ELECTRICALLY AND MAGNETICALLY CHARGED VORTICES IN THE CHERN–SIMONS–HIGGS THEORY

ROBIN MING CHEN, YUJIN GUO, DANIEL SPIRN AND YISONG YANG

ABSTRACT. In this paper, we prove the existence of finite-energy electrically and magnetically charged vortex solutions in the full Chern–Simons–Higgs theory for which both the Maxwell term and Chern–Simons term are present in the Lagrangian density. We consider both Abelian and non-Abelian cases. The solutions are smooth and satisfy natural boundary conditions. Existence is established via a constrained minimization procedure applied on indefinite action functionals. This work settles a long-standing open problem concerning the existence of dually charged vortices in the classical gauge field Higgs model minimally extended to contain a Chern–Simons term.

1. INTRODUCTION

In his celebrated work [24], Dirac showed that the existence of a magnetic monopole solution to the Maxwell equations has the profound implication that electric charges in the universe are all quantized. Later, Schwinger [64] further explored the idea of Dirac and proposed the existence of both electrically and magnetically charged particle-like solutions, called dyons and used them to model quarks. In particular, Schwinger [64] generalized the electric charge quantization condition of Dirac [24] to a quantization condition relating electric and magnetic charges of a dyon. In modern theoretical physics, dyons are considered as excited states of magnetic monopoles. Both magnetic monopoles and dyons, and their abundance, are predicted by grand unified theories [2, 31, 45, 46, 49, 60, 78]. The well-known finite-energy singularity-free magnetic monopole and dyon solutions in the Yang–Mills–Higgs theory include the monopole solutions due to Polyakov[58], t'Hooft [77], Bogomol'nyi [12], Prasad and Sommerfeld [59], Taubes [37, 76], and the dyon solutions due to Julia and Zee [38], Bogomol'nyi [12], Prasad and Sommerfeld [59]. See also [21, 85] for the construction of dyon solutions in the Weinberg–Salam electroweak theory. These are all static solutions of the governing gauge field equations in three-space dimensions.

Vortices arise as static solutions to gauge field equations in two-space dimensions. Unlike monopoles, magnetic vortices not only arise as theoretical constructs but also play important roles in areas such as superconductivity [1, 30, 37], electroweak theory [3, 4, 5, 6], and cosmology [81]. The mathematical existence and properties of such vortices have been well studied [7, 8, 9, 10, 25, 27, 37, 43, 44, 48, 50, 54, 55, 57, 65, 68, 69, 75, 86]. Naturally, it will be interesting and important to establish the existence of dyon-like vortices, simply called electrically charged vortices, carrying both electric and magnetic charges. Such dually charged vortices have applications in a wide range of areas including high-temperature superconductivity [40, 47], optics [11], the Bose–Einstein condensates [33, 39], the quantum Hall effect [66], and superfluids.

Surprisingly, unlike static gauge field theory in three-space dimensions, it is recognized that there can be no finite-energy electrically charged vortex solutions in two-space dimensions for the classical Yang–Mills–Higgs equations, Abelian or non-Abelian. The impossibility of finite-energy electrically charged solutions is known as the Julia–Zee theorem [38, 72]. Due to the pioneering studies of Jackiw–Templeton [34], Schonfeld [63], Deser–Jackiw–Templeton [22, 23], Paul–Khare [56], de Vega–Schaposnik [79, 80], and Kumar–Khare [41], it has become accepted that, in order to accommodate electrically charged vortices, one needs to introduce into the action Lagrangian a Chern–Simons topological term [19, 20], which has become a central structure in anyon physics [28, 82, 83]. Therefore, an imperative problem one encounters is to develop an existence theory for the solutions of the full Chern–Simons–Higgs equations [56, 79, 80] governing such electrically charged vortices.

This basic existence problem, however, has not yet been tackled in literature, despite of some successful numerical solutions reported [36]. In fact, the lack of understanding of the solutions of the full system of equations has led to some dramatic trade-wind changes in the research on the Chern–Simons vortices, starting from the seminal papers of Hong–Kim–Pac [32] and Jackiw–Weinberg [35], in which the Maxwell term is removed from the Lagrangian density while the Chern–Simons term stands out alone to govern the dynamics of electromagnetism. Physically, this procedure recognizes the dominance of the Chern–Simons term over the Maxwell term over large distances; mathematically, it allows one to pursue a Bogomol'nyi reduction [12] when the Higgs potential takes a critical form as that in the classical Abelian Higgs model [12, 37]. Such an approach triggered a wide range of exploration on the reduction of numerous Chern–Simons models, Abelian and non-Abelian, relativistic and non-relativistic (see [26] for a review) and a rich spectrum of mathematical existence results for the Bogomol'nyi type Chern–Simons vortex equations have been obtained [13, 15, 16, 17, 42, 52, 53, 61, 70, 71, 74, 75, 84]. We note that existence of planar Abelian Chern–Simons models with no Maxwell term for non Bogomol'nyi regimes has recently been established in [18, 67]. Although these contributions lead to considerable understanding of the properties of charged vortices when interaction between vortices is absent, the original problem of the existence of charged vortices, which are necessarily subject to interaction due to the lack of a Bogomol'nyi structure, in the Chern–Simons–Higgs theory containing a Maxwell term [41, 56, 79, 80] remains unsolved.

In the present paper, we will establish the existence of charged vortices in the full Chern–Simons–Higgs theory with the Maxwell term [41, 56, 79, 80] in both Abelian and non-Abelian cases.

The rest of the paper is organized as follows. In Section 2, we review the Abelian Chern–Simons–Higgs theory, discuss some basic properties of charged vortices and their governing equations, and state our main existence theorem. Then, we discuss the methods used in our proofs. In Section 3, we describe the basic setup of our problem and introduce our constraint space. In Section 4 through Section 6, we prove the existence of weak solutions. In Section 7, we show that our weak solutions are in fact classical solutions. In Section 8, we establish the quantization formulas (2.16) and (2.17) expected for the magnetic and electric charges. Finally, in Section 9, we apply our methods to solve the non-Abelian Chern–Simons–Higgs equations.

2. Abelian Chern–Simons–Higgs Equations and Main Existence Theorem

After adding a Chern–Simons term to the classical Abelian Higgs Lagrangian density [37, 51] and taking normalized units, the minimally extended action density, or the Chern–Simons–Higgs Lagrangian density introduced in [56, 79], defined over the Minkowski spacetime $\mathbb{R}^{2,1}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1)$, may be written in the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\varepsilon^{\mu\nu\alpha}A_\mu F_{\nu\alpha} + \frac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \frac{\lambda}{8}(|\phi|^2 - 1)^2, \tag{2.1}$$

where ϕ is a complex scalar function, the Higgs field, A_μ ($\mu = 0, 1, 2$) is a real-valued vector field, the Abelian gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the induced electromagnetic field, and $D_\mu = \partial_\mu + \mathrm{i}A_\mu$ is the gauge-covariant derivative, $\kappa > 0$ is a constant referred to as the Chern–Simons coupling parameter, $\varepsilon^{\mu\nu\gamma}$ is the Kronecker skewsymmetric tensor with $\varepsilon^{012} = 1$, and summation convention over repeated indices is observed. The extremals of the Lagrangian density (2.1) formally satisfy its Euler–Lagrange equations, or the Abelian Chern–Simons–Higgs equations [56],

$$D_\mu D^\mu\phi = \frac{\lambda}{2}\phi(1 - |\phi|^2), \tag{2.2}$$

$$\partial_\nu F^{\mu\nu} - \frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}F_{\nu\alpha} = -J^\mu. \tag{2.3}$$

in which (2.3) expresses the modified Maxwell equations so that the current density J^μ is given by

$$J^\mu = \frac{\mathrm{i}}{2}(\overline{\phi}D^\mu\phi - \phi\overline{D^\mu\phi}). \tag{2.4}$$

Recall that we may rewrite J^μ into a decomposed form, $J^\mu = (\rho, \mathbf{J})$ such that ρ represents electric charge density and $\mathbf{J} = J^k$ represents electric current density. Here, and in the sequel, we use the Latin letters $j, k = 1, 2$ to denote the indices of spatial components.

Therefore, since we will consider static configurations only so that all the fields are independent of the temporal coordinate, $t = x^0$, we have

$$\rho = J^0 = \frac{\mathrm{i}}{2}(\overline{\phi}D^0\phi - \phi\overline{D^0\phi}) = -A_0|\phi|^2, \tag{2.5}$$

which indicates that a nontrivial temporal component, A_0, of the gauge field A_μ is essential for the presence of electric charge. Besides, also recall that the electric field $\mathbf{E} = E^j$ (in the spatial plane) and magnetic fields H (perpendicular to the spatial plane) induced from the gauge field A_μ are

$$E^j = \partial_j A_0, \quad j = 1, 2; \quad H = F_{12}, \tag{2.6}$$

respectively. The static version of the Chern–Simons–Higgs equations (2.2) and (2.3) take the explicit form

$$D_j^2\phi = \frac{\lambda}{2}(|\phi|^2 - 1)\phi - A_0^2\phi, \tag{2.7}$$

$$\partial_k F_{jk} - \kappa\,\varepsilon_{jk}\partial_k A_0 = \frac{\mathrm{i}}{2}(\overline{\phi}D_j\phi - \phi\overline{D_j\phi}), \tag{2.8}$$

$$\Delta A_0 = \kappa F_{12} + |\phi|^2 A_0. \tag{2.9}$$

On the other hand, since the Chern–Simons term gives rise to a topological invariant, it makes no contribution to the energy-momentum tensor $T_{\mu\nu}$ of the action density (2.1) which may be calculated as

$$T_{\mu\nu} = -\eta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{2}([D_\mu\phi]\overline{[D_\nu\phi]} + \overline{[D_\mu\phi]}[D_\nu\phi]) - \eta_{\mu\nu}\mathcal{L}_0, \tag{2.10}$$

where $\mathcal{L}_0$ is obtained from the Lagrangian (2.1) by setting $\kappa = 0$. Hence, it follows that the Hamiltonian $\mathcal{H} = T_{00}$ or the energy density of the theory is given by

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}F_{01}^2 + \frac{1}{2}F_{02}^2 + \frac{1}{2}|\phi|^2 A_0^2 + \frac{1}{2}F_{12}^2 + \frac{1}{2}(|D_1\phi|^2 + |D_2\phi|^2) + \frac{\lambda}{8}(|\phi|^2 - 1)^2 \\ &= \frac{1}{2}|\nabla A_0|^2 + \frac{1}{2}|\phi|^2 A_0^2 + \frac{1}{2}F_{12}^2 + \frac{1}{2}(|D_1\phi|^2 + |D_2\phi|^2) + \frac{\lambda}{8}(|\phi|^2 - 1)^2, \end{aligned} \tag{2.11}$$

which is positive-definite and the terms in (2.11) not containing A_0 are exactly those appearing in the classical Abelian Higgs model [37, 51]. Thus, the finite-energy condition

$$E(A_0, A_j, \phi) = \int_{\mathbb{R}^2} \mathcal{H}(A_0, A_j, \phi)(x)\,\mathrm{d}x < \infty \tag{2.12}$$

leads us to arriving at the following natural asymptotic behavior of the fields A_0, A_j, and ϕ,

$$A_0,\ \partial_j A_0 \to 0, \tag{2.13}$$
$$F_{12} \to 0, \tag{2.14}$$
$$|\phi| \to 1,\ |D_A\phi| \to 0, \tag{2.15}$$

as $|x| \to \infty$. In analogue to the Abelian Higgs model [37, 51], we see that a finite-energy solution of the Chern–Simons–Higgs equations (2.7)–(2.9) should be classified by the winding number, say $N \in \mathbb{Z}$, of the complex scalar field ϕ near infinity, which is expected to give rise to the total quantized magnetic charge (or magnetic flux).

