ANALYSIS ON THE BLOW-UP OF SOLUTIONS TO A CLASS OF INTEGRABLE PEAKON EQUATIONS

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Abstract. We investigate the blow-up mechanism of solutions to a class of quasilinear integrable equations which could possess peakons. The dynamics of the blow-up quantity along the characteristics is established by the Riccati-type differential inequality which involves the interaction among three parts: a local nonlinearity, a nonlocal term, and a term stemming from the weak linear dispersion. To analyse the interplay among these quantities, we provide two different approaches. The first one is designed for the case when the equations do not exhibit a weak linear dispersion and hence focuses on the interplay between the first two parts. The method is based on a refined analysis on either evolution of the solution u and its gradient u_x, that is, $Cu \pm u_x$ or the growth rate of the relative ratio u_x/u. The second one handles the general situation when all of three parts are present. The idea is to extract the "truly" blow-up component from the Riccati-type differential inequality and utilizes the Morawetz-type identity or higher order conservation laws to show that such a component blows up in finite time before the other component degenerates.

Keywords: Degasperis-Procesi equation, Novikov equation, generalized modified Camassa-Holm equation, blow-up, peakon, wave-breaking.

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1. Introduction

Wave motion can be distinguished in two main classes: the hyperbolic waves and dispersive waves [30]. One of the common characteristics that both these wave models exhibit is the (finite-time) blow-up: there is a time $T < \infty$ such that the certain norm of solution of the model equations becomes unbounded as $t \uparrow T$. Some prototypes include the shock waves which are propagating discontinuities in the dependent variables – a nonlinear feature of the hyperbolic waves – which is caused by progressively nonlinear steepening of the wave profiles [14]; and the dispersive blow-up for dispersive waves, which is a focusing type of behavior that is due to propensity of the dispersion relation so that infinitely many, widely spaced small disturbances may coalesce locally in space-time [2].

On the one hand, dispersion is known to spread out waves and make them decay in time, delaying the onset of blow-up. In fact if the dispersion can be strong enough to overcome the nonlinear effects so that wave interaction takes place at a fast rate over a short time, then the smoothness of the waves can persist, excluding the possibility of blow-ups. One of the best known examples may be found in the context of water waves, namely the celebrated Korteweg-de Vries (KdV) equation [21]. On the other hand, it is observed that there is another effect, namely the nonlocal (smoothing) effect, which can help maintain

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the regularity while waves propagate and hence prevent them from blowing up, even when dispersion is weak or absent. See, for example, the Benjamin-Bona-Mahoney (BBM) equation [1]. As the nonlinearity becomes stronger and dominates over the dispersion and nonlocal effects singularities may occur in the sense of *wave-breaking*, i.e., the wave profile remains bounded, but its slope becomes unbounded. Examples can be found in the Whitham equation [9, 30], Camassa-Holm (CH) equation [5, 13], the Degasperis-Procesi (DP) equation[16, 13], and the Novikov equation [24], etc. When the dispersion is weak and the nonlinearity reaches a certain balance with the nonlocal terms, the *curvature* (the second derivative of the solution) may blow up in finite time. See, for instance, the modified Camassa-Holm (mCH) equation [25]. Understanding the wave-breaking mechanism such as when a singularity can form and what the nature of it is not only presents fundamental importance from mathematical point of view but also is of great physical interest, since it would help provide a key-mechanism for localizing energy in conservative systems by forming one or several small-scale spots. For instance, in fluid dynamics, the possible phenomenon of finite time breakdown for the incompressible Euler equations signifies the onset of turbulence in high Reynolds number flows.

The purpose of this paper is to study finite-time blow-up of solutions for a class of quasilinear dispersive equations. These equations include the DP equation, the Novikov equation, and the generalized modified Camass-Holm (gmCH) equation, all of which exhibit nonlocal nonlinearities and nonlinear dispersion. A distinctive feature, which is also a primary reason for the interest in these equations is that they are integrable models for the breakdown of regularity. Moreover, these equations admit a remarkable variety of the so-called "peakon" solutions – peaked traveling wave solutions with a discontinuous derivative at crest [15, 20, 26]. Physically, due to the relevance of many of the preceding equations to water waves, those peakons reveal some similarity to the well-known Stokes waves of greatest height – the traveling waves of maximum possible amplitude that are solutions to the governing equations for irrotational water waves [7, 29].

We would first like to review some basic integrability properties of the three equations mentioned above. The well-studied DP equation

$$m_t + um_x + 3u_x m = 0, \qquad m = u - u_{xx} \tag{1.1}$$

is completely integrable with the associated Lax pair and admits a bi-Hamiltonian structure [15, 16]

$$m_t = B_0 \frac{\delta \bar{H}_{-1}}{\delta m} = B_1 \frac{\delta \bar{H}_0}{\delta m},$$

where

$$B_0 = -\frac{1}{6}\partial_x(1-\partial_x^2)(4-\partial_x^2), \quad B_1 = -\frac{9}{2}m^{2/3}\partial_x m^{1/3}(\partial_x - \partial_x^3)^{-1}m^{1/3}\partial_x m^{2/3},$$
$$\bar{H}_{-1} = \int_{\mathbb{R}} u^3 \, dx, \qquad \bar{H}_0 = \int_{\mathbb{R}} m \, dx.$$

Furthermore, the DP equation admits the following conserved density [23]

$$\bar{H}_1[u] = \int_{\mathbb{R}} y(t,x)v(t,x)dx = \int_{\mathbb{R}} y_0(x)v_0(x)dx, \tag{1.2}$$

where $y = (1-\partial_x^2)u$, $v = (4-\partial_x^2)^{-1}u$.

In contrast to the DP equation, the Novikov equation [24] exhibits a cubic nonlinearity

$$m_t + u^2 m_x + 3uu_x m + \gamma u_x = 0, \qquad m = u - u_{xx}. \tag{1.3}$$

It can also be written in a bi-Hamiltonian form [20]

$$m_t = J_0 \frac{\delta \widetilde{H}_1}{\delta m} = J_1 \frac{\delta \widetilde{H}_0}{\delta m},$$

where

$$J_0 = \frac{1}{4}(1 - D_x^2) m^{-1} D_x m^{-1} (1 - D_x^2),$$
$$J_1 = -\frac{1}{3}(3mD_x + 2m_x)(4D_x - D_x^3)^{-1}(3mD_x + m_x) + \frac{1}{3}\gamma D_x,$$

and

$$\widetilde{H}_1 = \int_{\mathbb{R}} umdx, \qquad \widetilde{H}_0 = \int_{\mathbb{R}} \left(umD_x^{-1}(mu_t) - \frac{1}{2}\gamma umD_x^{-1}(m(1-D_x^2)^{-1}u_x) \right) dx.$$

The gmCH equation [18]

$$m_t + k_1 \left((u^2 - u_x^2)m \right)_x + k_2(2u_x m + um_x) + \gamma u_x = 0, \quad m = u - u_{xx} \tag{1.4}$$

involves both quadratic and cubic nonlinearities, and is obtained by applying tri-Hamiltonian duality to the bi-Hamiltonian Gardner equation. Note that equation (1.4) reduces to the CH equation when $k_1 = 0, k_2 = 1$, and to the mCH equation when $k_1 = 1, k_2 = 0$, respectively. The gmCH equation (1.4) also admits the Lax pair and has the bi-Hamiltonian form [25, 26]

$$m_t = J \frac{\delta H_1}{\delta m} = K \frac{\delta H_2}{\delta m},$$

where

$$J = -k_1 \partial_x m \partial_x^{-1} m \partial_x - \frac{1}{2}k_2(m\partial_x + \partial_x m) - \frac{1}{2}\gamma \partial_x, \quad \text{and} \quad K = \frac{1}{4}\left(\partial_x^3 - \partial_x\right)$$
$$H_1 = \int_{\mathbb{R}} umdx, \qquad \text{and} \quad H_2 = k_1 I_1 + 2k_2 I_2 + 2\gamma \int_{\mathbb{R}} u^2 dx \tag{1.5}$$

with

$$I_1 = \int_{\mathbb{R}} \left(u^4 + 2u^2 u_x^2 - \frac{1}{3}u_x^4 \right) dx, \quad I_2 = \int_{\mathbb{R}} \left(u^3 + uu_x^2 \right) \, dx.$$

Unlike the semilinear dispersive systems, for instance, the KdV and Schrödinger equations, where in many cases the linear dispersion dominates over the nonlinearity and a contraction principle can be applied even in a very low-regularity regime to obtain well-posedness, and consequently the smoothness of the solution propagates with the help of conservation laws, the equations considered here are all quasilinear with weak linear dispersion, suggesting that well-posedness can only be established in a high regularity regime, and the initial profile can determine the existence time and the regularity of the solution map rather strongly.

There has been many of study of the finite-time blow-up of equations (1.1), (1.3), (1.4) and some related models. We do not attempt to exhaust all the literatures. One can refer to [6, 8, 9, 10, 11, 12, 16, 19, 22, 28] and the references therein for details. The main idea used in the analysis is to trace the dynamics of the blow-up quantity along the characteristics. Due to the connection between u and the momentum density m, these equations inherit a nonlocal structure and can be reformulated in a weak form of nonlinear nonlocal transport type. From the transport theory, the blow-up criteria assert that singularities are caused by the focusing of characteristics, which involve the information on the gradient u_x and m. Roughly speaking, the dynamics of the blow-up quantity $B = B(u, u_x, m)$ along the characteristics is governed by an equation

$$B'(t) \lesssim -B^2 + f(u,m)\left(C^2u^2 - u_x^2\right) + \mathcal{N}(u, u_x, m) + \mathcal{D}_\gamma, \tag{1.6}$$

where $\mathcal{N}(u, u_x, m)$ is the nonlocal part, which usually consists of convolutions against the kernel $p(x) = \frac{1}{2}e^{-|x|}$, the fundamental solution of $(1 - \partial_x^2)^{-1}$ on $\mathbb{R}$, and $\mathcal{D}_\gamma$ comes from the weak linear dispersion and is of lower order. In many classical cases there is no linear dispersion, and hence $\mathcal{D}_\gamma = 0$. Standard approaches seek certain conservation laws (like, for instance, the conservation of the H^1-norm of u, the persistence of the sign of m, antisymmetry, etc.) to control the local quantities involving u, m as well as the nonlocal term $\mathcal{N}$ by constants. In particular, for the CH and DP equations, the term $f(u,m)\left(C^2u^2 - u_x^2\right)$ can be replaced by a function $f(u)$ depending solely on u, and hence the dynamics of B follows a Riccati-type inequality $B' \lesssim -B^2 + C$, which leads to a finite-time blow-up provided B is sufficiently negative initially. As such approaches make an intensive use of the "global" information of solutions, the blow-up mechanism ignores the *local* structure of the solutions.

