EXISTENCE AND SYMMETRY OF GROUND STATES TO THE BOUSSINESQ *abcd* SYSTEMS

ELLEN SHITING BAO, ROBIN MING CHEN, AND QING LIU

Abstract. We consider a four-parameter family of Boussinesq systems derived by Bona-Chen-Saut [7]. We establish the existence of the ground states which are solitary waves minimizing the action functional of the systems. We further show that in the presence of large surface tension the ground states are even up to translation.

1. Introduction

The dynamics of surface waves in an ideal fluid obeys complicated nonlinear and dispersive equations. To simplify them, multiscale asymptotic methods can be employed. One scaling regime that arises in practical situations is that of waves in a channel of approximately constant depth h that are uniform across the channel, and which are of small amplitude and long wavelength. Let a be a typical wave amplitude and λ a typical wavelength, the conditions above amount to

$$\varepsilon = \frac{a}{h} \ll 1, \quad \delta = \frac{h}{\lambda} \ll 1,$$

where the parameter ε measures the nonlinear effect and δ indicates the strength of dispersion. The equations within the above scaling regime couple the free surface elevation η to the horizontal component of the velocity v.

When the nonlinear and dispersive effects are balanced, that is $\varepsilon = \delta^2$, taking advantage of the freedom associated with the choice of the velocity variable and making full use of the lower-order relations (the wave equation written as a coupled system) in the dispersive terms, Bona-Chen-Saut [7] derived the following three-parameter family of Boussinesq systems (referred to as the *abcd* system) for one dimensional surface (generalized to include the surface tension in [18], and a two-dimensional analogue is derived in Bona-Colin-Lannes [9]),

$$\begin{cases} \eta_t + v_x + (\eta v)_x + a v_{xxx} - b\eta_{txx} = 0 \\ v_t + \eta_x + v v_x + c\eta_{xxx} - d v_{txx} = 0, \end{cases} \tag{1.1}$$

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which is formally equivalent models of solutions of the Euler equations. In the above system, $\eta(t,x)$ is proportional to the deviation of the free surface from its rest position, and $v(t,x)$ is proportional to the horizontal velocity taken at the scaled height θ with $0 \le \theta \le 1$ ($\theta = 1$ at the free surface and $\theta = 0$ at the bottom), and

$$a = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\nu, \quad b = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\nu), \quad c = \frac{(1-\theta^2)}{2}\mu - \tau, \quad d = \frac{(1-\theta^2)}{2}(1-\mu)$$

with ν and μ arbitrary real numbers, and $\tau \ge 0$ the surface tension.

The *abcd* system (1.1) carries a Hamiltonian structure when $b = d$, as it can be written in this case as

$$\partial_t \begin{pmatrix} \eta \\ u \end{pmatrix} = J\nabla\mathcal{H}(\eta, u),$$

where

$$\mathcal{H}(\eta,u) = \frac{1}{2}\int_{\mathbb{R}} \left[-c\eta_x^2 - au_x^2 + \eta^2 + (1+\eta)u^2\right]\, dx, \tag{1.2}$$

and

$$J = -(1-b\partial_x^2)^{-1}\partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

is a skew-adjoint operator.

Another conserved quantity when $b = d$ is the impulse functional

$$\mathcal{I}(\eta,u) = \int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx. \tag{1.3}$$

Throughout this paper we will just be dealing with the case $a, c < 0$ and $b = d$.

Among various topics of the water wave theory, solitary waves have their long-standing history dated back to Scott Russell's horseback observation and play a center role in understanding the wave coherent structure. In the context of two-dimensional full gravity-capillary water wave equations, it is shown that for large surface tension ($\tau > 1/3$), depression solitary waves exist (cf. [21, 31] for analysis and [20, 34, 19] for computational studies). Using asymptotic expansions, Ablowitz-Haut [1] find that for large enough surface tension the asymptotic series for the free surface monotonically increases to zero away from its unique minimum. A natural question is how well the Boussinesq model can capture this feature: are there one-dimensional depression solitary waves for system (1.1) that increasing from its minimum to zero at infinity?

Our goal here is to understand the aforementioned properties for a special class of solitary waves to system (1.1), namely the *ground states*, which carries the least Lagrangian action energy among all solitary waves (a precise definition is given in Definition 1.1). By a solitary wave, we mean a solution of (1.1) of the type

$$\eta(x,t) = \eta(x-\omega t) \in H^1(\mathbb{R}), \quad u(x,t) = u(x-\omega t) \in H^1(\mathbb{R}), \tag{1.4}$$

where ω denotes the traveling speed.

We are thus looking in the class of "localized" solutions to the system

$$\begin{cases} c\eta_{xx} + \eta - \omega u + b\omega u_{xx} + \dfrac{1}{2}u^2 = 0 \\ au_{xx} + u - \omega\eta + b\omega\eta_{xx} + \eta u = 0. \end{cases} \tag{1.5}$$

A variational approach has been adopted to prove the existence of solitary waves. In [15] the authors make use of the aforementioned two conserved quantities, namely the Hamiltonian $\mathcal{H}$ and the impulse $\mathcal{I}$, following the idea introduced by Buffoni [11] in dealing with the full water wave problem. A solitary wave is thus characterized as a critical point of the energy subject to the constraint of fixed impulse; it is therefore a critical point of the functional

$$S_\omega = \mathcal{H} - \omega\mathcal{I}, \tag{1.6}$$

where the Lagrange multiplier ω gives the speed of the wave; and the energetic stability of the set of such solitary waves follows from the Benjamins principle [3]. However, the solitary waves found there are in $X^2 = H^2(\mathbb{R}) \times H^2(\mathbb{R})$ instead of the energy space $X^1 = H^1(\mathbb{R}) \times H^1(\mathbb{R})$, and $\|\eta\|_{H^2}$ needs to be small to obtain the coercivity of $\mathcal{H}$. In addition, a large surface tension assumption is needed to prove the stability result.

Another variational formulation is established in [16] by utilizing the following two functionals

$$H_\omega = \frac{1}{2}\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx - \omega \int_{\mathbb{R}} \left(\eta u + b\eta_x u_x\right)\, dx \tag{1.7}$$

$$P = \frac{1}{2}\int_{\mathbb{R}} \eta u^2 \, dx, \tag{1.8}$$

which can be viewed as a quadratic-cubic splitting of the full action functional S_ω. An application of Lion's concentration compactness principle [23] leads to the existence of the minimization problem

$$\inf\{H_\omega(\eta, u) : \ (\eta, u) \in X^1, P(\eta, u) = p\}, \tag{1.9}$$

and hence the existence of solitary waves in X^1, without any assumption on the size of data and the strength of surface tension. But a small traveling speed assumption is needed to ensure the coercivity of H_ω.

We want to first establish an existence result of the ground states in the natural energy space, which is X^1. Now we give the definition of ground states as follows.

Definition 1.1. A *ground state* $\vec{u} = (\eta, u)$ of (1.1) is an X^1-solitary wave solution of (1.5) which minimizes

$$S_\omega = \mathcal{H} - \omega\mathcal{I}$$

among all nonzero solutions of (1.5).

Our main existence theorem is the following.

Theorem 1.1 (Existence of ground states)**.** *For $a, c < 0, b = d$ and $|\omega| < \omega_0$ where*

$$\omega_0 = \begin{cases} \min\left\{1, \frac{\sqrt{ac}}{|b|}\right\}, & \text{when } b \neq 0, \\ 1, & \text{when } b = 0, \end{cases} \tag{1.10}$$

then system (1.1) *has a ground state solution.*

We use a Nehari manifold approach [28, 29], which is to minimize S_ω over the natural constraint set

$$\mathcal{N} = \left\{\vec{u} \in X^1 \backslash \{0\} : \ \langle S'_\omega(\vec{u}), \vec{u}\rangle = 0\right\}.$$

We prove that this gives an equivalent formulation, and hence introduce the following constraint minimization (cf. Proposition 2.2)

$$\inf_{\vec{u} \in \mathcal{N}} S_\omega(\vec{u}).$$

The difficulties in the direct minimization of the above problem lie in the fact that S_ω is not coercive due to the indefinite cubic term, and that $\mathcal{N}$ may not be weakly closed. To get around these issues, we cut off the cubic part from S_ω and relax the constraint set.