The resolution of the aforementioned open problem for the existence of charged vortices in the full Chern–Simons–Higgs theory amounts to prove that, for any integer N, the coupled nonlinear elliptic equations (2.7)–(2.9) over $\mathbb{R}^2$ possess a smooth solution (A_0, A_j, ϕ) satisfying the finite-energy condition (2.12) and natural boundary conditions (2.13)–(2.15) so that the winding number of ϕ near infinity is N.

Here is our main existence theorem, which solves the above problem.

Theorem 2.1. *For any given integer N, the Chern–Simons–Higgs equations (2.7)–(2.9) over $\mathbb{R}^2$ have a smooth finite-energy solution (A_0, A_j, ϕ) satisfying the asymptotic properties (2.13)–(2.15) as $|x| \to \infty$ such that the winding number of ϕ near infinity is N, which is also the algebraic multiplicity of zeros of ϕ in $\mathbb{R}^2$, and the total magnetic charge or flux Φ and electric charge Q are given by the quantization formulas*

$$\Phi = \int_{\mathbb{R}^2} F_{12}\,\mathrm{d}x = 2\pi N, \tag{2.16}$$
$$Q = \int_{\mathbb{R}^2} \rho\,\mathrm{d}x = 2\pi N\kappa. \tag{2.17}$$

Such a solution represents an N-vortex soliton which is indeed both magnetically and electrically charged.

The proof of Theorem 2.1 is contained in the proofs of Propositions 5.2, 6.1, and 8.1. In the subsequent sections, we shall establish this theorem.

Methodology. We use the following standard ansatz to represent a radially symmetric N-vortex solution of the Abelian Chern–Simons–Higgs equations so that the N vortices are clustered at the origin:

$$\phi(x) = u(r)\mathrm{e}^{\mathrm{i}N\theta}, \tag{2.18}$$

$$A_j(x) = Nv(r)\varepsilon_{kj}\frac{x_k}{r^2}, \quad j,k=1,2, \tag{2.19}$$

$$A_0(x) = w(r). \tag{2.20}$$

As derived by Paul and Khare [56] (and also de Vega and Schaposnik [79]), the equations of motion (2.7)–(2.9) become

$$u'' + \frac{1}{r}u' = \frac{N^2}{r^2}(v-1)^2u - w^2u + \frac{\lambda}{2}u(u^2-1), \tag{2.21}$$

$$v'' - \frac{1}{r}v' = (v-1)u^2 + \frac{\kappa r}{N}w', \tag{2.22}$$

$$w'' + \frac{1}{r}w' = u^2w + \frac{\kappa N}{r}v'. \tag{2.23}$$

Regularity and finite-energy condition prompt us to impose the boundary conditions

$$\lim_{r\to 0} u(r) = \lim_{r\to 0} v(r) = \lim_{r\to\infty} w(r) = 0, \tag{2.24}$$

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = 1, \tag{2.25}$$

$$\lim_{r\to 0} w(r) = w_0. \tag{2.26}$$

Here w_0 is some finite constant, depending on N, λ, and κ, that should arise from our constrained minimization procedure.

In order to establish existence we note that (2.21)–(2.23) are the Euler–Lagrange equations of the indefinite action functional

$$\begin{aligned}I(u,v,w) &= \int_0^\infty \Big(r(u')^2 + \frac{N^2}{r}u^2(v-1)^2 + \frac{\lambda}{4}(1-u^2)^2r + \frac{N^2}{r}(v')^2\Big)\,\mathrm{d}r \\ &\quad - \int_0^\infty \Big(r(w')^2 + ru^2w^2 + 2\kappa Nv'w\Big)\,\mathrm{d}r \\ &= G(u,v) - J_{u,v}(w).\end{aligned}$$

Here $G(u,v)$ is the standard Ginzburg–Landau functional for radially symmetric vortices, studied by Plohr [57] and Berger–Chen [8]. The functional $J_{u,v}(w)$ is indefinite and a source of difficulty in our existence problem.

Notice that, in view of the radially symmetric ansatz (2.18)–(2.20), the total energy calculated from the Hamiltonian density (2.11) is

$$\begin{aligned}E(u,v,w) = \pi\int_0^\infty \Bigg(&r(u')^2 + \frac{N^2}{r}(v')^2 + r(w')^2 \\ &+\frac{N^2}{r}u^2(v-1)^2 + ru^2w^2 + \frac{\lambda}{4}(u^2-1)^2r\Bigg)\,\mathrm{d}r.\end{aligned} \tag{2.27}$$

In Section 3, we discuss some general notation and definitions used throughout the paper and set up our constrained minimization space. In particular, we will minimize $I(u,v,w)$ over the space $\mathcal{C}$, consisting of triples (u,v,w) such that w is a weak solution to (2.23) with u,v given. This approach is similar to those of Schechter–Weder [62] and Yang [86] for the dyon problem in three spaces.

In Section 4, we assume bounded $G(u,v)$ energy and we show that $J_{u,v}(w)$ has a minimizer, say $w_{u,v}$, among H^1_r functions, and this minimizer is the unique critical point of $J_{u,v}(w)$. Here we first show that we have a uniform control of the radius R such that

$|u(r)| > \frac{1}{2}$ (say) outside the ball B_R which implies both the boundedness of $J_{u,v}(w)$, $C \geq J_{u,v}(w) \geq -C$, and the control of the H^1 norm of w. Such boundedness and H^1 control give us the existence of a minimizer for $J_{u,v}$.

We prove the existence of weak solutions of (2.21)–(2.23) in Sections 5 and 6. To do so we show that $I(u,v,w) \geq G(u,v)$ for $(u,v,w) \in \mathcal{C}$, which implies the coercivity of $I(u,v,w)$. Once we have this coercivity behavior, we can take a minimizing sequence in $\mathcal{C}$ and obtain a constrained minimizer. Such a minimizer can be shown to solve the equations (2.21)–(2.23) at least in a weak sense. Here some extra attention will be given to proving the existence of a Fréchet derivative.

In Section 7, we establish the boundary conditions and expected full regularity of our solutions. In Seciton 8, we obtain the quantization formulas for the magnetic and electric charges. In Section 9, we construct non-Abelian Chern–Simons–Higgs vortex solutions using our methods presented in the previous sections.

3. Radial equations, action principle, and the constrained admissible space

Recall that a radially symmetric solution of the Chern–Simons–Higgs theory with N vortices clustered at the origin satisfies the equations (2.21)–(2.23) which can be derived from the indefinite action functional

$$\begin{aligned} I(u,v,w) = \int_0^\infty \Big(& r(u')^2 + \frac{N^2}{r} u^2 (v-1)^2 + \frac{\lambda}{4}(1-u^2)^2 r + \frac{N^2}{r}(v')^2 \\ & - r(w')^2 - r u^2 w^2 - 2\kappa N v' w \Big)\, \mathrm{d}r. \end{aligned} \tag{3.1}$$

Let

$$G(u,v) = \int_0^\infty \Big(r(u')^2 + \frac{N^2}{r} u^2 (v-1)^2 + \frac{\lambda}{4}(1-u^2)^2 r + \frac{N^2}{r}(v')^2 \Big)\, \mathrm{d}r, \tag{3.2}$$

$$J_{u,v}(w) = \int_0^\infty \Big(r(w')^2 + r u^2 w^2 + 2\kappa N v' w \Big)\, \mathrm{d}r. \tag{3.3}$$

Then $I(u,v,w) = G(u,v) - J_{u,v}(w)$. Notice that $G(u,v)$ does not depend on w and has the form of the Ginzburg–Landau energy, while $J_{u,v}(w)$ contains an indefinite part $\int_0^\infty 2\kappa N v' w \,\mathrm{d}r$.

The natural admissible space $\mathcal{A}$ is defined by

$$\mathcal{A} = \{(u,v,w) |\; E(u,v,w) < \infty \text{ and } u,v,w \text{ satisfy (2.24), (2.25)}\}. \tag{3.4}$$

Note that here we leave out the boundary condition (2.26) in the admissible set because it cannot be simply recovered from a finite energy requirement. However, condition (2.26) will be obtained when we construct a constrained admissible space.

Our goal is to find a critical point of the functional (3.1) in the admissible space $\mathcal{A}$. Yet the difficulty comes from both the negative definite energy part and the indefinite energy part, which is an obstacle to getting the coerciveness of $I(u,v,w)$. Motivated by the idea of the constrained minimization methods by Schechter–Weder [62] and Yang [86], we look for a suitable set of constraints to restrict the consideration of (3.1) over a smaller admissible space, say, $\mathcal{C}$. With this choice of $\mathcal{C}$, $I(u,v,w)$ becomes coercive on $\mathcal{C}$ and the minimizer of $I(u,v,w)$ over $\mathcal{C}$ can be shown to be a critical point over the original admissible $\mathcal{A}$, thus is a solution of (2.21)–(2.23).

