathbb{R}^4 : x_3 > 0\}.$$

We define the norm

$$||u||^2_{s,\gamma} := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(\gamma^2+|\xi|^2)^s|\widehat{u}(\xi)|^2 d\xi, \ \forall u \in H^s(\mathbb{R}^3),$$

with $\widehat{u}(\xi)$ being the Fourier transform of u with respect to (t,x_2,x_3). Setting $\tilde{u} = e^{-\gamma t}u$, we see that $||u||_{H^s_\gamma(\mathbb{R}^3)}$ and $||\tilde{u}||_{s,\gamma}$ are equivalent, denoted by $||u||_{H^s_\gamma(\mathbb{R}^3)} \simeq ||\tilde{u}||_{s,\gamma}$. Now, we can define the space $L^2(\mathbb{R}_+; H^s_\gamma(\mathbb{R}^3))$, endowed with the norm

$$|||v|||^2_{L^2(H^s_\gamma)} := \int_0^{+\infty} ||v(\cdot,x_3)||^2_{H^s_\gamma(\mathbb{R}^3)}dx_3.$$

We also have

$$|||v|||^2_{L^2(H^s_\gamma)} \simeq |||\tilde{v}|||^2_{s,\gamma} := \int_0^{+\infty} ||\tilde{v}(\cdot,x_3)||^2_{s,\gamma}dx_3.$$

It is easy to see that when $s=0$, $||\cdot||_{0,\gamma} = ||\cdot||_{L^2(\mathbb{R}^3)}$ and $|||\cdot|||_{0,\gamma}$ is the usual norm of $L^2(\mathbb{R}^4_+)$.

3. Linear Stability

The goal of this section is to study the linear stability of the three-dimensional vortex sheets in elastodynamics with the initial data around a constant background state given in (3.1). Sufficient and necessary conditions of weak stability and violent instability conditions are obtained in Theorem 3.1 and Theorem 3.2, motivated by the approach proposed by Coulombel–Secchi [7] and the upper triangulation method introduced in Chen–Hu–Wang [38], both of which rely on a delicate spectral analysis on constant coefficient problems. We want to emphasize that the stabilization phenomenon only occurs in a subsonic bubble, which is in stark contrast with the two-dimensional elastic case [38] where the vortex sheets are also stable in a supersonic region.

Note from Remark 2.2 that (2.7) admits piecewise constant solutions. Under a Galilean transformation and the change of the scale of measurement, without loss of generality we may assume that the piecewise constant background solution takes the following form:

$$\begin{aligned}
&\bar{U}^+ := (\bar{\rho}, u^r, 0, 0, F^r_{11}, F^r_{21}, 0, F^r_{12}, F^r_{22}, 0, F^r_{13}, F^r_{23}, 0)^T,\\
&\bar{U}^- := (\bar{\rho}, u^l, 0, 0, F^l_{11}, F^l_{21}, 0, F^l_{12}, F^l_{22}, 0, F^l_{13}, F^l_{23}, 0)^T,\\
&\bar{\Phi}^\pm(t,x_1,x_2,x_3) := \pm x_3,
\end{aligned} \tag{3.1}$$

where the constants $\bar{\rho}, u^r, u^l, F_{ij}^r, F_{ij}^l, i \in \{1,2\}, j \in \{1,2,3\}$ satisfy

$$u^r + u^l = 0, \quad F_{ij}^r + F_{ij}^l = 0, \quad \text{and} \quad u^r, F_{ij}^r \neq 0.$$

Remark 3.1. For the background solution, we assume the second and third direction of the velocities to be zero by using the Galilean transformation and rigid transformation. Compared with [44] and [46] for steady flows, this simplifies the linear constant coefficient analysis. As shown in [44], when the tangential velocities are parallel, the planar contact discontinuity is always linearly unstable. Therefore, in [46], only the case of non-parallelled tangential velocities is considered. In [6], a simple criterion predicting neutral stability or violent instability for two- or three-dimensional nonisentropic Euler equations is provided. Here in our paper, the tangential velocities are parallel and new stable zone occurs that is different from the steady three-dimensional flows [44] and [46].

Next, we linearize the system (2.7)-(2.10) around the background solution defined by (3.1). Let

$$\dot{U}^\pm = (\dot{\rho}^\pm, \dot{\mathbf{u}}^\pm, \dot{\mathrm{F}}^\pm) = U^\pm - \bar{U}^\pm, \quad \dot{\Phi}^\pm = \Phi^\pm - \bar{\Phi}^\pm$$

be some small perturbation of the constant solution. Then the perturbed linearized quantities satisfy:

$$\partial_t \dot{U}^\pm + A_1(\bar{U}^\pm)\partial_1 \dot{U}^\pm + A_2(\bar{U}^\pm)\partial_2 \dot{U}^\pm \pm A_3(\bar{U}^\pm)\partial_3 \dot{U}^\pm = 0,$$

in $x_3 > 0$, with the boundary condition at $x_3 = 0$:

$$\begin{cases} (u^r - u^l)\partial_1\varphi - (\dot{w}^+ - \dot{w}^-) = 0; \\ \partial_t\varphi + u^r\partial_1\varphi - \dot{w}^+ = 0; \\ \dot{\rho}^+ = \dot{\rho}^-, \end{cases}$$

where $\varphi = (\Phi^\pm - \dot{\Phi}^\pm)|_{x_3=0} = \psi$ at $x_3 = 0$. Therefore, we have

$$\begin{cases} \mathcal{L}'\dot{U} = 0, & \text{if } x_3 > 0, \\ \mathcal{B}(\dot{U}, \varphi) = 0, & \text{if } x_3 = 0, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \mathcal{L}'\dot{U} = \partial_t \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix} &+ \begin{bmatrix} A_1(\bar{U}^+) & 0 \\ 0 & A_1(\bar{U}^-) \end{bmatrix} \partial_1 \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix} \\ &+ \begin{bmatrix} A_2(\bar{U}^+) & 0 \\ 0 & A_2(\bar{U}^-) \end{bmatrix} \partial_2 \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix} + \begin{bmatrix} A_3(\bar{U}^+) & 0 \\ 0 & -A_3(\bar{U}^-) \end{bmatrix} \partial_3 \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix}, \end{aligned}$$

$$\mathcal{B}(\dot{U}, \varphi) = \begin{bmatrix} (u^r - u^l)\partial_1\varphi - (\dot{w}^+ - \dot{w}^-) \\ \partial_t\varphi + u^r\partial_1\varphi - \dot{w}^+ \\ \dot{\rho}^+ - \dot{\rho}^- \end{bmatrix}.$$

Next we need to symmetrize the system (3.2). Here, we consider the change of variables as follows,

$$W = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \dot{U}^+ \\ \dot{U}^- \end{bmatrix}, \tag{3.3}$$

where T is the given matrix:

$$T = \begin{bmatrix} -\frac{1}{2\bar{\rho}} & 0 & 0 & \frac{1}{2c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2\bar{\rho}} & 0 & 0 & \frac{1}{2c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.4}$$

$c = \sqrt{p'(\bar{\rho})}$ stands for the local sound speed of constant solutions. Denote the components of the new variable by

$$W = (W_1, W_2, \cdots, W_{26})^T,$$

and

$$\begin{aligned} &W^{tan} = (W_3, W_4, W_5, W_6, W_8, W_9, W_{11}, W_{12}, W_{16}, W_{17}, W_{18}, W_{19}, W_{21}, W_{22}, W_{24}, W_{25})^T, \\ &W^n = (W_1, W_2, W_7, W_{10}, W_{13}, W_{14}, W_{15}, W_{20}, W_{23}, W_{26})^T, \\ &W^c = (W_3, W_4, W_5, \cdots, W_{13}, W_{16}, W_{17}, W_{18}, \cdots, W_{26})^T, \\ &W^{nc} = (W_1, W_2, W_{14}, W_{15})^T. \end{aligned} \tag{3.5}$$

After performing variable transformation, we multiply the system (3.2) by a symmetrizer

$$\mathcal{A}_0 = \mathrm{diag}\{2c^2, 2c^2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2c^2, 2c^2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}.$$

Then, we obtain that

$$\begin{cases} \mathcal{L}W := \mathcal{A}_0 \partial_t W + \mathcal{A}_1 \partial_1 W + \mathcal{A}_2 \partial_2 W \pm \mathcal{A}_3 \partial_3 W = 0, \quad x_3 > 0, \\ \mathcal{B}(W^{nc}, \varphi) := \underline{M} W^{nc}|_{x_3=0} + \underline{b} \begin{bmatrix} \partial_t \varphi \\ \partial_1 \varphi \\ \partial_2 \varphi \end{bmatrix} = 0, \end{cases} \tag{3.6}$$

where

$$\mathcal{A}_i = \begin{bmatrix} \mathcal{A}_i^r & 0 \\ 0 & \mathcal{A}_i^l \end{bmatrix}, \quad i = 1, 2,$$

$$\mathcal{A}_1^{r,l} = \begin{bmatrix}
2c^2u^{r,l} & 0 & -c^2 & 0 & 0 & 0 & -cF_{11}^{r,l} & 0 & 0 & -cF_{12}^{r,l} & 0 & 0 & -cF_{13}^{r,l} \\
0 & 2c^2u^{r,l} & c^2 & 0 & 0 & 0 & -cF_{11}^{r,l} & 0 & 0 & -cF_{12}^{r,l} & 0 & 0 & -cF_{13}^{r,l} \\
-c^2 & c^2 & u^{r,l} & 0 & -F_{11}^{r,l} & 0 & 0 & -F_{12}^{r,l} & 0 & 0 & -F_{13}^{r,l} & 0 & 0 \\
0 & 0 & 0 & u^{r,l} & 0 & -F_{11}^{r,l} & 0 & 0 & -F_{12}^{r,l} & 0 & 0 & -F_{13}^{r,l} & 0 \\
0 & 0 & -F_{11}^{r,l} & 0 & u^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -F_{11}^{r,l} & 0 & u^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-cF_{11}^{r,l} & -cF_{11}^{r,l} & 0 & 0 & 0 & 0 & u^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -F_{12}^{r,l} & 0 & 0 & 0 & 0 & u^{r,l} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -F_{12}^{r,l} & 0 & 0 & 0 & 0 & u^{r,l} & 0 & 0 & 0 & 0 \\
-cF_{12}^{r,l} & -cF_{12}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^{r,l} & 0 & 0 & 0 \\
0 & 0 & -F_{13}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^{r,l} & 0 & 0 \\
0 & 0 & 0 & -F_{13}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^{r,l} & 0 \\
-cF_{13}^{r,l} & -cF_{13}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^{r,l}
\end{bmatrix},$$

$$\mathcal{A}_2^{r,l} = \begin{bmatrix}
0 & 0 & 0 & -c^2 & 0 & 0 & -cF_{21}^{r,l} & 0 & 0 & -cF_{22}^{r,l} & 0 & 0 & -cF_{23}^{r,l} \\
0 & 0 & 0 & c^2 & 0 & 0 & -cF_{21}^{r,l} & 0 & 0 & -cF_{22}^{r,l} & 0 & 0 & -cF_{23}^{r,l} \\
0 & 0 & 0 & 0 & -F_{21}^{r,l} & 0 & 0 & -F_{22}^{r,l} & 0 & 0 & -F_{23}^{r,l} & 0 & 0 \\
-c^2 & c^2 & 0 & 0 & 0 & -F_{21}^{r,l} & 0 & 0 & -F_{22}^{r,l} & 0 & 0 & -F_{23}^{r,l} & 0 \\
0 & 0 & -F_{21}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -F_{21}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-cF_{21}^{r,l} & -cF_{21}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -F_{22}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -F_{22}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-cF_{22}^{r,l} & -cF_{22}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -F_{23}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -F_{23}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-cF_{23}^{r,l} & -cF_{23}^{r,l} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$\mathcal{A}_3 = \operatorname{diag}\{-2c^3, 2c^3, 0,0,0,0,0,0,0,0,0,0,0, 2c^3, -2c^3, 0,0,0,0,0,0,0,0,0,0,0\},$$

and

$$\underline{M} = \begin{bmatrix} -c & -c & c & c \\ -c & -c & 0 & 0 \\ -1 & -1 & 1 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 & 2u^r & 0 \\ 1 & u^r & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.1. **Main result.** For $j = 1, 2, 3$ we denote

$$\mathrm{F}_j := \text{ the } j\text{th row of the deformation matrix } \mathbf{F}^r. \tag{3.7}$$

From (3.1) we know that $\mathrm{F}_3 = 0$. We further define the vector projections (see Fig. 1)

$$\begin{aligned} \Pi_b(a) &:= \text{ the parallel projection of } a \text{ onto } b, \\ \Pi_b^{\perp}(a) &:= a - \Pi_b(a) = \text{ the perpendicular projection of } a \text{ onto } b. \end{aligned} \tag{3.8}$$

Now we state our main result.