The second topic we address here is the symmetry of the ground states. It turns out that the ground states solve a weakly coupled elliptic system. We follow the moving-plane method as is used in Busca-Sirakov [12], which requires that the nonlinearities satisfy certain restrictions, and the solutions do not change sign as well. The non-sign-changing property of the ground states turns out to be rather delicate to check due to the appearance of the product of the derivatives in the minimizing functional. To resolve this difficulty we introduce a change-of-unknowns transformation (cf. (3.10), (3.12)) and then apply a "selection principle" (cf. Remark 3.1). Our symmetry results for the ground states are the follows.

Theorem 1.2 (Symmetry of ground states)**.** *Under the same assumptions as in Theorem 1.1, and assume further that*

$$0 \leq b \leq \max\{-a, -c\}.$$

Then any ground state (η^*, u^*) of (1.1) *is even up to translation, that is, there exists a point $x_0 \in \mathbb{R}$ such that $\eta^*(x) = \eta^*(|x - x_0|)$ and $u^*(x) = u^*(|x - x_0|)$. Moreover,*

(1) *when $\omega > 0$,*

$$\frac{d\eta^*}{dr} > 0 \quad \text{and} \quad \frac{du^*}{dr} > 0 \quad \text{for all } r = |x - x_0| > 0; \tag{1.11}$$

(2) *when $\omega < 0$,*

$$\frac{d\eta^*}{dr} > 0 \quad \text{and} \quad \frac{du^*}{dr} < 0 \quad \text{for all } r = |x - x_0| > 0; \tag{1.12}$$

(3) *when $\omega = 0$, then either* (1.11) *or* (1.12) *holds.*

It would certainly be interesting to study the situation when the parameters fall out of the regime considered here, while keeping the Hamiltonian structure (that is, $b = d$), so that ground states can still be defined. It is also nature to require that the system be linearly well-posed in the parameter regime [7]. The main difficulty now lies in the fact that the coercivity property is lost in the variational procedure, and hence it is hard to even search for solitary waves. For some special cases, solitary waves are found using a dynamical system approach (see, e.g. [14, 33]). However the existence of solitary waves in the general linearly well-posed regime still remains open.

The rest of the paper is organized as follows. In Section 2 we prove the existence of ground states to system (1.1) using a variational approach, and show the connection between the ground states and the solitary waves obtained in [16] (cf. Proposition 2.5). In Section 3 we establish a non-sign-changing condition for the ground states so that we can apply the "moving plane" method to prove the symmetry property of the ground states in Section 4. Finally in the Appendix we show that in fact all solitary waves of system (1.5) decay exponentially to zero at infinity.

2. Existence of ground states

Denote the set of ground states with traveling speed ω by

$$\mathcal{G}_\omega = \{\vec{u} \in X^1\backslash\{0\} : \ S_\omega(\vec{u}) \leq S_\omega(\vec{v}) \ \text{ for all } \vec{v} \in X^1\backslash\{0\} \text{ satisfying } \langle S'_\omega(\vec{v}), \vec{v}\rangle = 0\}. \tag{2.1}$$

The main goal of this section is to prove the existence result Theorem 1.1. First we state a result on the coercivity of the functional H_ω defined in (1.7)

Lemma 2.1. [16] *For $a, c < 0, b = d$ and $|\omega| < \omega_0$ where ω_0 is given in* (1.10)*, one has*

$$H_\omega(\vec{u}) \geq C_0 \|\vec{u}\|^2_{X^1}, \tag{2.2}$$

where

$$C_0 = \frac{1}{2}\min\left\{\left(1 - \frac{|\omega b|}{\sqrt{ac}}\right)|a|, \left(1 - \frac{|\omega b|}{\sqrt{ac}}\right)|c|, (1-|\omega|)\right\} > 0. \tag{2.3}$$

We next define the functional

$$N(\vec{u}) = \int_{\mathbb{R}} \left(-c\eta_x^2 - a u_x^2 + \eta^2 + u^2\right) \, dx - 2\omega \int_{\mathbb{R}} (\eta u + b\eta_x u_x) \, dx + \frac{3}{2}\int_{\mathbb{R}} \eta u^2 \, dx. \tag{2.4}$$

Therefore if $\vec{u}$ is a solitary wave solution, then $N(\vec{u}) = \langle S'_\omega(\vec{u}), \vec{u}\rangle = 0$. Moreover we have

Proposition 2.2. *Under the assumption of Lemma 2.1, $\vec{u} \in \mathcal{G}_\omega$ if and only if $\vec{u}$ solves the minimization problem*

$$J_\omega = \inf\{S_\omega(\vec{v}) : \ \vec{v} \in X^1\backslash\{0\}, N(\vec{v}) = 0\}. \tag{2.5}$$

Proof. "$\Rightarrow$": This is obvious since $S'_\omega(\vec{v}) = 0$ implies $N(\vec{v}) = 0$.

"$\Leftarrow$": If $S_\omega(\vec{u}) = J_\omega$ and $N(\vec{u}) = 0$, the $\vec{u}$ satisfies the Euler-Lagrange equation

$$S'_\omega(\vec{u}) + \lambda N'(\vec{u}) = 0$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$. Taking the L^2-inner product of the above equations with $\vec{u}$ and making use of the fact that $\langle S'_\omega(\vec{u}), \vec{u}\rangle = N(\vec{u}) = 0$ we have that $\lambda\langle N'(\vec{u}), \vec{u}\rangle = 0$. On the other hand, from Lemma 2.1

$$\begin{aligned}\langle N'(\vec{u}), \vec{u}\rangle &= 2\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx - 4\omega\int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx + 3\int_{\mathbb{R}} \frac{3}{2}\eta u^2\, dx \\ &= -\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx + 2\omega\int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx \\ &= -2H_\omega(\vec{u}) < 0.\end{aligned}$$

Hence $\lambda = 0$ and in turn $S'_\omega(\vec{u}) = 0$. Moreover since $N(\vec{u}) = \langle S'_\omega(\vec{u}), \vec{u}\rangle$ for any $\vec{u} \in X^1$ and $\vec{u}$ is a minimizer of J, thus $S_\omega(\vec{u}) \le S_\omega(\vec{v})$ for any solitary wave $\vec{v} \in X^1$ of (1.1). Therefore $\vec{u} \in \mathcal{G}_\omega$. $\square$

As is pointed out in the Introduction, there are two difficulties in minimizing J_ω directly. The first difficulty is that S_ω is not homogeneous, and moreover is not coercive. The second one is the lack of convergence of $N(\vec{u}_n)$, provided a weak X^1-convergence of $\vec{u}_n$. To handle the first issue, we remove the cubic terms from S_ω by using N, that is, we consider the functional

$$\tilde{S}_\omega = S_\omega - \frac{1}{3}N = \frac{1}{6}\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx - \frac{1}{3}\omega\int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx, \tag{2.6}$$

and look for minimizers of $\tilde{S}_\omega$. Note that when $N = 0$, $S_\omega = \tilde{S}_\omega$. To resolve the second issue, we relax the constraint $N = 0$ to $N \le 0$. In this way, we seek minimizers to the problem

$$\tilde{J}_\omega = \inf\{\tilde{S}_\omega(\vec{v}) :\ \vec{v} \in X^1\backslash\{0\}, N(\vec{v}) \le 0\}. \tag{2.7}$$

Proposition 2.3. *Under the same assumptions of Theorem 1.1, we have*

$$J_\omega = \tilde{J}_\omega. \tag{2.8}$$

Proof. From the assumption $|\omega| < \omega_0$ and Lemma 2.1 we obtain the coercivity of $\tilde{S}_\omega$:

$$\tilde{S}_\omega(\vec{u}) \ge \frac{1}{3}C_0\|\vec{u}\|^2_{X^1}. \tag{2.9}$$

For $\vec{u} \in X^1\backslash\{0\}$ with $N(\vec{u}) = 0$, it is straightforward that $\tilde{J}_\omega \le \tilde{S}_\omega(\vec{u}) = S_\omega(\vec{u})$. Hence $\tilde{J}_\omega \le J_\omega$. Next we claim that $J_\omega \le \tilde{J}_\omega$. Notice that for any $\vec{u} \in X^1\backslash\{0\}$ with

$$N(\vec{u}) = 2H_\omega(\vec{u}) + \frac{3}{2}\int_{\mathbb{R}} \eta u^2\, dx \le 0,$$

from Lemma 2.1 we know that $H_\omega \geq C_0\|\vec{u}\|^2_{X^1} > 0$ when $|\omega| < \omega_0$. Thus $\int_{\mathbb{R}} \eta u^2\, dx < 0$. So there exists some $k \in (0,1)$ sufficiently small such that

$$N(k\vec{u}) = k^2\left[\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx - 2\omega\int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx\right] + \frac{3}{2}k^3\int_{\mathbb{R}} \eta u^2\, dx > 0.$$