Recently Brandolese and Cortez [3, 4] introduced a new type of blow-up criteria in the study of the CH-type equations which highlights how local structure of the solution affects the blow-ups. Their argument relies heavily on the fact that the convolution terms are quadratic and positively definite, and that the convolution kernel $p(x)$ satisfies that $p \pm \kappa p_x \geq 0$ for $|\kappa| \leq 1$. For the DP equation considered here, however, one needs to deal with convolutions against $p \pm \sqrt{\frac{3}{2}}p_x$, and more seriously, for the Novikov and gmCH equations, the convolution contains cubic terms which do not have a lower bound in terms of the local terms. For this reason, it is not clear whether a purely local condition on the initial data can generate finite-time blow-ups.

We present two different ideas to investigate the breakdown mechanism of equations (1.1), (1.3) and (1.4). The first approach deals particularly with the dispersionless situation $\gamma = 0$. We look for some global property, namely the sign persistence of momentum density m, to bound the nonlocal term $\mathcal{N}$. Thus from (1.6), the blow-up can be deduced by the interplay between u and u_x. More precisely, this motivates us to carry out a refined analysis of the characteristic dynamics of $M = Cu - u_x$ and $N = Cu + u_x$. For the DP and Novikov equations, the estimates of M and N can be closed in the form of

$$M'(t) \geq -g(u)MN + \mathcal{N}_1, \quad N'(t) \leq g(u)MN + \mathcal{N}_2, \tag{1.7}$$

where $g(u) \geq 0$ and the nonlocal terms $\mathcal{N}_i$ $(i = 1, 2)$ can be bounded in terms of certain higher order conservation laws. From this the monotonicity of M and N can be established, and hence the finite-time blow-up follows. The situation for the gmCH equation is more delicate. The estimates (1.7) are not available for M and N. But note that an alternative way to show that $C^2u^2 - u_x^2 \leq 0$ is to track the relative ratio $|u_x/u|$, and to prove that the ratio stays sufficiently large. Physically this amounts to considering the local oscillation of solutions. Intuitively, one would expect that fast oscillation causes breakdown of solutions.

It turns out that the dynamics of u_x/u can be put in a rather clean form. However the inhomogeneity of the nonlinearities in the equation makes it difficult to extract a clear ratio condition out of the estimate. By performing a vertical shift of the solution, we are able to make the estimates homogeneous, which in turn provides the desired ratio property. The main theorems along this line are Theorem 2.1, Theorem 3.1 and Theorem 4.1.

The other approach we adopt has some similarity to the classical one and can be applied to the more general situations with weak linear dispersion. We will focus on the Novikov and the gmCH equations since this type of wave-breaking for the DP equation has been addressed in [23]. There are two sources of difficulties in this approach. Firstly, the convolution part in $\mathcal{N}$ contains cubic nonlinearities in u_x, and thus it requires some higher-order conservation laws. Fortunately, for the dispersionless Novikov and the gmCH equations, there exist some conserved quantities that bound the L^4 norm of u_x (cf. (3.4) for Novikov and (1.5) for gmCH). This together with the H^1 estimates of u controls the cubic terms in the convolution. For the general Novikov equation with linear dispersion, the previous functional in the dispersionless case is not conserved. Instead, we can still control the term $\|u_x\|_{L^4}^4$ by a Morawetz-type identity derived for a modified functional $I(t)$ in (3.13). Since the evolution of I is bounded by $\|u_x\|_{L^4}^2$, that is

$$\frac{d}{dt}\|u_x(t)\|_{L^4}^4 \lesssim \|u_x(t)\|_{L^4}^2,$$

this implies that $\|u_x\|_{L^4}$ is bounded by $\sqrt{t}$. In this way the cubic convolution can be bounded accordingly.

The second difficulty, which is more serious, lies in the fact that the needed local estimates in (1.6) involve the L^∞ control of u_x or m, which can not be inferred from the conservation laws. Our idea is to avoid examining the dynamic of B by extracting the "truly" blow-up component from it and to look at the dynamics of that component instead. More precisely, for the Novikov equation, the blow-up quantity is $B = uu_x$. The H^1 conservation indicates that u remains bounded for all time. Hence for B to blow up, it suffices to show that u_x blows up while u does not degenerate to zero in finite time. It turns out that the dynamics of u and u_x are much simpler, cf. (3.6), and the local estimates do not require an L^∞ bound on u_x. Therefore we are able to push u_x to infinity before u shrinks down to zero. The corresponding result is stated in Theorem 3.2.

On the other hand, the blow-up quantity for the gmCH equation is $B = (k_1 m + k_2)u_x$, in which both m and u_x can potentially blow up in finite time. By checking their dynamics individually we find that the dynamic equation for m has a very simple structure $m'(t) = q(m)u_x$, where $q(m)$ is a quadratic polynomial in m. Moreover, although the equation for u_x' still contains local terms involving u_x, it can be made of a definite sign with the help of the conservation laws. This way u_x will be monotone. With appropriate choice of initial data, the later dynamics of m satisfies $m'(t) \gtrsim q(m)$. Solving this differential inequality one obtains a finite-time blow-up of m. Due to the monotonicity of u_x and the proper choice of its initial value, u_x can be made uniformly away from zero. Hence B blows up. The details can be found in Theorem 4.2.

The rest of the paper is organized as follows. Section 2 deals with the DP equation and formulates the wave-breaking mechanism in Theorem 2.1. Sections 3 and Section 4 are devoted to the blow-up for the Novikov and gmCH equations respectively, with an emphasis on the role of the weak linear dispersion. The blow-up results in the absence of the weak dispersion are illustrated in Theorem 3.1 for the Novikov equation and Theorem 4.1 for the gmCH equation, respectively. In the general case when the weak linear dispersion is at present, the breakdown mechanisms are set up in Theorem 3.2 and Theorem 4.2.

Notation. In the sequel, we denote by $*$ the convolution. For $1 \le p < \infty$, the norms in the Lebesgue space $L^p(\mathbb{R})$ is $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}$, the space $L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue measurable functions f equipped with the norm $\|f\|_\infty = \inf_{\mu(e)=0} \sup_{x\in\mathbb{R}\setminus e} |f(x)|$. For a function f in the classical Sobolev spaces $H^s(\mathbb{R})$ $(s \ge 0)$ the norm is denoted by $\|f\|_{H^s}$. We denote $p(x) = \frac{1}{2}e^{-|x|}$ the fundamental solution of $1 - \partial_x^2$ on $\mathbb{R}$, and define the two convolution operators $p_+,\ p_-$ as

$$\begin{aligned} p_+ * f(x) &= \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{y} f(y)dy \\ p_- * f(x) &= \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} f(y)dy. \end{aligned} \tag{1.8}$$

Then we have the relations $p = p_+ + p_-, \quad p_x = p_- - p_+$.

2. Wave-breaking for the Degasperis-Procesi equation

We begin with the DP equation (1.1). The characteristics $q(t,x)$ associated to the DP equation is governed by

$$\begin{cases} q_t(t,x) = u(t, q(t,x)), \\ q(0,x) = x, \end{cases} \qquad x \in \mathbb{R}, \quad t \in [0,T). \tag{2.1}$$

One can easily check that $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$ with $q_x(t,x) > 0$ for all $(t,x) \in [0,T) \times \mathbb{R}$. Furthermore the potential $m = u - u_{xx}$ satisfies

$$m(t, q(t,x)) q_x^3(t,x) = m_0(x), \qquad (t,x) \in [0,T) \times \mathbb{R}, \tag{2.2}$$

which implies that the zeros and the sign of m are preserved under the flow.

The precise blow-up criterion for the DP equation can be formulated as

Lemma 2.1. [23] *Let $u_0 \in H^s(\mathbb{R})$ for $s > 3/2$. The blow-up of solution in finite time $T^* < +\infty$ occurs if and only if*

$$\lim_{t\to T^*} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty. \tag{2.3}$$

It is easy to check that the derivatives of u and u_x along the characteristics can be obtained from the following computations

$$u_t + u u_x = -p_x * \left(\frac{3}{2} u^2 \right), \tag{2.4}$$

$$u_{xt} + u u_{xx} = \frac{3}{2} u^2 - u_x^2 - p * \left(\frac{3}{2} u^2 \right). \tag{2.5}$$

For wave-breaking of the DP equation, one would like to know under what conditions on the initial data $u_0(x)$, u_x approaches $-\infty$ in finite time. From equation (2.5), it suffices to show that u_x can have super-linear decay rate along certain characteristics. Our goal is to find some initial data so that at later time along the corresponding characteristics u_x^2 outgrows $\frac{3}{2}u^2$. This can be done by tracking the dynamics of the two quantities $\sqrt{\frac{3}{2}}u \pm u_x$. Notice that the dynamic equation (2.4) consists only of the convolution term, which can be controlled by using the Young inequality and the uniform L^2 norm in the following

$$p_\pm * \left(\frac{3}{2}u^2\right) \le \|p_\pm\|_\infty \left\|\frac{3}{2}u^2\right\|_1 = \frac{3}{4}\|u\|_2^2 \le 3\bar{H}_1[u_0], \tag{2.6}$$

where $\bar{H}_1[u_0]$ is given in (1.2) and we have used the following estimate (see [23])

$$\frac{1}{4}\|u(t)\|_2^2 = \frac{1}{4}\int_{\mathbb{R}} \hat{u}^2(\xi)d\xi \le \int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}\hat{u}^2(\xi)d\xi = \int_{\mathbb{R}} yvdx = \bar{H}_1[u_0].$$

The wave-breaking result is now formulated as follows.