Theorem 3.1. *(i) Assume that the background solution defined by* (3.1) *satisfies* $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$. *If*

$$0 < (u^r)^2 < \mathcal{F}(\mathrm{F}_1, \mathrm{F}_2), \tag{3.9}$$

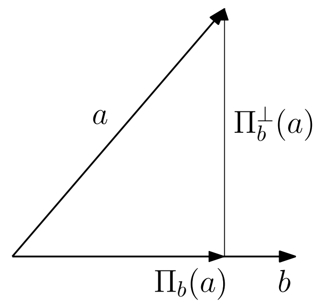

FIGURE 1. Vector projections

where $\mathcal{F}(\mathrm{F}_1, \mathrm{F}_2)$ *is defined in* (3.54), *then there is a positive constant* C *such that for all* $\gamma > 1, W \in H_\gamma^3(\mathbb{R}_+^4)$ *and* $\varphi \in H_\gamma^3(\mathbb{R}^3)$, *the following estimate holds:*

$$\begin{aligned}&\gamma|||W|||^2_{L^2(H^0_\gamma)} + ||W^{nc}|_{x_3=0}||^2_{0,\gamma} + ||\varphi||^2_{0,\gamma}\\ &\quad \leq C\Big(\frac{1}{\gamma^5}|||\mathcal{L}^\gamma W|||^2_{L^2(H^2_\gamma)} + \frac{1}{\gamma^4}||\mathcal{B}^\gamma(W^{nc}|_{x_3=0},\varphi)||^2_{H^2_\gamma(\mathbb{R}^3)}\Big).\end{aligned} \tag{3.10}$$

(ii) Assume that the background solution defined by (3.1) *satisfies* $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$. *If*

$$\mathcal{F}(\mathrm{F}_1, \mathrm{F}_2) \leq (u^r)^2 \leq \left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2 \tag{3.11}$$

then there is a positive constant C *such that for all* $\gamma > 1, W \in H_\gamma^4(\mathbb{R}_+^4)$ *and* $\varphi \in H_\gamma^4(\mathbb{R}^3)$, *the following estimate holds:*

$$\begin{aligned}&\gamma|||W|||^2_{L^2(H^0_\gamma)} + ||W^{nc}|_{x_3=0}||^2_{0,\gamma} + ||\varphi||^2_{0,\gamma}\\ &\quad \leq C\Big(\frac{1}{\gamma^7}|||\mathcal{L}^\gamma W|||^2_{L^2(H^3_\gamma)} + \frac{1}{\gamma^6}||\mathcal{B}^\gamma(W^{nc}|_{x_3=0},\varphi)||^2_{H^3_\gamma(\mathbb{R}^3)}\Big).\end{aligned} \tag{3.12}$$

(iii) Assume that the background solution defined by (3.1) *satisfies*

$$(u^r)^2 > \left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2, \tag{3.13}$$

then the constant vortex sheet solutions (3.1) *are linearly unstable.*

Remark 3.2. Case (i) and Case (ii) provide the linear stability of the background solution (3.1). The linear instability in Case (iii) is understood in the sense that the Lopatinskiĭ condition is violated.

Remark 3.3. The function $\mathcal{F}(\mathrm{F}_1, \mathrm{F}_2)$ is defined in (3.54), but its explicit expression is very complicated and thus not provided. On the other hand, a rough bound is given by (see (3.55))

$$\frac{\left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2}{4} \leq \mathcal{F}(\mathrm{F}_1, \mathrm{F}_2) \leq \frac{\left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2}{2}.$$

Therefore the results in Case (i) and Case (ii) confirm the stabilization from elasticity. More precisely, from the fact that

$$\mathcal{F}(\mathrm{F}_1, \mathrm{F}_2) = 0 \quad \Longleftrightarrow \quad \mathrm{F}_1 \times \mathrm{F}_2 = 0 \ \ (\text{or } \mathrm{F}_1 /\!/ \mathrm{F}_2) \quad \Longleftrightarrow \quad \Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1) = 0,$$

we see that the geometric property $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$ gives a sufficient condition for stability. Together with (3.13) in Case (iii) we further conclude that such a geometric condition is also necessary for stability. Moreover we see that the elastic stabilization is more pronounced in the sense that the critical sonic speed $\left|\Pi_{\mathrm{F}_2}^{\perp}(\mathrm{F}_1)\right|^2$ increases as F_1 and F_2 are closer to being orthogonal.

Remark 3.4. In Case (ii) we see that one only has a weak stability property (3.12) for velocity ranging in an interval defined by (3.11) rather than at some discrete points as in the two-dimensional case [38]. This stems from the stronger degeneracy of the Lopatinskiĭ determinant due to the increased spatial dimension.

Remark 3.5. From Case (iii) we see that for the three-dimensional compressible elastic vortex sheets there is only one stable region where the velocity is subsonic. This is very different from the two-dimensional situation where elasticity can also produce a supersonic stable region. Similar to Case (ii), this loss of stability is due to the the increased spatial dimension, which can potentially host more unstable directions.

Remark 3.6. From the non-parallel condition $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$ it follows that at the free boundary, $\mathrm{rank}\mathbf{F}^{r,l} = 2$. Interestingly, such a geometric condition also appears as a sufficient condition for stability in the study of a single-phase compressible elastodynamics free boundary problem [43]. However another stabilization criterion in the case when the non-parallel condition fails can be obtained in the form of the Rayleigh-Taylor sign condition. We want to point out that our free boundary problem is different from that of [43], and it is the different boundary conditions that allow us to further infer the necessity of the non-parallel condition for stability.

Remark 3.7. The subsonic condition $(u^r)^2 \leq \left|\Pi_{\mathrm{F}_2}^{\perp}(\mathrm{F}_1)\right|^2$ ensures that the projected fluid velocity is below the projected "elastic" sound speed (i.e. the first inequality in (1.2)) along any direction. This is consistent with the two-dimensional case [38], where $\vec{s}$ becomes a scalar, $\mathbf{F} \in \mathbf{M}^{2\times 2}$ and $\mathrm{F}_2 = 0$ in (1.2). To recover the 2D stable supersonic region in [38] from Theorem 3.1 is not so straightforward. See Remark 3.14 for a more detailed discussion.

Remark 3.8. Recall that (see, for example, [40]) three-dimensional compressible vortex sheets are violently unstable. This can be recovered from our result by taking $\mathbf{F} = 0$.

Remark 3.9. This paper focuses on the linear stability with constant coefficients of the three-dimensional elastic vortex sheets. The nonlinear stability is more challenging and will be addressed in future works.

Now, we perform the following transformation and simplification in our proof of Theorem 3.1.

$$\tilde{W} = e^{-\gamma t} W, \quad \tilde{\varphi} = e^{-\gamma t}\varphi,$$

with $\gamma > 1$. Introducing two new "γ-dependent" operators $\mathcal{L}^\gamma$ and $\mathcal{B}^\gamma$ by

$$\mathcal{L}^\gamma \tilde{W} = e^{-\gamma t}\mathcal{L}W = \gamma\mathcal{A}_0\tilde{W} + \mathcal{A}_0\partial_t\tilde{W} + \mathcal{A}_1\partial_1\tilde{W} + \mathcal{A}_2\partial_2\tilde{W} + \mathcal{A}_3\partial_3\tilde{W},$$

$$\mathcal{B}^\gamma(\tilde{W}^{nc}, \tilde{\varphi}) = e^{-\gamma t}\mathcal{B}(W^{nc}, \varphi) = \underline{M}\tilde{W}^{nc} + \underline{b}\begin{bmatrix} \gamma\tilde{\varphi} + \partial_t\tilde{\varphi} \\ \partial_1\tilde{\varphi} \\ \partial_2\tilde{\varphi} \end{bmatrix}.$$

Then, we have $|||e^{-\gamma t}v|||_{s,\gamma} \simeq |||v|||_{L^2(H^s_\gamma)}$ and $||e^{-\gamma t}u||_{s,\gamma} \simeq ||u||_{H^s_\gamma}$. A direct consequence of Theorem 3.1 is the following theorem.

Theorem 3.2. *(i) Assume that the background solution defined by* (3.1) *satisfies that* $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$*. If* (3.9) *is satisfied, then there is a positive constant* C *such that for all* $\gamma > 1, \tilde{W} \in H^3_\gamma(\mathbb{R}^4_+)$ *and* $\tilde{\varphi} \in H^3_\gamma(\mathbb{R}^3)$*, the following estimate holds:*

$$\begin{aligned}&\gamma|||\tilde{W}|||^2_{0,\gamma} + ||\tilde{W}^{nc}|_{x_3=0}||^2_{0,\gamma} + ||\tilde{\varphi}||^2_{0,\gamma} \\ &\leq C\Big(\frac{1}{\gamma^5}|||\mathcal{L}^\gamma \tilde{W}|||^2_{2,\gamma} + \frac{1}{\gamma^4}||\mathcal{B}^\gamma(\tilde{W}^{nc}|_{x_3=0}, \tilde{\varphi})||^2_{2,\gamma}\Big).\end{aligned} \tag{3.14}$$

(ii) Assume that the background solution defined by (3.1) *satisfies that* $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$*. If* (3.11) *is true, then there is a positive constant* C *such that for all* $\gamma > 1, \tilde{W} \in H^4_\gamma(\mathbb{R}^4_+)$ *and* $\tilde{\varphi} \in H^4_\gamma(\mathbb{R}^3)$*, the following estimate holds:*

$$\begin{aligned}&\gamma|||\tilde{W}|||^2_{0,\gamma} + ||\tilde{W}^{nc}|_{x_3=0}||^2_{0,\gamma} + ||\tilde{\varphi}||^2_{0,\gamma} \\ &\leq C\Big(\frac{1}{\gamma^7}|||\mathcal{L}^\gamma \tilde{W}|||^2_{3,\gamma} + \frac{1}{\gamma^6}||\mathcal{B}^\gamma(\tilde{W}^{nc}|_{x_3=0}, \tilde{\varphi})||^2_{3,\gamma}\Big).\end{aligned} \tag{3.15}$$

(iii) Assume that the background solution defined by (3.1) *satisfies* (3.13)*, then the constant vortex sheet solutions* (3.1) *are linearly unstable.*

3.2. **Partial homogenization of the system and front elimination.** In this section, we perform certain transformation and simplification to eliminate the unknown wave front $\tilde{\varphi}$ from the linearized problem. Consider the following problem for $\tilde{W}$ and $\tilde{\varphi}$ on $\mathbb{R}^4_+$:

$$\begin{cases}\mathcal{L}^\gamma \tilde{W} = \tilde{f}, & \text{if } x_3 > 0,\\ \mathcal{B}^\gamma(\tilde{W}^{nc}, \tilde{\varphi}) = \tilde{g}, & \text{if } x_3 = 0,\end{cases} \tag{3.16}$$

where $\tilde{f}$ and $\tilde{g}$ are given source terms.

We can decompose the system (3.16) into two subsystems by observing the linear structure. First, we consider the following auxiliary problem for V :

$$\begin{cases}\mathcal{L}^\gamma V = \tilde{f}, & \text{if } x_3 > 0,\\ M_1 V^{nc} = 0, & \text{if } x_3 = 0,\end{cases} \tag{3.17}$$

where

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{3.18}$$

From the symmetric hyperbolic theory introduced by Lax and Phillips [29], the boundary condition is maximally dissipative and thus (3.17) has a solution such that the following estimate holds:

$$\gamma|||V|||^2_0 \leq \frac{C}{\gamma}|||\tilde{f}|||^2_0, \quad ||V^{nc}|_{x_3=0}||^2_{j,\gamma} \leq \frac{C}{\gamma}|||\tilde{f}|||^2_{j,\gamma},$$

for any nonnegative integer j. Then let us define $W := \tilde{W} - V$. It satisfies the following homogenous equations:

$$\begin{cases}\mathcal{L}^\gamma W = 0, & \text{if } x_3 > 0,\\ \mathcal{B}^\gamma(W^{nc}, \varphi) = g := \tilde{g} - \underline{M} V^{nc}, & \text{if } x_3 = 0.\end{cases} \tag{3.19}$$

Remark 3.10. We shall use the same notation to consider W as a solution to (3.19) rather than the perturbation of planar vortex sheets in (3.5) for simplicity.

From standard energy estimates, we obtain

$$\gamma|||W|||_0^2 \leq C||W^{nc}|_{x_3=0}||_0^2.$$

Then it remains to prove the following estimate on W:

$$||W^{nc}|_{x_3=0}||_0^2 + ||\varphi||_{0,\gamma}^2 \leq \frac{C}{\gamma^{2k}}||g||_{k,\gamma}^2, \tag{3.20}$$

where $k = 2$ or $k = 3$ will be discussed separately in the Section 3.6. In such a way, we could achieve all the estimates in Theorem 3.2. Now, we perform the Fourier-Laplace transform to the system (3.19), Laplace in time and Fourier in the tangential directions of the hyperplane $x_3 = 0$. Denote the variable in the frequency space by $(\delta, \eta, \tilde{\eta})$. Let $\tau = \gamma + i\delta$. Then, the PDE system (3.19) is transformed into the following ODE system for $\widehat{W}$:

$$\begin{cases} (\tau\mathcal{A}_0 + i\eta\mathcal{A}_1 + i\tilde{\eta}\mathcal{A}_2)\widehat{W} + \mathcal{A}_3\frac{d\widehat{W}}{dx_3} = 0, & \text{if } x_3 > 0, \\ b(\tau, \eta, \tilde{\eta})\widehat{\varphi} + \underline{M}\widehat{W}^{nc} = \widehat{g}, & \text{if } x_3 = 0, \end{cases} \tag{3.21}$$

where

$$b(\tau, \eta, \tilde{\eta}) = \underline{b} \cdot \begin{bmatrix} \tau \\ i\eta \end{bmatrix} = \begin{bmatrix} 2iu^r\eta \\ \tau + iu^r\eta \\ 0 \end{bmatrix}.$$