In order to make $I(u,v,w)$ coercive over a properly constrained admissible space $\mathcal{C}$, we need to control $J_{u,v}(w)$. To do so we need to “freeze" the unknown w, which certainly cannot be done arbitrarily since we are looking for a solution of (2.21)–(2.23) eventually. Hence we naturally require w satisfy (2.23) in a suitable weak sense for given u,v. In this way, we are led to considering seeking for each fixed pair (u,v), a critical point of the functional $J_{u,v}(w)$. In order to get a good convergence result, we restrict further to considering $w \in H^1_r(\mathbb{R}^2)$, where

$$H^1_r(\mathbb{R}^2) = \{f \in H^1(\mathbb{R}^2) |\ f \text{ is radially symmetric about the origin } \}. \tag{3.5}$$

We often use $f(r)$ to unambiguously denote the radial dependence of the function f over $\mathbb{R}^2$ which is symmetric about the origin of $\mathbb{R}^2$.

Notice that $w \in H^1_r(\mathbb{R}^2)$ implies $w(\infty) = 0$ [73]. In fact, it is easily seen that the set of all $w \in H^1_r(\mathbb{R}^2)$ so that $J_{u,v}(w) < \infty$ is an affine linear space. Besides, since $J_{u,v}$ is strictly convex with respect to w for each given pair (u,v),

$$J_{u,v} \text{ can at most have one critical point.} \tag{3.6}$$

If w is a critical point then

$$\int_0^\infty (rw'\bar{w}' + ru^2 w\bar{w} + \kappa N v'\bar{w})\, \mathrm{d}r = 0, \tag{3.7}$$

for all $\bar{w} \in H^1_r(\mathbb{R}^2)$ such that

$$J_{u,v}(w + \bar{w}) < \infty.$$

In this way we may define the constrained admissible space

$$\mathcal{C} = \Big\{(u,v,w) \in \mathcal{A} \mid w \in H^1_r(\mathbb{R}^2), (u,v,w) \text{ satisfies (3.7)}\Big\}. \tag{3.8}$$

We need to make sure that $\mathcal{C}$ is not empty. A natural way is to use the variational approach, that is, to consider minimizing $J_{u,v}(w)$ over $w \in H^1_r(\mathbb{R}^2)$ for certain fixed (u,v). The major difficulty is that when it comes to minimizing $I(u,v,w)$, one is looking at a class of (u,v). Moreover, $J_{u,v}(w)$ contains an indefinite part which, after applying Cauchy–Schwartz, introduces a term $\|w\|^2_{L^2(\mathbb{R}^2)}$ which cannot be controlled by $\|w'\|^2_{L^2(\mathbb{R}^2)}$ only. Therefore we have to enlist the second term $\|uw\|^2_{L^2(\mathbb{R}^2)}$ in $J_{u,v}(w)$ to help control the H^1 norm of w.

4. Minimization of $J_{u,v}(w)$

Since u may vanish in a finite-energy setting, we need to control the size of the set in which $|u| \le \frac{1}{2}$.

Proposition 4.1. *Suppose that (u,v) satisfies that $G(u,v) \le M < \infty$. Then there exists an R independent of u such that $\{x:\ |u(x)| \le \frac{1}{2}\} \subset B_R$, where B_R is a ball in $\mathbb{R}^2$ of radius R centered at the origin.*

Proof. Consider a pair (u,v) such that $G(u,v) \le M < \infty$. Then using the result in Ginzburg–Landau theory [8] we know that $1 - |u| \in H^1_r(\mathbb{R}^2)$. We also know that $\big||u|'\big| \le$

$|u'|$ a.e. [29]. Hence we have

$$\begin{aligned}\Big(1-|u(r)|\Big)^2 &\le 2\int_r^\infty \big|1-|u(\rho)|\big||u'(\rho)|\,\mathrm{d}\rho\\ &\le \frac{2}{r}\Big(\int_r^\infty \big(1-u(\rho)\big)^2\rho\,\mathrm{d}\rho\Big)^{1/2}\Big(\int_r^\infty \big(u'(\rho)\big)^2\rho\,\mathrm{d}\rho\Big)^{1/2}\\ &\le \frac{2}{r}\Big(\int_r^\infty \big(1-u^2(\rho)\big)^2\rho\,\mathrm{d}\rho\Big)^{1/2}\Big(\int_r^\infty \big(u'(\rho)\big)^2\rho\,\mathrm{d}\rho\Big)^{1/2}\\ &\le \frac{4}{r\sqrt{\lambda}}G(u,v)\le \frac{4M}{r\sqrt{\lambda}}.\end{aligned}$$

In this way, we may choose

$$R=\frac{16M}{\sqrt{\lambda}} \tag{4.1}$$

so that $|u(x)|>\frac{1}{2}$ for $|x|\ge R$. □

We are now ready to study the minimization problem for $J_{u,v}(w)$ over $H_r^1(\mathbb{R}^2)$, for a fixed pair (u,v) such that $G(u,v)<\infty$, $u(0)=v(0)=0$, $u(\infty)=v(\infty)=1$.

Lemma 4.2. *For each (u,v) with $G(u,v)<\infty$, the following minimization problem*

$$\min\{J_{u,v}(w)|\ w\in H_r^1(\mathbb{R}^2)\} \tag{4.2}$$

has a unique solution. Hence $\mathcal{C}\neq\emptyset$.

Proof. The uniqueness of the minimizer can be seen from the fact that the functional $J_{u,v}(w)$ is strictly convex.

In order to prove the existence of minimizer, we need to first show that $J_{u,v}(w)$ is bounded from below, provided that $G(u,v)\le M<\infty$. Using Cauchy–Schwartz we have

$$\begin{aligned}J_{u,v}(w) &\ge \int_0^\infty \Big(r(w')^2+ru^2w^2-\varepsilon rw^2-\frac{1}{\varepsilon}\frac{\kappa^2N^2}{r}(v')^2\Big)\,\mathrm{d}r\\ &= \int_0^\infty r(w')^2\,\mathrm{d}r+\int_{|u|>\frac12} r(u^2-\varepsilon)w^2\,\mathrm{d}r\\ &\quad +\int_{|u|\le\frac12} r(u^2-\varepsilon)w^2\,\mathrm{d}r-\frac{1}{\varepsilon}\int_0^\infty \frac{\kappa^2N^2}{r}(v')^2\,\mathrm{d}r\\ &\ge \int_0^\infty r(w')^2\,\mathrm{d}r+\int_{|u|>\frac12}\big(\frac14-\varepsilon\big)rw^2\,\mathrm{d}r\\ &\quad -\int_{|u|\le\frac12}\varepsilon rw^2\,\mathrm{d}r-\frac{1}{\varepsilon}\int_0^\infty \frac{\kappa^2N^2}{r}(v')^2\,\mathrm{d}r\\ &\ge \int_0^\infty r(w')^2\,\mathrm{d}r+\int_R^\infty\big(\frac14-\varepsilon\big)rw^2\,\mathrm{d}r\\ &\quad -\int_0^R\varepsilon rw^2\,\mathrm{d}r-\frac{1}{\varepsilon}\int_0^\infty \frac{\kappa^2N^2}{r}(v')^2\,\mathrm{d}r,\end{aligned}$$

where the R in the last inequality is defined by (4.1).

Take a smooth function $\eta(x)$ on $\mathbb{R}^2$ such that supp $\eta\subset B_{2R}$, $0\le\eta(x)\le 1$ and $\eta\equiv 1$ on B_R. Let $\tilde{w}=\eta w$. Then $\tilde{w}\in H_0^1(\mathbb{R}^2)$. Hence using Poincaré's inequality we have

$$\int_{B_R} w^2\,\mathrm{d}x\le\int_{B_{2R}}\tilde{w}^2\,\mathrm{d}x\le CR\int_{B_{2R}}|\nabla\tilde{w}|^2\,\mathrm{d}x\le CR\|\nabla\tilde{w}\|^2_{L^2(\mathbb{R}^2)}.$$

However,

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla \tilde{w}|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^2} |(\nabla\eta) w + \eta \nabla w|^2 \,\mathrm{d}x \\ &\le 2\int_{\mathbb{R}^2} \Big(|(\nabla\eta)w|^2 + |\eta\nabla w|^2\Big)\,\mathrm{d}x \\ &\le 2\Big(C\int_{B_R^c} w^2\,\mathrm{d}x + \int_{\mathbb{R}^2} |\nabla w|^2\,\mathrm{d}x\Big).\end{aligned}$$

Therefore

$$\int_{B_R} w^2\,\mathrm{d}x = \int_0^R w^2\, r\,\mathrm{d}r \le CR\Big(\int_0^\infty r(w')^2\,\mathrm{d}r + \int_R^\infty rw^2\,\mathrm{d}r\Big). \tag{4.3}$$

Hence, we obtain

$$\begin{aligned}J_{u,v}(w) &\ge (1-\varepsilon CR)\int_0^\infty r(w')^2\,\mathrm{d}r + \Big(\frac{1}{4} - (1+CR)\varepsilon\Big)\int_R^\infty rw^2\,\mathrm{d}r \\ &\quad - \frac{1}{\varepsilon}\int_0^\infty \frac{\kappa^2 N^2}{r}(v')^2\,\mathrm{d}r \\ &\quad \text{(now choosing } \varepsilon = 1/8(1+CR)\text{)} \\ &\ge \frac{7}{8}\int_0^\infty r(w')^2\,\mathrm{d}r + \frac{1}{8}\int_R^\infty rw^2\,\mathrm{d}r - 8(1+CR)\int_0^\infty \frac{\kappa^2N^2}{r}(v')^2\,\mathrm{d}r \\ &\ge \frac{7}{8}\int_0^\infty r(w')^2\,\mathrm{d}r + \frac{1}{8}\int_R^\infty rw^2\,\mathrm{d}r - 8(1+CR)\kappa^2 M.\end{aligned}$$

From (4.3) and the above inequality we can also derive the following control of H^1-norm of w in terms of $J_{u,v}(w)$:

$$\begin{aligned}\|w\|^2_{H^1_r(\mathbb{R}^2)} &= \int_0^\infty r(w')^2\,\mathrm{d}r + \int_0^\infty rw^2\,\mathrm{d}r \\ &\le (1+CR)\Big(\int_0^\infty r(w')^2\,\mathrm{d}r + \int_R^\infty rw^2\,\mathrm{d}r\Big) \\ &\le 8(1+CR)\Big[J_{u,v}(w) + 8(1+CR)\kappa^2 M\Big].\end{aligned} \tag{4.4}$$

Now we can take a minimizing sequence $\{w_n\}$ in $H^1_r(\mathbb{R}^2)$. Then by (4.4), $\|w_n\|_{H^1_r(\mathbb{R}^2)}$ is uniformly bounded. Hence (up to a subsequence)

$$w_n \rightharpoonup w \ \text{ in } \ H^1_r(\mathbb{R}^2).$$

Then by the compactness lemma in [73] we know that

$$w_n \to w \ \text{ a.e. on } (0,\infty).$$

From [8] we know that $G(u,v) < \infty$ implies that $1 - |u| \in H^1_r(\mathbb{R}^2)$ and $\frac{v'}{r} \in L^2(\mathbb{R}^2)$. Thus the weak lower semicontinuity of L^2-norm, the Fatou's lemma, and the weak convergence of w_n imply that

$$J_{u,v}(w) = \int_0^\infty \Big(r(w')^2 + ru^2w^2 - 2\kappa N v' w\Big)\,\mathrm{d}r \le \liminf_{n\to\infty} J_{u,v}(w_n).$$

Therefore w solves (4.2).