Hence by continuity we can find a $k_0 \in (k,1)$ such that $N(k_0\vec{u}) = 0$. Therefore

$$\begin{aligned} J_\omega \leq S(k_0\vec{u}) &= \frac{k_0^2}{2}\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx - k_0^2\omega\int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx + \frac{k_0^3}{2}\int_{\mathbb{R}} \eta u^2\, dx \\ &= k_0^2\left[\frac{1}{6}\int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right)\, dx - \frac{1}{3}\omega\int_{\mathbb{R}} (\eta u + b\eta_x u_x)\, dx\right] \\ &= \frac{k_0^2}{3}\tilde{S}_\omega(\vec{u}) \leq \tilde{S}_\omega(\vec{u}), \end{aligned}$$

where we have used the coercivity of $\tilde{S}_\omega$ in the last inequality. Hence $J_\omega \leq \tilde{J}_\omega$, and therefore $J_\omega = \tilde{J}_\omega$ □

From the above discussion we arrive at the minimization of $\tilde{J}_\omega$. The following Lemma ensures the existence of such minimizers, and hence completes the proof of Theorem 1.1.

Lemma 2.4. *Under the assumption of Theorem 1.1, the minimization problem* (2.5) *has a solution.*

Proof. We first consider the minimization of $\tilde{J}_\omega$. From (2.9) we know that $\tilde{S}_\omega$ is coercive. Hence there exists a minimizing sequence $\{\vec{u}_n = (\eta_n, u_n)\}$ for $\tilde{S}_\omega$ satisfying

$$\vec{u}_n \neq 0, \quad N(\vec{u}_n) \leq 0, \quad \text{and } \lim_{n\to\infty} \tilde{S}_\omega(\vec{u}_n) = \tilde{J}_\omega. \tag{2.10}$$

The coercivity of $\tilde{S}_\omega$ also implies that $\{\vec{u}_n\}$ is bounded in X^1, therefore it has a subsequence, still denoted by $\{\vec{u}_n\}$, converges weakly to some $\vec{u}^* = (\eta^*, u^*) \in X^1$.

1°. We claim that

$$\inf_n \left|\int_{\mathbb{R}} \eta_n u_n^2\, dx\right| = \alpha > 0.$$

Suppose not. Then there is a subsequence, still denoted by $\{\vec{u}_n\}$, such that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \eta_n u_n^2\, dx = 0.$$

Since $N(\vec{u}_n) \leq 0$, it follows Lemma 2.1 that

$$\begin{aligned} 2C_0\|\vec{u}_n\|^2_{X^1} \leq 2H_\omega(\vec{u}_n) &= \int_{\mathbb{R}} \left(-c\eta_{nx}^2 - au_{nx}^2 + \eta_n^2 + u_n^2\right)\, dx - 2\omega\int_{\mathbb{R}} (\eta_n u_n + b\eta_{nx}u_{nx})\, dx \\ &\leq -\frac{3}{2}\int_{\mathbb{R}} \eta_n u_n^2\, dx \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{2.11}$$

and thus $\|\vec{u}^*\|_{X^1}^2 \le 0$. On the other hand, from Sobolev embedding we know that

$$\left|\int_{\mathbb{R}} \eta_n u_n^2 \; dx\right| \le C\|\vec{u}_n\|_{X^1}^3, \qquad \text{for } C > 0.$$

Combing this with the previous estimate (2.11) leads to

$$C_0\|\vec{u}_n\|_{X^1}^2 \le C\|\vec{u}_n\|_{X^1}^3.$$

Moreover $\vec{u}_n \neq 0$, then $\|\vec{u}_n\|_{X^1} \ge C_0/C > 0$, contradicting (2.11), hence we prove the claim.

2°. We prove that $\vec{u}^* \neq 0$. We perform the following splitting

$$\begin{aligned}
\alpha &\le \int_{\mathbb{R}} \left|\eta_n u_n^2\right| \; dx \le \left(\int_{|\eta_n|\le\epsilon} + \int_{\epsilon<|\eta_n|,|u_n|<1/\epsilon} + \int_{|u_n|\le\epsilon} + \int_{|u_n|\ge 1/\epsilon} + \int_{|\eta_n|\ge 1/\epsilon}\right) \left|\eta_n u_n^2\right| \; dx \\
&\le \epsilon\|\vec{u}_n\|_{X^1}^2 + \frac{1}{\epsilon^3}\left|\{|\eta_n|,|u_n| > \epsilon\}\right| + \epsilon\|\vec{u}_n\|_{X^1}^2 + \epsilon^2\int_{\mathbb{R}} \left|\eta_n u_n^4\right| \; dx + \epsilon\int_{\mathbb{R}} \eta_n^2 u_n^2 \; dx \\
&\le C(\epsilon+\epsilon^2) + \frac{1}{\epsilon^3}\left|\{|\eta_n|,|u_n| > \epsilon\}\right|
\end{aligned}$$

for any $\epsilon > 0$. Choosing ϵ small enough so that $C(\epsilon+\epsilon^2) < \alpha/2$ we thus have

$$\left|\{|\eta_n|,|u_n| > \epsilon\}\right| > \frac{\alpha\epsilon^3}{2} = \delta > 0.$$

Since $\|\vec{u}_n\|_{X^1}$ is bounded, a one-dimensional analogue of Lemma 4 in [25] leads to

$$\left|B \cap \{|\eta_n| > \epsilon/2, |u_n| > \epsilon/2\}\right| > \delta_0 > 0$$

for some $\delta_0 > 0$, where B is a ball in $\mathbb{R}$ of unit radius. Then using the fact that $\vec{u}_n \to \vec{u}^*$ a.e. we prove the assertion.

3°. Now we show that $\vec{u}^*$ is a minimizer of (2.7), that is, $N(\vec{u}^*) \le 0$ and $\tilde{S}_\omega(\vec{u}^*) = \tilde{J}_\omega$. Denote

$$\begin{aligned}
G(\vec{u}) &= \int_{\mathbb{R}} \left(-c\eta_x^2 - au_x^2 + \eta^2 + u^2\right) \; dx, \\
K(\vec{u}) &= \int_{\mathbb{R}} \left(\eta u + b\eta_x u_x\right) \; dx.
\end{aligned}$$

Then $\tilde{S}_\omega = \frac{1}{6}G - \frac{1}{3}\omega K$. Since $\vec{u}_n \rightharpoonup \vec{u}^*$ in X^1, $\vec{u}_n \to \vec{u}^*$ a.e., it is easy to deduce that

$$\begin{aligned}
G(\vec{u}_n) - G(\vec{u}_n - \vec{u}^*) - G(\vec{u}^*) &\to 0, \\
K(\vec{u}_n) - K(\vec{u}_n - \vec{u}^*) - K(\vec{u}^*) &\to 0 \qquad \text{as } n \to \infty,
\end{aligned}$$

and so

$$\tilde{S}_\omega(\vec{u}_n) - \tilde{S}_\omega(\vec{u}_n - \vec{u}^*) - \tilde{S}_\omega(\vec{u}^*) \to 0. \tag{2.12}$$

We claim that the same limit holds for P (defined (1.8)):

$$P(\vec{u}_n) - P(\vec{u}_n - \vec{u}^*) - P(\vec{u}^*) \to 0. \tag{2.13}$$

To prove (2.13), we use the idea from Brezis-Lieb [10]. Consider a function $j : \mathbb{R}^2 \to \mathbb{R}$, $j(\vec{a}) = a_1 a_2^2$, and in this way $P(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}} j(\vec{u})\, dx$. Then one can easily verify the following decomposition estimate

$$|j(\vec{a}+\vec{b}) - j(\vec{a})| \leq \epsilon\varphi(\vec{a}) + \psi_\epsilon(\vec{b}), \quad \text{ for all } \vec{a}, \vec{b} \in \mathbb{R}^2,$$

where

$$\varphi(\vec{a}) = a_1^2 a_2^2 + a_1^2 + a_2^4 + a_2^2, \quad \psi_\epsilon(\vec{b}) = C_\epsilon(b_1^2 + b_1^2 b_2^2 + b_2^2 + b_2^4) + b_1 b_2^2.$$

Then the functions $j, \varphi, \psi, \vec{u}_n$, and $\vec{u}$ satisfy the conditions in Theorem 2 of [10], and thus

$$\int_{\mathbb{R}} |j(\vec{u}_n) - j(\vec{u}_n - \vec{u}^*) - j(\vec{u}^*)| \to 0,$$

implying (2.13).