Due to the homogeneity of the equations (3.21) , we define a hemisphere

$$\Sigma = \{(\tau, \eta, \tilde{\eta}) : |\tau|^2 + \eta^2 + \tilde{\eta}^2 = 1, \text{ and } \Re\tau \geq 0\}$$

in the whole frequency space $\Pi := \{(\tau, \eta, \tilde{\eta}) : \tau \in \mathbb{C}, \eta, \tilde{\eta} \in \mathbb{R}, |\tau|^2 + \eta^2 + \tilde{\eta}^2 \neq 0, \Re\tau \geq 0\}$. It is noted that $\Pi = \{k \cdot (\tau, \eta, \tilde{\eta}) : k > 0, (\tau, \eta, \tilde{\eta}) \in \Sigma\} = (0, \infty) \cdot \Sigma$. Our argument will be casted on the hemisphere Σ and then be extended to the whole frequency space Π by applying the homogeneity property. Different from the two-dimensional elastic flows, due to the extra frequency variable $\tilde{\eta}$, the boundary symbol $b(\tau, \eta, \tilde{\eta})$ can vanish on Σ if and only if $\eta = 0$ and $\tau = 0$. We can rewrite the boundary conditions in (3.21) as follows:

$$\begin{bmatrix} 2iu^r\eta \\ \tau + iu^r\eta \\ 0 \end{bmatrix} \widehat{\varphi} + \begin{bmatrix} -c & -c & c & c \\ -c & -c & 0 & 0 \\ -1 & -1 & 1 & -1 \end{bmatrix} \widehat{W}^{nc} = \widehat{g} := \begin{bmatrix} \widehat{g}_1 \\ \widehat{g}_2 \\ \widehat{g}_3 \end{bmatrix}.$$

We see that

$$(\tau + iu^r\eta)\widehat{\varphi} - c\widehat{W}_1(0) - c\widehat{W}_2(0) = \widehat{g}_2,$$

then we have

$$\gamma^2|\widehat{\varphi}|^2 \leq C(|\widehat{g}|^2 + |\widehat{W}^{nc}|_{x_3=0}|^2), \quad \forall(\tau, \eta, \tilde{\eta}) \in \Pi.$$

If $\tau = \eta = 0$, there is one-order of degeneracy in the front φ, which yields the estimate for the front

$$||\varphi||_{0,\gamma}^2 \leq \frac{C}{\gamma^2}(||\hat{W}^{nc}|_{x_3=0}||^2 + ||g||_{0,\gamma}^2). \tag{3.22}$$

Lemma 3.1. *There exists a C^∞ mapping $Q : \Pi \to \mathbb{C}^{3\times 3}$, which is homogeneous of degree 0, such that*

$$Q(\tau, \eta, \tilde{\eta})b(\tau, \eta, \tilde{\eta}) = \begin{bmatrix} 0 \\ 0 \\ l(\tau, \eta, \tilde{\eta}) \end{bmatrix},$$

where $l(\tau, \eta, \tilde{\eta}) = 4(u^r)^2\eta^2 + |\tau + iu^r\eta|^2$.

Proof. Define the map Q as

$$Q(\tau,\eta,\tilde{\eta}) = \begin{bmatrix} 0 & 0 & 1 \\ \tau + iu^r\eta & -2iu^r\eta & 0 \\ -2iu^r\eta & \bar{\tau} - iu^r\eta & 0 \end{bmatrix}, \quad \forall(\tau,\eta,\tilde{\eta}) \in \Sigma,$$

and extend Q to the whole frequency space by homogeneity of order zero. A simple calculation concludes the proof of the lemma. □

Remark 3.11. We see that $l(\tau,\eta,\tilde{\eta}) = \det Q(\tau,\eta,\tilde{\eta})$ can vanish on Σ when $(\tau,\eta,\tilde{\eta}) = (0,0,\pm 1)$. This implies that the additional direction of velocity leads to extra possible direction of degeneracy.

Remark 3.12. Here, we obtain L^2 estimates for the wave fronts in our main theorem. In fact, the non-parallel condition $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$ enhances the regularity (derivatives estimates) of φ in the nonlinear analysis. This can be understood as an "ellipticity" property of the front symbol and will play a key role in the forthcoming nonlinear analysis. Such a property also appears in the study of a single-phase compressible elastic fluid [43] as well as in the MHD vortex sheets [4, 42].

Multiplying (3.21) by $Q(\tau,\eta,\tilde{\eta})$ yields the new boundary conditions:

$$Qb\widehat{\varphi} + Q\underline{M}\widehat{W}^{nc} = Q\widehat{g}, \text{ at } x_3 = 0. \tag{3.23}$$

Simple calculation tells us that

$$Q\underline{M} = \begin{bmatrix} -1 & 1 & 1 & -1 \\ -c(\tau - iu^r\eta) & -c(\tau - iu^r\eta) & c(\tau + iu^r\eta) & c(\tau + iu^r\eta) \\ -c(\bar{\tau} - 3iu^r\eta) & -c(\bar{\tau} - 3iu^r\eta) & -2ciu^r\eta & -2ciu^r\eta \end{bmatrix} \tag{3.24}$$

on Σ, where $\bar{\tau}$ denotes the complex conjugate number of τ. It is noted that Q is homogeneous of degree 0 in $(\tau,\eta,\tilde{\eta})$. Then, we consider the first two rows in the new boundary condition (3.23) at $x_3 = 0$ and the equation of (3.21) for $x_3 > 0$. After eliminating the front function φ, we have

$$\begin{cases} (\tau\mathcal{A}_0 + i\eta\mathcal{A}_1 + i\tilde{\eta}\mathcal{A}_2)\widehat{W} + \mathcal{A}_3\frac{d\widehat{W}}{dx_3} = 0, & \text{if } x_3 > 0, \\ \beta\widehat{W}^{nc}|_{x_3=0} = H, \end{cases}$$

where we denote H to be a function that contains the first two rows of $Q(\tau,\eta,\tilde{\eta})\widehat{g}$ and

$$\beta := \begin{bmatrix} -1 & 1 & 1 & -1 \\ -c(\tau - iu^r\eta) & -c(\tau - iu^r\eta) & c(\tau + iu^r\eta) & c(\tau + iu^r\eta) \end{bmatrix}$$

on Σ and is a function with homogeneity of degree 0 after extension to the whole frequency space Π. Now, our goal is to obtain the estimate of $||\widehat{W}^{nc}|_{x_3=0}||_0^2$ from the system (3.21).

3.3. **Normal mode analysis.** We will perform a mode-separation procedure as in [38] to separate the outgoing modes and incoming modes from the system. This separation can provide delicate estimates for the outgoing modes. With this we will show in Section 3.6 that the outgoing modes are indeed zero.

In order to obtain an estimate of $||\widehat{W}^{nc}|_{x_3=0}||_0^2$ with respect to the source term in the boundary conditions, we need to derive a system of $\widehat{W}^{nc}$. To this end, we will choose twenty-two algebraic equations from (3.21):

$$
\begin{aligned}
&c^2 i\eta(-\widehat{W}_1+\widehat{W}_2)+(\tau+i\eta u^r)\widehat{W}_3\\
&\qquad -i(\eta F^r_{11}+\tilde\eta F^r_{21})\widehat{W}_5-i(\eta F^r_{12}+\tilde\eta F^r_{22})\widehat{W}_8-i(\eta F^r_{13}+\tilde\eta F^r_{23})\widehat{W}_{11}=0,\\
&c^2 i\tilde\eta(-\widehat{W}_1+\widehat{W}_2)+(\tau+i\eta u^r)\widehat{W}_4\\
&\qquad -i(\eta F^r_{11}+\tilde\eta F^r_{21})\widehat{W}_6-i(\eta F^r_{12}+\tilde\eta F^r_{22})\widehat{W}_7-i(\eta F^r_{13}+\tilde\eta F^r_{23})\widehat{W}_{12}=0,\\
&-i(\eta F^r_{11}+\tilde\eta F^r_{21})\widehat{W}_3+(\tau+i\eta u^r)\widehat{W}_5=0,\\
&-i(\eta F^r_{11}+\tilde\eta F^r_{21})\widehat{W}_4+(\tau+i\eta u^r)\widehat{W}_6=0,\\
&-ic(\eta F^r_{11}+\tilde\eta F^r_{21})\widehat{W}_1-ic(\eta F^r_{11}+\tilde\eta F^r_{21})\widehat{W}_2+(\tau+iu^r\eta)\widehat{W}_7=0,\\
&-i(\eta F^r_{12}+\tilde\eta F^r_{22})\widehat{W}_3+(\tau+i\eta u^r)\widehat{W}_8=0,\\
&-i(\eta F^r_{12}+\tilde\eta F^r_{22})\widehat{W}_4+(\tau+i\eta u^r)\widehat{W}_9=0,\\
&-ic(\eta F^r_{12}+\tilde\eta F^r_{22})\widehat{W}_1-ic(\eta F^r_{12}+\tilde\eta F^r_{22})\widehat{W}_2+(\tau+iu^r\eta)\widehat{W}_{10}=0,\\
&-i(\eta F^r_{13}+\tilde\eta F^r_{23})\widehat{W}_3+(\tau+i\eta u^r)\widehat{W}_{11}=0,\\
&-i(\eta F^r_{13}+\tilde\eta F^r_{23})\widehat{W}_4+(\tau+i\eta u^r)\widehat{W}_{12}=0,\\
&-ic(\eta F^r_{13}+\tilde\eta F^r_{23})\widehat{W}_1-ic(\eta F^r_{13}+\tilde\eta F^r_{23})\widehat{W}_2+(\tau+iu^r\eta)\widehat{W}_{13}=0,\\
&c^2 i\eta(-\widehat{W}_{14}+\widehat{W}_{15})+(\tau+i\eta u^l)\widehat{W}_{16}\\
&\qquad -i(\eta F^l_{11}+\tilde\eta F^l_{21})\widehat{W}_{18}-i(\eta F^l_{12}+\tilde\eta F^l_{22})\widehat{W}_{21}-i(\eta F^l_{13}+\tilde\eta F^l_{23})\widehat{W}_{24}=0,\\
&c^2 i\tilde\eta(-\widehat{W}_{14}+\widehat{W}_{15})+(\tau+i\eta u^l)\widehat{W}_{17}\\
&\qquad -i(\eta F^l_{11}+\tilde\eta F^l_{21})\widehat{W}_{19}-i(\eta F^l_{12}+\tilde\eta F^l_{22})\widehat{W}_{20}-i(\eta F^l_{13}+\tilde\eta F^l_{23})\widehat{W}_{25}=0,\\
&-i(\eta F^l_{11}+\tilde\eta F^l_{21})\widehat{W}_{16}+(\tau+i\eta u^l)\widehat{W}_{18}=0,\\
&-i(\eta F^l_{11}+\tilde\eta F^l_{21})\widehat{W}_{17}+(\tau+i\eta u^l)\widehat{W}_{19}=0,\\
&-ic(\eta F^l_{11}+\tilde\eta F^l_{21})\widehat{W}_{14}-ic(\eta F^l_{11}+\tilde\eta F^l_{21})\widehat{W}_{15}+(\tau+iu^r\eta)\widehat{W}_{20}=0,\\
&-i(\eta F^l_{12}+\tilde\eta F^l_{22})\widehat{W}_{16}+(\tau+i\eta u^l)\widehat{W}_{21}=0,\\
&-i(\eta F^l_{12}+\tilde\eta F^l_{22})\widehat{W}_{17}+(\tau+i\eta u^l)\widehat{W}_{22}=0,\\
&-ic(\eta F^l_{12}+\tilde\eta F^l_{22})\widehat{W}_{14}-ic(\eta F^l_{12}+\tilde\eta F^l_{22})\widehat{W}_{15}+(\tau+iu^l\eta)\widehat{W}_{23}=0,\\
&-i(\eta F^l_{13}+\tilde\eta F^l_{23})\widehat{W}_{16}+(\tau+i\eta u^l)\widehat{W}_{24}=0,\\
&-i(\eta F^l_{13}+\tilde\eta F^l_{23})\widehat{W}_{17}+(\tau+i\eta u^l)\widehat{W}_{25}=0,\\
&-ic(\eta F^l_{13}+\tilde\eta F^l_{23})\widehat{W}_{14}-ic(\eta F^l_{13}+\tilde\eta F^l_{23})\widehat{W}_{15}+(\tau+iu^l\eta)\widehat{W}_{26}=0.
\end{aligned}
$$