Theorem 2.1. *Let $u_0 \in H^s(\mathbb{R})$ for $s > 3/2$. Suppose that there exists a point $x_1 \in \mathbb{R}$ such that*

$$u_{0,x}(x_1) < -\sqrt{\frac{3}{2}}|u_0(x_1)| - C, \tag{2.7}$$

where

$$C = \sqrt{3\left(\sqrt{\frac{3}{2}} - 1\right)\bar{H}_1[u_0]}. \tag{2.8}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \le \frac{1}{2C}\log\left(\frac{\sqrt{u_{0,x}^2(x_1) - \frac{3}{2}u_0^2(x_1)} + C}{\sqrt{u_{0,x}^2(x_1) - \frac{3}{2}u_0^2(x_1)} - C}\right).$$

Proof. We track the dynamics of $M_1(t) = \left(\sqrt{\frac{3}{2}}u - u_x\right)(t, q(t,x_1))$ and $N_1(t) = \left(\sqrt{\frac{3}{2}}u + u_x\right)(t, q(t,x_1))$ along the characteristics and apply the convolution estimate (2.6) to obtain

$$\begin{aligned}
M_1'(t) &= -M_1N_1 + \left(\sqrt{\frac{3}{2}} + 1\right)p_+ * \left(\frac{3}{2}u^2\right) - \left(\sqrt{\frac{3}{2}} - 1\right)p_- * \left(\frac{3}{2}u^2\right)\\
&\ge -M_1N_1 - \left(\sqrt{\frac{3}{2}} - 1\right)p_- * \left(\frac{3}{2}u^2\right)\\
&\ge -M_1N_1 - 3\left(\sqrt{\frac{3}{2}} - 1\right)\bar{H}_1[u_0] = -M_1N_1 - C^2,\\
N_1'(t) &= M_1N_1 + \left(\sqrt{\frac{3}{2}} - 1\right)p_+ * \left(\frac{3}{2}u^2\right) - \left(\sqrt{\frac{3}{2}} + 1\right)p_- * \left(\frac{3}{2}u^2\right)\\
&\le M_1N_1 + 3\left(\sqrt{\frac{3}{2}} - 1\right)\bar{H}_1[u_0] = M_1N_1 + C^2,
\end{aligned}$$

where C is defined in (2.8), and $'$ denotes the derivative $\partial_t + u\partial_x$. Then we have

$$M_1'(t) \geq -M_1N_1 - C^2, \quad N_1'(t) \leq M_1N_1 + C^2.$$

The expected monotonicity conditions on M and N indicate that we would like to have

$$M_1N_1(t) + C^2 < 0.$$

Therefore it is found from (2.7) that the initial data satisfies

$$\frac{3}{2}u_0^2(x_1) - u_{0,x}^2(x_1) < -C^2, \quad \sqrt{\frac{3}{2}}u_0(x_1) + u_{0,x}(x_1) < 0, \tag{2.9}$$

In this way, we know that along the characteristics emanating from x_1,

$$M_1(0) > 0, \quad N_1(0) < 0, \quad M_1'(0) > 0, \quad N_1'(0) < 0.$$

Therefore over the time of existence it always holds that

$$M_1'(t) > 0, \quad N_1'(t) < 0, \quad M_1N_1(t) < -C^2.$$

Hence we may consider the evolution of the quantity $h_1(t) = \sqrt{-M_1N_1(t)}$. By using the estimate $\dfrac{M_1 - N_1}{2} \geq h_1$, we have

$$\begin{aligned} h_1' &= -\frac{M_1'N_1 + M_1N_1'}{2\sqrt{-M_1N_1}} \geq \frac{-(M_1N_1 + C^2)(M_1 - N_1)}{2\sqrt{-M_1N_1}} \\ &= (h_1^2 - C^2)\frac{(M_1 - N_1)}{2\sqrt{-M_1N_1}} \geq h_1^2 - C^2, \end{aligned}$$

which implies that $h_1(t) \to +\infty$ as $t \to T^*$ with estimate on T^* given by

$$T^* \leq \frac{1}{2C}\log\frac{h_1(0) + C}{h_1(0) - C}.$$

Note that $h_1(t) \leq -u_x(t, q(t, x_1))$. Hence $h_1(t) \to \infty$ as $t \to T^*$ implies the finite time blow up $u_x(t, q(t, x_1)) \to -\infty$ as $t \to T^*$. □

3. Wave-breaking for the Novikov equation

The Novikov equation (1.3) can also be written as

$$u_t - u_{txx} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx} - \gamma u_x. \tag{3.1}$$

The associated characteristics is

$$\begin{cases} q_t(t, x) = u^2(t, q(t, x)), \\ q(0, x) = x, \end{cases} \quad x \in \mathbb{R}, \quad t \in [0, T). \tag{3.2}$$

If $\gamma = 0$, then the dynamics of the momentum density is

$$m(t, q(t, x))q_x^{3/2}(t, x) = m_0(x), \qquad (t, x) \in [0, T) \times \mathbb{R}. \tag{3.3}$$

In this case, equation (3.1) admits the following two conserved densities, which will be important in our blow-up analysis.

$$\begin{aligned} \widetilde{H}_1[u(t)] &= \int_{\mathbb{R}} (u^2 + u_x^2)\ dx = \widetilde{H}_1[u_0], \\ \widetilde{H}_2[u(t)] &= \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right)\ dx = \widetilde{H}_2[u_0]. \end{aligned} \tag{3.4}$$

The blow-up criterion for the Novikov equation is formulated as follows.

Lemma 3.1. [28] *Let $u_0 \in H^s(\mathbb{R})$ for $s > 3/2$. The solution of equation* (3.1) *with initial data u_0 blows up in finite time T^* if and only if*

$$\lim_{t\to T^*} \inf_{x\in\mathbb{R}} \{u(t,x)u_x(t,x)\} = -\infty. \tag{3.5}$$

3.1. **Dynamics along the characteristics.** Let us compute the dynamics of a few important quantities along the characteristics $q(t,x)$. Denote $'$ to be the derivative $\partial_t + u^2\partial_x$ along the characteristics.

Lemma 3.2. *Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$. Then $u(t,q(t,x))$, $u_x(t,q(t,x))$ and $(uu_x)(t,q(t,x))$ satisfy the following integro-differential equations*

$$\begin{aligned}
u'(t) &= -\gamma p * u_x + \frac{1}{2}\left[p_+ * (u-u_x)^3 - p_- * (u+u_x)^3\right],\\
u_x'(t) &= \frac{u}{2}(u^2-u_x^2) + \gamma\,(u - p*u)\\
&\quad - \frac{1}{2}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right],\\
(uu_x)'(t) &= \frac{u^2}{2}(u^2-u_x^2) + \gamma u^2 - \gamma\left[u_x(p*u_x) + u(p*u)\right]\\
&\quad - \frac{1}{2}\left[(u-u_x)p_+ * (u-u_x)^3 + (u+u_x)p_- * (u+u_x)^3\right].
\end{aligned} \tag{3.6}$$

Proof. The first one we look at is

$$\begin{aligned}
u'(t) &= -p * (3uu_xu_{xx} + 2u_x^3 + 3u^2u_x + \gamma u_x)\\
&= -p_x * \left(\frac{3}{2}uu_x^2 + u^3\right) - \frac{1}{2}p * u_x^3 - \gamma p * u_x\\
&= p_+ * \left(\frac{3}{2}uu_x^2 + u^3 - \frac{1}{2}u_x^3\right) - p_- * \left(\frac{3}{2}uu_x^2 + u^3 + \frac{1}{2}u_x^3\right) - \gamma p * u_x.
\end{aligned}$$

The first term can be calculated in the following.

$$\begin{aligned}
p_+ * \left(\frac{3}{2}uu_x^2 + u^3 - \frac{1}{2}u_x^3\right) &= \frac{1}{2}p_+ * (u-u_x)^3 + \frac{1}{2}p_+ * \left(u^3 + 3u^2u_x\right)\\
&= \frac{1}{2}p_+ * \left((u-u_x)^3 + u^3\right) + \frac{1}{4}u^3 - \frac{1}{2}p_+ * u^3\\
&= \frac{1}{2}p_+ * (u-u_x)^3 + \frac{1}{4}u^3.
\end{aligned}$$

Similarly, the second term is

$$p_- * \left(\frac{3}{2}uu_x^2 + u^3 + \frac{1}{2}u_x^3\right) = \frac{1}{2}p_- * (u+u_x)^3 + \frac{1}{4}u^3.$$

Putting together, we obtain

$$u'(t) = -\gamma p * u_x + \frac{1}{2}\left(p_+ * (u-u_x)^3 - p_- * (u+u_x)^3\right). \tag{3.7}$$

Next we estimate u_x'.

$$u_x'(t) = -\frac{1}{2}uu_x^2 + u^3 - p * \left(\frac{3}{2}uu_x^2 + u^3\right) - \frac{1}{2}p_x * u_x^3 - \gamma p_x * u_x$$

$$
\begin{aligned}
&= -\frac{1}{2}uu_x^2 + u^3 + \gamma(u - p * u) - p_+ * \left(\frac{3}{2}uu_x^2 + u^3 - \frac{1}{2}u_x^3\right) \\
&\quad - p_- * \left(\frac{3}{2}uu_x^2 + u^3 + \frac{1}{2}u_x^3\right).
\end{aligned}
$$

The above computation then leads to

$$
u_x'(t) = \frac{u}{2}(u^2 - u_x^2) + \gamma(u - p * u) - \frac{1}{2}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right]. \tag{3.8}
$$

Finally we turn our attention to the blow-up quantity uu_x.

$$
\begin{aligned}
(uu_x)'(t) &= u^2\left(u^2 - \frac{1}{2}u_x^2\right) + \gamma u^2 - \gamma[u_x(p * u_x) + u(p * u)] \\
&\quad - (u - u_x)p_+ * \left(\frac{3}{2}uu_x^2 + u^3 - \frac{1}{2}u_x^3\right) - (u + u_x)p_- * \left(\frac{3}{2}uu_x^2 + u^3 + \frac{1}{2}u_x^3\right).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
(uu_x)'(t) &= \frac{u^2}{2}(u^2 - u_x^2) + \gamma u^2 - \gamma\left[u_x(p * u_x) + u(p * u)\right] \\
&\quad - \frac{1}{2}\left[(u - u_x)p_+ * (u - u_x)^3 + (u + u_x)p_- * (u + u_x)^3\right].
\end{aligned} \tag{3.9}
$$

□

3.2. **Wave-breaking data.** In this section we will consider two different classes of initial data related to the appearance of the weak linear dispersion, and establish the needed estimates for the convolution terms. The following result for ODE theory will be useful in our proof of blow-up.

Lemma 3.3. *Let $f \in C^1(\mathbb{R})$, $a > 0, b > 0$ and $f(0) > \sqrt{\frac{b}{a}}$. If $f'(t) \geq af^2(t) - b$, then*

$$
f(t) \to +\infty \qquad as \qquad t \to t^* \leq \frac{1}{2\sqrt{ab}} \log\left(\frac{f(0) + \sqrt{\frac{b}{a}}}{f(0) - \sqrt{\frac{b}{a}}}\right).
$$

3.2.1. *Sign-changing momentum.* First we consider the case when there is no weak linear dispersion and so the momentum density preserves its sign along the characteristics. We choose the initial momentum density m_0 that changes sign at exactly one point.