Since critical points of $J_{u,v}(w)$ of course satisfy (3.7), $\mathcal{C} \ne \emptyset$, and the lemma is proved. $\square$

Remark 4.3. From the above Proposition, we understand the structure of $\mathcal{C}$ explicitly: for any pair (u, v) satisfying $G(u, v) < \infty$, (2.24) and (2.25), then $(u, v, w) \in \mathcal{C}$ is the unique triplet such that w is the unique solution to (4.2), and in fact minimizing $J_{u,v}(w)$. Thus each pair u, v unambiguously defines $w = w_{(u,v)}$ and $\mathcal{C}$ looks like the image of the map $(u, v) \mapsto w_{(u,v)}$ in $\mathcal{A}$.

5. MINIMIZATION OF $I(u, v, w)$

In this section we try to solve the minimization problem of the full energy $I(u, v, w)$ over the constrained admissible space $\mathcal{C}$. We first show that $I(u, v, w)$ is positive definite and coercive with respect to u, v on $\mathcal{C}$.

Proposition 5.1. *For* $(u, v, w) \in \mathcal{C}$*,*

$$I(u, v, w) \geq G(u, v). \tag{5.1}$$

Proof. Considering (3.7) for (u, v, w) and taking $\bar{w} = w$, we get

$$\int_0^\infty \Big(r(w')^2 + ru^2w^2 + \kappa N v' w \Big) \, \mathrm{d}r = 0.$$

Therefore

$$J_{u,v}(w) = -\int_0^\infty \Big(r(w')^2 + ru^2w^2 \Big) \, \mathrm{d}r \leq 0. \tag{5.2}$$

Hence we have (5.1). □

Proposition 5.2. *The minimization problem*

$$\min\{I(u, v, w) | \, (u, v, w) \in \mathcal{C}\} \tag{5.3}$$

has a solution.

Proof. By Lemma 4.2, we can take a minimizing sequence $\{(u_n, v_n, w_n)\}$ of (5.3). Since all terms involving function u appear in a quadratic form, we may take all $u_n \geq 0$. From (5.1) we know $\{G(u_n, v_n)\}$ is uniformly bounded. Therefore from [8] we know that $\|1 - u_n\|_{H_r^1(\mathbb{R}^2)}$ and $\|v_n\|_{C_S} = \|(1/r)v'\|_{L^2(\mathbb{R}^2)}$ are uniformly bounded, where

$$C_S = \left\{ \begin{array}{l} \text{the set of real-valued radially symmetric functions } v(|x|) \text{ on } \mathbb{R}^2 \\ \text{such that } (1/r)v \in L^2_{\text{loc}}(\mathbb{R}^2) \text{ and } (1/r)v' \in L^2(\mathbb{R}^2) \text{ where} \\ \text{the derivative } v' \text{ is in the distributional sense.} \end{array} \right\} \tag{5.4}$$

Hence

$$1 - u_n \rightharpoonup 1 - u \quad \text{in } H_r^1(\mathbb{R}^2), \quad v_n \rightharpoonup v \quad \text{in } C_S.$$

From (5.2) we know that $J_{u_n,v_n}(w_n) \leq 0$. So by (4.4), $\|w_n\|_{H_r^1(\mathbb{R}^2)}$ is uniformly bounded. Therefore

$$w_n \rightharpoonup w \quad \text{in } H_r^1(\mathbb{R}^2).$$

Moreover, we have

$$u_n \to u, \;\; v_n \to v, \;\; w_n \to w, \;\; \text{a.e. on } (0, \infty).$$

Next we check that $(u, v, w) \in \mathcal{C}$, that is, (3.7) is satisfied for all $\bar{w} \in H_r^1(\mathbb{R}^2)$ with $J_{u,v}(w + \bar{w}) < \infty$.

Weak convergence of $\{w_n'\}$ in L^2 and $\{v_n\}$ in C_S imply that

$$\lim_{n\to\infty}\int_0^\infty rw_n'\bar{w}'\,\mathrm{d}r=\int_0^\infty rw'\bar{w}'\,\mathrm{d}r,$$
$$\lim_{n\to\infty}\int_0^\infty v_n'\bar{w}\,\mathrm{d}r=\int_0^\infty v'\bar{w}\,\mathrm{d}r.$$

As for the second term in (3.7),

$$\begin{aligned}&\int_0^\infty ru_n^2w_n\bar{w}\,\mathrm{d}r-\int_0^\infty ru^2w\bar{w}\,\mathrm{d}r\\&=\int_0^\infty r(u_n-u)u_nw_n\bar{w}\,\mathrm{d}r+\int_0^\infty ruu_n(w_n-w)\bar{w}\,\mathrm{d}r+\int_0^\infty ru(u_n-u)w\bar{w}\,\mathrm{d}r\\&\equiv\ T_1+T_2+T_3.\end{aligned}$$

Using the compact embedding of $H_r^1(\mathbb{R}^2)\subset\subset L_r^p(\mathbb{R}^2)$ for any $p>2$ [14],

$$\begin{aligned}|T_1|=&\Big|\int_0^\infty r(u_n-u)(u_n-1)w_n\bar{w}\,\mathrm{d}r+\int_0^\infty r(u_n-u)w_n\bar{w}\,\mathrm{d}r\Big|\\&\le\|(1-u_n)-(1-u)\|_{L^4}\|1-u_n\|_{L^4}\|w_n\|_{L^4}\|\bar{w}\|_{L^4}\\&\quad+\|(1-u_n)-(1-u)\|_{L^3}\|w_n\|_{L^3}\|\bar{w}\|_{L^3}\\&\longrightarrow 0,\quad\text{as } n\to\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}|T_2|=&\Big|\int_0^\infty r(1-u)(1-u_n)(w_n-w)\bar{w}\,\mathrm{d}r+\int_0^\infty r(w_n-w)\bar{w}\,\mathrm{d}r\\&+\int_0^\infty r(u-1)(w_n-w)\bar{w}\,\mathrm{d}r+\int_0^\infty r(u_n-1)(w_n-w)\bar{w}\,\mathrm{d}r\Big|\\&\longrightarrow 0,\quad\text{as } n\to\infty,\end{aligned}$$

and

$$\begin{aligned}|T_3|=&\Big|\int_0^\infty r(u-1)\big((1-u)-(1-u_n)\big)w\bar{w}\,\mathrm{d}r\\&+\int_0^\infty r\big((1-u)-(1-u_n)\big)w\bar{w}\,\mathrm{d}r\Big|\\&\longrightarrow 0,\quad\text{as } n\to\infty.\end{aligned}$$

Therefore we have proved that $(u,v,w)\in\mathcal{C}$. To show that the limiting configuration (u,v,w) is a minimizer of (5.3), we consider (3.7) for $(u,v,w)=(u_n,v_n,w_n)$ and take $\bar{w}=w_n$. Then

$$\int_0^\infty\Big(r(w_n')^2+ru_n^2w_n^2+\kappa Nv_n'w_n\Big)\,\mathrm{d}r=0.$$

In the same way, considering (3.7) for (u,v,w) and taking $\bar{w}=w$, we get

$$\int_0^\infty\Big(r(w')^2+ru^2w^2+\kappa Nv'w\Big)\,\mathrm{d}r=0.$$

Therefore

$$J_{u_n,v_n}(w_n)=-\int_0^\infty\Big(r(w_n')^2+ru_n^2w_n^2\Big)\,\mathrm{d}r,$$
$$J_{u,v}(w)=-\int_0^\infty\Big(r(w')^2+ru^2w^2\Big)\,\mathrm{d}r.$$

Thus, using weak lower semicontinuity and Fatou's lemma, we have

$$\begin{aligned}
I(u,v,w) &= G(u,v) + \int_0^\infty \Big(r(w')^2 + ru^2w^2 \Big)\,\mathrm{d}r \\
&\le \liminf_{n\to\infty} \Big(G(u_n,v_n) + \int_0^\infty \Big(r(w_n')^2 + ru_n^2w_n^2 \Big)\,\mathrm{d}r \Big) \\
&= \liminf_{n\to\infty} I(u_n,v_n,w_n).
\end{aligned}$$

Hence we conclude that such a limit (u,v,w) satisfies (5.3). $\square$

6. Weak solutions of the governing equations

We use the idea developed in [86] to establish the existence of weak solutions.

Proposition 6.1. *The action minimizing solution* (u,v,w) *of the problem* (5.3) *is a weak solution of equations* (2.21)–(2.23) *subject to the partial boundary conditions* (2.24) *and* (2.25).

Proof. As discussed earlier, with

$$\mathcal{S} = \{(u,v)|\, G(u,v) < \infty, u(0) = v(0) = 0, u(\infty) = v(\infty) = 1\}, \tag{6.1}$$

the constrained set $\mathcal{C}$ may be viewed as the image of the map $\chi : \mathcal{S} \to \mathcal{A}$, $(u,v) \mapsto (u,v,w_{(u,v)})$ with $w = w_{(u,v)}$ being determined by (3.7), which is the weak form of equation (2.23). Consequently, χ is a differentiable map in an obvious sense. Besides, the minimizer (u,v,w) of the constrained problem (5.3) obtained in Proposition 5.2 may simply be viewed as the image under χ of an absolute minimizer (u,v), of the functional $I(u,v,w_{(u,v)})$ over the unconstrained class $\mathcal{S}$.