From (3.5), we can solve the above system by using "noncharacteristic" terms $\widehat{W}^{nc} = (W_1, W_2, W_{14}, W_{15})^T$:

$$
\begin{aligned}
\widehat{W}_3 &= \frac{ic^2 k_1^r \eta(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r}, \quad \widehat{W}_4 = \frac{ic^2 \tilde{\eta}(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r},\\
\widehat{W}_5 &= \frac{-c^2 \eta(\eta F_{11}^r + \tilde{\eta} F_{21}^r)(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r}, \quad \widehat{W}_6 = \frac{-c^2 \tilde{\eta}(\eta F_{11}^r + \tilde{\eta} F_{21}^r)(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r},\\
\widehat{W}_7 &= \frac{ic(\eta F_{11}^r + \tilde{\eta} F_{21}^r)}{k_1^r}(\widehat{W}_1 + \widehat{W}_2), \quad \widehat{W}_8 = \frac{-c^2 \eta(\eta F_{12}^r + \tilde{\eta} F_{22}^r)(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r},\\
\widehat{W}_9 &= \frac{-c^2 \tilde{\eta}(\eta F_{12}^r + \tilde{\eta} F_{22}^r)(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r}, \quad \widehat{W}_{10} = \frac{ic(\eta F_{12}^r + \tilde{\eta} F_{22}^r)}{k_1^r}(\widehat{W}_1 + \widehat{W}_2),\\
\widehat{W}_{11} &= \frac{-c^2 \eta(\eta F_{13}^r + \tilde{\eta} F_{23}^r)(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r}, \quad \widehat{W}_{12} = \frac{-c^2 \tilde{\eta}(\eta F_{13}^r + \tilde{\eta} F_{23}^r)(\widehat{W}_1 - \widehat{W}_2)}{(k_1^r)^2 + k_2^r},\\
\widehat{W}_{13} &= \frac{ic(\eta F_{13}^r + \tilde{\eta} F_{23}^r)}{k_1^r}(\widehat{W}_1 + \widehat{W}_2),\\
\widehat{W}_{16} &= \frac{ic^2 k_1^l \eta(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l}, \quad \widehat{W}_{17} = \frac{ic^2 \tilde{\eta}(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l},\\
\widehat{W}_{18} &= \frac{-c^2 \eta(\eta F_{11}^l + \tilde{\eta} F_{21}^l)(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l}, \quad \widehat{W}_{19} = \frac{-c^2 \tilde{\eta}(\eta F_{11}^l + \tilde{\eta} F_{21}^l)(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l},\\
\widehat{W}_{20} &= \frac{ic(\eta F_{11}^l + \tilde{\eta} F_{21}^l)}{k_1^l}(\widehat{W}_{14} + \widehat{W}_{15}), \quad \widehat{W}_{21} = \frac{-c^2 \eta(\eta F_{12}^l + \tilde{\eta} F_{22}^l)(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l},\\
\widehat{W}_{22} &= \frac{-c^2 \tilde{\eta}(\eta F_{12}^l + \tilde{\eta} F_{22}^l)(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l}, \quad \widehat{W}_{23} = \frac{ic(\eta F_{12}^l + \tilde{\eta} F_{22}^l)}{k_1^l}(\widehat{W}_{14} + \widehat{W}_{15}),\\
\widehat{W}_{24} &= \frac{-c^2 \eta(\eta F_{13}^l + \tilde{\eta} F_{23}^l)(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l}, \quad \widehat{W}_{25} = \frac{-c^2 \tilde{\eta}(\eta F_{13}^l + \tilde{\eta} F_{23}^l)(\widehat{W}_{14} - \widehat{W}_{15})}{(k_1^l)^2 + k_2^l},\\
\widehat{W}_{26} &= \frac{ic(\eta F_{13}^l + \tilde{\eta} F_{23}^l)}{k_1^l}(\widehat{W}_{14} + \widehat{W}_{15}),
\end{aligned}
$$

where $k_1^{r,l} := \tau + i\eta u^{r,l}$ and $k_2^{r,l} := (\eta F_{11}^{r,l} + \tilde{\eta} F_{21}^{r,l})^2 + (\eta F_{12}^{r,l} + \tilde{\eta} F_{22}^{r,l})^2 + (\eta F_{13}^{r,l} + \tilde{\eta} F_{23}^{r,l})^2$. Taking advantage of differential equations in (3.21), we obtain the ODE for $\widehat{W}^{nc}$ in the following form:

$$
\frac{d}{dx_3}\widehat{W}^{nc} = A\widehat{W}^{nc}, \tag{3.25}
$$

where

$$
A := \begin{bmatrix} n^r & -m^r & 0 & 0 \\ m^r & -n^r & 0 & 0 \\ 0 & 0 & -n^l & m^l \\ 0 & 0 & -m^l & n^l \end{bmatrix} \tag{3.26}
$$

with

$$n^{r,l} := \frac{2(k_1^{r,l})^2 + k_2^{r,l}}{2ck_1^{r,l}} + \frac{c(\eta^2+\tilde{\eta}^2)k_1^{r,l}}{2((k_1^{r,l})^2 + k_2^{r,l})}, \quad m^{r,l} := \frac{c(\eta^2+\tilde{\eta}^2)k_1^{r,l}}{2((k_1^{r,l})^2 + k_2^{r,l})} - \frac{k_2^{r,l}}{2ck_1^{r,l}}.$$

Consider a background solution defined by (3.1). For a given direction vector $\mathbf{s} := (\cos\theta, \sin\theta) \in \mathbb{S}^1$, we define

$$g_{r,l}(\theta) := (\cos\theta F_{11}^{r,l} + \sin\theta F_{21}^{r,l})^2 + (\cos\theta F_{12}^{r,l} + \sin\theta F_{22}^{r,l})^2 + (\cos\theta F_{13}^{r,l} + \sin\theta F_{23}^{r,l})^2. \quad (3.27)$$

We will see in the later discussion that the function g plays a role of a projected "elastic" sound speed along the direction $\vec{s}$.

From the classical theory of hyperbolic conservation laws [16], we need to bound the components of $\widehat{W}^{nc}$ on the stable subspace of A by estimating $\|\widehat{W}^{nc}|_{x_3=0}\|_0^2$. We first provide the following Hersh-type Lemma [19] which describes the stable subspace of A defined on Σ explicitly.

Lemma 3.2. *For $(\tau, \eta, \tilde{\eta}) \in \Sigma$ and $\Re\tau > 0$, the matrix A defined in (3.26) admits four eigenvalues $\pm\omega^r$ and $\pm\omega^l$, where $\Re\omega^r$ and $\Re\omega^l$ are negative. Moreover, the following dispersion relations hold:*

$$(\omega^{r,l})^2 = (n^{r,l})^2 - (m^{r,l})^2 = \frac{1}{c^2}\left[(\tau + iu^{r,l}\eta)^2 + (\eta^2+\tilde{\eta}^2)g_{r,l}(\theta)\right] + \eta^2 + \tilde{\eta}^2, \qquad (3.28)$$

where $\cos\theta = \frac{\eta}{\sqrt{\eta^2+\tilde{\eta}^2}}$ and $\sin\theta = \frac{\tilde{\eta}}{\sqrt{\eta^2+\tilde{\eta}^2}}$. The eigenvectors of $\omega^r, -\omega^r, \omega^l, -\omega^l$ take the following forms:

$$\begin{aligned} E_-^r &= (a^r, b^r, 0, 0)^T, \quad E_+^r = (a^r, c^r, 0, 0)^T, \\ E_-^l &= (0, 0, b^l, a^l)^T, \quad E_-^l = (0, 0, c^l, a^l)^T, \end{aligned} \qquad (3.29)$$

where

$$\begin{aligned} a^{r,l} &= m^{r,l}\alpha^{r,l}, \quad b^{r,l} = (n^{r,l} - \omega^{r,l})\alpha^{r,l}, \quad c^{r,l} = (n^{r,l} + \omega^{r,l})\alpha^{r,l}, \\ \alpha^{r,l} &= (\tau + iu^{r,l}\eta)[(\tau + iu^{r,l}\eta)^2 + (\eta^2+\tilde{\eta}^2)g_r(\theta)]. \end{aligned}$$

Both ω^r and ω^l can admit a continuous extension to all the points such that $\Re\tau = 0$, and $(\tau, \eta, \tilde{\eta}) \in \Sigma$. so can $E_\pm^r$ and $E_\pm^l$. Moreover, two vectors E_-^r and E_-^l are linearly independent for all frequency $(\tau, \eta, \tilde{\eta}) \in \Sigma$.

According to the definition (3.26) of A, (3.29) holds on Π, and we cannot diagonalize A smoothly near the neighborhood of some "singular" points $(\tau, \eta, \tilde{\eta}) \in \Sigma$ satisfying $m^{r,l} = 0$ or $\omega^{r,l} = 0$, or $\tau = \pm iu^r\eta$, or

$$\tau = i\left(\pm u^r\eta \pm \sqrt{(\eta^2+\tilde{\eta}^2)g_r(\theta)}\right).$$

It is noted that E_-^r and E_+^r or E_-^l and E_+^l become parallel at these points. For $\tau = \pm iu^r\eta$, or $\tau = i(\pm u^r\eta \pm \sqrt{(\eta^2+\tilde{\eta}^2)g_r(\theta)})$, we name these points $(\tau, \eta, \tilde{\eta})$ the poles of A. Next we adopt the methodology of upper triangularization of matrix A in [38] when performing separation of modes.

3.4. **Separation of modes.** In this section, we need to concentrate our analysis in a microlocal manner due to the degeneracy of the eigenbasis of A at some points on Σ mentioned above. Following the argument in [38], for each point $(\tau_0, \eta_0, \tilde{\eta}_0) \in \Sigma$, we will single out the outgoing modes of A in the neighborhood $\mathcal{V} \in \Sigma$ of the point $(\tau_0, \eta_0, \tilde{\eta}_0)$ from the system (3.25). For the outgoing modes of A, they are exactly the components of $\widehat{W}^{nc}$ which are not in the stable subspace of A. Applying this separation, we will prove in Section 3.6 that for every $(\tau_0, \eta_0, \tilde{\eta}_0) \in \Sigma$, these outgoing modes are all zeros in $\mathcal{V} \cap \{(\tau, \eta, \tilde{\eta}) : \Re\tau > 0\}$. The compactness of Σ then allows us to propagate this vanishing of outgoing modes to the entire $\Sigma \cap \{(\tau, \eta, \tilde{\eta}) : \Re\tau > 0\}$.

Compared with the two-dimensional elastic vortex sheets, the additional dimension makes it more challenging to perform the separation of modes. The non-parallel condition $\text{rank}\mathbf{F}^r = 2$ is essential to ensure $(\tau, \eta, \tilde{\eta}) \in \Sigma$. Meanwhile, the extra dimension in the frequency space could potentially increase the possibility of instability.

Now we prove a proposition which is useful in the mode-separation for all points on Σ.

Proposition 3.1. *For $\omega^{r,l}$ defined in Lemma 3.2, we have*

$$(\tau + iu^{r,l}\eta)\omega^{r,l} - c((\omega^{r,l})^2 - \eta^2 - \tilde{\eta}^2) \neq 0, \quad \forall(\tau, \eta, \tilde{\eta}) \in \Sigma.$$

Proof. We will check the signs of the real and imaginary parts of $\omega^{r,l}$ at the point $(\tau, \eta, \tilde{\eta}) \in \Sigma$ with $\Re\tau > 0$. To this end, as in the 2D case, we consider $(x+iy)^2 = p+iq$ for $x, y, p, q \in \mathbb{R}$, and $x \leq 0$. Solving this equation leads to the solution formula

$$x = -\sqrt{\frac{p + \sqrt{p^2 + q^2}}{2}}, \quad y = -\text{sgn}(q)\sqrt{\frac{\sqrt{p^2 + q^2} - p}{2}} \tag{3.30}$$

for $p, q \in \mathbb{R}^2 \backslash \{p < 0, q = 0\}$.

Let $\omega^{r,l} = x^{r,l} + iy^{r,l}$ and $(\omega^{r,l})^2 = p^{r,l} + iq^{r,l}$, where $x^{r,l}, y^{r,l}, p^{r,l}, q^{r,l} \in \mathbb{R}$. From the definition (3.28) of $\omega^{r,l}$, we can obtain that $x^{r,l} \leq 0$ and

$$p^{r,l} = \frac{\gamma^2 - (\delta + u^{r,l}\eta)^2 + (\eta^2 + \tilde{\eta}^2)g_{r,l}(\theta)}{c^2} + \eta^2 + \tilde{\eta}^2, \tag{3.31}$$

$$q^{r,l} = \frac{2\gamma(\delta + u^{r,l}\eta)}{c^2}, \tag{3.32}$$

where we recall the definition of the function $g_{r,l}$ in (3.27). From (3.30), we can obtain that when $(p^{r,l}, q^{r,l}) \notin \{p < 0, q = 0\}$ and $\delta + u^{r,l}\eta \neq 0$, $y^{r,l}$ and $\delta + u^{r,l}\eta$ are of opposite signs. Here, we use the fact that $\gamma = \Re\tau > 0$. On the other hand, (3.30) does not serve as a solution formula when $(p^{r,l}, q^{r,l}) \in \{p < 0, q = 0\}$. At these points we have

$$\gamma = 0, \quad \delta + u^{r,l}\eta \neq 0, \quad \text{and} \quad p^{r,l} < 0.$$

Therefore, these points are on the boundary of Σ. By Lemma 3.2, the values of $\omega^{r,l}$ at the boundary of Σ are defined as the continuous extension limits of the interior values of $\omega^{r,l}$. Similarly, the signs of $x^{r,l}$ and $y^{r,l}$ can be treated by a continuity argument.

Compared with the 2D case, the way to extend $\omega^{r,l}$ from the interior to the boundary Σ is different in 3D. With the goal of still being able to determine the signs of $x^{r,l}$ and $y^{r,l}$ through continuity, we will continuously extend $\omega^{r,l}$ along a frequency path where the ratio between δ and a certain linear combination of $\{\eta, \tilde{\eta}\}$ is fixed. This way the signs of

$y^{r,l}$ and $\delta+u^{r,l}\eta$ are opposite correspondingly at those exceptional points $(p^{r,l},q^{r,l})\in\{p<0,q=0\}$. Hence we obtain that

$$\text{If } \delta+u^{r,l}\eta\neq 0, y^{r,l} \text{ and } \delta+u^{r,l}\eta \text{ are of opposite signs, respectively.} \tag{3.33}$$

Now, we continue to prove the proposition in the case $(\tau+iu^r\eta)\omega^r-c((\omega^r)^2-\eta^2-\tilde{\eta}^2)\neq 0$ on Σ. The other case, $c((\omega^l)^2-\eta^2-\tilde{\eta}^2)-(\tau+iu^l\eta)\omega^l\neq 0$ can be dealt with using a similar argument. We prove this by contradiction, i.e., assuming

$$(\tau+iu^r\eta)\omega^r-c((\omega^r)^2-\eta^2-\tilde{\eta}^2)=0 \tag{3.34}$$

holds. If $\tau+iu^r\eta=0$, the equation (3.34) becomes $(\omega^r)^2-\eta^2-\tilde{\eta}^2=0$. Combining this with (3.28) we obtain that

$$(\eta^2+\tilde{\eta}^2)g_r(\theta)=0.$$

Note that from (3.9), one has that rank $\mathbf{F}^r=2$. Therefore, $\eta=\tilde{\eta}=0$. And hence, $\tau=0$. This contradicts with $(\tau,\eta,\tilde{\eta})\in\Sigma$. Thus, we assume that $\tau+iu^r\eta\neq 0$, and we obtain that

$$\begin{aligned}\omega^r&=\frac{c((\omega^r)^2-\eta^2-\tilde{\eta}^2)}{\tau+iu^r\eta}\\&=\frac{1}{c}\left[(\tau+iu^r\eta)+\frac{(\eta^2+\tilde{\eta}^2)g_r(\theta)}{\tau+iu^r\eta}\right].\end{aligned} \tag{3.35}$$

If $\Re\tau=\gamma>0$, we can obtain that the real part of the right hand side of (3.35) is positive. This is in contradiction with the definition $\Re\omega^r<0$.