Theorem 3.1. *Let $\gamma = 0$ and $m_0 \in H^s(\mathbb{R})$ for $s > 1/2$. Assume that there exists some point $x_2 \in \mathbb{R}$ such that $u_0(x_2) > 0$ and*

$$
m_0(x) > 0,\ x < x_2; \quad m_0(x_2) = 0; \quad m_0(x) < 0,\ x > x_2.
$$

Moreover

$$
\frac{u_0(x_2)}{2}\left[u_0^2(x_2) - u_{0,x}^2(x_2)\right] + K_1 < 0,
$$

where

$$
K_1 = 2\left(\sqrt{\frac{\widetilde{H}_1^3[u_0]}{2}} + \sqrt{3\widetilde{H}_1[u_0]\left(\widetilde{H}_1^2[u_0] - \widetilde{H}_2[u_0]\right)}\right). \tag{3.10}
$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \le \frac{1}{\sqrt{2K_1 u_0(x_2)}} \log\left(\frac{\sqrt{u_{0,x}^2(x_2)-u_0^2(x_2)}+\sqrt{2K_1/u_0(x_2)}}{\sqrt{u_{0,x}^2(x_2)-u_0^2(x_2)}-\sqrt{2K_1/u_0(x_2)}}\right).$$

Proof. From the evolution equation (3.3) we know that along the characteristics $q := q(t,x_2)$ emanating from x_2 we have

$$m(t,q)=0.$$

Moreover

$$m(t,x)>0,\ x<q(t,x_2); \qquad m(t,x)<0,\ x>q(t,x_2).$$

Using the identities

$$u+u_x = 2p_- * m; \quad u-u_x = 2p_+ * m,$$

we obtain

$$(u+u_x)(t,x)<0,\ x\ge q(t,x_2);\ \ (u-u_x)(t,x)>0,\ x\le q(t,x_2).$$

In view of the above sign conditions we infer that along the characteristics $q(t,x_2)$,

$$\left(p_+ * (u-u_x)^3\right)(t,q)>0, \quad \left(p_- * (u+u_x)^3\right)(t,q)<0.$$

Furthermore from (3.7) we conclude that

$$u(t,q)>0, \quad u'(t,q)>0. \tag{3.11}$$

Now we use the conservation laws (3.4) to derive the needed convolution estimates. Note that from (3.4) it follows that

$$\begin{aligned}\|u_x\|_4^4 &= 3\int_{\mathbb{R}}\left(u^4+2u^2u_x^2\right)\,dx - 3\widetilde{H}_2[u_0]\\ &\le 3\left(\|u\|_\infty^2\|u\|_2^2+2\|u\|_\infty^2\|u_x\|_2^2\right)-3\widetilde{H}_2[u_0]\\ &\le 3\left(\widetilde{H}_1^2[u_0]-\widetilde{H}_2[u_0]\right),\end{aligned}$$

where we have used the estimate $\|u\|_\infty \le \frac{1}{\sqrt{2}}\|u\|_{H^1}$. Hence we can further estimate the two convolution terms

$$\begin{aligned}\left|p_\pm * (u\mp u_x)^3\right| &\le \|p_\pm\|_\infty \left\|(u\mp u_x)^3\right\|_1 \le \frac{1}{2}\left\|(|u|+|u_x|)^3\right\|_1\\ &\le 2\left(\|u\|_3^3+\|u_x\|_3^3\right)\\ &\le 2\left(\sqrt{\frac{\widetilde{H}_1^3[u_0]}{2}}+\sqrt{3\widetilde{H}_1[u_0]\left(\widetilde{H}_1^2[u_0]-\widetilde{H}_2[u_0]\right)}\right)\\ &=: K_1.\end{aligned} \tag{3.12}$$

Using the above convolution estimates, the finite time blow-up is argued as follows. Denote

$$\begin{aligned} M_2(t)&=(u-u_x)(t,q), & N_2(t)&=(u+u_x)(t,q),\\ \widehat{u}(t)&=u(t,q), & \widehat{u_x}(t)&=u_x(t,q).\end{aligned}$$

Then from the previous calculation we have

$$
\begin{aligned}
M_2'(t) &= -\frac{\widehat{u}}{2}(\widehat{u}^2 - \widehat{u_x}^2) + \left[p_+ * (u-u_x)^3\right](t,q) > -\frac{\widehat{u}}{2} M_2 N_2(t), \\
N_2'(t) &= \frac{\widehat{u}}{2}(\widehat{u}^2 - \widehat{u_x}^2) - \left[p_- * (u+u_x)^3\right](t,q) \le \frac{\widehat{u}}{2} M_2 N_2(t) + K_1.
\end{aligned}
$$

So using the monotonicity of u (c.f. (3.11)), if the initial data satisfies

$$
\frac{1}{2} u_0(x_2) M_2(0) N_2(0) + K_1 < 0,
$$

then it implies that

$$
M_2'(t) > 0, \quad N_2'(t) < 0, \quad \frac{\widehat{u}}{2}(M_2 N_2)(t) + K_1 < 0.
$$

Let $h_2(t) = \sqrt{-M_2 N_2}(t)$. It in turn follows that

$$
\begin{aligned}
h_2'(t) &= -\frac{M_2' N_2 + M_2 N_2'}{2\sqrt{-M_2 N_2}} \ge -\frac{\left(\frac{u}{2} M_2 N_2 + K_1\right)(M_2 - N_2)}{2\sqrt{-M_2 N_2}} \\
&= \left(\frac{u}{2} h_2^2 - K_1\right) \frac{(M_2 - N_2)}{2\sqrt{-M_2 N_2}} \ge \frac{u}{2} h_2^2 - K_1 \ge \frac{u_0(x_2)}{2} h_2^2 - K_1.
\end{aligned}
$$

It is deduced from Lemma 3.3 that $h_2(t) \to +\infty$ as $t \to T^*$ with estimate on T^* given by

$$
T^* \le \frac{1}{\sqrt{2K_1 u_0(x_2)}} \log \left(\frac{h_2(0) + \sqrt{2K_1/u_0(x_2)}}{h_2(0) - \sqrt{2K_1/u_0(x_2)}} \right).
$$

Since the conservation of $\widetilde{H}_1$ implies that $\|u\|_\infty$ stays bounded, therefore $h_2(t) \to +\infty$ implies that $\widehat{u_x}^2(t) \to +\infty$ as $t \to T^*$. Moreover from (3.8)

$$
\widehat{u_x}'(t) \le \frac{\widehat{u}}{2} M_2 N_2 + K_1 < 0,
$$

together with the fact that $\widehat{u}'(t) > 0$ we know that

$$
u u_x(t,q) \to -\infty, \quad \text{as} \ \ t \to T^*.
$$

$\square$

Remark 3.1. *Using a similar argument one can prove the finite time blow-up for data satisfying* $u_0(\tilde{x}_2) < 0$ *and*

$$
\begin{aligned}
& m_0(x) < 0, \ x < \tilde{x}_2; \quad m_0(\tilde{x}_2) = 0; \quad m_0(x) > 0, \ x > \tilde{x}_2, \\
& \frac{u_0(\tilde{x}_2)}{2} \left[u_0^2(\tilde{x}_2) - u_{0,x}^2(\tilde{x}_2) \right] - K_1 > 0.
\end{aligned}
$$

3.2.2. *Non-sign-changing momentum.* Next we consider the Novikov equation with a weak linear dispersion and also allow for a general initial momentum density m_0. For a general $\gamma \in \mathbb{R}$, the sign-preservation of m does not hold. Moreover $\widetilde{H}_2$ as in (3.4) may not be conserved. Thus $\widetilde{H}_2$ does not necessarily provide a bound for $\|u_x\|_4$ directly. However, we can find a substitute by slightly modifying $\widetilde{H}_2$. More precisely, we have

Proposition 3.1. *Denote $I(t)$ by*

$$I(t)=\int_{\mathbb{R}}\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4+\frac{4}{3}\gamma u^2\right)dx. \tag{3.13}$$

Then there holds the following Morawetz-type identity

$$\frac{dI}{dt}=-\frac{4}{3}\gamma\int_{\mathbb{R}}uu_x^3dx. \tag{3.14}$$

Proof. Note that the Novikov equation (1.3) can be rewritten as

$$u_t+u^2u_x+(1-\partial_x^2)^{-1}\partial_x\left(u^3+\frac{3}{2}uu_x^2+\gamma u\right)+\frac{1}{2}(1-\partial_x^2)^{-1}u_x^3=0. \tag{3.15}$$

Differentiating (3.15) with respect to x leads to the expression for u_{xt}

$$\begin{aligned}u_{xt}=&-u^2u_{xx}-\frac{1}{2}uu_x^2+u^3+\gamma u-(1-\partial_x^2)^{-1}\left(u^3+\frac{3}{2}uu_x^2+\gamma u\right)\\&-\frac{1}{2}(1-\partial_x^2)^{-1}\partial_xu_x^3.\end{aligned}$$

A direct computation using (3.13) yields

$$\frac{dI}{dt}=\int_{\mathbb{R}}\left(4u^3+4uu_x^2+4u_x^2u_{xx}+\frac{8}{3}\gamma u\right)u_tdx-\frac{4}{3}\int_{\mathbb{R}}u^3u_{xxt}\,dx. \tag{3.16}$$

Plugging

$$u_{xxt}=u_t+4u^2u_x-3uu_xu_{xx}-u^2u_{xxx}+\gamma u_x$$

into (3.16), and using equation (3.15) to replace u_t, the Morawetz-type identity (3.14) is then obtained after integration by parts. ☐

We will see later that the quantity $I(t)$ will play a similar role as $\widetilde{H}_2$ to control the L^4 norm of u_x. Before we state the blow-up result, let us introduce some notations for convenience. Let $I_0=I(0)$ ($I(t)$ is defined in (3.13)), and define

$$\begin{aligned}&C_1=\frac{2\sqrt{2}}{3}|\gamma|\sqrt{\widetilde{H}_1[u_0]},\quad C_2=-I_0+\frac{4}{3}|\gamma|\widetilde{H}_1[u_0]+\widetilde{H}_1^2[u_0],\\&C_3=\frac{3}{2}C_1\sqrt{\widetilde{H}_1[u_0]},\qquad C_4=\sqrt{2}\widetilde{H}_1^{\frac{3}{2}}[u_0]+\left(\frac{1}{2}|\gamma|+2\sqrt{3C_2}\right)\sqrt{\widetilde{H}_1[u_0]},\ \textit{and}\\&K_2(t)=\sqrt{2}\widetilde{H}_1^{\frac{3}{2}}[u_0]+\left(3C_1t+2\sqrt{3C_2}\right)\sqrt{\widetilde{H}_1[u_0]}.\end{aligned} \tag{3.17}$$

Theorem 3.2. *Let $u_0\in H^s(\mathbb{R})$ for $s>\frac{5}{2}$. Suppose there exist some $0<\beta<1$ and $x_3\in\mathbb{R}$ such that $u_0(x_3)>0$ and*

$$u_{0,x}(x_3)<-\frac{B+1}{B-1}\sqrt{\frac{2K_3(T_2)}{\beta u_0(x_3)}}, \tag{3.18}$$

where

$$\begin{aligned}B&=\exp\left(T_2\sqrt{2\beta u_0(x_3)K_3(T_2)}\right),\\T_2&=\frac{-C_4+\sqrt{C_4^2+4C_3(1-\beta)u_0(x_3)}}{2C_3},\qquad \textit{and}\end{aligned}$$

$$K_3(t) = \left(1+\frac{\sqrt{2}}{2}\right)|\gamma|\sqrt{\widetilde{H}_1[u_0]} + \frac{(2-\beta)^3}{2}u_0^3(x_3) + K_2(t),$$

with C_3, C_4 and $K_2(t)$ defined in (3.17)*. Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as*

$$T^* \le T_3 := \frac{1}{\sqrt{2\beta u_0(x_3)K_3(T_2)}} \log\left(\frac{u_{0,x}(x_3) - \sqrt{\frac{2K_3(T_2)}{\beta u_0(x_3)}}}{u_{0,x}(x_3) + \sqrt{\frac{2K_3(T_2)}{\beta u_0(x_3)}}}\right). \tag{3.19}$$