So combining the above convergence results and using the identity that $N = G - 2\omega K + 3P$ we have

$$N(\vec{u}_n) - N(\vec{u}_n - \vec{u}^*) - N(\vec{u}^*) \to 0. \tag{2.14}$$

If $N(\vec{u}^*) > 0$, from the constraint $N(\vec{u}_n) \leq 0$ and (2.14) we know $N(\vec{u}_n - \vec{u}*) < 0$ as $n \to \infty$ and hence also in the constraint class of $\tilde{S}_\omega$, and subsequently when n is large, $\tilde{S}_\omega(\vec{u}_n - \vec{u}^*) \geq \tilde{J}_\omega$. On the other hand, by definition $\tilde{S}_\omega(\vec{u}_n) \to \tilde{J}_\omega$ as $n \to \infty$. Together with (2.12) we infer that $\tilde{S}_\omega(\vec{u}^*) \leq 0$. Then from the coercivity of $\tilde{S}_\omega$, (2.9) we know $\|\vec{u}^*\|_{X^1} \leq 0$, implying that $\vec{u}^* = 0$ a.e., which contradicts the result in 2°. Therefore $N(\vec{u}^*) \leq 0$.

Moreover from (2.12) we have

$$\frac{1}{3}C_0\|\vec{u}_n - \vec{u}^*\|_{X^1}^2 \leq \tilde{S}_\omega(\vec{u}_n - \vec{u}^*) \to \tilde{S}_\omega(\vec{u}_n) - \tilde{S}_\omega(\vec{u}^*) \to \tilde{J}_\omega - \tilde{S}_\omega(\vec{u}^*).$$

Hence $\tilde{S}_\omega(\vec{u}^*) \leq \tilde{J}_\omega$, and $\tilde{J}_\omega > 0$.

4°. Now it remains to check that indeed $N(\vec{u}^*) = 0$. If not, then $N(\vec{u}^*) < 0$. A scaling argument similar to the one in Proposition 2.3 indicates that $N(k\vec{u}^*) > 0$ for some $0 < k < 1$ sufficiently small, and by continuity of N there is a $k_0 \in (0, k)$ such that $N(k_0\vec{u}^*) = 0$. Accordingly,

$$\tilde{J}_\omega \leq \tilde{S}_\omega(k_0\vec{u}^*) = k_0^2\tilde{S}_\omega(\vec{u}^*) = k_0^2\tilde{J}_\omega < \tilde{J}_\omega,$$

a contradiction. Therefore $\tilde{S}_\omega(\vec{u}^*) = \tilde{J}_\omega$ and $N(\vec{u}^*) = 0$. From Proposition 2.3 we know that $\vec{u}^*$ solves (2.5). Hence proves the lemma. ☐

Proof of Theorem 1.1. The existence of the ground states can be inferred from Propositions 2.2, 2.3, and Lemma 2.4. ☐

The proposition below gives several characterizations of a ground state of (1.5).

Proposition 2.5 (Characterization of ground states)**.** *Under the same assumptions as in Theorem 1.1, the following statements are equivalent:*

(i) $\vec{u}^*$ *is a ground state;*
(ii) $N(\vec{u}^*)=0$ *and* $H_\omega(\vec{u}^*)=\inf\{H_\omega(\vec{u}):P(\vec{u})=\lambda^*\}$, *where* $\lambda^*=P(\vec{u}^*)<0$*;*
(iii) $N(\vec{u}^*)=0$ *and* $P(\vec{u}^*)=\sup\{P(\vec{u}):\vec{u}\neq 0, N(\vec{u})=0\}$*;*
(iv) $N(\vec{u}^*)=0=\inf\{N(\vec{u}):\vec{u}\neq 0, P(\vec{u})=P(\vec{u}^*)\}$.

Proof. It is clear that (i) and (iii) are equivalent. Indeed, by Proposition 2.2, $\vec{u}^*\in\mathcal{G}_\omega$ if and only if $\vec{u}^*$ is a minimizer of (2.5), where $\vec{u}^*$ is required to satisfy $N(\vec{u}^*)=0$. But since $N=2S_\omega+P$, it is equivalent to saying that any ground state $\vec{u}^*$ maximizes P under the same requirement.

Let us next prove (ii) implies (iii). Let $\vec{u}^*$ satisfy $N(\vec{u}^*)=0$, $P(\vec{u}^*)=\lambda^*$ and $H_\omega(\vec{u}^*)=\min\{H_\omega(\vec{u}):P(\vec{u})=\lambda^*\}$. For any $\vec{u}\neq 0$ such that $N(\vec{u})=0$, $G(\vec{u})-2\omega K(\vec{u})>0$ and therefore $P(\vec{u})<0$. We take

$$b=\left(\frac{P(\vec{u}^*)}{P(\vec{u})}\right)^{1/3}.$$

Our goal is to show that $b\leq 1$.

By direct calculation, we get

$$N(b\vec{u})=b^2(G(\vec{u})-2\omega K(\vec{u}))+3b^3P(\vec{u})=(b^2-b^3)(G-2\omega K)(\vec{u}) \tag{2.15}$$

and

$$P(b\vec{u})=b^3P(\vec{u})=P(\vec{u}^*)=\lambda^*.$$

By definition of $\vec{u}^*$, we have $H_\omega(b\vec{u})\geq H_\omega(\vec{u}^*)$. We are then led to

$$0=N(\vec{u}^*)=H_\omega(\vec{u}^*)+3P(\vec{u}^*)\leq H_\omega(b\vec{u})+3P(b\vec{u})=N(b\vec{u}).$$

Combining the relation above with (2.15), we have

$$(b^2-b^3)(G-2\omega K)(\vec{u})\geq 0,$$

which yields that $b\leq 1$ due to the positivity of $(G-2\omega K)(\vec{u})$.

We next show that (i) implies (ii). Let $\vec{u}^*$ be a ground state of (1.5). Then we have

$$N(\vec{u}^*)=(G-2\omega K)(\vec{u}^*)+3P(\vec{u}^*)=0.$$

We let $\lambda^*=P(\vec{u}^*)(<0)$ and in consequence $(G-2\omega K)(\vec{u}^*)=-3\lambda^*$. Since

$$\begin{aligned}&\min\{S_\omega(\vec{v}):\ \vec{v}\in X^1\backslash\{0\}, N(\vec{v})=0\}\\ =&\min\left\{\frac{1}{2}G(\vec{v})-\omega K(\vec{v})-\frac{1}{3}(G-2\omega K)(\vec{v}):\ \vec{v}\in X^1\backslash\{0\}, N(\vec{v})=0\right\}\\ =&\min\left\{\frac{1}{6}(G-2\omega K)(\vec{v}):\ \vec{v}\in X^1\backslash\{0\}, N(\vec{v})=0\right\},\end{aligned}$$

$\vec{u}^*$ in fact minimizes $G-2\omega K$ among all $\vec{u}$ satisfying $N(\vec{u})=0$. Our goal is to prove $\vec{u}^*$ minimizes $G-2\omega K$ among all $\vec{u}$ satisfying $P(\vec{u})=\lambda^*$.