Thus we focus on the case $\gamma=0$. In this case, we obtain $\tau+iu^r\eta=i(\delta+u^r\eta)\neq 0$. By (3.35), we know that $\Re\omega^r=0$ and hence $q^r=0$ and $p^r\leq 0$. It is noted that $p^r\neq 0$. If $p^r=0$, from $q^r=0$, we obtain that $\omega^r=0$. Then, from (3.34), we have $\eta=\tilde{\eta}=0$. By (3.31), it follows that $\delta=0$, and then $\tau=0$. It contradicts with the fact that $(\tau,\eta,\tilde{\eta})\in\Sigma$. Therefore, we shall focus on $(\tau,\eta,\tilde{\eta})\in\Sigma$, when $\tau+iu^r\eta\neq 0,\gamma=0$ and $p^r<0$. This yields that $\delta+u^r\eta\neq 0$. However, by (3.35) and the fact that $\Re\tau=0$,

$$\Im\omega^r=\frac{(\delta+u^r\eta)^2-(\eta^2+\tilde{\eta}^2)g_r(\theta)}{c(\delta+u^r\eta)}.$$

Since $p^r<0$, using (3.31), we obtain that $(\delta+u^r\eta)^2-(\eta^2+\tilde{\eta}^2)g_r(\theta)>0$. Then, the sign of $\Im\omega^r$ is the same as the sign of $\delta+u^r\eta$, which leads to the contradiction with (3.33). Therefore, $(\tau+iu^r\eta)\omega^r-c((\omega^r)^2-\eta^2-\tilde{\eta}^2)\neq 0$ on Σ. This completes the proof of the proposition. $\square$

We will prove that the eigenvector $E_-^{r,l}$ can not vanish at any point in Σ using the above proposition. Otherwise, if $E_-^{r,l}=0$, we have $m^{r,l}\alpha^{r,l}=0$ and $(n^{r,l}-\omega^{r,l})\alpha^{r,l}=0$. Direct calculation tells us that $\alpha^{r,l}\neq 0$. Then, $m^{r,l}\alpha^{r,l}=0$ implies that $m^{r,l}=0$. From the definition of $m^{r,l}$, we obtain that

$$\frac{c}{2}\cdot\frac{(\tau+iu^r\eta)(\eta^2+\tilde{\eta}^2)}{(\tau+iu^r\eta)^2+(\eta^2+\tilde{\eta}^2)g_r(\theta)}=\frac{(\eta^2+\tilde{\eta}^2)g_r(\theta)}{2c(\tau+iu^r\eta)}.$$

Together with $(n^{r,l}-\omega^{r,l})\alpha^{r,l}=0$ and (3.28), we obtain that

$$(\tau+iu^{r,l}\eta)\omega^{r,l}-c((\omega^{r,l})^2-\eta^2-\tilde{\eta}^2)=0, n^{r,l}=\omega^{r,l}.$$

This contradicts with Proposition 3.1. Construct a new matrix T

$$T = \{E_-^r, F^r, E_-^l, F^l\},$$

by observing that vectors $E_-^{r,l}$ are not degenerate in the neighborhood of $(\tau_0, \eta_0, \tilde{\eta}_0) \in \Sigma$ with

$$F^r = \begin{cases} (0,1,0,0)^T, & \text{if } m^r\alpha^r \neq 0 \text{ at } (\tau_0, \eta_0, \tilde{\eta}_0), \\ (1,0,0,0)^T, & \text{if } (n^r - \omega^r)\alpha^r \neq 0 \text{ at } (\tau_0, \eta_0, \tilde{\eta}_0), \end{cases}$$

and likewise

$$F^l = \begin{cases} (0,0,1,0)^T, & \text{if } m^l\alpha^l \neq 0 \text{ at } (\tau_0, \eta_0, \tilde{\eta}_0), \\ (0,0,0,1)^T, & \text{if } (n^l - \omega^l)\alpha^l \neq 0 \text{ at } (\tau_0, \eta_0, \tilde{\eta}_0). \end{cases}$$

Therefore, for any point $(\tau_0, \eta_0, \tilde{\eta}_0) \in \Sigma$, there is an open neighborhood $\mathcal{V}$ of $(\tau_0, \eta_0, \tilde{\eta}_0)$ where T is invertible on $\mathcal{V}$. Then, we have finished upper triangularization procedures:

$$T^{-1}AT = \begin{bmatrix} \omega^r & z^r & 0 & 0 \\ 0 & -\omega^r & 0 & 0 \\ 0 & 0 & \omega^l & z^l \\ 0 & 0 & 0 & -\omega^l \end{bmatrix} \tag{3.36}$$

on $\mathcal{V}$ where A is a block matrix given in (3.26) and

$$z^{r,l} = \begin{cases} -\frac{1}{\alpha^{r,l}}, & \text{if } m^{r,l}\alpha^{r,l} \neq 0 \text{ at } (\tau_0, \eta_0, \tilde{\eta}_0), \\ \frac{m^{r,l}}{(n^{r,l} - \omega^{r,l})\alpha^{r,l}}, & \text{if } (n^{r,l} - \omega^{r,l})\alpha^{r,l} \neq 0 \text{ at } (\tau_0, \eta_0, \tilde{\eta}_0). \end{cases}$$

3.5. **Lopatinskiĭ determinant.** In this section, we aim to estimate the components of $\widehat{W}^{nc}|_{x_3=0}$ in the stable subspace of A combining the boundary conditions. This leads us to study the invertibility of the matrix $\beta(E_-^r, E_-^l)$ associated with the boundary condition (see, for example, [33]), which results in analyzing the roots of the Lopatinskiĭ determinant:

$$\begin{aligned} \Delta &:= \det(\beta(E_-^r, E_-^l)) \\ &= c^4\big(\tau + iu^r\eta\big)\big(\tau + iu^l\eta\big)\,\big((\tau + iu^r\eta)\omega^r - c((\omega^r)^2 - \eta^2 - \tilde{\eta}^2)\big) \\ &\quad \times \big(c((\omega^l)^2 - \eta^2 - \tilde{\eta}^2) - (\tau + iu^l\eta)\omega^l\big)\big(\omega^l\omega^r - \eta^2 - \tilde{\eta}^2\big)\big(\omega^r + \omega^l\big). \end{aligned} \tag{3.37}$$

It is easily seen that the Lopatinskiĭ determinant Δ can vanish at certain points in Σ. Therefore the uniform Lopatinskiĭ condition fails.

Lemma 3.3 (Root distribution)**.** *Consider a particular solution defined by* (3.1). *The roots of the Lopatinskiĭ determinant Δ are distributed in the following ways:*

(Case1) *If there exists an $\mathbf{s}_0 = (\cos\theta_0, \sin\theta_0) \in \mathbb{S}^1$ with $\cos\theta_0 \neq 0$, such that*

$$\frac{g_r(\theta_0)}{\cos^2\theta_0} < (u^r)^2 < 2c^2 + \frac{g_r(\theta_0)}{\cos^2\theta_0}, \tag{3.38}$$

where g_r is defined in (3.27). *Then it holds that some roots of the Lopatinskiĭ determinant are in the interior of Σ, and hence the Lopatinskiĭ condition fails.*

(Case2) *If*

$$0 < (u^r)^2 < \inf_{\cos\theta\neq 0} \frac{g_r(\theta)}{\cos^2\theta}, \tag{3.39}$$

and

$$(u^r\cos\theta)^2 \neq \frac{g_r(\theta)\left(g_r(\theta)+2c^2\right)}{4\left(g_r(\theta)+c^2\right)}, \tag{3.40}$$

then all roots are at most double and on the boundary of Σ*, and the Lopatinskiĭ condition holds. Specifically, the roots* $(\tau,\eta,\tilde\eta)\in\Sigma$ *satisfy*

(i) $\tau = \pm i u^{r,l}\eta$*, (at most double) or*

(ii) $\tau = \pm iV_1\sqrt{\eta^2+\tilde\eta^2}$*, where*

$$V_1^2 := (u^r\cos\theta)^2 + c^2 + g_r(\theta) - \sqrt{c^4 + 4(u^r\cos\theta)^2\left(g_r(\theta)+c^2\right)} \tag{3.41}$$

with $\cos\theta = \frac{\eta}{\sqrt{\eta^2+\tilde\eta^2}}$ *and* $\sin\theta = \frac{\tilde\eta}{\sqrt{\eta^2+\tilde\eta^2}}$.

(Case3) *If for all* $\mathbf{s}\in\mathbb{S}^1$*,* (3.39) *holds and there exists an* $\mathbf{s}_0 = (\cos\theta_0,\sin\theta_0)\in\mathbb{S}^1$ *such that*

$$(u^r\cos\theta_0)^2 = \frac{g_r(\theta_0)\left(g_r(\theta_0)+2c^2\right)}{4\left(g_r(\theta_0)+c^2\right)}, \tag{3.42}$$

then all roots are also on the boundary of Σ*, and the Lopatinskiĭ condition holds. Now the roots* $(\tau,\eta,\tilde\eta)\in\Sigma$ *are at most tripled and satisfy*

$$\tau = \pm i u^{r,l}\eta.$$

(Case4) *If* $\inf_{\cos\theta\neq 0} \frac{g_r(\theta)}{\cos^2\theta}$ *is attainable and*

$$(u^r)^2 = \inf_{\cos\theta\neq 0} \frac{g_r(\theta)}{\cos^2\theta}, \tag{3.43}$$

then all roots are on the boundary of Σ*, and the Lopatinskiĭ condition holds. More precisely, the roots are* $(\tau,\eta,\tilde\eta)\in\Sigma$ *such that*

(i) $\tau = \pm i u^{r,l}\eta$ *(at most double roots) or*

(ii) $\tau = 0$ *(double root).*

Remark 3.13. One can verify (see (3.52) in Lemma 3.4) that condition (3.39) makes sense when $\mathrm{F}_1\times\mathrm{F}_2\neq 0$.

Proof. The proof of the above lemma depends on a careful analysis on each factor of the Lopatinskiĭ determinant. In the following, we will divide our analysis for each factor step by step.

Step 1: The third and fourth factors $(\tau + iu^{r,l}\eta)\omega^{r,l} - c((\omega^{r,l})^2 - \eta^2 - \tilde\eta^2)$.
These two factors are nonzero, since they have exactly the same form in Proposition 3.1.

Step 2: The first and second factors $\tau + iu^{r,l}\eta$**.**
Different from the two-dimensional elastic flows, by checking the two directions $(\tau,\eta,\tilde\eta) = (0,0,\pm1)$ we see that $\tau = -iu^{r,l}\eta$ are not always the simple roots of $\tau + iu^{r,l}\eta = 0$, respectively.

Step 3: The fifth factor $\omega^r\omega^l - \eta^2 - \tilde\eta^2$.
Now we assume that

$$\omega^r\omega^l - \eta^2 - \tilde\eta^2 = 0. \tag{3.44}$$

If $\eta = \tilde{\eta} = 0$, we have $\omega^r = \omega^l = -\frac{\tau}{c}$, then $\omega^r\omega^l \neq 0$. Hence, $\omega^r\omega^l - \eta^2 - \tilde{\eta}^2 \neq 0$. Therefore, at least one of η and $\tilde{\eta}$ is not zero. We introduce the following two variables.

$$V = \frac{\tau}{i\sqrt{\eta^2+\tilde{\eta}^2}}, \qquad \Omega^{r,l} = \frac{\omega^{r,l}}{i\sqrt{\eta^2+\tilde{\eta}^2}}. \tag{3.45}$$

From (3.44), we have that $\Omega^r\Omega^l = -1$, and hence $(\Omega^r)^2(\Omega^l)^2 = 1$. By (3.28), we obtain that

$$(\Omega^{r,l})^2 = \frac{1}{c^2}\left[(V + u^{r,l}\cos\theta)^2 - g_{r,l}(\theta)\right] - 1. \tag{3.46}$$

Hence, we have

$$\left[(V+su^r)^2 - g_r(\theta)\right]\left[(V+u^l\cos\theta)^2 - g_l(\theta)\right] = c^4.$$

Solving the above equation for V^2, and using the quadratic formula, we obtain two roots of the above equation:

$$V_1^2 = (u^r\cos\theta)^2 + g_r(\theta) + c^2 - \sqrt{c^4 + 4(u^r\cos\theta)^2(g_r(\theta)+c^2)}, \tag{3.47}$$

$$V_2^2 = (u^r\cos\theta)^2 + g_r(\theta) + c^2 + \sqrt{c^4 + 4(u^r\cos\theta)^2(g_r(\theta)+c^2)}. \tag{3.48}$$

We will prove that the points $(\tau,\eta,\tilde{\eta}) \in \Sigma$ with $\tau = \pm iV_2\sqrt{\eta^2+\tilde{\eta}^2}$ are not the roots of (3.44). We assume $V_2 > 0$. Simple calculation yields that

$$V_2 + u^{r,l}\cos\theta > \sqrt{c^2 + g_r(\theta)}, \quad \text{and} \quad -V_2 + u^{r,l}\cos\theta < -\sqrt{c^2+g_r(\theta)}.$$

If $\tau = iV_2\sqrt{\eta^2+\tilde{\eta}^2}$, we have $\gamma = \Re\tau = 0$, and $\delta + u^{r,l}\eta = V_2\sqrt{\eta^2+\tilde{\eta}^2} + u^{r,l}\eta$. We claim that in this situation $\delta + u^{r,l}\eta \neq 0$. If $\delta + u^{r,l}\eta = 0$, say, for instance, $\delta + u^r\eta = 0$, then from (3.28), we have that ω^r is real and negative. If $\eta = 0$, $\tilde{\eta} \neq 0$, then we have $\delta + u^l\eta = \delta \neq 0$. If $\eta \neq 0$, then we also have $\delta + u^l\eta \neq 0$. All of these cases lead to $\Im w^l \neq 0$. Thus $\omega^r\omega^l$ can not be a real number, which violates (3.44). Therefore $\delta + u^{r,l}\eta \neq 0$, $\Im\omega^{r,l}$ and $\delta + u^{r,l}\eta$ are of opposite signs respectively, and ω^r and ω^l are purely imaginary and

$$\Omega^{r,l} = \Im\omega^{r,l}\sqrt{\eta^2+\tilde{\eta}^2} \in \mathbb{R},$$

from which we deduce that

$$\operatorname{sgn}(\Omega^{r,l}) = -\operatorname{sgn}\left(V_2 + \frac{u^{r,l}\eta}{\sqrt{\eta^2+\tilde{\eta}^2}}\right) = -\operatorname{sgn}(V_2 + u^{r,l}\cos\theta) = -1.$$

Therefore, $\Omega^r\Omega^l \neq -1$ and (3.44) is not satisfied. Similarly we can show that $(\tau,\eta,\tilde{\eta}) \in \Sigma$ with $\tau = -iV_2\sqrt{\eta^2+\tilde{\eta}^2}$ are also not the roots for (3.44).