Let h be a real parameter confined in a small interval, say, $|h| < 1$, and $\tilde{u} \in C_0^1(0,\infty)$ (functions with compact supports). Set $w_h = w_{(u+h\tilde{u},v)}$. We use the following notations

$$\Delta w = w_h - w, \quad Dw = \lim_{h\to 0} \frac{\Delta w}{h} = \frac{\mathrm{d}\Delta w}{\mathrm{d}h}\Big|_{h=0}.$$

Then we use Δw as a test function in (3.7) to get

$$\int_0^\infty \Big(rw'(\Delta w)' + ru^2 w\Delta w + \kappa N v'\Delta w \Big)\,\mathrm{d}r = 0.$$

We also have

$$\int_0^\infty \Big(rw_h'(\Delta w)' + r(u+h\tilde{u})^2 w_h\Delta w + \kappa N v'\Delta w \Big)\,\mathrm{d}r = 0.$$

Subtracting the first equality from the second one we get

$$\int_0^\infty \Big(r\big[(\Delta w)'\big]^2 + ru^2(\Delta w)^2 \Big)\,\mathrm{d}r = -h\int_0^\infty \Big(2ru\tilde{u}w_h\Delta w + hr\tilde{u}^2 w_h\Delta w \Big)\,\mathrm{d}r.$$

Using Cauchy–Schwartz we obtain

$$\int_0^\infty \Big(r\big[\frac{(\Delta w)'}{h}\big]^2 + ru^2\big(\frac{\Delta w}{h}\big)^2 \Big)\,\mathrm{d}r \le \int_0^\infty \Big(\frac{1}{2}ru^2\big(\frac{\Delta w}{h}\big)^2 + 2r\tilde{u}^2w_h^2 + r\tilde{u}^2|w_h\Delta w| \Big)\,\mathrm{d}r.$$

Hence

$$\begin{aligned}
\int_0^\infty \Big(r\big[\frac{(\Delta w)'}{h}\big]^2 + ru^2\big(\frac{\Delta w}{h}\big)^2 \Big)\,\mathrm{d}r &\le 5\int_0^\infty \Big(r\tilde{u}^2 w_h^2 + r\tilde{u}^2(\Delta w)^2 \Big)\,\mathrm{d}r \\
&\le 15\int_0^\infty \Big(r\tilde{u}^2 w_h^2 + r\tilde{u}^2 w^2 \Big)\,\mathrm{d}r. \qquad (6.2)
\end{aligned}$$

From (4.4) and (5.2) we know that

$$\|w_h\|_{L^2}^2 \le \|w_h\|_{H^1}^2 \le C,$$

where C depends on $G(u + h\tilde{u}, v)$. By assumption we know $|h| < 1$. Then

$$\begin{aligned}\int_0^\infty r\big((u+h\tilde{u})'\big)^2 \,\mathrm{d}r &\le 2\int_0^\infty \Big(r(u')^2 + rh^2(\tilde{u}')^2\Big)\,\mathrm{d}r\\ &\le 2\int_0^\infty \Big(r(u')^2 + r(\tilde{u}')^2\Big)\,\mathrm{d}r,\end{aligned}$$

$$\int_0^\infty \frac{N^2}{r}(u+h\tilde{u})^2(v-1)^2\,\mathrm{d}r \le 2\int_0^\infty \frac{N^2}{r}(u^2+\tilde{u}^2)(v-1)^2\,\mathrm{d}r,$$

$$\begin{aligned}\int_0^\infty \Big(1-(u+h\tilde{u})^2\Big)^2\,\mathrm{d}r &= \int_0^\infty \Big((1-u^2) + 2h\tilde{u}(1-u) - 2h\tilde{u} - h^2\tilde{u}^2)\Big)^2\,\mathrm{d}r\\ &\le 2(1+\|\tilde{u}\|_{L^\infty}^2)^2\int_0^\infty ((1-u^2)^2\mathrm{d}r\\ &\quad + 2\|\tilde{u}\|_{L^\infty}^2\int_{\text{supp }\tilde{u}}(2+|\tilde{u}|)^2\,\mathrm{d}r.\end{aligned}$$

Hence

$$G(u+h\tilde{u}, v) \le CG(u,v) + C,$$

where C depends on $\tilde{u}$, not on h.

Similarly we obtain that

$$\|w\|_{L^2}^2 \le C,$$

where C is independent of h. Thus

$$\int_0^\infty \Big(r\big[\frac{(\Delta w)'}{h}\big]^2 + ru^2\big(\frac{\Delta w}{h}\big)^2\Big)\,\mathrm{d}r \le C, \tag{6.3}$$

where C is independent of h. Taking $h \to 0$ in (6.3) we have

$$\int_0^\infty \Big(r\big[(Dw)'\big]^2 + ru^2\big(Dw\big)^2\Big)\,\mathrm{d}r \le C.$$

The the third term in $J_{u,v}(Dw)$ can be bounded as follows by using Proposition 9.1 and (4.4).

$$\begin{aligned}\int_0^\infty 2\kappa N v' Dw\,\mathrm{d}r &\le \int_0^\infty \Big(\kappa^2\frac{N^2}{r}(v')^2 + r(Dw)^2\Big)\,\mathrm{d}r\\ &\le \kappa^2 G(u,v) + \|Dw\|_{H^1(\mathbb{R}^2)}^2\\ &\le \kappa^2 G(u,v) + (1+CR)\Big(\int_0^\infty r[(Dw)']^2\,\mathrm{d}r + \int_R^\infty r(Dw)^2\,\mathrm{d}r\Big)\\ &\le \kappa^2 G(u,v) + (1+CR)\Big(\int_0^\infty r[(Dw)']^2\,\mathrm{d}r + 4\int_R^\infty ru^2(Dw)^2\,\mathrm{d}r\Big)\\ &\le \kappa^2 G(u,v) + 4(1+CR)\int_0^\infty \Big(r\big[(Dw)'\big]^2 + ru^2\big(Dw\big)^2\Big)\,\mathrm{d}r \le C,\end{aligned}$$

where we have used (4.4) from the second inequality to the third and Proposition 9.1 from the third to the fourth. Therefore we get $J_{u,v}(Dw) < \infty$. Hence

$$J_{u,v}(w + Dw) < \infty.$$

Further more from the above estimates we also obtain that $Dw \in H^1_r(\mathbb{R}^2)$. Therefore (3.7) is satisfied with $\bar{w} = Dw$.

Since (u, v) minimizes $I(u, v, w_{(u,v)})$, we have

$$\frac{\mathrm{d}}{\mathrm{d}h} I(u + h\tilde{u}, v, w_h)\Big|_{h=0} = 0,$$

which gives

$$\begin{aligned} &\int_0^\infty \Big(ru'\tilde{u}' - \frac{\lambda}{2} ru(1-u^2)\tilde{u} + \frac{N^2}{r} u(v-1)^2 \tilde{u} - ruw^2\tilde{u} \Big)\,\mathrm{d}r \\ &= \int_0^\infty \Big(rw'(Dw)' + ru^2 wDw + \kappa N v' Dw \Big)\,\mathrm{d}r \\ &= 0. \end{aligned} \tag{6.4}$$

The left-hand side of the above leads to the validity of a weak form of equation (2.21).

Similarly we fix a compactly supported test function $\tilde{v}$ and consider $w_h = w_{(u,v+h\tilde{v})}$ as before. We can show in a similar way as we did for (6.4) that

$$\int_0^\infty \Big(\frac{N^2}{r} v'\tilde{v}' + \frac{N^2}{r} u^2 (v-1)\tilde{v} + \kappa N w'\tilde{v} \Big)\,\mathrm{d}r = 0, \tag{6.5}$$

which is the weak form of (2.22). Therefore the proof of the proposition is complete. □

7. Full set of boundary conditions and regularity

In this section we show that the remaining boundary condition (2.26) also holds for the solution (u, v, w) obtained in the last section and then prove that the solution (u, v, w) is indeed a classical solution to equations (2.21)–(2.23).

Lemma 7.1. *Let (u, v, w) be the solution of (2.21)–(2.23) obtained in the last section. Then (2.26) holds for a certain suitable w_0.*

Proof. From the finite energy configuration we know

$$\int_0^\infty \frac{(v')^2}{r}\,\mathrm{d}r < \infty,$$

we have

$$\liminf_{r\to 0} \{|v'(r)|\} = 0. \tag{7.1}$$

We rewrite (2.22) as

$$(rv')' = 2v' + r(v-1)u^2 + \frac{\kappa}{N} r^2 w'. \tag{7.2}$$

Integrating (7.2) and using (7.1), we have

$$rv'(r) = 2v(r) + \int_0^r \rho\Big(v(\rho) - 1\Big) u^2(\rho)\,\mathrm{d}\rho + \frac{\kappa}{N}\int_0^r \rho^2 w'(\rho)\,\mathrm{d}\rho. \tag{7.3}$$

On the other hand, the condition

$$\int_0^\infty r(w')^2\,\mathrm{d}r < \infty$$

implies that

$$\liminf_{r\to 0}\{r|w'(r)|\} = 0. \tag{7.4}$$

Using (7.4) to integrate (2.23), we obtain in view of (7.2) that

$$\begin{aligned} w'(r) &= \frac{1}{r}\int_0^r \rho u^2(\rho) w(\rho)\,\mathrm{d}\rho + \kappa N \frac{v(r)}{r} \\ &= \frac{1}{r}\int_0^r \rho u^2(\rho) w(\rho)\,\mathrm{d}\rho + \frac{\kappa N}{2} v'(r) - \frac{\kappa N}{2r}\int_0^r \rho (v(\rho)-1) u^2(\rho)\,\mathrm{d}\rho \\ &\quad + \frac{\kappa^2}{2r}\int_0^r \rho^2 w'(\rho)\,\mathrm{d}\rho \\ &\equiv \frac{1}{r} I_1(r) + \frac{\kappa N}{2} v'(r) - \frac{\kappa N}{2r} I_2(r) + \frac{\kappa^2}{2r} I_3(r), \quad r > 0. \end{aligned} \tag{7.5}$$

For $I_1(r)$, the Schwartz inequality gives us

$$|I_1(r)| \le Cr\left(\int_0^r \rho u^2(\rho) w^2(\rho)\,\mathrm{d}\rho\right)^{1/2}, \tag{7.6}$$

where C may depend on the upper bound of $|u|$. Similarly, for $I_2(r)$ and $I_3(r)$, we have

$$|I_2(r)| \le Cr^2\left(\int_0^r \frac{1}{\rho}(v(\rho)-1)^2 u^2(\rho)\,\mathrm{d}\rho\right)^{1/2}, \tag{7.7}$$

and

$$|I_3(r)| \le Cr^2\left(\int_0^r \rho (w'(\rho))^2\,\mathrm{d}\rho\right)^{1/2}. \tag{7.8}$$

Note that each of the right-hand sides of (7.6)–(7.8) appears in the energy functionals.