Let $\vec{u}_1 = (\eta_1, u_1)$ be such a minimizer. It suffices to show that $(G-2\omega K)(\vec{u}_1) = (G-2\omega K)(\vec{u}^*)$. We apply Lagrange multiplier method to get a $\theta \in \mathbb{R}$ such that $(H'_\omega + \theta P')(\vec{u}_1) = 0$, which implies that η_1 and u_1 satisfy

$$\begin{cases} c\eta_{xx} + \eta - \omega u + b\omega u_{xx} + \dfrac{1}{2}\theta u^2 = 0 \\ au_{xx} + u - \omega\eta + b\omega\eta_{xx} + \theta\eta u = 0. \end{cases}$$

This yields $(G-2\omega K)(\vec{u}_1) = -3\theta P(\vec{u}_1) = -3\theta\lambda^*$. It remains to prove $\theta = 1$. To this end, we first apply the definition of $\vec{u}_1$ to get

$$(G-2\omega K)(\vec{u}_1) \le (G-2\omega K)(\vec{u}^*),$$

which means $-3\theta\lambda^* \le -3\lambda^*$ and therefore $\theta \le 1$.

On the other hand, if we let $\vec{u_2} = \theta\vec{u_1}$, then $\vec{u_2} = (\eta_2, u_2)$ satisfies (1.5) and therefore $N(\vec{u_2}) = 0$. Since $\vec{u}^*$ minimizes $G - 2\omega K$ among all $\vec{u}$ satisfying $N(\vec{u}) = 0$, we obtain

$$(G-2\omega K)(\vec{u}^*) \le (G-2\omega K)(\vec{u_2}) = (G-2\omega K)(\theta\vec{u}_1) = \theta^2(G-2\omega K)(\vec{u}_1) = -3\theta^3\lambda^*.$$

This gives us $-3\lambda^* \le -3\theta^3\lambda^*$, which implies $\theta \ge 1$.

We next prove the equivalence between (iii) and (iv). We first show (iii) implies (iv). Suppose that $\vec{u}^*$ satisfies (iii). Then for any $\vec{u}$ such that $P(\vec{u}) = P(\vec{u}^*)$, we must have $N(\vec{u}) \ge 0$. If it is not the case, i.e., $N(\vec{u}) < 0$ and hence $P(\vec{u}) < 0$, we get

$$N(b\vec{u}) = b^2(G-2\omega K)(\vec{u}) + 3b^3 P(\vec{u}) > 0$$

for some $b \in (0,1)$ sufficiently small. Then there exists $b_0 \in (0,1)$ such that $N(b_0\vec{u}) = 0$. Meanwhile, $P(b_0\vec{u}) = b_0^3 P(\vec{u}) > P(\vec{u}^*)$. This contradicts the fact that $\vec{u}^*$ satisfies (iii).

We finally show (iv) implies (iii). Suppose that $\vec{u}^*$ satisfies (iv). Assume by contradiction that there is some $\vec{u} \ne 0$ satisfying $N(\vec{u}) = 0$ but $P(\vec{u}) > P(\vec{u}^*)$. It is clear that $P(\vec{u}) < 0$. Then we can find $b_0 > 1$ such that $P(b_0\vec{u}) = b_0^3 P(\vec{u}) = P(\vec{u}^*)$. On the other hand, $N(b_0\vec{u}) = b_0^2(G-2\omega K)(\vec{u}) + 3b_0^3 P(\vec{u}) < 0$. This is a contradiction to (iv). □

Remark 2.1. The existence of $\vec{u}^*$ satisfying the condition (ii) can be derived by scaling the result of [16]. Indeed, it is known that there exists $\vec{u}_0$ such that $H_\omega(\vec{u}_0) = \min\{H_\omega(\vec{u}) : P(\vec{u}) = \lambda_0\}$, where $\lambda_0 = P(\vec{u}_0)$. Then there exists a Lagrange multiplier $\theta \in \mathbb{R}$ such that $(G-2\omega K)(\vec{u}_0) + 3\theta P(\vec{u}_0) = 0$. The proof is trivial if $\theta = 1$ or $\theta = 0$.

We thus assume $\theta \ne 1$ and $\theta \ne 0$. By letting $\vec{u}^* := \theta\vec{u}_0$, we have

$$N(\vec{u}^*) = \theta^2(G-2\omega K)(\vec{u}_0) + 3\theta^3 P(\vec{u}_0) = 0.$$

Moreover, it is easily seen that $H_\omega(\vec{u}^*) = \min\{H_\omega(\vec{u}) : P(\vec{u}) = \lambda^*\}$, where $\lambda^* = P(\vec{u}^*)$. In fact, for any $\vec{u}$ such that $P(\vec{u}) = \lambda^*$, $P(\frac{\vec{u}}{\theta}) = \lambda_0$. Then in view of the definition of $\vec{u}_0$, we have

$$H_\omega(\vec{u}_0) \le H_\omega(\vec{u}/\theta),$$

which implies

$$H_\omega(\vec{u}^*) = \theta^2 H_\omega(\vec{u}_0) \le \theta^2 H_\omega(\vec{u}/\theta) = H_\omega(\vec{u}).$$

Remark 2.2. From (ii) in Proposition 2.5 and Proposition 3.2 in [16] we see that the ground states are all smooth classical solutions, that is $(\eta^*, u^*) \in H^\infty(\mathbb{R}) \times H^\infty(\mathbb{R})$.

3. Positivity/Negativity of the ground states

We prove in this section that in a certain parameter regime the ground states obtained in the previous section have fixed sign.

From Proposition 2.5 we know that if $\vec{u}^* = (\eta^*, u^*)$ is a ground state, then

$$P(\vec{u}^*) = \lambda^* < 0, \quad H_\omega(\vec{u}^*) = -\frac{3}{2}\lambda^*, (\text{since } Q(\vec{u}^*) = 2H_\omega(\vec{u}^*) + 3P(\vec{u}^*) = 0.) \tag{3.1}$$

and

$$S_\omega(\vec{u}^*) = H_\omega(\vec{u}^*) + P(\vec{u}^*) = -\frac{1}{2}\lambda^*. \tag{3.2}$$

The following lemma is crucial to our analysis.

Lemma 3.1. *Under the same assumptions as in Theorem 1.1, if $\vec{u} = (\eta, u) \in X^1$ satisfies that*

$$P(\vec{u}) \le \lambda^*, \quad H_\omega(\vec{u}) \le -\frac{3}{2}\lambda^*, \tag{3.3}$$

then $\vec{u} \in \mathcal{G}_\omega$, that is, $\vec{u}$ is a ground state.

Proof. For $t \in \mathbb{R}$, consider

$$Q(t\vec{u}) = 2H_\omega(t\vec{u}) + 3P(t\vec{u}) = t^2\left[2H_\omega(\vec{u}) + 3tP(\vec{u})\right].$$

So choosing

$$t = \frac{2H_\omega(\vec{u})}{-3P(\vec{u})} \tag{3.4}$$

we have $Q(t\vec{u}) = 0$. From assumption 3.3 we know that

$$0 < t \le 1. \tag{3.5}$$

Now using the definition of ground states and (3.2)-(3.5) we deduce that

$$\begin{aligned} -\frac{1}{2}\lambda^* &\le S_\omega(t\vec{u}) = t^2\left[H_\omega(\vec{u}) + tP(\vec{u})\right] \\ &\le t^2\left(-\frac{3}{2}\lambda^* + t\lambda^*\right) = -\frac{\lambda^*}{2}t^2(3-2t). \end{aligned} \tag{3.6}$$

Hence

$$t^2(3-2t) \ge 1,$$

which only holds when $t = 1$ and the inequality becomes an equality. Therefore $Q(\vec{u}) = 0$ and $S_\omega(\vec{u}) = -\frac{1}{2}\lambda^*$, which shows that $\vec{u}$ is a ground state. $\square$

Remark 3.1. The importance of Lemma 3.1 is that it provides a "selection principle" of the ground states in the sense that it asserts that the following cases cannot happen.

$$P(\vec{u}) < \lambda^*, \quad H_\omega(\vec{u}) \le -\frac{3}{2}\lambda^*; \tag{3.7}$$

$$P(\vec{u}) \le \lambda^*, \quad H_\omega(\vec{u}) < -\frac{3}{2}\lambda^*. \tag{3.8}$$