If the particular solution defined by (3.1) satisfies (3.38), then, together with (3.45), we obtain that $\tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}$ are real for a certain choice of $(\eta,\tilde{\eta})$. Therefore it follows that $\delta = 0$. Since $\eta \neq 0$ and $\Re\tau \neq 0$, we have $\delta + u^r\eta \neq 0$. From (3.31) and (3.32), we obtain that $p^r = p^l$ and $q^r = -q^l \neq 0$. Using (3.30), we have $x^r = x^l, y^r = -y^l$. Note that ω^r is the conjugate of ω^l. Then $\omega^r\omega^l > 0$. This implies that $\tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}$ are the roots of (3.44). Hence, we can find a root $(\tau,\eta,\tilde{\eta})$ with $\Re\tau > 0$. This violates the Lopatinskiĭ conditions, which proves (Case1).

On the other hand, if (3.39) holds, then we have $V_1^2 > 0$ for all $\theta \in [0, 2\pi]$. This implies that $\tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}$ are purely imaginary. For simplicity, we consider $\tau = iV_1\sqrt{\eta^2+\tilde{\eta}^2}$, then $\Re\tau = 0$, but $\delta \neq 0$, and $\sqrt{\eta^2+\tilde{\eta}^2} \neq 0$. We then have

$$|V_1 + u^{r,l}\cos\theta| < \sqrt{g_r(\theta)+c^2}. \tag{3.49}$$

By (3.28) and (3.49), it follows that $(\omega^{r,l})^2$ are both real and positive, and hence $\omega^r\omega^l > 0$. Thus $(\tau, \eta, \tilde{\eta}) \in \Sigma$ with $\tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}$ being the roots of (3.44). Now, we prove that the roots to (3.44) are simple. Since (3.44) does not admit a root at $\eta = \tilde{\eta} = 0$, the points $(\tau, \eta, \tilde{\eta}) \in \Sigma$ satisfying $\omega^{r,l} = 0$ are not the roots of $\omega^r\omega^l - \eta^2 - \tilde{\eta}^2 = 0$. From (3.28), $\omega^{r,l}$ are analytic near the points where $\omega^{r,l}$ do not vanish. We can differentiate (3.46) with respect to V at $V = V_1$ to obtain that

$$\frac{d\Omega^{r,l}}{dV}\Big|_{V=V_1} = \frac{V_1 + u^{r,l}\cos\theta}{\Omega^{r,l}c^2}.$$

Thus,

$$\frac{d(\Omega^r\Omega^l+1)}{dV}\Big|_{V=V_1} = \frac{(V_1+u^r\cos\theta)(\Omega^l)^2 + (V_1+u^l\cos\theta)(\Omega^r)^2}{c^2\Omega^r\Omega^l}.$$

Using (3.46) and (3.47), we have

$$\frac{d(\Omega^r\Omega^l+1)}{dV}\Big|_{V=V_1} = \frac{2V_1[V_1^2 - g_r(\theta) - c^2]}{c^4\Omega^r\Omega^l} \neq 0. \tag{3.50}$$

Hence, we have showed that $(\tau, \eta, \tilde{\eta}) \in \Sigma$ with $\tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}$ are all simple roots of (3.44) under the condition (3.9). We also have

$$\omega^r\omega^l - \eta^2 - \tilde{\eta}^2 = (\tau \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2})h^{\pm}(\tau, \eta, \tilde{\eta})$$

for some continuous function $h^{\pm}(\tau, \eta, \tilde{\eta}) \neq 0$ respectively.

If the borderline relation (3.43) holds, say $\inf\limits_{\cos\theta\neq 0} g_r(\theta)/\cos^2\theta$ is attained at some $\mathbf{s}_* = (\cos\theta_*, \sin\theta_*) \in \mathbb{S}^1$. Then at such an $\mathbf{s}_*$ with the corresponding $(\eta, \tilde{\eta})$ we have $\tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2} = 0$. So $\Re\tau = \delta = 0$. From (3.31) and (3.32) it follows that $p^{r,l} = \eta^2+\tilde{\eta}^2 > 0$ and $q^{r,l} = 0$, which implies that $\omega^{r,l}$ are both negative and real, leading to $\omega^r\omega^l > 0$. Therefore such a point $(0, \eta, \tilde{\eta}) \in \Sigma$ gives a root of (3.44). Now, we need to check the multiplicity of this point, from (3.50) we know that the first derivative vanishes at $V = V_1$. Direct calculation indicates that the second derivative is non-degenerate there. Thus we conclude (Case4).

Step 4: The last factor $\omega^r + \omega^l$.

We turn to the last factor in (3.37):

$$\omega^r + \omega^l = 0. \tag{3.51}$$

It is easy to see from (3.28) that if $\Re\tau > 0$ then $\Re\omega^r < 0$ and $\Re\omega^l < 0$, and thus $\omega^r+\omega^l \neq 0$. So we will consider the case $\Re\tau = 0$. Using (3.32), we have $q^{r,l} = 0$. Condition (3.51) the definition of (3.45) infer that $(\Omega^r)^2 = (\Omega^l)^2$, which leads to $p^r = p^l$. Using (3.31), we obtain that

$$2u^r\delta\eta = 2u^l\delta\eta,$$

which implies $\delta\eta = 0$. If $\delta = \eta = 0$, then we have $\tilde{\eta} \neq 0$, and hence $p^{r,l} > 0$. Hence, $\omega^{r,l}$ are real and negative, which contradicts with (3.51). If $\delta \neq 0$ and $\eta = 0$, then we can assume $\delta > 0$, and it follows that $\delta + u^{r,l}\eta > 0$. From (3.33), we have $\Im\omega^r = \Im\omega^l < 0$. This contradicts with (3.51). Finally, we are left with $\eta \neq 0$ and $\delta = 0$. In this case we have

$$p^{r,l} = \left(\eta^2 + \tilde{\eta}^2\right)\left[\frac{-(u^{r,l}\cos\theta)^2 + g_r(\theta)}{c^2} + 1\right].$$

Under the assumption (3.39), we have $p^r = p^l > 0, q^r = q^l = 0$. Thus $\omega^{r,l}$ are both real and negative, contradicting (3.51).

We further remark that under the assumption (3.39), $\pm u^r\eta \neq \pm V_1\sqrt{\eta^2+\tilde{\eta}^2}$, the roots will not coincide. Therefore, we have derived the possible roots $(\tau, \eta, \tilde{\eta})$ of the Lopatinskiĭ determinant

$$\tau = -iu^{r,l}\eta \text{ (at most double roots)}, \quad \tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}.$$

This proves (Case2).

The above argument reveals all the possibilities for the roots $(\tau, \eta, \tilde{\eta})$ of the Lopatinskiĭ determinant, that is,

$$\tau = -iu^{r,l}\eta, \quad \tau = \pm iV_1\sqrt{\eta^2+\tilde{\eta}^2}, \quad \text{or} \quad \tau = 0$$

with the assumption that $u^r > 0$. We have discussed the possibility when $V_1 = 0$. The final left-over case is when $u^r\eta = V_1\sqrt{\eta^2+\tilde{\eta}^2}$, i.e., $u^r\cos\theta = V_1$. Solving this relation directly we find the condition (3.42), and finally we conclude Lemma 3.3. ☐

Remark 3.14. As is pointed out in Remark 3.7, to recover the 2D result of [38, Lemma 5.1], we will take $\mathbf{F} \in \mathbf{M}^{2\times 2}$ with $\mathrm{F}_2 = 0$, and $\tilde{\eta} = 0$ in the computation. Thus $\mathbf{s}$ degenerates to a scalar, i.e. $\cos\theta = 1$. This way the elastic sound speed becomes a constant

$$g_{r,l} = (F_{11}^{r,l})^2 + (F_{12}^{r,l})^2,$$

which gives the stable subsonic region, with the "degenerate" elastic sound speed in (3.42) being

$$\sqrt{\frac{\left((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2\right)\left((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2 + 2c^2\right)}{4\left((F_{11}^{r,l})^2 + (F_{12}^{r,l})^2 + c^2\right)}}.$$

On the other hand, the stable supersonic threshold in (3.38), which is unbounded in the three-dimensional case, also becomes a constant given by

$$(F_{11}^{r,l})^2 + (F_{12}^{r,l})^2 + 2c^2,$$

and hence agrees with [38, Lemma 5.1].

Lemma 3.3 provides a detailed description of the root distribution of the Lopatinskiĭ determinant under certain algebraic relation between the tangential velocity u^r of the flow and the projected elastic sound speed $g_r(\theta)$. The following lemma further unravels such relation in terms of the elastic deformation.

Lemma 3.4. *Consider a particular solution defined by* (3.1) *and recall the definitions* (3.27) *and* (3.7). *Then*

$$\inf_{\cos\theta\neq 0}\frac{g_r(\theta)}{\cos^2\theta} = |\mathrm{F}_1|^2 - \frac{|\mathrm{F}_1\cdot\mathrm{F}_2|}{|\mathrm{F}_2|^2} = \left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2, \tag{3.52}$$

where we recall the projections operators in (3.8). *This minimum is attained at* $\mathbf{s}_* = (\cos\theta_*, \sin\theta_*) \in \mathbb{S}^1$ *where*

$$\tan\theta_* = -\frac{|\Pi_{\mathrm{F}_2}(\mathrm{F}_1)|}{|\mathrm{F}_2|}. \tag{3.53}$$

Moreover, for $\mathrm{F}_1, \mathrm{F}_2 \in \mathbb{R}^3$*, set*

$$\mathcal{F}(\mathrm{F}_1, \mathrm{F}_2) := \inf_{\cos\theta\neq 0} \frac{g_r(\theta)\left(g_r(\theta) + 2c^2\right)}{4\cos^2\theta\,(g_r(\theta) + c^2)}. \tag{3.54}$$

Then $\mathcal{F}$ *is well-defined, and*

$$\frac{\left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2}{4} \le \mathcal{F}(\mathrm{F}_1, \mathrm{F}_2) \le \frac{\left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2}{2}. \tag{3.55}$$

Proof. Using $\mathrm{F}_1, \mathrm{F}_2$ as in (3.7) we can write

$$\frac{g_r(\theta)}{\cos^2\theta} = |\mathrm{F}_1|^2 + |\mathrm{F}_2|^2\tan^2\theta + 2(\mathrm{F}_1\cdot\mathrm{F}_2)\tan\theta.$$

Further introducing notations

$$t := \frac{|\mathrm{F}_2|}{|\mathrm{F}_1|}, \quad \alpha := \text{ the angle between } \mathrm{F}_1 \text{ and } \mathrm{F}_2,$$

the above becomes

$$\begin{aligned}\frac{g_r(\theta)}{\cos^2\theta} &= |\mathrm{F}_1|^2\left(1 + 2t\cos\alpha\tan\theta + t^2\tan^2\theta\right)\\ &= |\mathrm{F}_1|^2\left(\sin^2\alpha + (\cos\alpha + t\tan\theta)^2\right) =: |\mathrm{F}_1|^2 f(\theta).\end{aligned} \tag{3.56}$$

It is obvious that

$$\begin{aligned}\frac{g_r(\theta)}{\cos^2\theta} &= |\mathrm{F}_1|^2\left(t^2\left(\tan\theta + \frac{\cos\alpha}{t}\right)^2 + 1 - \cos^2\alpha\right)\\ &\ge |\mathrm{F}_1|^2(1-\cos^2\alpha) = \left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2,\end{aligned}$$

giving (3.52), where equality holds if and only if $\theta = \theta_*$ where $\tan\theta_* = -\cos\alpha/t$, which is (3.53).