Proof. Note that $\widetilde{H}_1[u]$ is conserved and $\|u\|_\infty \le \frac{1}{\sqrt{2}}\|u\|_{H^1}$. A simply computation shows that

$$\int_{\mathbb{R}} u^2u_x^2dx \le \|u\|_\infty^2 \int_{\mathbb{R}} u_x^2dx \le \|u\|_\infty^2\|u\|_{H^1}^2 \le \frac{1}{2}\|u\|_{H^1}^4 = \frac{1}{2}\widetilde{H}_1^2[u_0] \tag{3.20}$$

and

$$\begin{aligned}\int_{\mathbb{R}}\left(u^4+2u^2u_x^2+\frac{4}{3}\gamma u^2\right)dx &\le \|u\|_\infty^2\|u\|_2^2 + \frac{4}{3}|\gamma|\|u\|_2^2 + 2\int_{\mathbb{R}} u^2u_x^2dx \\ &\le \frac{4}{3}|\gamma|\widetilde{H}_1[u_0] + \widetilde{H}_1^2[u_0].\end{aligned} \tag{3.21}$$

We now use the quantity $I(t)$ to derive some convolution estimates. Let $f(t) = \left(\int_{\mathbb{R}} u_x^4dx\right)^{\frac{1}{2}}$. By (3.14), (3.20) and the definition of C_1, we get

$$-\frac{dI}{dt} = \frac{4}{3}\gamma\int_{\mathbb{R}} uu_x^3dx \le \frac{4}{3}|\gamma|\left(\int_{\mathbb{R}} u^2u_x^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} u_x^4\right)^{\frac{1}{2}} \le \frac{2\sqrt{2}}{3}|\gamma|\sqrt{\widetilde{H}_1[u_0]}f(t) = C_1f(t),$$

then

$$-I(t) \le C_1\int_0^t f(\tau)d\tau - I_0.$$

According to the definition of $I(t)$, we deduce

$$\frac{1}{3}f^2(t) \le C_1\int_0^t f(\tau)d\tau - I_0 + \int_{\mathbb{R}}\left(u^4+2u^2u_x^2+\frac{4}{3}\gamma u^2\right)dx.$$

This in turn implies that

$$f^2(t) \le 3C_1\int_0^t f(\tau)d\tau + 3C_2, \tag{3.22}$$

where we have used (3.21) and the definition of C_2. Denote the right-hand side of (3.22) by $g(t)$. It is then inferred from (3.22) that

$$g'(t) = 3C_1f(t) \le 3C_1g^{\frac{1}{2}}(t).$$

Integration over the interval $[0,t]$ gives

$$\sqrt{g(t)} \le \frac{3}{2}C_1t + \sqrt{g(0)} = \frac{3}{2}C_1t + \sqrt{3C_2},$$

so

$$f(t) \le \sqrt{g} \le \frac{3}{2}C_1t + \sqrt{3C_2}.$$

With the above inequality in hand, we can derive the following convolution estimates

$$\begin{aligned}\left|p_\pm * (u \mp u_x)^3\right| &\le \|p_\pm\|_\infty \left\|(u \mp u_x)^3\right\|_1 \le\ 2\left(\|u\|_3^3 + \|u_x\|_3^3\right)\\ &\le 2\left(\sqrt{\frac{\widetilde{H}_1^3[u_0]}{2}} + \|u_x\|_4^2\sqrt{\widetilde{H}_1[u_0]}\right)\\ &\le 2\left(\sqrt{\frac{\widetilde{H}_1^3[u_0]}{2}} + \sqrt{\widetilde{H}_1[u_0]}\left(\frac{3}{2}C_1 t + \sqrt{3C_2}\right)\right)\\ &=\ : K_2(t).\end{aligned} \tag{3.23}$$

We are now in the position to prove the blow-up result. In view of (3.6) and (3.23), it follows that

$$\begin{aligned}|u'(t)| &\le |\gamma||p * u_x| + \frac{1}{2}\left|p_+ * (u-u_x)^3 - p_- * (u+u_x)^3\right|\\ &\le |\gamma|\|p\|_2\|u_x\|_2 + K_2(t) \le\ \frac{1}{2}|\gamma|\|u\|_{H^1} + K_2(t)\\ &= \frac{1}{2}|\gamma|\sqrt{\widetilde{H}_1[u_0]} + K_2(t)\\ &=: 2C_3 t + C_4,\end{aligned} \tag{3.24}$$

where C_3 and C_4 are defined in (3.17). Integration over the time interval $[0,t]$ yields

$$u_0(x_3) - C_3t^2 - C_4 t \le \hat{u}(t) := u\big(t, q(t,x_3)\big) \le u_0(x_3) + C_3 t^2 + C_4 t, \tag{3.25}$$

so

$$\widehat{u}(t) > 0 \quad \text{for} \quad 0 \le t < T_1 := \frac{-C_4 + \sqrt{C_4^2 + 4C_3 u_0(x_3)}}{2C_3}.$$

Applying the convolution estimate (3.23) to the dynamics of u_x in (3.6), we have

$$\begin{aligned}\widehat{u_x}'(t) &\le \frac{\widehat{u}}{2}\left(\widehat{u}^2 - \widehat{u}_x^2\right) + |\gamma||u - p * u| + K_2(t)\\ &\le \frac{\widehat{u}}{2}\left(\widehat{u}^2 - \widehat{u}_x^2\right) + |\gamma|\left(\|u\|_\infty + \|p\|_1\|u\|_2\right) + K_2(t)\\ &\le \frac{\widehat{u}}{2}\left(\widehat{u}^2 - \widehat{u}_x^2\right) + |\gamma|\left(\frac{\sqrt{2}}{2}+1\right)\sqrt{\widetilde{H}_1[u_0]} + K_2(t).\end{aligned} \tag{3.26}$$

We now consider inequality (3.25) on the time interval $0 \le t \le T_2 = \frac{-C_4 + \sqrt{C_4^2 + 4C_3(1-\beta)u_0(x_3)}}{2C_3}$. It in turn implies that

$$0 < \beta u_0(x_3) \le \widehat{u}(t) \le (2-\beta)u_0(x_3).$$

Therefore, for $0 \le t \le T_2$, we deduce from (3.26) that

$$\begin{aligned}\widehat{u_x}'(t) &\le\ -\frac{\beta}{2}u_0(x_3)\widehat{u_x}^2 + \frac{1}{2}(2-\beta)^3 u_0^3(x_3) + |\gamma|\left(\frac{\sqrt{2}}{2}+1\right)\sqrt{\widetilde{H}_1[u_0]} + K_2(t)\\ &=:\ -\frac{\beta}{2}u_0(x_3)\widehat{u_x}^2 + K_3(t) \le -\frac{\beta}{2}u_0(x_3)\widehat{u_x}^2 + K_3(T_2).\end{aligned} \tag{3.27}$$

It is observed from (3.18) that $u_{0,x}(x_3) < -\sqrt{\frac{2K_3(T_2)}{\beta u_0(x_3)}}$ and $T_3 < T_2$. Then applying Lemma 3.3 to (3.27) implies $\widehat{u_x}(t) \to -\infty$ as $t \to T^*$, where T^* is estimated in (3.19). Finally notice that as $t \to T^*$, $\widehat{u}(t) \geq \beta u_0(x_3) > 0$. The blow-up of $\widehat{u_x}$ thus implies the blow-up of $\widehat{u u_x}$. This completes the proof of Theorem 3.2. $\square$

In the same spirit we can establish the following result for the special case $\gamma = 0$.

Corollary 3.1. *Let $\gamma = 0$, $u_0 \in H^s(\mathbb{R})$ for $s > 5/2$. Suppose there exist some $0 < \delta < 1$ and $\tilde{x}_3 \in \mathbb{R}$ such that $u_0(\tilde{x}_3) > 0$ and*

$$u_{0,x}(\tilde{x}_3) \leq -\frac{A+1}{A-1}\sqrt{\frac{(2-\delta)^3 u_0^3(\tilde{x}_3) + 2K_1}{\delta u_0(\tilde{x}_3)}}, \tag{3.28}$$

where K_1 is defined in (3.10) and

$$A = \exp\left[\frac{(1-\delta)u_0(\tilde{x}_3)\sqrt{\delta u_0(\tilde{x}_3)\left((2-\delta)^3 u_0^3(\tilde{x}_3) + 2K_1\right)}}{K_1}\right].$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq T_0 := \frac{1}{\sqrt{\delta u_0(\tilde{x}_3)[(2-\delta)^3 u_0^3(\tilde{x}_3) + 2K_1]}} \log\left(\frac{u_{0,x}(\tilde{x}_3) - \sqrt{\frac{(2-\delta)^3 u_0^3(\tilde{x}_3)+2K_1}{\delta u_0(\tilde{x}_3)}}}{u_{0,x}(\tilde{x}_3) + \sqrt{\frac{(2-\delta)^3 u_0^3(\tilde{x}_3)+2K_1}{\delta u_0(\tilde{x}_3)}}}\right). \tag{3.29}$$

Proof. Taking account of (3.7) and (3.12), we conclude that

$$u_0(\tilde{x}_3) - K_1 t \leq \widehat{u}(t) := u(t, q(t, \tilde{x}_3)) \leq u_0(\tilde{x}_3) + K_1 t. \tag{3.30}$$

This then implies that $\widehat{u}(t) > 0$ for $0 \leq t < T_+ := u_0(\tilde{x}_3)/K_1$. Using the convolution estimates in (3.12) to the dynamics (3.8), we have

$$\widehat{u_x}'(t) \leq \frac{\widehat{u}}{2}(\widehat{u}^2 - \widehat{u_x}^2) + K_1.$$

Consider the bounds (3.30) on the time interval $0 \leq t \leq (1-\delta)T_+$ we know that

$$0 < \delta u_0(\tilde{x}_3) \leq \widehat{u}(t) \leq (2-\delta)u_0(\tilde{x}_3). \tag{3.31}$$

Consequently,

$$\widehat{u_x}'(t) \leq -\frac{\delta}{2}u_0(\tilde{x}_3)\widehat{u_x}^2 + \left(\frac{(2-\delta)^3 u_0^3(\tilde{x}_3)}{2} + K_1\right). \tag{3.32}$$

Applying Lemma 3.3 to (3.32) we conclude that

$$\widehat{u_x}(t) \to -\infty \quad \text{as} \quad t \to T^*,$$

where T^* is estimated in (3.29), provided that

$$u_{0,x}(\tilde{x}_3) < -\sqrt{\frac{(2-\delta)^3 u_0^3(\tilde{x}_3) + 2K_1}{\delta u_0(\tilde{x}_3)}}, \text{ and } T_0 \leq (1-\delta)T_+.$$

Solving the above we obtain the condition on the initial data as given in (3.28). Finally notice that as $t \to T^*$, $\widehat{u}(t) \geq \delta u_0(\tilde{x}_3) > 0$, hence the blow-up of $\widehat{u_x}$ implies the blow-up of $\widehat{u u_x}$, which completes the proof of the corollary. $\square$

4. Blow-up for the gmCH equation

In this section we focus on the finite-time blow-up of waves for the gmCH equation (1.4). Similar to the proof of Theorem 6.2 in [27], we can establish the following blow-up criterion for the gmCH equation.