Integrating (7.5) and using (7.6)–(7.8), we see that the limit

$$w_0 = \lim_{r\to 0} w(r)$$

exists as hoped. □

Lemma 7.2. *Through the ansatz (2.18)–(2.20), the solution* (u, v, w) *of the radial equations* (2.21)–(2.23) *obtained in the last section gives rise to a classical (smooth) solution* (ϕ, A_j, A_0) *of the static Chern–Simons–Higgs equations (2.7)–(2.9) over* $\mathbb{R}^2$.

Proof. We first prove the interior regularity of solutions. From the minimization procedure we obtain that the weak solution lives in the space: $1-u \in H_r^1$, $v \in C_S$, where C_S is defined in (5.4), and $w \in H_r^1$. For any $0 < \delta < R$, let $\Omega = B_R \backslash B_\delta$, then (u, v, w) is a generalized solution of the system

$$\begin{aligned} -\Delta u &= \frac{\lambda}{2} u(1-u^2) - \frac{N^2}{r^2}(v-1)^2 u + w^2 u, \\ -\Delta v &= (1-v)u^2 - \frac{2}{r} v' - \frac{\kappa r}{N} w', \\ -\Delta w &= -u^2 w - \frac{\kappa N}{r} v', \end{aligned}$$

on Ω. The right-hand side of the third equation is in $L^2(\Omega)$. Hence $w \in H^2(\Omega)$. In the second equation,

$$\begin{aligned} \|(1-v)u^2\|^2_{L^2(\Omega)} &= \int_\Omega (v-1)^2 u^4\,\mathrm{d}x \\ &\le \|ru\|^2_{L^\infty(\Omega)} \int_{\mathbb{R}^2} \frac{(v-1)^2}{r^2} u^2\,\mathrm{d}x < \infty. \end{aligned}$$

Hence we also have $v \in H^2(\Omega)$. In the first equation,

$$\begin{aligned}\left\|\frac{(v-1)^2}{r^2}u\right\|^2_{L^2(\Omega)} &= \int_\Omega \frac{(v-1)^2}{r^2}u^2 \cdot \frac{(v-1)^2}{r^2}\,\mathrm{d}x \\ &\le \sup_{\delta<r<R}\left|\frac{v-1}{r}\right|^2 \int_{\mathbb{R}^2} \frac{(v-1)^2}{r^2}u^2\,\mathrm{d}x \\ &\le \left(C + \|v'/r\|^2_{L^2(\mathbb{R}^2)}\right)\int_{\mathbb{R}^2} \frac{(v-1)^2}{r^2}u^2\,\mathrm{d}x < \infty,\end{aligned}$$

where we have used the fact [8]

$$\sup_{0<r<\infty}\left|\frac{1}{r}v\right| \le \|v'/r\|^2_{L^2(\mathbb{R}^2)}.$$

In this way, $u \in H^2(\Omega)$. Therefore by standard regularity theory of elliptic equations and using the iterative bootstrap argument we conclude that (u, v, w) is a classical solution of (2.21)–(2.23) on Ω.

Since both u and w satisfy the property that

$$\lim_{r\to 0}\frac{u}{\ln r} = \lim_{r\to 0}\frac{w}{\ln r} = 0, \tag{7.9}$$

by the removable singularity theorem, the regularity of u, w extends to the origin, so does the regularity of $\phi(x)$ and $A_0(x)$ as in (2.18) and (2.20).

As for v, we first look at $A_j(x)$. Since $\partial_j A_j(x) = 0$ (divergence free) away from the origin, we know that in Ω, $A_j(x)$ satisfies

$$\Delta A_j = h_j \tag{7.10}$$

for some $h_j \in L^2(\Omega)$. Since $A_j(x)$ is an $H^1(\Omega)$ solution, from the previous interior regularity argument we know that it is also an $H^2(\Omega)$ solution. Hence we may apply the same removable singularity theorem to extend the regularity of $A_j(x)$ to the origin.

Bootstrap then shows that ϕ, A_j, A_0 are all smooth across the origin. For example, for ϕ, we notice that (2.7) may be rewritten as

$$\Delta\phi - \mathrm{i}2A_j\partial_j\phi - \mathrm{i}(\partial_j A_j)\phi = (A_1^2 + A_2^2 - A_0^2)\phi + \frac{\lambda}{2}(|\phi|^2 - 1)\phi. \tag{7.11}$$

Therefore we know that (u, v, w) gives rise to a classical solution. $\square$

8. Quantization of magnetic flux and electrostatic charge

We finish with the proof of (2.16) and (2.17).

Proposition 8.1. *The solution satisfies the quantization relationship*

$$Q = \kappa\Phi = 2\pi N$$

where Q is the electrostatic charge and Φ is the magnetic flux.

Proof. In the static case, the $\mu = 0$ component of (2.3) is the Gauss law,

$$\Delta A_0 = \kappa F_{12} + |\phi|^2 A_0, \quad \text{where } \rho = J^0 = -|\phi|^2 A_0 = \text{electric charge density}. \tag{8.1}$$

On the other hand, within the radial ansatz (2.19), we know that the magnetic field is represented by

$$F_{12} = N\frac{v'(r)}{r}, \quad r > 0.$$

Therefore (2.23) is exactly the radial form of the Gauss law (8.1) which correctly relates the magnetic magnetic field F_{12} to the electric charge density J^0 and implies that electricity and magnetism must coexist when the Chern–Simons coupling parameter is nontrivial, $\kappa \neq 0$. Thus the total magnetic charge (flux) is given by

$$\Phi = \int_{\mathbb{R}^2} F_{12}\,\mathrm{d}x = 2\pi N \int_0^\infty v'(r)\,\mathrm{d}r = 2\pi N. \tag{8.2}$$

Since $\int_0^\infty r(w'(r))^2 dr < \infty$, then

$$\liminf_{r\to 0}\{r|w'(r)|\} = \liminf_{r\to\infty}\{r|w'(r)|\} = 0. \tag{8.3}$$

Multiplying (2.23) by r, integrating, and using (8.3), we get

$$\int_0^\infty r u^2(r) w(r)\,\mathrm{d}r = \kappa N \int_0^\infty v'(r)\,\mathrm{d}r = \kappa N.$$

In particular,

$$Q = \int_{\mathbb{R}^2} J^0\,\mathrm{d}x = \kappa \int_{\mathbb{R}^2} F_{12}\,\mathrm{d}x = \kappa\Phi = 2\pi\kappa N,$$

which explicitly shows how electric charge is proportional to magnetic flux.

□

9. Application to non-Abelian Chern–Simons–Higgs equations

We start from the simplest non-Abelian case [79] where the gauge group is $SU(2)$ and the scalar fields are two scalar fields represented adjointly. For convenience, use isovectors. The Chern–Simons–Higgs field-theoretical Lagrangian density reads [79]

$$\begin{aligned}\mathcal{L} \;&=\; -\frac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu} + \frac{1}{2}D_\mu\phi\cdot D^\mu\phi + \frac{1}{2}D_\mu\psi\cdot D^\mu\psi \\ &\quad +\frac{1}{4}\kappa\varepsilon_{\mu\nu\alpha}(\mathbf{F}^{\alpha\mu}\cdot\mathbf{A}^\nu - \frac{2}{3}\mathbf{A}^\alpha\cdot[\mathbf{A}^\mu\times\mathbf{A}^\nu]) - V(\phi,\psi),\end{aligned} \tag{9.1}$$

where $\mathbf{A}_\mu = (A_\mu^1, A_\mu^2, A_\mu^3)$ $(\mu = 0,1,2)$, $\phi = (\phi^1,\phi^2,\phi^3)$, $\psi = (\psi^1,\psi^2,\psi^3)$ are isovectors,

$$\begin{aligned} D_\mu\phi &= \partial_\mu\phi + \mathbf{A}_\mu\times\phi, \quad D_\mu\psi = \partial_\mu\psi + \mathbf{A}_\mu\times\psi, \\ \mathbf{F}_{\mu\nu} &= \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + \mathbf{A}_\mu\times\mathbf{A}_\nu,\end{aligned}$$

and the Higgs potential density is chosen to be

$$V(\phi,\psi) = \frac{1}{8}\lambda(|\phi|^2-1)^2 + \frac{1}{8}\lambda_1(|\psi|^2-1)^2 + \frac{1}{2}\lambda_2(\phi\cdot\psi)^2. \tag{9.2}$$

The equations of motion of (9.1) are

$$D_\mu D^\mu\phi = -\frac{\delta V}{\delta\phi}, \quad D_\mu D^\mu\psi = -\frac{\delta V}{\delta\psi}, \tag{9.3}$$

$$D_\mu\mathbf{F}^{\mu\nu} = D^\nu\phi\times\phi + D^\nu\psi\times\psi + \frac{1}{2}\kappa\varepsilon^{\nu\alpha\beta}\mathbf{F}_{\alpha\beta}. \tag{9.4}$$

Following [79], we take the following radially symmetric ansatz for an electrically charged static vortex solution so that ϕ and ψ are orthogonal in isospace,

$$\phi = u(r)(\cos\theta,\sin\theta,0), \quad \psi = (0,0,1), \tag{9.5}$$

$$\mathbf{A}_r = 0, \quad \mathbf{A}_\theta = -v(r)(0,0,1), \quad \mathbf{A}_0 = w(r)(0,0,1), \tag{9.6}$$

where u, v, w are real-valued functions. Then the governing equations (9.3)–(9.4) become (2.21)–(2.23) when $N = 1$,

$$u'' + \frac{1}{r}u' = \frac{1}{r^2}(v-1)^2u - w^2u + \frac{\lambda}{2}(u^2-1)u, \tag{9.7}$$

$$v'' - \frac{1}{r}v = (v-1)u^2 + \kappa r w', \tag{9.8}$$

$$w'' + \frac{1}{r}w' = u^2w + \frac{\kappa}{r}v', \tag{9.9}$$

subject to the boundary conditions (2.24)–(2.26). (Note that, in [79], (2.26) is stated in a stronger form that the constant w_0 assumes zero value. However, we have seen in our present study that w_0 cannot be determined by the structure of the governing equations. This undeterminedness does affect the regularity, finiteness of energy, and quantization of electric and magnetic charges, of solutions.) Thus, the existence of electrically and magnetically charged static vortex solutions as described in [79] follows.