For (3.54) we write

$$\frac{g_r(\theta)\left(g_r(\theta) + 2c^2\right)}{4\cos^2\theta\,(g_r(\theta) + c^2)} = \frac{|\mathrm{F}_1|^2}{4}\cdot\frac{f(\theta)\left(f(\theta)\cos^2\theta + 2\tilde{c}^2\right)}{f(\theta)\cos^2\theta + \tilde{c}^2} =: h(\theta), \tag{3.57}$$

where $\tilde{c} = \frac{c}{|\mathrm{F}_1|}$. Note that

$$\frac{g_r(\theta)}{4\cos^2\theta} = \frac{|\mathrm{F}_1|^2}{4}f(\theta) < h(\theta) < \frac{g_r(\theta)}{2\cos^2\theta} = \frac{|\mathrm{F}_1|^2}{2}f(\theta). \tag{3.58}$$

Taking infimum and using (3.52) it holds that

$$\frac{\left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2}{4} \le \inf_{\cos\theta\neq 0} h(\theta) \le \frac{\left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2}{2}. \tag{3.59}$$

Redefining f and h to be functions of $x := \tan\theta$ we see that both f and h are defined for all $x \in \mathbb{R}$, and $f \to +\infty$ as $|x| \to \infty$. From (3.58) it follows that $h \to +\infty$ as $|x| \to \infty$.

Therefore $\inf_{x\in\mathbb{R}} h$, and thus $\inf_{\cos\theta\neq 0} h(\theta)$ exists. This means that the function $\mathcal{F}$ in (3.54) is well-defined. Finally (3.55) follows from (3.59). $\square$

With the help of the above lemma we can interpret Lemma 3.3 in a geometrical way.

Lemma 3.5. *Consider a particular solution defined by* (3.1) *and recall the definition* (3.7). *Then the conditions in Lemma 3.3 can be equivalently stated as follows:*

$$(\text{Case1}) \Longleftrightarrow (u^r)^2 > \left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2; \tag{3.60}$$

$$(\text{Case2}) \Longleftrightarrow 0 < (u^r)^2 < \mathcal{F}(\mathrm{F}_1, \mathrm{F}_2); \tag{3.61}$$

$$(\text{Case3}) \Longleftrightarrow (u^r)^2 = \mathcal{F}(\mathrm{F}_1, \mathrm{F}_2); \tag{3.62}$$

$$(\text{Case4}) \Longleftrightarrow (u^r)^2 = \left|\Pi^{\perp}_{\mathrm{F}_2}(\mathrm{F}_1)\right|^2. \tag{3.63}$$

Proof. First we know from (3.52) that (3.63) holds.

From (3.56) we see that

$$(\text{Case1}) \Longrightarrow (u^r)^2 > |\mathrm{F}_1|^2 f(\theta_*).$$

To check the converse, since $f(\theta)$ is quadratic in $\tan\theta$ and $f(\theta) \to +\infty$ as $\theta \to \pi/2$, we know that there exists some $\tilde{\theta} \in (\theta_*, \pi/2)$ such that

$$(u^r)^2 = |\mathrm{F}_1|^2 f(\tilde{\theta}).$$

Continuity of f implies the existence of some θ_0 close to $\tilde{\theta}$ such that

$$|\mathrm{F}_1|^2 f(\theta_0) < (u^r)^2 < 2c^2 + |\mathrm{F}_1|^2 f(\theta_0),$$

which implies (Case1).

For (3.61) and (3.62), it suffices to prove (3.62). The argument goes in a similar way as we proved (3.60). The "$\Longrightarrow$" part follows easily from (3.54). For the "$\Longleftarrow$" part, i.e.,

$$|\mathrm{F}_1|^2 f(\theta_*) > (u^r)^2 \geq h(\theta_*),$$

using $h(\theta)$ in (3.57) we find that

$$\frac{|\mathrm{F}_1|^2 f(\theta)}{2} > h(\theta) > \frac{|\mathrm{F}_1|^2 f(\theta)}{4} \to +\infty \quad \text{as} \quad \theta \to \frac{\pi}{2}.$$

Continuity of $h(\theta)$ implies the existence of some θ_0 such that $(u^r)^2 = h(\theta_0)$, leading to (Case3). $\square$

Using Lemma 3.3, we have the following property on the stable subspace of A near the the roots of the Lopatinskiĭ determinant.

Lemma 3.6. *Let* $(\tau_0, \eta_0, \tilde{\eta}_0) \in \Sigma$ *be a root of the Lopatinskiĭ determinant* Δ. *If* $\mathrm{F}_1 \times \mathrm{F}_2 \neq 0$ *and* (3.9) *holds, then there is a neighborhood of* $(\tau_0, \eta_0, \tilde{\eta}_0)$ *which excludes any other roots of* Δ *and a constant* κ_0, *such that for any* $(\tau, \eta, \tilde{\eta}) \in \mathcal{V}$ *and* $X^- \in \mathbb{R}^2$,
(i) If (3.39) *and* (3.40) *hold, then*

$$|\beta(E^r, E^l)X^-|^2 \geq \kappa_0 \gamma^4 |X^-|^2.$$

(ii) Assume (3.39) *and suppose* (3.42) *holds for some* θ_0, *then*

$$|\beta(E^r, E^l)X^-|^2 \geq \kappa_0 \gamma^6 |X^-|^2.$$

(iii) If $\inf_{\cos\theta\neq 0}\dfrac{g_r(\theta)}{\cos^2\theta}$ *is attainable and* (3.43) *holds, then when* $\tau_0 = -iu^{r,l}\eta_0$ *or* $\tau_0 = 0$ *we have*

$$|\beta(E^r, E^l)X^-|^2 \geq \kappa_0\gamma^4|X^-|^2;$$

Proof. We rewrite the Lopatinskiĭ matrix as follows:

$$\beta(E^r, E^l) = \begin{bmatrix} -a^r + b^r & b^l - a^l \\ -c(\tau - iu^r\eta)(a^r + b^r) & c(\tau + iu^r\eta)(a^l + b^l) \end{bmatrix} =: \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}.$$

First, we observe that each component of $\beta(E^r, E^l)$ is continuous. We know that if one element of this matrix is not zero at $(\tau_0, \eta_0, \tilde{\eta}_0)$, then there is a neighborhood $\mathcal{V}$ of $(\tau_0, \eta_0, \tilde{\eta}_0)$ such that the matrix can be diagonalized by non-singular matrices $\mathcal{P}, \mathcal{Q}$ in $\mathcal{V}$, that is

$$\mathcal{P}\beta(E^r_-, E^l_-)\mathcal{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \Delta \end{bmatrix}. \tag{3.64}$$

For instance, if $d_{11} \neq 0$, then we have obtained the identity as follows

$$\begin{bmatrix} \frac{1}{d_{11}} & 0 \\ -\frac{d_{21}}{d_{11}} & 1 \end{bmatrix} \beta(E^r, E^l) \begin{bmatrix} 1 & -d_{12} \\ 0 & d_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \Delta \end{bmatrix}.$$

We note that in our case

$$\begin{aligned} d_{11} = &- [(\tau + iu^r\eta)^2 + (\eta F^r_{11} + \tilde{\eta}F^r_{21})^2 + (\eta F^r_{12} + \tilde{\eta}F^r_{22})^2 + (\eta F^r_{13} + \tilde{\eta}F^r_{23})^2] \\ &\times [(\tau + iu^r\eta)\omega^r - c((\omega^r)^2 - \eta^2 - \tilde{\eta}^2)], \\ d_{12} = &- [(\tau + iu^l\eta)^2 + (\eta F^l_{11} + \tilde{\eta}F^l_{21})^2 + (\eta F^l_{12} + \tilde{\eta}F^l_{22})^2 + (\eta F^l_{13} + \tilde{\eta}F^l_{23})^2] \\ &\times [(\tau + iu^l\eta)\omega^l - c((\omega^l)^2 - \eta^2 - \tilde{\eta}^2)]. \end{aligned}$$

From previous argument, we note that $(\tau + iu^{r,l}\eta)\omega^{r,l} - c((\omega^{r,l})^2 - \eta^2 - \tilde{\eta}^2) \neq 0$. Therefore,

$d_{11} = 0$ if and only if $\tau = -iu^r\eta \pm i\sqrt{(\eta F^r_{11} + \tilde{\eta}F^r_{21})^2 + (\eta F^r_{12} + \tilde{\eta}F^r_{22})^2 + (\eta F^r_{13} + \tilde{\eta}F^r_{23})^2}$,

$d_{12} = 0$ if and only if $\tau = iu^r\eta \pm i\sqrt{(\eta F^r_{11} + \tilde{\eta}F^r_{21})^2 + (\eta F^r_{12} + \tilde{\eta}F^r_{22})^2 + (\eta F^r_{13} + \tilde{\eta}F^r_{23})^2}$.

Suppose $d_{11} = d_{12} = 0$, we have

$$\tau = 0, \text{ and } u^r\eta = \pm\sqrt{(\eta F^r_{11} + \tilde{\eta}F^r_{21})^2 + (\eta F^r_{12} + \tilde{\eta}F^r_{22})^2 + (\eta F^r_{13} + \tilde{\eta}F^r_{23})^2}.$$

If $\eta = 0$, then we have $\tilde{\eta} = 0$. This again leads to a contradiction with $(\tau, \eta, \tilde{\eta}) \in \Sigma$. Hence, $\eta \neq 0$. Therefore,

$$\tau \pm iu^r\eta \neq 0, \quad \omega^{r,l} \neq 0.$$

Simple calculation shows that $d_{21} \neq 0$ and $d_{22} \neq 0$. This argument is different from the 2D case, because of addition frequency directions. Hence, after performing diagonalzation of the matrix $\beta(E^r, E^l)$, for any $(\tau_0, \eta_0, \tilde{\eta}_0)$, the matrix locally and continuously transform into diag$\{1, \Delta\}$ in an open neighborhood $\mathcal{V}$ of $(\tau_0, \eta_0, \tilde{\eta}_0)$. Therefore, by utilizing the continuity and boundedness of d_{ij}, the equation (3.64) implies that

$$|\beta(E^r_-, E^l_-)X^-|^2 \geq \kappa\min(1, |\Delta|^2)|X^-|^2,$$

in $\mathcal{V}$, where $\kappa > 0$ depends only on the boundary point $(\tau_0, \eta_0, \tilde{\eta}_0)$. $\mathcal{V}$ can only be taken as the neighborhood that contains the only one root of Δ. This finishes the proof. $\square$

3.6. **Energy estimates.** With all the preparation in the previous discussion, we are ready to derive the desired energy estimates. For a generating point $(\tau_0, \eta_0, \tilde{\eta}_0)$, we can find a neighborhood $\mathcal{V}$ where we have separated different modes of A and the estimates of the Lopatinskiĭ determinant. Note that $\Delta(\tau_0, \eta_0, \tilde{\eta}_0) \neq 0$. Hence $\Delta \neq 0$ at every point of $\mathcal{V}$. Repeating this throughout Σ yields a finite covering $\{\mathcal{V}_i\}_{i=1}^N$ of Σ with generating points $\{(\tau_i, \eta_i, \tilde{\eta}_i)\}_{i=1}^N$, from which we can construct a smooth partition of the unity with cut-off functions $\chi_i \in C_0^\infty(\mathcal{V}_i)$ for $i = 1, \cdots, N$ associated with this covering such that $\sum_{i=1}^N \chi_i^2 = 1$ on Σ. Such a covering includes all the neighborhoods $\mathcal{V}$ of $(\tau_0, \eta_0, \tilde{\eta}_0)$ such that $\Delta(\tau_0, \eta_0, \tilde{\eta}_0) = 0$.

We start by deriving the energy estimate in each conic zone $\Pi_i := \{(\tau, \eta, \tilde{\eta}) : k \cdot (\tau, \eta, \tilde{\eta}) \in \mathcal{V}_i, \text{ for some } k > 0\}$. In each neighborhood $\mathcal{V}_i$ of $(\tau_i, \eta_i, \tilde{\eta}_i)$, we extend χ_i and the transformation matrix T_i to the whole region Π_i as homogeneous mappings of degree 0 with respect to $(\tau, \eta, \tilde{\eta})$. Then we focus on

$$X = \chi_i T_i^{-1} \widehat{W}^{nc} \tag{3.65}$$

for all $(\tau, \eta, \tilde{\eta}) \in \Pi_i$. $X = (X_1, X_2, X_3, X_4)^T$ satisfies the following system of ODEs

$$\frac{dX}{dx_3} = (T_i^{-1} A T_i) X.$$

Then we only need to estimate X for $\Re\tau > 0$. From (3.36), the second and fourth equations are

$$\frac{dX_2}{dx_3} = -\omega^r X_2, \qquad \frac{dX_4}{dx_3} = -\omega^l X_4, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i,$$

with $\Re\tau > 0$. By (3.28), we obtain that $\Re\omega^{r,l}(\tau, \eta, \tilde{\eta}) < 0$ provided that $\Re\tau > 0$. Besides, since $\widehat{W}(\tau, \eta, \tilde{\eta}, \cdot) \in L^2$ and T_i^{-1} is a smooth invertible mapping and bounded from above in Π_i, we obtain that $X(\tau, \eta, \tilde{\eta}, \cdot) \in L^2, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i$. Hence solving the above ODE, we obtain that

$$X_2 = 0, \qquad X_4 = 0, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i, \tag{3.66}$$

with $\Re\tau > 0$. For X_1 and X_3, from (3.65) and (3.66) we have

$$\chi_i \widehat{W}^{nc} = T_i X = (E_-^r, E_-^l) \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i,$$

with $\Re\tau > 0$. Then, the boundary conditions become

$$\chi_i H = \chi_i \beta \widehat{W}^{nc}|_{x_3=0} = \beta(E_-^r, E_-^l) \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}\Big|_{x_3=0}, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i, \tag{3.67}$$

with $\Re\tau > 0$.