Lemma 4.1. *Suppose that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then the corresponding solution u to the initial value problem* (1.4) *blows up in finite time $T > 0$ if and only if*

$$\lim_{t\to T} \inf_{x\in\mathbb{R}} \left(k_1 m(t,x) + k_2\right) u_x(t,x) = -\infty.$$

The characteristics associated to the gmCH equation (1.4) is determined as follows

$$\begin{cases} \dfrac{dq(t,x)}{dt} = \left(k_1(u^2 - u_x^2) + k_2 u\right)(t, q(t,x)), \\ q(0,x) = x, \end{cases} \qquad x \in \mathbb{R}, \quad t \in [0,T). \tag{4.1}$$

4.1. **Dynamics along the characteristics.** We now compute the dynamics of some important quantities along the characteristic $q(t,x)$ associated to the gmCH equation. Denote $'$ to be the derivative $\partial_t + [k_1(u^2 - u_x^2) + k_2 u]\partial_x$ along the characteristic.

Lemma 4.2. *Let $u_0 \in H^s(\mathbb{R}), s \geq 3$. Then $u(t,q(t,x)), u_x(t,q(t,x))$ and $m(t,q(t,x))$ satisfy the following integro-differential equations*

$$\begin{aligned} u'(t) = &-\frac{2}{3}k_1 u_x^3 - \gamma p_x * u + \frac{k_1}{3}\left[p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right] \\ &- k_2 p_x * \left(u^2 + \frac{1}{2}u_x^2\right), \end{aligned} \tag{4.2}$$

$$\begin{aligned} u_x'(t) = {} & k_1\left(\frac{1}{3}u^3 - u u_x^2\right) + k_2\left(u^2 - \frac{1}{2}u_x^2\right) + \gamma(u - p * u) \\ &- \frac{k_1}{3}\left[p_+ * (u - u_x)^3 + p_- * (u + u_x)^3\right] - k_2 p * \left(u^2 + \frac{1}{2}u_x^2\right), \end{aligned} \tag{4.3}$$

$$m'(t) = -\left(2k_1 m^2 + 2k_2 m + \gamma\right) u_x. \tag{4.4}$$

Proof. In view of (1.4), we have

$$\begin{aligned} &(1 - \partial_x^2)\left[u_t + \left(k_1(u^2 - u_x^2) + k_2 u\right) u_x\right] \\ &\quad = k_1\left[-\left((u^2 - u_x^2)m\right)_x + u_x(u^2 - u_x^2) - \left(u_{xx}(u^2 - u_x^2)\right)_x - 2\left(u_x^2 m\right)_x\right] \\ &\qquad + k_2[-2u_x m - u m_x + u u_x - \partial_x(u_x^2 + u u_{xx})] - \gamma u_x \\ &\quad = -k_1\left(2m u u_x + 2(u_x^2 m)_x\right) - k_2(2u u_x + u_x u_{xx}) - \gamma u_x, \end{aligned}$$

which implies that

$$u'(t) = -\gamma p * u_x - k_1 p * \left(2m u u_x + 2\left(u_x^2 m\right)_x\right) - k_2 p * (2u u_x + u_x u_{xx}). \tag{4.5}$$

Plugging the identity (see (3.6) in [6])

$$\begin{aligned} 2p * (m u u_x) + 2p * \left(\left(u_x^2 m\right)_x\right) &= 2p * (m u u_x) + 2p_x * \left(u_x^2 m\right) \\ &= \frac{2}{3}u_x^3 - \frac{1}{3}\left(p_+ * (u - u_x)^3 - p_- * (u + u_x)^3\right) \end{aligned}$$

into (4.5), we obtain (4.2). Next we calculate u_x'. Differentiating (4.5) with respect to x, we have

$$\begin{aligned} u_x' = & -\gamma p_x * u_x - k_1\left[\left(u^2-u_x^2\right)_x u_x + p_x * \left(2muu_x + 2\left(u_x^2 m\right)_x\right)\right] \\ & - k_2\left[u_x^2 + p_x * (2uu_x + u_x u_{xx})\right] \\ = & \gamma(u - p*u) - 2k_1\left[p_x * (muu_x) + p * \left(u_x^2 m\right)\right] \\ & - k_2\left[\frac{1}{2}u_x^2 - u^2 + p * \left(u^2 + \frac{1}{2}u_x^2\right)\right]. \end{aligned} \tag{4.6}$$

Plugging the identity (see (3.7) in [6])

$$2p_x * (muu_x) + 2p * \left(u_x^2 m\right) = uu_x^2 - \frac{1}{3}u^3 + \frac{1}{3}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right],$$

we obtain (4.3). Finally our attention is turned to the estimate of m. On account of (1.4), we deduce that

$$\begin{aligned} m'(t) &= m_t + [k_1(u^2-u_x^2) + k_2 u]m_x \\ &= k_1\left[(u^2-u_x^2)m_x - \left((u^2-u_x^2)m\right)_x\right] + k_2[um_x - 2u_x m - um_x] - \gamma u_x \\ &= -2k_1 u_x m^2 - 2k_2 u_x m - \gamma u_x, \end{aligned}$$

thereby concluding the proof of Lemma 4.2. □

4.2. **Non-sign-changing momentum.** In this subsection, we derive some sufficient conditions for the blow-up of the initial-value problem (1.4) when the parameter $\gamma = 0$. The following lemma shows that, if $m_0 = (1-\partial_x^2)u_0$ does not change sign, then $m(t,x)$ will not change sign for any $t \in [0,T)$. This conservative property of the momentum m will be crucial in the proof of our blow-up result.

Lemma 4.3. *Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, and let $T > 0$ be the maximal existence time of the corresponding strong solution u to (1.4). Then (4.1) has a unique solution $q \in C^1([0,T)\times\mathbb{R},\mathbb{R})$ such that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t \left(2k_1 m + k_2\right)u_x(s,q(s,x))ds\right) > 0,\ \forall\ (t,x) \in [0,T)\times\mathbb{R}. \tag{4.7}$$

Furthermore, for all $(t,x) \in [0,T)\times\mathbb{R}$ it holds that

$$m\big(t,q(t,x)\big) = m_0(x)\exp\left(-2\int_0^t (k_1 m + k_2)u_x\big(s,q(s,x)\big)ds\right). \tag{4.8}$$

Proof. Since $u \in C^1\left([0,T), H^{s-1}(\mathbb{R})\right)$ and $H^s(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, both $u(t,x)$ and $u_x(t,x)$ are bounded, Lipschitz in the space variable x, and of class C^1 in time. Therefore, by the well-known classical results in the theory of ordinary differential equations, the initial value problem (4.1) has a unique solution $q(t,x) \in C^1\left([0,T)\times\mathbb{R}\right)$.

Differentiating (4.1) with respect to x yields

$$\begin{cases} \dfrac{d}{dt}q_x = \big(2k_1 m + k_2\big)u_x(t,q)q_x, & x\in\mathbb{R}, \quad t\in[0,T). \\ q_x(0,x) = 1, & \end{cases} \tag{4.9}$$

The solution to (4.9) is given by (4.7). For every $T' < T$, it follows from the Sobolev embedding that

$$\sup_{(s,x)\in[0,T')\times\mathbb{R}} |\big(2k_1 m + k_2\big)u_x(s,x)| < \infty.$$

It is inferred from (4.7) that there exists a constant $K > 0$ such that $q_x(t,x) \geq e^{-Kt}$, $(t,x) \in [0,T)\times\mathbb{R}$, which implies that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ before the blow-up time.

On the other hand, by (1.4) and (4.1), we have

$$\begin{aligned}\frac{d}{dt}m\big(t,q(t,x)\big) =& \ (m_t + m_x q_t)\left(t,q(t,x)\right)\\ =& \ m_t\big(t,q(t,x)\big) + \left[k_1\left(u^2-u_x^2\right)+k_2 u\right] m_x\big(t,q(t,x)\big)\\ =& \ -2\left(k_1 m u_x + k_2 u_x\right) m\big(t,q(t,x)\big).\end{aligned}$$

Therefore, solving the equation with regard to $m\big(t,q(t,x)\big)$ leads to (4.8). This completes the proof of Lemma 4.3. $\square$

We now state the following result on the blow-up for a non-changing-sign momentum.

Theorem 4.1. *Let $k_1 > 0$, $k_2 \geq 0$, $u_0 \in H^s(\mathbb{R})$ for $s > \frac{5}{2}$ and $m_0 \geq 0$. Suppose that there exists a point $x_4 \in \mathbb{R}$ such that*

$$m_0(x_4) > 0 \quad \textit{and} \quad u_{0,x}(x_4) < -\frac{1}{\sqrt{2}}\left(u_0(x_4)+\frac{3k_2}{2k_1}\right). \tag{4.10}$$

Then the corresponding solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1 m_0(x_4) u_{0,x}(x_4)}.$$

Proof. As before, we will trace the dynamics along the characteristics emanating from x_4. Denote

$$\begin{aligned}&\widehat{u}(t) = u\left(t,q(t,x_4)\right), \widehat{u_x}(t) = u_x\left(t,q(t,x_4)\right),\\ &\widehat{m}(t) = m\left(t,q(t,x_4)\right), \widehat{M}(t) = \left(m u_x\right)\left(t,q(t,x_4)\right).\end{aligned}$$

Since we know that $m_0 \geq 0$, in particular, $m_0(x_4) > 0$, so from (4.8), we know that $m(t,x) \geq 0$ and $\widehat{m}(t) > 0$. Therefore from the identities

$$u(t,x) = p * m(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} m(t,y)dy, \quad u_x(t,x) = p_x * m(t,x),$$

we have $u(t,x) \geq 0$ and $\widehat{u}(t) > 0$. Moreover

$$u - u_x = 2p_+ * m \geq 0, \quad u + u_x = 2p_- * m \geq 0. \tag{4.11}$$

Hence we know $|u_x(t,x)| \leq u(t,x)$. Therefore u_x does not blow up. From the blow-up criterion in Lemma 4.1, it suffices to consider the quantity $M = m u_x$. Using (4.3) and (4.4), a simple calculation then gives

$$\begin{aligned}\widehat{M}'(t) =& \ -2k_1\widehat{M}^2 + \frac{k_1}{3}\widehat{m}\widehat{u}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right) + k_2\left(\widehat{m}\widehat{u}^2 - \frac{5}{2}\widehat{m}\widehat{u_x}^2\right)\\ & -\frac{k_1}{3}\widehat{m}\left[p_+ * (u-u_x)^3 + p_- * (u+u_x)^3\right] - k_2\widehat{m}p * \left(u^2+\frac{1}{2}u_x^2\right).\end{aligned}$$