We next describe how to apply our work to the study of the dually charged vortex solutions in the general non-Abelian Chern–Simons–Higgs gauge field theory. To be specific, we consider the $SU(n)$ ($n \geq 3$) theory formulated in [80]. We use $su(n)$ to denote the Lie algebra of $SU(n)$ consisting of n by n Hermitian matrices with vanishing trace. The inner product over $su(n)$ is then defined by $(A, B) = \text{Tr}(AB^\dagger) = \text{Tr}(AB)$ ($A, B \in su(n)$). Recall that the dimension of the Cartan subalgebra, or the rank, of $SU(n)$ is $n-1$. Following [80], we consider the Chern–Simons–Higgs field theory housing $2(n-1)$ Higgs scalar particles ϕ^a, ψ^a ($a = 1, 2, \cdots, n-1$) in the adjoint representation of $SU(n)$ given by the Lagrangian density

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\text{Tr}F_{\mu\nu}F^{\mu\nu} + \text{Tr}\sum_{a=1}^{n-1} D_\mu\phi^a D^\mu\phi^a + \text{Tr}\sum_{a=1}^{n-1} D_\mu\psi^a D^\mu\psi^a \\ &+ \frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}\text{Tr}\Big(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha\Big) - V(\phi,\psi),\end{aligned} \tag{9.10}$$

where $A_\mu \in su(n)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, $D_\mu = \partial_\mu + [A_\mu,]$, and the potential density may be chosen to take the typical form

$$V(\phi,\psi) = \sum_{a=1}^{n-1}\frac{\lambda_a}{8}(|\phi^a|^2 - \eta_a^2)^2 + \sum_{a=1}^{n-1}\frac{\mu_a}{8}(|\psi^a|^2-\gamma_a^2)^2 + \sum_{a,b=1}^{n-1} V_{ab}(\text{Tr}(\phi^a\psi^b)), \tag{9.11}$$

in which V_{ab}'s are some functions satisfying $V_{ab} \geq 0$ and $V_{ab}(0) = 0$ ($1 \leq a, b \leq n-1$) and $\lambda_a, \eta_a, \mu_a, \gamma_a$ ($1 \leq a \leq n-1$) are positive coupling constants.

Recall that we can use the Cartan–Chevalley–Weyl basis $\{H_a, E_R\}$ to decompose $su(n)$, where $\{H_a \,|\, a = 1, 2, \cdots, n-1\}$ is a basis of the (Abelian) Cartan subalgebra and $R = (R_1, \cdots, R_{n-1})$ are root vectors, so that the spaces H and E, spanned by $\{H_a\}$ and $\{E_R\}$, respectively, satisfy $H \perp E, [H, H] = \{0\}, [H, E] \subset E, [E, E] \subset H$. With these facts, it is consistent to impose the condition that the gauge field A_μ lies in H and the scalar fields ϕ^a and the scalar fields ψ^a stay in E and H, respectively, for which ψ^a takes a constant value in H ($a = 1, 2, \cdots, n-1$). Therefore, the equations of motion of (9.10) contain A_μ and Φ^α only which are rewritten as [80]

$$D_\mu D^\mu \phi^a = \frac{\delta V}{\delta \phi^a}, \tag{9.12}$$

$$D_\nu F^{\mu\nu} - \frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}F_{\nu\alpha} = J^\mu, \tag{9.13}$$

where $J^\mu = \mathrm{i}\sum_{a=1}^{n-1}[D_\mu\phi^a, \phi^a]$ is the matter current generated from the Higgs particles.

To proceed, we follow [80] to write down the group element

$$\Omega_m(\theta) = \mathrm{diag}\{\mathrm{e}^{\mathrm{i}m\theta/n}, \mathrm{e}^{\mathrm{i}m\theta/n}, \cdots, \mathrm{e}^{\mathrm{i}m\theta/n}, \mathrm{e}^{-\mathrm{i}([n-1]/n)m\theta}\}, \tag{9.14}$$

$m = 0, 1, \ldots, n-1$, which lies in the Cartan subgroup and is responsible for the degeneracy of vacuum space. Then set

$$M = -\frac{\mathrm{i}}{m}\Omega_m^{-1}\partial_\theta\Omega_m = \mathrm{diag}\left\{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}, \frac{1-n}{n}\right\}, \quad 1 \le m \le n-1. \tag{9.15}$$

The radially symmetric static vortex solutions of the $SU(n)$ Chern–Simons–Higgs theory formulated in [80] are given by the ansatz

$$\phi^a = \frac{u_a(r)}{\sqrt{n}}\Omega_m^{-1}(\theta)E_{R_a}\Omega_m(\theta), \quad a = 1, 2, \cdots, n-1, \tag{9.16}$$

$$A_\theta = v(r)mM, \quad A_r = 0, \quad A_0 = w(r)mM, \tag{9.17}$$

realizing a solution asymptotically associated with the mth nontrivial vacuum state represented by an integral class in the fundamental group of the coset space of center Z_n of $SU(n)$, that is, by $m \in \pi_1(SU(n)/Z_n) = \mathbb{Z}_n$, where the ladder generators $\{E_{R_a}\}$ are chosen to assume the normalized forms [80]

$$(E_{R_a})_{jk} = \frac{1}{\sqrt{2}}\delta_{ja}\delta_{kn}, \quad a = 1, 2, \cdots, n-1, \quad j, k = 1, 2, \cdots, n. \tag{9.18}$$

Inserting (9.16) and (9.17) into (9.12) and (9.13) and using (9.18), we arrive at the radial version of the equations of motion [80]:

$$u_a'' + \frac{1}{r}u_a' = \frac{m^2}{r^2}(v-1)^2u_a - m^2w^2u_a + \frac{\lambda_a}{2}(u_a^2 - \eta_a^2)u_a, \tag{9.19}$$

$$v'' - \frac{1}{r}v' = \frac{1}{n-1}\Big(\sum_{a=1}^{n-1}u_a^2\Big)(v-1) + \kappa r w', \tag{9.20}$$

$$w'' + \frac{1}{r}w' = \frac{1}{n-1}\Big(\sum_{a=1}^{n-1}u_a^2\Big)w + \frac{\kappa}{r}v', \tag{9.21}$$

subject to the boundary condition consisting of

$$\lim_{r\to 0}u_a(r) = \lim_{r\to 0}v(r) = \lim_{r\to\infty}w(r) = 0, \text{ for } a = 1, 2, \cdots, n-1. \tag{9.22}$$

$$\lim_{r\to\infty}u_a(r) = \eta_a, \text{ for } a = 1, 2, \cdots, n-1, \ \lim_{r\to\infty}v(r) = 1, \tag{9.23}$$

$$\lim_{r\to 0}w(r) = w_0. \tag{9.24}$$

The associated action functional to the above equations is

$$\begin{aligned}\tilde{I}(u, v, w) = \int_0^\infty \Big(& r\Big(\sum_{a=1}^{n-1}\frac{(u_a')^2}{n-1}\Big) + \frac{m^2}{r}\Big(\sum_{a=1}^{n-1}\frac{u_a^2(v-1)^2}{n-1}\Big) \\ & + \sum_{a=1}^{n-1}\frac{\lambda_a}{4}\frac{(\eta_a^2 - u_a^2)^2}{n-1}r + \frac{m^2}{r}(v')^2 \\ & - rm^2(w')^2 - r\Big(\sum_{a=1}^{n-1}\frac{u_a^2}{n-1}\Big)m^2w^2 - 2\kappa m^2 v'w\Big)\,\mathrm{d}r,\end{aligned} \tag{9.25}$$

which is again indefinite, of course. Here and in the sequel, we use the vector notation $u = (u_a) = (u_1, \cdots, u_{n-1})$. Let

$$\tilde{G}(u,v) = \int_0^\infty \Big(r\Big(\sum_{a=1}^{n-1} \frac{(u_a')^2}{n-1}\Big) + \frac{m^2}{r}\Big(\sum_{a=1}^{n-1} \frac{u_a^2(v-1)^2}{n-1}\Big) + \sum_{a=1}^{n-1} \frac{\lambda_a}{4}\frac{(\eta_a^2 - u_a^2)^2}{n-1} r + \frac{m^2}{r}(v')^2 \Big) \mathrm{d}r,$$

$$\tilde{J}_{u,v}(w) = m^2 \int_0^\infty \Big(r(w')^2 + r\Big(\sum_{a=1}^{n-1} \frac{u_a^2}{n-1}\Big) w^2 + 2\kappa v' w \Big)\, \mathrm{d}r.$$

Then it is clear that $\tilde{I}(u,v,w) = \tilde{G}(u,v) - \tilde{J}_{u,v}(w)$.

The total energy is

$$\begin{aligned}\tilde{E}(u,v,w) = \int_0^\infty \Big(& r\Big(\sum_{a=1}^{n-1} \frac{(u_a')^2}{n-1}\Big) + \frac{m^2}{r}(v')^2 + r m^2 (w')^2 \\ & + \frac{m^2}{r}\Big(\sum_{a=1}^{n-1} \frac{u_a^2}{n-1}\Big)(v-1)^2 + r\Big(\sum_{a=1}^{n-1} \frac{u_a^2}{n-1}\Big) m^2 w^2 \\ & + r \sum_{a=1}^{n-1} \frac{\lambda_a}{4}\frac{(\eta_a^2 - u_a^2)^2}{n-1} \Big) \mathrm{d}r. \end{aligned} \tag{9.26}$$

Thus the natural admissible space $\tilde{\mathcal{A}}$ is

$$\tilde{\mathcal{A}} = \{(u,v,w) \mid \tilde{E}(u,v,w) < \infty \text{ and } (9.22),\ (9.23) \text{ hold}\}. \tag{9.27}$$

We will first minimize $\tilde{J}_{u,v}(w)$ for (u,v) such that $\tilde{G}(u,v) \le M < \infty$ in order to construct our constraint set. From the argument before, we need to control the size of the set where $|u_a(x)| \le \frac{\eta_a}{2}$ for each $a = 1, 2, \ldots, n-1$.