If $\det(\beta(E_-^r, E_-^l)) \neq 0$ at $(\tau_i, \eta_i, \tilde{\eta}_i)$, we obtain that

$$|\beta(E_-^r, E_-^l) X^-|^2 \geq \kappa_i |X^-|^2,$$

where $(\tau, \eta, \tilde{\eta}) \in \mathcal{V}_i$ and $X^- \in \mathbb{R}^2$, and κ_i is a positive constant depending on $(\tau_i, \eta_i, \tilde{\eta}_i)$. Since β is homogeneous of degree 0, we obtain that

$$|\beta(E_-^r, E_-^l) X^-|^2 \geq \kappa_i |X^-|^2,$$

where $(\tau, \eta, \tilde{\eta}) \in \Pi_i$ and $X^- \in \mathbb{R}^2$. By (3.67), we obtain that

$$\left| \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}_{x_3=0} \right|^2 \leq \frac{\chi_i^2}{\kappa_i} |H|^2, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i, \tag{3.68}$$

with $\Re\tau > 0$.

If $(\tau_i, \eta_i, \tilde{\eta}_i)$ is a simple root of Δ, from the proof of Lemma 3.6, we obtain that

$$\left| \beta(E_-^r, E_-^l) \right|^2 \geq \kappa_i \gamma^2 |X^-|^2$$

for all $(\tau, \eta, \tilde{\eta}) \in \mathcal{V}_i$ and $X^- \in \mathbb{R}^2$. Since β is homogeneous of degree 0, we obtain that

$$(|\tau|^2 + \eta^2 + \tilde{\eta}^2) |\beta(E_-^r, E_-^l)|^2 \geq \kappa_i \gamma^2 |X^-|^2,$$

where $(\tau, \eta, \tilde{\eta}) \in \Pi_i$ and $X^- \in \mathbb{R}^2$. Using (3.67), we have

$$\left| \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}_{x_3=0} \right|^2 \leq \frac{\chi_i^2 (|\tau|^2 + \eta^2 + \tilde{\eta}^2)}{\kappa_i \gamma^2} |H|^2, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i, \tag{3.69}$$

with $\Re\tau > 0$.

If $(\tau_i, \eta_i, \tilde{\eta}_i)$ is a double root of Δ we obtain that

$$\left| \beta(E_-^r, E_-^l) X^- \right|^2 \geq \kappa_i \gamma^4 |X^-|^2, \quad \forall (\tau, \eta, \tilde{\eta}) \in \mathcal{V}_i$$

and $X^- \in \mathbb{R}^2$. Since β is homogeneous of degree 0, we obtain that

$$(|\tau|^2 + \eta^2 + \tilde{\eta}^2)^2 |\beta(E_-^r, E_-^l) X^-|^2 \geq \kappa_i \gamma^4 |X^-|^2,$$

where $(\tau, \eta, \tilde{\eta}) \in \Pi_i$ and $X^- \in \mathbb{R}^2$. Using (3.67), we have

$$\left| \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}_{x_3=0} \right|^2 \leq \frac{\chi_i^2 (|\tau|^2 + \eta^2 + \tilde{\eta}^2)^2}{\kappa_i \gamma^4} |H|^2, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i \tag{3.70}$$

with $\Re\tau > 0$. From (3.66), (3.68)-(3.70) we have the following estimate for X in Π_i,

$$|X|_{x_3=0}|^2 \leq \chi_i^2 \frac{(|\tau|^2 + \eta^2 + \tilde{\eta}^2)^j}{\kappa_i \gamma^{2j}} |H|^2. \tag{3.71}$$

Here, $j = 1, 2$ represents the multiplicity of the roots of Lopatinskiĭ determinant.

Now, we prove the main theorem.

Proof of Theorem 3.1. When (3.9) holds, from Lemma 3.6, we know that either $\beta(E_-^r, E_-^l)$ is invertible at $(\tau_i, \eta_i, \tilde{\eta}_i)$ or $(\tau_i, \eta_i, \tilde{\eta}_i)$ is root of Δ with multiplicity at most two. From (3.71) we know that for $i = 1, \cdots, N$,

$$|X|_{x_3=0}|^2 \leq \chi_i^2 \frac{(|\tau|^2 + \eta^2 + \tilde{\eta}^2)^2}{\kappa_i \gamma^4} |H|^2, \quad \forall (\tau, \eta, \tilde{\eta}) \in \Pi_i$$

holds with $\Re\tau > 0$. From (3.65), we have

$$\chi_i^2 \left| T_i^{-1} \widehat{W}^{nc} \right|_{x_3=0}^2 \leq \chi_i^2 \frac{(|\tau|^2 + \eta^2 + \tilde{\eta}^2)^2}{\kappa_i \gamma^4} |H|^2.$$

Combining the boundedness of T_i in Π_i and adding all the estimates over all the conic zones $\{\Pi_i\}_{i=1}^N$, we have

$$\left|\widehat{W}^{nc}\Big|_{x_3=0}\right|^2 \leq C\frac{(|\tau|^2+\eta^2+\tilde{\eta}^2)^2}{\gamma^4}|H|^2, \quad \forall(\tau,\eta,\tilde{\eta}) \in \Pi \tag{3.72}$$

with $\Re\tau > 0$. Integrating the inequality (3.72) with respect to $(\delta,\eta,\tilde{\eta})$ over $\mathbb{R}^3$ yields

$$\left\|\widehat{W}^{nc}\Big|_{x_3=0}\right\|_0^2 \leq \frac{C}{\gamma^4}\|g\|_{2,\gamma}^2,$$

which gives (3.10).

When (3.11) holds, from Lemma 3.6, it follows that either $\beta(E_-^r, E_-^l)$ is an invertible matrix at $(\tau_i,\eta_i,\tilde{\eta}_i)$ or $(\tau_i,\eta_i,\tilde{\eta}_i)$ is a root of Δ with multiplicity at most three. Thus a similar argument to the above proves (3.12). □

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References

[1] M. Renardy A. Kaffel. On the stability of plane parallel viscoelastic shear flows in the limit of infinite weissenberg and reynolds numbers. *J. Non-Newton. Fluid Mech.*, 165(23-24):1670–1676, 2010.

[2] Y. Trakhinin A. Morando and P. Trebeschi. Structural stability of shock waves in 2d compressible elastodynamics. *Math. Ann.*, 378(3):1471–1504, 2020.

[3] J. Azaiez and G. Homsy. Linear stability of free shear flow of viscoelastic liquids. *J.Fluid Mech.*, 268:37–69, 1994.

[4] G.-Q. Chen and Y.-G. Wang. Existence and stability of compressible current-vortex sheets in three-dimensional magnetohydrodynamics. *Arch. Ration. Mech. Anal.*, 187(3):369–408, 2008.

[5] J.F. Coulombel. Weakly stable multidimensional shocks. In *Ann. Inst. H. Poincaré Anal. Non linéaire*, volume 21, pages 401–443. Elsevier, 2004.

[6] J.F. Coulombel and A. Morando. Stability of contact discontinuities for the nonisentropic euler equations. *Ann. Univ. Ferrara.*, 50(1):79–90, 2004.

[7] J.F. Coulombel and P. Secchi. The stability of compressible vortex sheets in two space dimensions. *Indiana Univ. Math. J.*, 53:941–1012, 2004.

[8] J.F. Coulombel and P. Secchi. Nonlinear compressible vortex sheets in two space dimensions. In *Ann. Sci. Ec. Norm. Super*, volume 41, pages 85–139, 2008.

[9] T. Funada D. Joseph and J. Wang. *Potential Flows of Viscous and Viscoelastic Liquids*. Cambridge Aerospace Series. Cambridge University Press, 2007.

[10] C.M. Dafermos. Hyperbolic conservation laws in continuum physics. *Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences)*, 2010.

[11] D. Wang F. Huang and D. Yuan. Nonlinear stability and existence of vortex sheets for inviscid liquid-gas two-phase flow. *Discrete Contin. Dyn. Syst. Ser. A.*, 39(6):3535, 2019.

[12] J.A. Fejer and W. Miles. On the stability of a plane vortex sheet with respect to three-dimensional disturbances. *J. Fluid Mech.*, 15(3):335–336, 1963.

[13] J. Francheteau and G. Metivier. Existence of weak shocks for multidimenional quasilinear hyperbolic systems. *Asterisque*, 268:1–198, 01 2000.
[14] P. Secchi G.-Q. Chen and T. Wang. Nonlinear stability of relativistic vortex sheets in three-dimensional minkowski spacetime. *Arch. Ration. Mech. Anal.*, 232(2):591–695, 2019.
[15] P. Secchi G.-Q. Chen and T. Wang. Stability of multidimensional thermoelastic contact discontinuities. *Arch. Ration. Mech. Anal.*, 237(3):1271–1323, 2020.
[16] S.B. Gavage and D. Serre. *Multi-dimensional hyperbolic partial differential equations: First-order Systems and Applications*. OUP Oxford, 2006.
[17] M.E. Gurtin. An introduction to continuum mechanics. *Mathematics in science and engineering*, 1981.
[18] W. Wang H. Li and Z. Zhang. Well-posedness of the free boundary problem in incompressible elastodynamics. *J. Differential Equations*, 267(11):6604–6643, 2019.
[19] R. Hersh. Mixed problems in several variables. *J. Math. Mech.*, pages 317–334, 1963.
[20] X. Hu and Y. Huang. Well-posedness of the free boundary problem for incompressible elastodynamics. *J. Differential Equations*, 266(12):7844–7889, 2019.
[21] X. Hu and D. Wang. Local strong solution to the compressible viscoelastic flow with large data. *J. Differential Equations*, 249(5):1179–1198, 2010.
[22] X. Hu and D. Wang. Formation of singularity for compressible viscoelasticity. *Acta Math. Sci. Ser. B Engl. Ed.*, 32(1):109–128, 2012.
[23] X. Hu and W. Zhao. Global existence for the compressible viscoelastic system with zero shear viscosity in three dimensions. *J. Differential Equations*, 268(4):1658–1685, 2020.
[24] X. Hu and W. Zhao. Global existence of compressible dissipative elastodynamics systems with zero shear viscosity in two dimensions. *Arch. Ration. Mech. Anal.*, 235(2):1177–1243, 2020.
[25] R.R. Huilgol. Propagation of a vortex sheet in viscoelastic liquids—the rayleigh problem. *J. Non-Newton. Fluid Mech.*, 8(3-4):337–347, 1981.
[26] R.R. Huilgol. *Fluid mechanics of viscoplasticity*. Springer, 2015.
[27] D. Joseph. *Fluid dynamics of viscoelastic liquids*, volume 84. Springer-Verlag, New York, 1990.
[28] S. Weng L. Ruan, D. Wang and C. Zhu. Rectilinear vortex sheets of inviscid liquid-gas two-phase flow: linear stability. *Commun. Math. Sci.*, 14(3):735–776, 2016.
[29] P.D. Lax and R.S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure. Appl. Math.*, 13(3):427–455, 1960.
[30] C. Liu and N.J. Walkington. An eulerian description of fluids containing visco-elastic particles. *Arch. Ration. Mech. Anal.*, 159(3):229–252, 2001.
[31] A.J. Majda. *The existence of multi-dimensional shock fronts*, volume 281. American Mathematical Soc., 1983.
[32] A.J. Majda. *The stability of multi-dimensional shock fronts*, volume 275. American Mathematical Soc., 1983.
[33] A.J. Majda and S. Osher. Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary. *Comm. Pure. Appl. Math.*, 28(5):607–675, 1975.
[34] J.W. Miles. On the reflection of sound at an interface of relative motion. *J. Acoust. Soc. Am.*, 29(2):226–228, 1957.
[35] J.W. Miles. On the disturbed motion of a plane vortex sheet. *J. Fluid. Mech.*, 4(5):538–552, 1958.
[36] J.G. Oldroyd. Non-newtonian effects in steady motion of some idealized elastico-viscous liquids. *Proc. R. Soc. Lond. Ser. A.*, 245(1241):278–297, 1958.
[37] D. Wang T. Wang R.M. Chen, J. Hu and D. Yuan. Nonlinear stability and existence of compressible vortex sheets in 2d elastodynamics. *J. Differential Equations*, 269(9):6899–6940, 2020.
[38] J. Hu R.M. Chen and D. Wang. Linear stability of compressible vortex sheets in two-dimensional elastodynamics. *Adv. Math.*, 311:18–60, 2017.
[39] J. Hu R.M. Chen and D. Wang. Linear stability of compressible vortex sheets in 2d elastodynamics: variable coefficients. *Math. Ann.*, 376(3):863–912, 2020.
[40] D. Serre. *Systems of Conservation Laws 2: Geometric Structures, Oscillations, and Initial-Boundary Value Problems*, volume 2. Cambridge University Press, 1999.
[41] A.W. Bargteil T.G. Goktekin and J.F. O'Brien. A method for animating viscoelastic fluids. In *ACM SIGGRAPH 2004 Papers*, pages 463–468. 2004.

[42] Y. Trakhinin. The existence of current-vortex sheets in ideal compressible magnetohydrodynamics. *Arch. Ration. Mech. Anal.*, 191(2):245–310, 2009.
[43] Y. Trakhinin. Well-posedness of the free boundary problem in compressible elastodynamics. *J. Differential Equations*, 264(3):1661–1715, 2018.
[44] Y.-G. Wang and F. Yu. Stability of contact discontinuities in three-dimensional compressible steady flows. *J. Differential Equations*, 255(6):1278–1356, 2013.
[45] Y.-G. Wang and F. Yu. Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets. *Arch. Ration. Mech. Anal.*, 208(2):341–389, 2013.
[46] Y.-G. Wang and F. Yu. Structural stability of supersonic contact discontinuities in three-dimensional compressible steady flows. *SIAM J. Math. Anal.*, 47(2):1291–1329, 2015.
[47] Y.-G. Wang and H. Yuan. Weak stability of transonic contact discontinuities in three-dimensional steady non-isentropic compressible euler flows. *Z. Angew. Math. Phys.*, 66(2):341–388, 2015.

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