Taking account of (4.11) and the inequality $p * \left(u^2 + \frac{1}{2}u_x^2\right) \geq \frac{1}{2}u^2$, it is then adduced that

$$\widehat{M}'(t) \leq -2k_1\widehat{M}^2 + \frac{k_1}{3}\widehat{m}\widehat{u}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right) + \frac{k_2}{2}\widehat{m}\left(\widehat{u}^2 - 5\widehat{u_x}^2\right).$$

Our argument is to find certain conditions on the initial data under which there holds the Riccati-like inequality $\widehat{M}'(t) \leq -C\widehat{M}^2$. Thus from the sign conditions $\widehat{u}(t) > 0$ and $\widehat{m}(t) > 0$, we would like to have $\widehat{u}^2 - 3\widehat{u_x}^2 \leq 0$, and this would also imply that $\widehat{u}^2 - 5\widehat{u_x}^2 \leq 0$. That is to say, it suffices to recognize finite-time blow-up of $\widehat{M}(t)$ if the ratio $|u_x/u|$ stays big along the characteristics. This suggests us to trace the dynamics of $\widehat{u_x}/\widehat{u}$. However, due to the inhomogeneity of the nonlinearities, one can only show that

$$\left(\frac{\widehat{u_x}}{\widehat{u}}\right)'(t) \leq C_1\left(\widehat{u}^2 - 2\widehat{u_x}^2 + C_2\widehat{u}\right).$$

Thus a large (negative) ratio $\widehat{u_x}/\widehat{u}$ is not enough to make the right-hand side negative. The way to resolve this is to absorb the linear term $C_2\widehat{u}$ into the quadratic one by replacing $\widehat{u}$ by a vertical shift $\widehat{u} + a$. Therefore instead, we will track the dynamics of $\widehat{u_x}/(\widehat{u} + a)$ along the characteristics, where $a \geq 0$ will be chosen later.

$$\begin{aligned}
\left(\frac{\widehat{u_x}}{\widehat{u}+a}\right)'(t) = {} & \frac{k_1}{3(\widehat{u}+a)^2}\left(\widehat{u}^2 - \widehat{u_x}^2\right)\left(\widehat{u}^2 - 2\widehat{u_x}^2\right) + \frac{k_2}{(\widehat{u}+a)^2}\left(\widehat{u}^3 - \frac{1}{2}\widehat{u}\widehat{u_x}^2\right) \\
& + \frac{ak_1\widehat{u}}{3(\widehat{u}+a)^2}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right) + \frac{ak_2}{2(\widehat{u}+a)^2}\left(2\widehat{u}^2 - \widehat{u_x}^2\right) \\
& - \frac{k_1}{3(\widehat{u}+a)^2}\left(\widehat{u} + \widehat{u_x} + a\right) p_+ * (u - u_x)^3\big(t, q(t, x_4)\big) \\
& - \frac{k_1}{3(\widehat{u}+a)^2}\left(\widehat{u} - \widehat{u_x} + a\right) p_- * (u + u_x)^3\big(t, q(t, x_4)\big) \\
& - \frac{k_2}{(\widehat{u}+a)^2}\left(\widehat{u} + \widehat{u_x} + a\right) p_+ * \left(u^2 + \frac{1}{2}u_x^2\right)\big(t, q(t, x_4)\big) \\
& - \frac{k_2}{(\widehat{u}+a)^2}\left(\widehat{u} - \widehat{u_x} + a\right) p_- * \left(u^2 + \frac{1}{2}u_x^2\right)\big(t, q(t, x_4)\big).
\end{aligned}$$

Regrouping the terms and using the fact that $p_\pm * \left(u^2 + \frac{1}{2}u_x^2\right) \geq \frac{1}{4}u^2$ we have

$$\begin{aligned}
\left(\frac{\widehat{u_x}}{\widehat{u}+a}\right)'(t) \leq {} & \frac{k_1}{3(\widehat{u}+a)^2}\left(\widehat{u}^2 - \widehat{u_x}^2\right)\left(\widehat{u}^2 - 2\widehat{u_x}^2\right) + \frac{k_2}{2(\widehat{u}+a)^2}\left(\widehat{u}^3 - \widehat{u}\widehat{u_x}^2\right) \\
& + \frac{ak_1\widehat{u}}{3(\widehat{u}+a)^2}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right) + \frac{ak_2}{2(\widehat{u}+a)^2}\left(\widehat{u}^2 - \widehat{u_x}^2\right) \\
= {} & \frac{k_1}{3(\widehat{u}+a)^2}\left[\left(\widehat{u}^2 - \widehat{u_x}^2\right)\left(\widehat{u}^2 - 2\widehat{u_x}^2 + \frac{3k_2}{2k_1}\widehat{u} + \frac{3ak_2}{2k_1}\right) + a\widehat{u}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right)\right] \\
\leq {} & \frac{k_1}{3(\widehat{u}+a)^2}\left\{\left(\widehat{u}^2 - \widehat{u_x}^2\right)\left[\left(\widehat{u} + \frac{3k_2}{2k_1}\right)^2 - 2\widehat{u_x}^2 + \frac{3k_2}{2k_1}\left(a - \frac{3k_2}{2k_1}\right)\right]\right. \\
& \left. + a\widehat{u}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right)\right\}.
\end{aligned}$$

So now choose $a = \frac{3k_2}{2k_1} \geq 0$, then the above reads

$$\left(\frac{\widehat{u_x}}{\widehat{u}+a}\right)'(t) \leq \frac{k_1}{3(\widehat{u}+a)^2}\left\{\left(\widehat{u}^2 - \widehat{u_x}^2\right)\left[\left(\widehat{u}+\frac{3k_2}{2k_1}\right)^2 - 2\widehat{u_x}^2\right] + a\widehat{u}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right)\right\}.$$

By condition (4.10), we have chosen the initial data such that

$$\widehat{u_x}(0) < -\frac{1}{\sqrt{2}}\left(\widehat{u}(0) + \frac{3k_2}{2k_1}\right) \leq -\frac{1}{\sqrt{2}}\widehat{u}(0) < -\frac{1}{\sqrt{3}}\widehat{u}(0).$$

From which we see that the right-hand side of the above is negative initially. Hence $\widehat{u_x}/\left(\widehat{u}+\frac{3k_2}{2k_1}\right)$ decreases, and thus

$$\frac{\widehat{u_x}}{\widehat{u}+\frac{3k_2}{2k_1}}(t) < \frac{\widehat{u_x}}{\widehat{u}+\frac{3k_2}{2k_1}}(0) < -\frac{1}{\sqrt{2}}.$$

So $\widehat{u}(t) + \frac{3k_2}{2k_1} + \sqrt{2}\widehat{u_x}(t) < 0$. Therefore

$$\widehat{u}(t) + \sqrt{3}\widehat{u_x}(t) < 0 \text{ and } \widehat{u}(t) + \sqrt{5}\widehat{u_x}(t) < 0. \tag{4.12}$$

Meanwhile we also have

$$\widehat{u}(t) - \sqrt{3}\widehat{u_x}(t) > 0 \quad \text{and} \quad \widehat{u}(t) - \sqrt{5}\widehat{u_x}(t) > 0. \tag{4.13}$$

Now plugging (4.12) and (4.13) into (4.2) we obtain

$$\widehat{M}'(t) \leq -2k_1\widehat{M}^2 + \frac{k_1}{3}\widehat{m}\widehat{u}\left(\widehat{u}^2 - 3\widehat{u_x}^2\right) + \frac{k_2}{2}\widehat{m}\left(\widehat{u}^2 - 5\widehat{u_x}^2\right) \leq -2k_1\widehat{M}^2,$$

and hence $\widehat{M}(t)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq -\frac{1}{2k_1\widehat{M}(0)} = -\frac{1}{2k_1 m_0(x_4)u_{0,x}(x_4)},$$

This completes the proof of Theorem 4.1. □

4.3. **Blow-up for a general momentum.** In this subsection we turn our attention to the general case where γ needs not equal to zero, and therefore there is no sign-preservation for the momentum density as in the previous case.

We first derive some convolution estimates for later use. Our assumptions for the parameters throughout this subsection are $k_1 > 0,\ k_2 \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. From the conservation laws (1.5) we infer that

$$\begin{aligned}
\|u_x\|_4^4 &= 3\int_{\mathbb{R}}\left(u^4 + 2u^2u_x^2\right)dx + \frac{6k_2}{k_1}\int_{\mathbb{R}}\left(u^3 + uu_x^2\right)dx \\
&\quad + \frac{6\gamma}{k_1}\int_{\mathbb{R}} u^2 dx - \frac{3}{k_1}H_2[u_0] \\
&\leq \left(4\|u\|_\infty^2 + \left|\frac{6k_2}{k_1}\right|\|u\|_\infty + \left|\frac{6\gamma}{k_1}\right|\right)H_1[u_0] - \frac{3}{k_1}H_2[u_0] \\
&\leq \left(2H_1[u_0] + 3\sqrt{2}\left|\frac{k_2}{k_1}\right|H_1^{\frac{1}{2}}[u_0] + \left|\frac{6\gamma}{k_1}\right|\right)H_1[u_0] - \frac{3}{k_1}H_2[u_0]
\end{aligned} \tag{4.14}$$

and this in turn gives the convolution estimates

$$
\begin{aligned}
&|p_\pm * (u \mp u_x)^3| \le \|p_\pm\|_\infty \| (u \mp u_x)^3 \|_1 \le 2\left(\|u\|_3^3 + \|u_x\|_3^3\right) \\
&\le\ 2\left[\left(2H_1[u_0] + 3\sqrt{2}\left|\frac{k_2}{k_1}\right| H_1^{\frac{1}{2}}[u_0] + \left|\frac{6\gamma}{k_1}\right|\right) H_1^2[u_0] - \frac{3}{k_1}H_1[u_0]H_2[u_0]\right]^{\frac{1}{2}} \\
&\quad + \sqrt{2}H_1^{\frac{3}{2}}[u_0] =:\ Q.
\end{aligned}
\tag{4.15}
$$

We can further utilize the conservation of $H_1[u_0]$ to bound

$$
\begin{aligned}
|p_x * u| &= |(p_- - p_+) * u| \le (\|p_-\|_1 + \|p_+\|_1)\|u\|_\infty \\
&\le \frac{1}{\sqrt{2}}\|u\|_{H^1} = \sqrt{\frac{H_1[u_0]}{2}}, \\
|u - p * u| &\le 2\|u\|_\infty \le \frac{2}{\sqrt{2}}\|u\|_{H^1} = \sqrt{2H_1[u_0]}. \\
\left|p * \left(u^2 + \frac{1}{2}u_x^2\right)\right| &\le \|p\|_\infty \left\|u^2 + \frac{1}{2}u_x^2\right\|_1 \le \frac{1}{2}H_1[u_0].
\end{aligned}
\tag{4.16}
$$

Our blow-up theorem for the gmCH equation (1.4) with general γ can be stated as follows.