Proposition 9.1. *Suppose that (u,v) satisfies that $\tilde{G}(u,v) \le M < \infty$. Then there exists an R independent of u_a such that $\{x : \ |u_a(x)| \le \frac{\eta_a}{2}\} \subset B_R$ for all $a = 1, 2, \ldots, n-1$, where B_R is a ball in $\mathbb{R}^2$ of radius R centered at the origin.*

Proof. Consider (u,v) such that $\tilde{G}(u,v) \le M < \infty$. From the result on Ginzburg-Landau theory we know that $\eta_a - |u_a| \in H^1(\mathbb{R}^2)$. Hence we have

$$\begin{aligned}\Big(1 - \frac{|u_a(r)|}{\eta_a}\Big)^2 &\le 2\int_r^\infty \Big|1 - \frac{|u_a(\rho)|}{\eta_a}\Big| \cdot \Big|\frac{|u_a'(\rho)|}{\eta_a}\Big|\, \mathrm{d}\rho \\ &\le \frac{2}{r}\Big(\int_r^\infty \Big(1 - \frac{|u_a(\rho)|}{\eta_a}\Big)^2 \rho\, \mathrm{d}\rho\Big)^{1/2} \Big(\int_r^\infty \frac{|u_a'(\rho)|^2}{\eta_a^2}\rho\, \mathrm{d}\rho\Big)^{1/2} \\ &\le \frac{2}{r}\Big(\int_r^\infty \Big(1 - \frac{|u_a(\rho)|^2}{\eta_a^2}\Big)^2 \rho\, \mathrm{d}\rho\Big)^{1/2} \Big(\int_r^\infty \frac{|u_a'(\rho)|^2}{\eta_a^2}\rho\, \mathrm{d}\rho\Big)^{1/2} \\ &\le \frac{4(n-1)}{r\eta_a^3\sqrt{\lambda_a}} \tilde{G}(u,v) \le \frac{4(n-1)M}{r\eta_a^3\sqrt{\lambda_a}}. \end{aligned}$$

In this way, we may choose

$$R = \max_{a=1,2,\ldots,n-1} \frac{16(n-1)M}{\eta_a^3\sqrt{\lambda_a}}. \tag{9.28}$$

Then $|u_a(x)| > \frac{\eta_a}{2}$ for $|x| \ge R$, $a = 1, \cdots, n-1$. $\square$

In particular, denote

$$\eta = \min_{a=1,2,\dots,n-1} \eta_a.$$

Hence from the above proposition $|u_a| > \frac{\eta}{2}$ for $|x| \geq R$. Then using the same argument as in Lemma 4.2, we have

Lemma 9.2. *For each (u, v) with $\tilde{G}(u, v) \leq M < \infty$, the following minimization problem*

$$\min\{\tilde{J}_{u,v}(w) |\ w \in H^1_r(\mathbb{R}^2)\} \tag{9.29}$$

has a unique solution.

Proof. The uniqueness of the minimizer can be seen from the fact that the functional $\tilde{J}_{u,v}(w)$ is strictly convex in w.

First we derive the lower bound for $\tilde{J}_{u,v}(w)$. Using Cauchy–Schwartz we have

$$\begin{aligned}
\frac{1}{m^2}\tilde{J}_{u,v}(w) &\geq \int_0^\infty \Big(r(w')^2 + r\sum_{a=1}^{n-1} \frac{u_a^2}{n-1} w^2 - \varepsilon r w^2 - \frac{\kappa^2 (v')^2}{\varepsilon r}\Big)\,\mathrm{d}r \\
&\geq \int_0^\infty r(w')^2\,\mathrm{d}r + \sum_{a=1}^{n-1}\int_{|u_a|>\frac{\eta}{2}} \Big(\frac{\eta^2}{4(n-1)}\Big) r w^2\,\mathrm{d}r \\
&\quad - \varepsilon\int_0^\infty r w^2\,\mathrm{d}r - \frac{\kappa^2}{\varepsilon}\int_0^\infty \frac{(v')^2}{r}\,\mathrm{d}r \\
&\geq \int_0^\infty r(w')^2\,\mathrm{d}r + \int_R^\infty \big(\frac{\eta^2}{4} - \varepsilon\big) r w^2\,\mathrm{d}r - \int_0^R \varepsilon r w^2\,\mathrm{d}r - \frac{\kappa^2}{m^2\varepsilon}\tilde{G}(u,v) \\
&\geq (1 - \varepsilon C R)\int_0^\infty r(w')^2\,\mathrm{d}r + \Big(\frac{\eta^2}{4} - (1+CR)\varepsilon\Big)\int_R^\infty r w^2\,\mathrm{d}r - \frac{\kappa^2 M}{m^2\varepsilon},
\end{aligned}$$

where R in the last inequality is defined by (9.28) and we have used (4.3) to obtain the last inequality. Choosing ε to satisfy

$$\varepsilon = \min\Big\{\frac{\eta^2}{8(1+CR)}, \frac{1}{2CR}\Big\}, \tag{9.30}$$

we obtain

$$\frac{1}{m^2}\tilde{J}_{u,v}(w) \geq \frac{1}{2}\int_0^\infty r(w')^2\,\mathrm{d}r + \frac{\eta^2}{8}\int_R^\infty r w^2\,\mathrm{d}r - \frac{\kappa^2 M}{m^2\varepsilon}.$$

From this and (4.3) we also obtain the control of $\|w\|_{H^1_r(\mathbb{R}^2)}$

$$\|w\|^2_{H^1_r(\mathbb{R}^2)} \leq \frac{1+CR}{\min\{\frac{1}{2}, \frac{\eta^2}{8}\}}\Big(\frac{1}{m^2}\tilde{J}_{u,v}(w) + \frac{\kappa^2 M}{m^2\varepsilon}\Big), \tag{9.31}$$

where ε satisfies (9.30). The rest of the proof will be the same as in Lemma 4.2. □

In this way we can construct the constraint set to the minimization problem of $\tilde{I}(u, v, w)$ to be

$$\begin{aligned}
\tilde{\mathcal{C}} = \Big\{(u,v,w) \in \tilde{\mathcal{A}} \,\Big|\, w \in H^1_r(\mathbb{R}^2), \int_0^\infty \Big(r w'\bar{w}' + r\sum_{a=1}^{n-1}\frac{u_a^2}{n-1} w\bar{w} + \kappa v'\bar{w}\Big)\,\mathrm{d}r = 0 \\
\text{for all } \bar{w} \in H^1_r(\mathbb{R}^2) \text{ such that } J_{u,v}(w+\bar{w}) < \infty\Big\}.
\end{aligned} \tag{9.32}$$

Therefore from Lemma 9.2 we know that $\tilde{\mathcal{C}} \neq \emptyset$, and for each minimizer w we have

$$\tilde{J}_{u,v}(w) = -m^2 \int_0^\infty \Big(r(w')^2 + \sum_{a=1}^{n-1} \frac{u_a^2}{n-1} w^2 \Big) \, \mathrm{d}r \leq 0. \tag{9.33}$$

Therefore the minimization of $\tilde{I}(u,v,w)$ can be done via the similar method as before.

Proposition 9.3. *The minimization problem*

$$\min\{\tilde{I}(u,v,w) | \, (u,v,w) \in \tilde{\mathcal{C}}\} \tag{9.34}$$

has a solution.

Furthermore, the existence and regularity of solutions, the verification of the boundary conditions can all be achieved by the same argument as in Sections 6, 7. As for the quantization relations, following [80], we introduce an electromagnetic tensor

$$\mathcal{F}_{\mu\nu} = \frac{\mathrm{Tr}[MF_{\mu\nu}]}{\mathrm{Tr}[M^2]}. \tag{9.35}$$

Then the quantization of the magnetic flux and the electric charge can also be obtained by the same method as in Section 8.

Summing up all of the above, we have

Theorem 9.4. *For any given integer $m \in \{1, \ldots, n-1\}$, the non-Abelian Chern–Simons–Higgs equations expressed in (9.12)–(9.13) over $\mathbb{R}^2$ have a smooth finite-energy solution (A_0, A, ϕ), where $\phi = (\phi^a)$ represents a multiplet of $n-1$ Higgs fields each lying in the Cartan subalgebra of $su(n)$, satisfying the asymptotic properties*

$$F_{\mu\nu} \to 0, \;\; D_\mu \phi^a \to 0, \;\; |\phi^a| \to \eta_a, \;\; a = 1, \cdots, n-1, \;\; A_0 \to 0, \;\; \partial_j A_0 \to 0,$$

as $|x| \to \infty$. Moreover, the total magnetic flux Φ and electric charge Q are given respectively by the quantization formulas

$$\Phi = \int_{\mathbb{R}^2} \frac{\mathrm{Tr}[MF_{12}]}{\mathrm{Tr}[M^2]} \, \mathrm{d}x = 2\pi m, \tag{9.36}$$

$$Q = \int_{\mathbb{R}^2} \frac{\mathrm{Tr}[MJ^0]}{\mathrm{Tr}[M^2]} \, \mathrm{d}x = 2\pi m\kappa. \tag{9.37}$$

Such a magnetically and electrically charged solution realizes an $SU(n)$ vortex configuration asymptotically and topologically represented by the mth integral class in the classification space of the vortex vacuum manifold $SU(n)/Z_n$, that is, by $m \in \pi_1(SU(n)/Z_n) = \mathbb{Z}_n$ for $m = 1, \cdots, n-1$.

To conclude, in this paper, we have developed an existence theory for the electrically and magnetically charged vortex solutions arising in the classical Abelian and non-Abelian Chern–Simons–Higgs models using a constrained variational approach. Such a construction is of a general nature and does not rely on exploring the self-dual or BPS formulation of the problem.

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ROBIN MING CHEN
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: `chenm@math.umn.edu`

YUJIN GUO
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: `yjguo@math.umn.edu`

DANIEL SPIRN
SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: `spirn@math.umn.edu`

YISONG YANG
DEPARTMENT OF MATHEMATICS, YESHIVA UNIVERSITY, NEW YORK, NEW YORK 10033
E-mail address: `yisong.yang@yu.edu`