An easy situation is when $b = d = 0$, using the above lemma, we have

Lemma 3.2. *Suppose $a, c < 0, b = d = 0$, $|\omega| < \omega_0$, and $\vec{u}^* = (\eta^*, u^*) \in \mathcal{G}_\omega$. Then*

1. *if $\omega > 0$, then $(\eta^*, u^*) = (-|\eta^*|, -|u^*|)$;*
2. *if $\omega < 0$, then $(\eta^*, u^*) = (-|\eta^*|, |u^*|)$;*
3. *if $\omega = 0$, then $(\eta^*, u^*) = (-|\eta^*|, -|u^*|)$ or $(\eta^*, u^*) = (-|\eta^*|, |u^*|)$.*

Proof. It is easy to see that (3.3) is satisfied for the new functions in (1), (2) and (3). Hence from Lemma 3.1, they are also ground states. Next we show that they are the same as $\vec{u}^*$. We will only discuss case (1), since the other two can be proved the same way. Note that now (η^*, u^*) solve the following equations.

$$\begin{cases} c\eta_{xx} + \eta - \omega u + \dfrac{1}{2}u^2 = 0 \\ au_{xx} + u - \omega\eta + \eta u = 0. \end{cases} \tag{3.9}$$

Hence it is obvious that $\eta^*, u^* \not\equiv 0$ (this can also be inferred from Proposition 2.5 (ii)).

First we show that $\eta^* = -|\eta^*|$. If not, then from the smoothness of the ground states (see Remark 2.2), there exists an open interval $(p, q), p < q$ such that $\eta^* > 0$ on (p, q). Then we must have $u^* \equiv 0$ on (p, q). Otherwise we would have

$$H_\omega(-|\eta^*|, -|u^*|) \le H_\omega(\eta^*, u^*), \quad P(-|\eta^*|, -|u^*|) < P(\eta^*, u^*),$$

and then the scaling parameter t given in (3.4) indicates that

$$0 < t = \frac{2H_\omega(-|\eta^*|, -|u^*|)}{-3P(-|\eta^*|, -|u^*|)} < 1.$$

Therefore inequality (3.6) is not valid, which is a contradiction. So $u^* \equiv 0$ on (p, q), and then $u^*_{xx} = 0$ on (p, q). Now plugging (η^*, u^*) into the second equation of (3.9) on (p, q) we obtain that $\eta^* \equiv 0$ on (p, q), which is a contradiction. So $\eta^* \le 0$ on $\mathbb{R}$, that is, $\eta^* = -|\eta^*|$.

Next we consider u^*. For simplicity we only discuss the case when $\omega > 0$ since the other cases can be done similarly. We need to prove that $u^* = -|u^*|$. If not, then $u^* > 0$ on some (p, q) with $p < q$.

Case 1: if $u^* > 0$ on $\mathbb{R}$. Since $\eta^* \not\equiv 0$, we have

$$H_\omega(\eta^*, -|u^*|) < H_\omega(\eta^*, u^*), \quad P(\eta^*, -|u^*|) = P(\eta^*, u^*),$$

and the scaling parameter t in (3.4) satisfies

$$0 < t < 1.$$

So (3.6) cannot hold, which is a contradiction.

Case 2: if $u^* > 0$ on (p,q) and $u^*(p) = u^*(q) = 0$ (p may be $-\infty$ and q may be $+\infty$). Similarly as in Case 1, if $\eta^* \not\equiv 0$ on (p,q), then let

$$\bar{u}^* = \begin{cases} -u^* & \text{on } (p,q) \\ u^* & \text{else.} \end{cases}$$

Then

$$H_\omega(\eta^*, \bar{u}^*) < H_\omega(\eta^*, u^*), \quad P(\eta^*, -\bar{u}^*) = P(\eta^*, u^*),$$

which again leads to a contradiction. Therefore we must have $\eta^* = 0$ on (p,q). But then the second equation of (3.9) indicates that on (p,q)

$$au^*_{xx} + u^* = 0,$$

together with $u^*(p) = u^*(q) = 0$, we infer that $u^* \equiv 0$ on (p,q), which is a contradiction. Thus we conclude that $u^* \leq 0$ on $\mathbb{R}$ and hence $u^* = -|u^*|$.

□

Remark 3.2. It can be further proved by using a strong maximum principle, as in Theorem 3.1, that η^* and u^* can not attain zero.

As is explained in the Introduction, when $b = d \neq 0$ and $\omega \neq 0$, it is difficult to analyze the sign of the ground state directly due to the appearance of the product of the derivatives in the functional. So instead, we use the trick of "completing the squares" and then make a change-of-unknowns transformation to remove the mixed derivative term. There are two ways to perform the transformation, as indicated in the following.

1. $H_\omega(\vec{u})$ and $P(\vec{u})$ can be written as

$$H_\omega(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}} (\eta - \omega u)^2 + \frac{1-\omega^2}{\omega^2}(\omega u)^2 - c\left(\eta + \frac{b}{c}\omega u\right)_x^2 + \frac{ac - b^2\omega^2}{-c\omega^2}(\omega u)_x^2 \, dx$$

$$P(\vec{u}) = \frac{1}{2\omega^2}\int_{\mathbb{R}} \eta(\omega u)^2 \, dx.$$

Let

$$w = \omega u, \quad \zeta = \eta + \frac{b}{c}w. \tag{3.10}$$

Then the corresponding functionals become

$$\begin{aligned} H^{(1)}_\omega(\zeta, w) &= \frac{1}{2}\int_{\mathbb{R}} \left(\zeta - \frac{b+c}{c}w\right)^2 + \frac{1-\omega^2}{\omega^2}w^2 - c\zeta_x^2 + \frac{ac - b^2\omega^2}{-c\omega^2}w_x^2 \, dx \\ P^{(1)}(\zeta, w) &= \frac{1}{2\omega^2}\int_{\mathbb{R}} \left(\zeta - \frac{b}{c}w\right) w^2 \, dx. \end{aligned} \tag{3.11}$$

2. Another way to write $H_\omega(\vec{u})$ and $P(\vec{u})$

$$H_\omega(\vec{u}) = \frac{1}{2}\int_{\mathbb{R}} (u-\omega\eta)^2 + \frac{1-\omega^2}{\omega^2}(\omega\eta)^2 - a\left(u + \frac{b}{a}\omega\eta\right)_x^2 + \frac{ac-b^2\omega^2}{-a\omega^2}(\omega\eta)_x^2\, dx$$

$$P(\vec{u}) = \frac{1}{2\omega}\int_{\mathbb{R}} (\omega\eta)u^2\, dx.$$

Now the transformation is

$$\xi = \omega\eta, \quad v = u + \frac{b}{a}\xi, \tag{3.12}$$

and the corresponding functionals are

$$\begin{aligned} H^{(2)}_\omega(\xi, v) &= \frac{1}{2}\int_{\mathbb{R}} \left(v - \frac{b+a}{a}\xi\right)^2 + \frac{1-\omega^2}{\omega^2}\xi^2 - a v_x^2 + \frac{ac-b^2\omega^2}{-a\omega^2}\xi_x^2\, dx \\ P^{(2)}(\xi, v) &= \frac{1}{2\omega}\int_{\mathbb{R}} \xi\left(v - \frac{b}{a}\xi\right)^2\, dx. \end{aligned} \tag{3.13}$$

Note that under the assumptions of Theorem 1.1, in both (3.11) and (3.13), the coefficients of the square terms in functional H_ω are all positive.

Now we can state our theorem on the sign of the ground states.

Theorem 3.1. *Under the same assumptions as in Theorem 1.1, and assume further that*

$$0 \le b \le \max\{-a, -c\}. \tag{3.14}$$

Let (η^*, u^*) *be a ground state of system* (1.1). *Then*

(1) *if* $\omega > 0$, *then* $\eta^*, u^* < 0$.
(2) *if* $\omega < 0$, *then* $\eta^* < 0, u^* > 0$.
(3) *if* $\omega = 0$, *then* $\eta^* < 0$, $u^* < 0$; *or* $\eta^* < 0$, $u^* > 0$. *This is true even without the assumption* (3.14).