Theorem 4.2. *Let $\gamma \in \mathbb{R}$, $k_1 > 0$ and $u_0 \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Assume that there exists an $x_5 \in \mathbb{R}$ such that*

$$
\begin{aligned}
&u_0(x_5) > \max\left\{0, -\frac{k_2}{2k_1}\right\}, \quad m_0(x_5) > \max\{\alpha, 0\}, \\
&u_{0,x}(x_5) \le \min\left\{-(A_1)^{\frac{1}{3}}, -\sqrt{\frac{A_2}{k_1 u_0(x_5) + \frac{k_2}{2}}}\right\},
\end{aligned}
\tag{4.17}
$$

where

$$
\begin{aligned}
A_1 &= Q + \frac{3|\gamma|}{2k_1}\sqrt{\frac{H_1[u_0]}{2}} + \frac{3|k_2|}{4k_1}H_1[u_0], \\
A_2 &= \frac{2}{3}k_1 Q + |\gamma|\sqrt{2H_1[u_0]} + |k_2|H_1[u_0] + \frac{k_1}{6\sqrt{2}}H_1^{\frac{3}{2}}[u_0],
\end{aligned}
$$

with Q defined in (4.15), and

$$
\alpha = \begin{cases} \dfrac{-k_2 + \sqrt{\Delta}}{2k_1}, & \text{if } \Delta = (k_2^2 - 2k_1\gamma) > 0, \\ -\dfrac{k_2}{2k_1}, & \text{if } \Delta = (k_2^2 - 2k_1\gamma) \le 0. \end{cases}
$$

Then the solution $u(t,x)$ blows up in finite time with an estimate of the blow-up time T^* as

$$T^* \leq \begin{cases} -\dfrac{1}{2u_{0,x}(x_5)\sqrt{\Delta}} \log\left(\dfrac{m_0(x_5)+\frac{k_2+\sqrt{\Delta}}{2k_1}}{m_0(x_5)+\frac{k_2-\sqrt{\Delta}}{2k_1}}\right), & \textit{if} \ \ \Delta>0, \\ -\dfrac{1}{2k_1 u_{0,x}(x_5)\left(m_0(x_5)+\frac{k_2}{2k_1}\right)}, & \textit{if} \ \ \Delta=0, \\ -\dfrac{\frac{\pi}{2}-\arctan\dfrac{2k_1 m_0(x_5)+k_2}{\sqrt{-\Delta}}}{u_{0,x}(x_5)\sqrt{-\Delta}}, & \textit{if} \ \ \Delta<0. \end{cases} \tag{4.18}$$

Proof. Plugging the estimates (4.15)-(4.16) into (4.2) and (4.3), we deduce that

$$\begin{aligned} \widehat{u}'(t) &= u'\left(t, q(t,x_5)\right) \geq -\frac{2}{3}k_1\widehat{u_x}^3 - |\gamma|\sqrt{\frac{H_1[u_0]}{2}} - \frac{2}{3}k_1 Q - \frac{1}{2}|k_2|H_1[u_0], \\ \widehat{u_x}'(t) &= u_x'\left(t, q(t,x_5)\right) \leq -\left(k_1\widehat{u}+\frac{1}{2}k_2\right)\widehat{u_x}^2 + \frac{2}{3}k_1 Q + |\gamma|\sqrt{2H_1[u_0]} \\ &\quad + |k_2|H_1[u_0] + \frac{k_1}{6\sqrt{2}}H_1^{\frac{3}{2}}[u_0]. \end{aligned}$$

So we know that $\widehat{u}(t)$ is increasing when

$$\widehat{u_x}^3(t) \leq -\left(Q + \frac{3|\gamma|}{2k_1}\sqrt{\frac{H_1[u_0]}{2}} + \frac{3|k_2|}{4k_1}H_1[u_0]\right) =: -A_1$$

and $\widehat{u_x}(t)$ is decreasing when

$$\left(k_1\widehat{u}+\frac{1}{2}k_2\right)\widehat{u_x}^2 \geq \frac{2}{3}k_1 Q + |\gamma|\sqrt{2H_1[u_0]} + |k_2|H_1[u_0] + \frac{k_1}{6\sqrt{2}}H_1^{\frac{3}{2}}[u_0] =: A_2.$$

Hence from the assumption on the initial data (4.17), we know that over the time of existence of solutions $\widehat{u}(t)$ is increasing and $\widehat{u_x}(t)$ is decreasing. In particular,

$$\widehat{u}(t) \geq u_0(x_5) > 0, \quad \widehat{u_x}(t) \leq u_{0,x}(x_5) < 0. \tag{4.19}$$

By the assumption (4.17) on $m_0(x_5)$, we know that

$$2k_1 m_0^2(x_5) + 2k_2 m_0(x_5) + \gamma > 0.$$

Using (4.19), we deduce from (4.4) that over the time of solutions $m(t)$ is increasing, so $m(t) > m_0(x_5)$ and

$$\begin{aligned} \widehat{m}'(t) &:= m'\left(t, q(t,x_5)\right) = -\widehat{u_x}(t)\left(2k_1\widehat{m}^2(t) + 2k_2\widehat{m}(t) + \gamma\right) \\ &\geq \ -u_{0,x}(x_5)\left(2k_1\widehat{m}^2(t) + 2k_2\widehat{m}(t) + \gamma\right) > \ 0. \end{aligned} \tag{4.20}$$

We show that blow-up must occur in finite time. The proof is divided into three cases.
Case 1: $\Delta > 0$. Let y_1 and y_2 be the two distinct real roots of the equation $2k_1y^2 + 2k_2y + \gamma = 0$, so

$$y_{1,2} = \frac{-k_2 \mp \sqrt{\Delta}}{2k_1}, \quad y_1 < y_2,$$
$$2k_1y^2 + 2k_2y + \gamma = 2k_1(y-y_1)(y-y_2).$$

Integrating inequality (4.20) over $[0,t]$ gives

$$\widehat{m}(t) \geq \frac{y_2 - y_1 E(t)}{1-E(t)} \to +\infty, \quad \text{as} \ \ t \to -\frac{1}{2k_1 u_{0,x}(x_5)(y_2-y_1)} \log\left(\frac{m_0(x_5)-y_1}{m_0(x_5)-y_2}\right),$$

where

$$E(t) = \frac{m_0(x_5)-y_2}{m_0(x_5)-y_1} \exp\Big(-2k_1 u_{0,x}(x_5)(y_2-y_1)t\Big).$$

Notice that $\widehat{u}_x(t) \leq u_{0,x}(x_5) < 0$. It is then deduced that in this case

$$(k_1 m + k_2) u_x\big(t, q(t,x_5)\big) \to -\infty, \quad \text{as} \ \ t \to -\frac{1}{2k_1 u_{0,x}(x_5)(y_2-y_1)} \log\left(\frac{m_0(x_5)-y_1}{m_0(x_5)-y_2}\right).$$

Case 2: $\Delta = 0$. In this case the equation $2k_1y^2 + 2k_2 y + \gamma = 0$ has the unique real root $y = -\frac{k_2}{2k_1}$, and so

$$2k_1 y^2 + 2k_2 y + \gamma = 2k_1\left(y + \frac{k_2}{2k_1}\right)^2.$$

Integrating inequality (4.20) over the interval $[0,t]$ gives

$$\widehat{m}(t) \geq \frac{m_0(x_5) + \frac{k_2}{2k_1}}{2k_1 u_{0,x}(x_5)\left(m_0(x_5)+\frac{k_2}{2k_1}\right)t+1} - \frac{k_2}{2k_1} \to +\infty,$$

$$\text{as} \ \ t \to -\frac{1}{2k_1 u_{0,x}(x_5)\left(m_0(x_5)+\frac{k_2}{2k_1}\right)},$$

which, similar to Case 1, implies that

$$(k_1 m + k_2) u_x\big(t, q(t,x_5)\big) \to -\infty, \quad \text{as} \ \ t \to -\frac{1}{2k_1 u_{0,x}(x_5)\left(m_0(x_5)+\frac{k_2}{2k_1}\right)}.$$

Case 3: $\Delta < 0$. In this case we have

$$2k_1 y^2 + 2k_2 y + \gamma = 2k_1\left(m + \frac{k_2}{2k_1}\right)^2 - \frac{\Delta}{2k_1}.$$

Integrating inequality (4.20) over the interval $[0,t]$ gives

$$\widehat{m}(t) \geq -\frac{k_2}{2k_1} + \frac{\sqrt{-\Delta}}{2k_1} \tan\left[-u_{0,x}(x_5)\sqrt{-\Delta}t + \arctan\frac{2k_1\left(m_0(x_5)+\frac{k_2}{2k_1}\right)}{\sqrt{-\Delta}}\right],$$

which goes to $+\infty$ as

$$t \to \frac{\frac{\pi}{2} - \arctan\frac{2k_1 m_0(x_5)+k_2}{\sqrt{-\Delta}}}{-u_{0,x}(x_5)\sqrt{-\Delta}}.$$

Hence, we deduce that

$$(k_1 m + k_2) u_x\big(t, q(t,x_5)\big) \to -\infty, \quad \text{as} \ \ t \to \frac{\frac{\pi}{2} - \arctan\frac{2k_1 m_0(x_5)+k_2}{\sqrt{-\Delta}}}{-u_{0,x}(x_5)\sqrt{-\Delta}}.$$

This completes the proof of Theorem 4.2. □

Following the proof of Theorem 4.2, we can deal with the case of $\gamma = 0$ and $k_2 < 0$. Indeed, we have the following result, which can not be obtained from Theorem 4.1 .

Corollary 4.1. *Let* $k_1 > 0,\ k_2 < 0,\ \gamma = 0$ *and* $u_0 \in H^s(\mathbb{R})$ *with* $s > \frac{5}{2}$ *and* $m_0(x) \geq 0, \forall x \in \mathbb{R}$. *Assume that there exists a point* $x_6 \in \mathbb{R}$ *such that*

$$
\begin{aligned}
& u_0(x_6) > -\frac{k_2}{2k_1}, \quad m_0(x_6) > -\frac{k_2}{k_1}, \\
& u_{0,x}(x_6) \leq \min\left\{ -(A_3)^{\frac{1}{3}}, -\sqrt{\frac{A_4}{k_1 u_0(x_6) + \frac{k_2}{2}}} \right\},
\end{aligned}
$$

where

$$
A_3 = \frac{Q}{2} - \frac{3k_2}{4k_1} H_1[u_0], \quad A_4 = \frac{k_1}{6\sqrt{2}} H_1^{3/2}[u_0] - \frac{k_2}{2} H_1[u_0],
$$

and Q *is defined by* (4.15). *Then the solution* $u(t,x)$ *blows up in finite time with an estimate of the blow-up time* T^* *as*

$$
T^* \leq \frac{1}{2k_2 u_{0,x}(x_6)} \log \frac{m_0(x_6)}{m_0(x_6) + \frac{k_2}{k_1}}.
$$

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