Proof. To prove (1) and (2), we will use the new functionals given in (3.11) and (3.13) with unknowns (ζ, w) and (ξ, v) defined in (3.10) and (3.12) respectively, where (η, u) is replaced by (η^*, u^*).

We claim that

$$\text{if } 0 \le b \le -c, \text{ then } \zeta, w \le 0, \tag{3.15}$$

$$\text{if } 0 \le b \le -a, \text{ then } \begin{cases} \xi, v \le 0, & \text{when } \omega > 0, \\ \xi, v \ge 0, & \text{when } \omega < 0. \end{cases} \tag{3.16}$$

In fact, when $0 \le b \le -c$, and if $w > 0$ on some (p, q), then

$$\begin{aligned} &H^{(1)}_\omega(-|\zeta|, -|w|) \le H^{(1)}_\omega(\zeta, w), \quad P^{(1)}(-|\zeta|, -|w|) < P^{(1)}(\zeta, w), \quad \text{if } 0 < b \le -c, \\ &H^{(1)}_\omega(-|\zeta|, -|w|) < H^{(1)}_\omega(\zeta, w), \quad P^{(1)}(-|\zeta|, -|w|) \le P^{(1)}(\zeta, w), \quad \text{if } 0 \le b < -c, \end{aligned}$$

and

$$H_\omega^{(1)}(\zeta,w)=H_\omega(\eta^*,u^*)=-\frac{3}{2}\lambda^*,\quad P^{(1)}(\zeta,w)=P(\eta^*,u^*)=\lambda^*.$$

Since the transformation (3.10) does not change the scaling of the two functionals H_ω and P, and

$$S_\omega(\eta^*,u^*)=H_\omega^{(1)}(\zeta,w)+P^{(1)}(\zeta,w),\quad Q(\eta^*,u^*)=2H_\omega^{(1)}(\zeta,w)+3P^{(1)}(\zeta,w),$$

we can apply the same argument as in Lemma 3.1. Define

$$\eta=-|\zeta|+\frac{b\omega}{c}|u^*|,$$

then $\eta\in H^1(\mathbb{R})$ and

$$H_\omega(\eta,u^*)=H_\omega^{(1)}(-|\zeta|,-|w|),\quad P(\eta,u^*)=P^{(1)}(-|\zeta|,-|w|),$$

and

$$Q(t\eta,tu^*)=0,\quad \text{for } 0<t=\frac{2H_\omega(\eta,u^*)}{-3P(\eta,u^*)}=\frac{2H_\omega^{(1)}(-|\zeta|,-|w|)}{-3P^{(1)}(-|\zeta|,-|w|)}<1.$$

So

$$\begin{aligned}-\frac{1}{2}\lambda^*=S_\omega(\eta^*,u^*)\le S_\omega(t\eta,tu^*)&=t^2H_\omega^{(1)}(-|\zeta|,-|w|)+t^3P^{(1)}(-|\zeta|,-|w|)\\&<-\frac{3}{2}\lambda^*t^2+\lambda^*t^3=-\frac{1}{2}\lambda^*t^2(3-2t),\end{aligned}$$

which cannot hold because $0<t<1$. This proves that $w\le 0$.

If $\zeta>0$ on some (p,q), then

$$H_\omega^{(1)}(-|\zeta|,-|w|)\le H_\omega^{(1)}(\zeta,w),\quad P^{(1)}(-|\zeta|,-|w|)<P^{(1)}(\zeta,w).$$

Then a similar argument as above leads to a contradiction. Therefore $\zeta\le 0$.

The other case when $0\le b\le -a$ can be done in the same way. When $\omega>0$, we consider $(-|\xi|,-|v|)$. When $\omega<0$, we use $(|\xi|,|v|)$. Hence we proved (3.15) and (3.16).

From the definitions (3.10), (3.12) and the sign conditions (3.15), (3.16) we infer that $\eta^*,u^*\le 0$ when $\omega>0$; and $\eta^*\le 0,u^*\ge 0$ when $\omega<0$. From Lemma 3.2 we know that when $\omega=0$ then either $\eta^*,u^*\le 0$ or $\eta^*\le 0,u^*\ge 0$.

Finally we prove that η^*,u^* do not attain zero. We only consider the case when $\omega>0$. The other cases can be treated the same way. Multiplying the first equation of (1.5) by a, and the second equation by $b\omega$, then subtracting we get

$$(ac-b^2\omega^2)\eta^*_{xx}+(a+b\omega^2)\eta^*-(a+b)\omega u^*+\frac{a}{2}(u^*)^2-b\omega\eta^*u^*=0. \tag{3.17}$$

Similarly, Multiplying the second equation of (1.5) by c, and the first equation by $b\omega$, then subtracting we get

$$(ac-b^2\omega^2)u^*_{xx}+(c+b\omega^2)u^*-(c+b)\omega\eta^*-\frac{b\omega}{2}(u^*)^2+c\eta^*u^*=0. \tag{3.18}$$

The following strong maximum principle is a classical case of Proposition 2.6 in Da Lio-Sirakov [17].

Proposition 3.3. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a smooth domain and let $b, c \in L^\infty(\overline{\mathcal{O}})$. Suppose $w \in C^2(\overline{\mathcal{O}})$ is a classical solution of*

$$\begin{cases} \Delta w - b(x)|\nabla w| + c(x)w \geq 0 & \text{in } \mathcal{O}, \\ w \leq 0 & \text{in } \mathcal{O}. \end{cases}$$

Then either $w \equiv 0$ in $\mathcal{O}$ or $w < 0$ in $\mathcal{O}$ and at any point $x_0 \in \partial\mathcal{O}$ where $w(x_0) = 0$ we have

$$\frac{\partial w}{\partial \vec{\nu}}(x_0) > 0,$$

where $\vec{\nu}$ is the outward normal to $\partial\mathcal{O}$ at x_0.

From the assumption of the theorem and that $\eta^*, u^* \leq 0$ we can rewrite equations (3.17) and (3.18) as

$$\begin{aligned} \eta^*_{xx} + \left[\frac{(a+b\omega^2) - b\omega u^*}{ac - b^2\omega^2}\right]\eta* &= \frac{(a+b)\omega u^* - \frac{a}{2}(u^*)^2}{ac - b^2\omega^2} \geq 0, \\ u^*_{xx} + \left[\frac{(c+b\omega^2) - \frac{b\omega}{2}u^* + c\eta^*}{ac - b^2\omega^2}\right]u^* &= \frac{(c+b)\omega\eta^*}{ac - b^2\omega^2} \geq 0. \end{aligned}$$

Note that η^*, u^* are smooth and in $H^1(\mathbb{R})$, and hence are both L^∞. So applying Proposition 3.3 to the above two equations we know that for any interval (p, q), either $\eta^* = u^* \equiv 0$ or $\eta^*, u^* < 0$ on (p, q) and

(i) if $\eta^*(p) = 0$ ($u^*(p) = 0$ resp.), then $\eta^*_x(p) < 0$ ($u^*_x(p) < 0$ resp.).
(ii) if $\eta^*(q) = 0$ ($u^*(q) = 0$ resp.), then $\eta^*_x(q) > 0$ ($u^*_x(q) > 0$ resp.).

However, since $\eta^*, u^* \leq 0$ and $\eta^*, u^* \not\equiv 0$ on all of $\mathbb{R}$, if $\eta^*(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then $\eta^*(x_0) = \max \eta^*$ and hence $\eta^*_x(x_0) = 0$. The same result holds for u^*. Therefore we must have $\eta^*, u^* < 0$ on all of $\mathbb{R}$.

□

4. Symmetry

In this section we discuss the symmetry property of the ground states. We prove that all ground states of (1.1) are even up to translation. The method we employ here stems from the general framework of Busca-Sirakov [12].

4.1. **Weakly coupled version of** (1.5)**.** We first rewrite our system (1.5) to the fit in the setting in [12]. For simplicity of discussion. we only consider the case when $\omega > 0$.

From (3.17) and (3.18) and Theorem 3.1 we obtain

(4.1)
$$\begin{cases} \eta^*_{xx} + \left[\dfrac{(a+b\omega^2)}{ac-b^2\omega^2}\right]\eta^* - \left[\dfrac{(a+b)\omega}{ac-b^2\omega^2}\right]u^* + \left[\dfrac{a}{2(ac-b^2\omega^2)}\right](u^*)^2 - \left[\dfrac{b\omega}{ac-b^2\omega^2}\right]\eta^* u^* = 0, \\ u^*_{xx} + \left[\dfrac{(c+b\omega^2)}{ac-b^2\omega^2}\right]u^* - \left[\dfrac{(c+b)\omega}{ac-b^2\omega^2}\right]\eta^* - \left[\dfrac{b\omega}{2(ac-b^2\omega^2)}\right](u^*)^2 + \left[\dfrac{c}{ac-b^2\omega^2}\right]\eta^* u^* = 0. \\ \eta^*, u^* < 0, \quad \eta^*, u^* \to 0 \text{ as } |x| \to \infty. \end{cases}$$

Denote

$$A = \frac{a}{ac-b^2\omega^2}, \ B = \frac{b\omega}{ac-b^2\omega^2}, \text{ and } C = \frac{c}{ac-b^2\omega^2}, \tag{4.2}$$

and let $U = -\eta^*, V = -u^*$. Then (4.1) becomes

$$\begin{cases} U_{xx} + (A+\omega B)U - (A\omega + B)V - \dfrac{1}{2}AV^2 + BUV = 0, \\ V_{xx} + (C+\omega B)V - (C\omega + B)U + \dfrac{1}{2}BV^2 - CUV = 0, \\ U, V > 0, \quad U, V \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{4.3}$$

Notice that (4.3) is a weakly coupled semilinear elliptic system. From the assumptions in Theorem 3.1 we have

$$B \geq 0, \quad A, C, A+\omega B, C+\omega B < 0, AC - B^2 > 0. \tag{4.4}$$

We next set

$$g(U,V) = (A+\omega B)U - (A\omega + B)V - \frac{1}{2}AV^2 + BUV,$$
$$f(U,V) = (C+\omega B)V - (C\omega + B)U + \frac{1}{2}BV^2 - CUV.$$

Then we get

$$\frac{\partial g}{\partial U} = A + \omega B + BV,$$
$$\frac{\partial g}{\partial V} = -(A\omega + B) - AV + BU,$$
$$\frac{\partial f}{\partial U} = -(C\omega + B) - CV,$$
$$\frac{\partial f}{\partial V} = C + \omega B + BV - CU.$$

Let us recall the symmetry result [12, Theorem 2], which requires that

(1) $\dfrac{\partial g}{\partial V}(U,V) \geq 0$ and $\dfrac{\partial f}{\partial U}(U,V) \geq 0$ for all $(U,V) \in [0,\infty) \times [0,\infty)$;
(2) $\dfrac{\partial g}{\partial U}(0,0) < 0$ and $\dfrac{\partial f}{\partial V}(0,0) < 0$;
(3) $\det X > 0$, where

$$X = \begin{pmatrix} \dfrac{\partial g}{\partial U} & \dfrac{\partial g}{\partial V} \\ \dfrac{\partial f}{\partial U} & \dfrac{\partial f}{\partial V} \end{pmatrix}(0,0).$$

From (3.14) we know that

$$A\omega + B, C\omega + B \leq 0,$$

and hence condition (1) is satisfied. From (4.4), we know that condition (2) is satisfied. A direct computation shows that

$$X = \begin{pmatrix} A + \omega B & -(A\omega + B) \\ -(C\omega + B) & C + \omega B \end{pmatrix}$$

From (1.10)

$$\det X = (1 - \omega^2)(AC - B^2) > 0.$$

Remark 4.1. In the case when $\omega \leq 0$, we know from Theorem 3.1 that $\eta^* < 0$, either $u^* > 0$ or $u^* < 0$. When $u^* > 0$, one may consider $(U, -V)$ and it is easy to see that conditions (1)–(3) are also satisfied for $(U, -V)$.

Therefore applying Theorem 2 in [12] we obtain

Lemma 4.1. *Under the same assumption as in Theorem 3.1, when $\omega > 0$, there exist points $x_1, x_2 \in \mathbb{R}$ such that $U(x) = U(|x - x_1|)$ and $V(x) = V(|x - x_2|)$. Moreover,*

$$\frac{dU}{dr_1} < 0 \quad \textit{and} \quad \frac{dV}{dr_2} < 0$$

for all $r_1 = |x - x_1| > 0$ and $r_2 = |x - x_2| > 0$. Therefore $\eta^*(x) = \eta^*(|x - x_1|)$, $u^*(x) = u^*(|x - x_2|)$ and

$$\frac{d\eta^*}{dr_1} > 0, \quad \frac{du^*}{dr_2} > 0.$$

Similar results hold when $\omega \leq 0$.

From [12], a sufficient condition so that $x_1 = x_2$ is that V (reps. U) appears in a non-zero term in the first (reps. second) equation in (4.3), that is,

(i) either $\partial g/\partial V$ or $\partial f/\partial U$ is positive in a neighborhood of $(0,0)$, except possibly on $\{U = 0\} \cup \{V = 0\}$. This is the case when $\omega \neq 0$;

or

(ii) either $\partial g/\partial V$ or $\partial f/\partial U$ does not depend on one of its variables and is not identically zero in every neighborhood of $(0,0)$. This is true when $\omega = 0$.

Putting all the above arguments together, we obtain our symmetry result for (η^*, u^*) as in Theorem 1.2.

Appendix A. Exponential decay of solitary waves

In this appendix, we concern the decay property of solitary waves to system (1.1). We prove, in the following theorem, an exponential decay estimate for the solitary waves under the condition $a, c < 0$, $b = d$, and $|\omega| < \omega_0$ where ω_0 is defined in (1.10).

Theorem A.1. *Let $a, c < 0$, $b = d$, and $|\omega| < \omega_0$ where ω_0 is defined in* (1.10)*. Then for any solitary wave (η, u) of* (1.1) *having the form* (1.4)*, we have*

$$\eta(x), u(x) = O(e^{-\alpha|x|}), \tag{A.1}$$

with α a positive constant depending on a, b, c and ω.

Proof. Since system (1.5) can be regarded as a four-dimensional ODE system, we may employ the Stable Manifold Theorem to obtain the exponential decay.

From definition, all solitary waves decay at infinity to zero, which is the trivial equilibrium of the system. Thus we first check the hyperbolicity at the trivial equilibrium. Let $\mathbf{x} = (\eta, u, \xi, v)^T$, and then it satisfies

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(A + B\omega) & A\omega + B & 0 & 0 \\ C\omega + B & -(C + B\omega) & 0 & 0 \end{pmatrix},$$

with A, B, C defined in (4.2). A short computation shows that the characteristic equation is

$$\lambda^4 + (A + C + 2\omega B)\lambda^2 + (1 - \omega^2)(AC - B^2) = 0. \tag{A.2}$$

From the assumptions in Theorem 3.1 we easily see that

$$A + C + 2\omega B < 0, \quad (1 - \omega^2)(AC - B^2) > 0. \tag{A.3}$$

Thus $\mathrm{Re}(\lambda^2) > 0$ and hence the system is hyperbolic. This way the exponential decay of solitary waves follows from the standard result on the exponential convergence to hyperbolic equilibria, cf. [30, p.115, Corollary]. $\square$

Remark A.1. In the parameter regime where our symmetry results hold, cf. Theorem 1.2, we can further quantify the decay rate of the corresponding ground states. Notice that in this regime we have (3.14), which leads to

$$(A\omega + B)(C\omega + B) \geq 0.$$

Therefore the discriminant of (A.2) satisfies

$$\Delta = (A - C)^2 + 4(A\omega + B)(C\omega + B) \geq 0.$$

Hence $\lambda^2 > 0$. Using the sign conditions (A.3) we infer that the decay exponent α can be bounded by

$$0 < \alpha \leq \sqrt{\frac{-(A + C + 2\omega B) - \sqrt{\Delta}}{2}}. \tag{A.4}$$

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Ellen ShiTing Bao
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: ellenbao@pitt.edu

Robin Ming Chen
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: mingchen@pitt.edu

Qing Liu
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: qingliu@pitt.edu